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BIEQUIVALENCES IN TRICATEGORIES

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Abstract. We show that every internal biequivalence in a tricategory T is part of a biadjoint biequivalence. We give two applications of this result, one for transporting monoidal structures and one for equipping a monoidal bicategory with invertible objects with a coherent choice of those inverses.

Introduction

It is common in mathematics to regard two objects X and Y as being the same if there is an isomorphism between them, that is a pair of maps $f : X \to Y$ and $g : Y \to X$ such that $fg = 1_Y$ and $gf = 1_X$. More importantly, any specific isomorphism $f : X \to Y$ gives an explicit means for transporting information about X to information about Y. The choice of ambient category plays a very important role in this process, especially when we examine two categories with the same objects but different morphisms. For example, first take the category of CW-complexes and continuous maps. In this category, an isomorphism between X and Y gives a formula for transporting cell structures from X to Y. If we now take the category of CW-complexes and homotopy classes of continuous maps, then isomorphisms are now homotopy equivalences. An isomorphism then no longer gives a recipe for transporting cell structures, but instead only gives a recipe for transporting homotopical information like homotopy and homology groups.

An important property of an isomorphism $f : X \to Y$ is that it uniquely determines the inverse $g : Y \to X$ by the formulas $fg = 1_Y$ and $gf = 1_X$; the proof is exactly the same as that showing that any group element has a unique inverse. Thus we have two concepts which are *a priori* different:

- the property that $f : X \to Y$ is an isomorphism, and
- the structure consisting of a pair (f, g) of morphisms $f : X \to Y$, $g : Y \to X$ such that $fg = 1_Y$ and $gf = 1_X$.

We have then uncovered that a morphism f has the property of being an isomorphism if and only if there is a pair (f, g) with the isomorphism structure, and that moreover the pair (f, g) is uniquely determined by f alone. This is an example of a structure (being part of an isomorphism pair) that is determined in a unique way by a property (the existence of an inverse).

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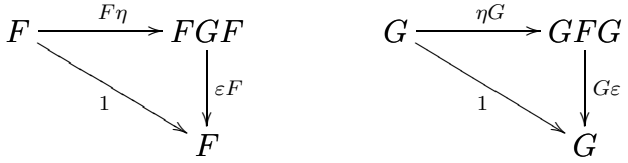
Moving up to the case in which the objects of study are now the 0-cells of some 2-category (or more generally, some bicategory), there are now more possible notions of sameness. While we can ask if two 0-cells of a 2-category are isomorphic, it is much more common to ask if they are equivalent. The canonical example of a 2-category is that of categories, functors, and natural transformations, and in this 2-category we see from experience that equivalence is the natural notion of sameness.

A functor $F : X \to Y$ is often defined to be an equivalence if it is essentially surjective, full, and faithful. It is then shown that a functor F is an equivalence if and only if there exists a functor $G : Y \to X$ such that the composites FG, GF are naturally isomorphic to the identity functors $1_Y, 1_X$, respectively. This definition identifies a property of a functor F that allows us to conclude that two categories X and Y are, in some sense, the same.

On the other hand, we can make the definition of the structure of adjoint equivalence $F \dashv_{eq} G$ which consists of a functor $F : X \to Y$, a functor $G : Y \to X$, and two natural isomorphisms $\eta : 1_X \Rightarrow GF, \varepsilon : FG \Rightarrow 1_Y$ such that the following two diagrams commute.



It is clear that if $F \dashv_{eq} G$ is an adjoint equivalence, then F is an equivalence. It is also well-known [Mac] that every equivalence F can be completed to an adjoint equivalence. Now F no longer determines G, η, ε uniquely, but instead only determines them up to a unique isomorphism preserving the adjoint equivalence structure. We refer the reader to the paper [KL] for a general discussion of this phenomenon in the setting of algebras for a 2-monad.

The aim of this paper is to establish an analogous result in three-dimensional category theory. We define two different notions of sameness internal to a tricategory, one a property of a 1-cell and the other a structure involving 1-, 2-, and 3-cells satisfying certain axioms. These two notions are that of a 1-cell having the property "is a biequivalence" on the one hand, and the structure of a biadjoint biequivalence on the other hand. Our main result is that, in any tricategory T, every 1-cell which is a biequivalence is part of a biadjoint biequivalence.

The proof of this result proceeds in three steps. First, we show that it is true for the special case when $T = \mathbf{Bicat}$, the tricategory of bicategories, functors, pseudo-natural transformations, and modifications. Second, we prove a result about transporting biadjoint biequivalences; more precisely, we show that if a functor $F : S \to T$ satisfies a kind of local embedding condition and T has the property that every biequivalence is part of a biadjoint biequivalence, then S has that property as well. Finally, we prove that this property is inherited by functor tricategories from the target, so that if T is a tricategory in which every biequivalence is part of a biadjoint biequivalence then the same holds for the functor tricategory $\mathbf{Tricat}(S, T)$ for any S. The main result then follows from these theorems and the Yoneda embedding for cubical tricategories.

It should be noted that the Gray-categorical analogue of this result appears in [L2], and that the full result can then be deduced by coherence for tricategories. The purpose of this paper, though, is not merely to prove this theorem, but to show that its proof can be deduced from the Yoneda Lemma together with some computations in the most obvious case. When considering similar theorems for higher dimensional categories, it might be possible to give a Yoneda embedding without necessarily having a good coherence theorem; such a scenario appears entirely plausible in the case of tetracategories.

We also give two applications of this result. The first is a transport-of-structure result, showing how biequivalences $F : X \to Y$ between bicategories can be used to transport monoidal structures from X to Y. This relies on choosing a weak inverse $G : Y \to X$ for the definition of the tensor product on Y, and then requires the rest of the biadjoint biequivalence structure in order to define the higher cells that are part of the definition of a monoidal bicategory and to check that they satisfy the necessary axioms. We also indicate how to prove similar results when the monoidal structure is replaced with a braided monoidal, sylleptic monoidal, or symmetric monoidal one.

The second application of the main result is to an elucidation of those monoidal bicategories in which every object is weakly invertible, called Picard 2-categories here. We show that every Picard 2-category is monoidally biequivalent to one in which a coherent choice of inverses has been made. In fact, the result is much stronger in that we show that the forgetful functor from coherent Picard 2-categories (those with a choice of inverses) to Picard 2-categories (those monoidal bicategories which merely have the property that every object is invertible) is a triequivalence. We go on to improve this result by defining functors that preserve a given coherent structure up to equivalence, and show that every monoidal functor between Picard 2-categories can be given the structure of such.

The paper is organized as follows. The first section is a warm-up in which we give a proof of the fact that every equivalence in a bicategory is part of an adjoint equivalence. We do this to give the reader a taste of the strategy that will be used later in the tricategorical case so as to clearly indicate the crucial points. This material is well-known although I am unaware of a reference that presents this result in full detail using the argument we give below.

The second section gives the definitions of biequivalence and biadjoint biequivalence that are at the heart of this paper. Both of these we express in the completely general case, working in an arbitrary tricategory T. The definition of biadjoint biequivalence has two forms, with and without the "horizontal cusp" axioms, and we discuss briefly why these two definitions are logically equivalent using the calculus of mates.

Section 3 gives a proof of our main result in the special case where $T = \mathbf{Bicat}$. This proof is largely calculation, much as the proof that every equivalence in **Cat** is part of an adjoint equivalence is done by straightforward calculation in Section 1. We use that a biequivalence in **Bicat** can be characterized in two different ways: as a functor having a weak inverse or as a functor which is biessentially surjective and a local equivalence. The proof largely consists of using this alternate characterization of biequivalences, results from section 1, and the biadjoint biequivalence axioms to construct the other cells of the

biadjoint biequivalence. This is done by constructing the components on objects first, and then building up the rest of the structure afterwards. The reader will note that the technical difficulties lie in two places: in constructing these cells once the components on objects are given, and in checking that our constructions satisfy all of the biadjoint biequivalence axioms since we only need a subset of them in order to define all the needed cells.

The fourth section provides the proof of our main result, that every biequivalence in a tricategory is part of a biadjoint biequivalence. The proofs in this section rely very heavily on coherence for tricategories in the form "every diagram in a free tricategory commutes" to simplify the pasting diagrams required. We often leave the particulars of checking axioms to the reader as the diagrams are very large, but we state exactly which axioms are needed in each case. The bulk of the technical work goes into showing that the components we construct satisfy the axioms required to be transformations, modifications, or perturbations, while the biadjoint biequivalence axioms are immediate.

Sections 5 and 6 give the two applications mentioned above. Both of these results should be thought of in the form "a certain 3-dimensional monad has property P" in each case. In the case of lifting monoidal structures, the monad would be the free monoidal bicategory monad on the tricategory **Bicat**, and the property P would be a 3-dimensional version of flexibility [BKP]. In the case of Picard 2-categories, the monad would be the free Picard 2-category monad on the tricategory of monoidal bicategories, and the property P would be a kind of 3-dimensional idempotency [KL]. This is the proper way to view these results, although we do not pursue the details here because of the lack of groundwork on 3-dimensional monads on tricategories.

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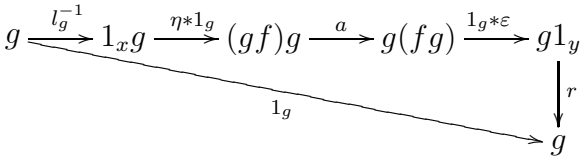
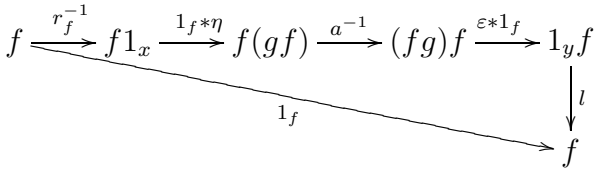
1. Equivalences and adjoint equivalences in bicategories

We begin by reviewing the relevant results for bicategories that we will later generalize to tricategories. We assume that the reader is familiar with bicategories and the coherence theorem for bicategories (see [MP] for coherence for bicategories, or [JS] for a discussion of coherence, including functors, for the case of monoidal categories instead of bicategories). We begin with some basic definitions.

1.1. Definition. *Let B be a bicategory, and let $f : x \to y$ and $g : y \to x$ be 1-cells in B. An adjunction $f \dashv g$ consists of a 2-cell $\varepsilon : fg \Rightarrow 1_y$ and a 2-cell $\eta : 1_x \Rightarrow gf$ such that the following two diagrams (the triangle identities) commute.*





We then say that f is left adjoint to g, or that g is right adjoint to f.

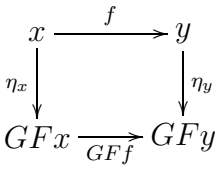
1.2. Remark. In the bicategory **Cat**, the associativity and unit isomorphisms are all identities. In that case, this definition reduces to the usual definition of an adjunction between functors.

1.3. Definition. *An adjunction $f \dashv g$ is an adjoint equivalence if ε and η are invertible. In this case, we write $f \dashv_{eq} g$.*

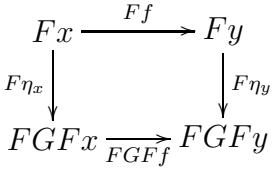
1.4. Theorem. *Let $F : X \to Y$ and $G : Y \to X$ be functors, and let $\alpha : FG \Rightarrow 1_Y$ and $\beta : 1_X \Rightarrow GF$ be natural isomorphisms. Then there is a unique adjoint equivalence $(F, G, \varepsilon, \eta)$ in* **Cat** *such that $\varepsilon = \alpha$.*

Proof. Let $\varepsilon = \alpha$. The second triangle identity states that $\varepsilon F \circ F\eta = 1_F$. By the invertibility of ε, this equation is the same as $F\eta = (\varepsilon F)^{-1}$. The righthand side of this equation is well-defined and F is full and faithful since it is an equivalence of categories, so we define $\eta_x : x \to GFx$ to be the unique arrow such that $F\eta_x = (\varepsilon_{Fx})^{-1}$.

We must now check that η is natural and that the first triangle identity holds. For naturality, we consider the square below.



Applying F to the diagram and using functoriality gives this square.



By the definition of η, this square is the naturality square of $(\varepsilon F)^{-1}$ and thus must commute. By the faithfulness of F, the original square commutes as well and so η is natural.

For the first triangle identity, we consider the composite

$$Gy \xrightarrow{\eta_{Gy}} GFGy \xrightarrow{G\varepsilon_y} Gy.$$

Applying F to this yields $FG\varepsilon_y \circ F\eta_{Gy}$, which is by definition $FG\varepsilon_y \circ \varepsilon^{-1}_{FGy}$. Now the following square commutes by the naturality of ε.

$$\begin{array}{ccc} FGFGy & \xrightarrow{\varepsilon_{FGy}} & FGy \\ {\scriptstyle FG\varepsilon_y}\downarrow & & \downarrow{\scriptstyle \varepsilon_y} \\ FGy & \xrightarrow[\varepsilon_y]{} & y \end{array}$$

By the invertibility of ε_y, we get that $\varepsilon_{FGy} = FG\varepsilon_y$. Therefore $FG\varepsilon_y \circ \varepsilon^{-1}_{FGy} = 1_{FGy}$. Once again by the faithfulness of F, $G\varepsilon_y \circ \eta_{Gy} = 1_y$ and thus the first triangle identity is satisfied. ∎

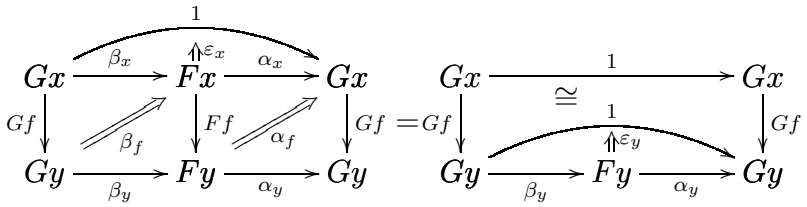
1.5. Remark. We could have just as easily constructed an adjoint equivalence with $\eta = \beta$ instead of $\varepsilon = \alpha$. In general it is not possible to require both of these conditions, though, as the choice of either η or ϵ fixes the other.

It would be possible at this point to prove, by a series of calculations, an analogous result with **Cat** replaced by any bicategory, but instead we choose a different approach that generalizes more easily to the case of tricategories.

1.6. Definition. *Let B be a bicategory, and let $f : x \to y$ be a 1-cell in B. Then f is an equivalence if there exists a $g : y \to x$ such that $fg \cong 1_y$ and $gf \cong 1_x$.*

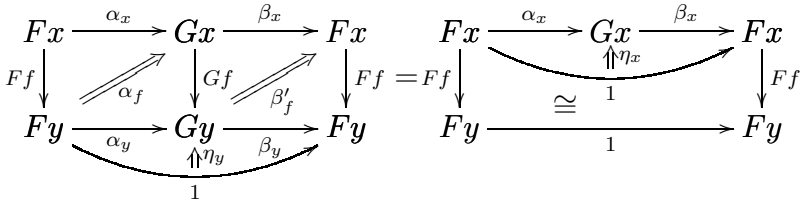
1.7. Lemma. *Let B, C be bicategories, and assume that every equivalence f in C is part of an adjoint equivalence $f \dashv_{eq} g$. Then every equivalence in the functor bicategory $[B, C]$ is part of an adjoint equivalence.*

Proof. Let $\alpha : F \Rightarrow G$ be an equivalence in the functor category. Then each 1-cell $\alpha_x : Fx \to Gx$ is an equivalence in C, thus we can produce adjoint equivalences $\alpha_x \dashv_{eq} \beta_x$ in C for every object x in B. We will now define a transformation $\beta : F \Rightarrow G$ using the β_x constructed above as the components on objects. Now given a morphism $f : x \to y$ in B, we must also produce an invertible 2-cell $\beta_f : \beta_y \circ Gf \Rightarrow Ff \circ \beta_x$ in C, subject to the transformation axioms. We define β_f by the requirement that it provides the equality of pasting diagrams given below.

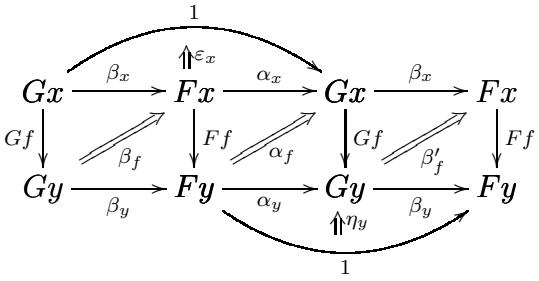


Here, ε_x is the counit of the adjoint equivalence $\alpha_x \dashv_{eq} \beta_x$. This gives a well-defined β_f as follows. Since all the 2-cells in this pasting diagram are invertible, this equality determines $1_{\alpha_y} * \beta_f$. But α_y is an equivalence 1-cell, hence the functor $\alpha_y \circ -$ is an equivalence of categories, so that $1_{\alpha_y} * \beta_f$ determines β_f.

In a similar fashion, we can also construct an invertible 2-cell β'_f with the same source and target as β_f by requiring it provides the equality of pasting diagrams given below.



After applying left and right unit isomorphisms, the pasting diagram below can be shown to be equal to both β_f and β'_f using the triangle identities, so $\beta_f = \beta'_f$.



This defines the components of β on both objects and morphisms.

Now we check that these components satisfy the axioms for a transformation. First, we show that β_f is natural in f. Given a 2-cell $\delta : f \Rightarrow g$ in B, we must show that

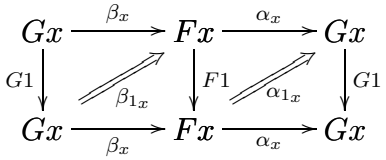
$$\beta_g \circ (1_{\beta_y} * G\delta) = (F\delta * 1_{\beta_x}) \circ \beta_f.$$

This follows from the naturality of both α_f and the coherence isomorphisms used in the definition of β_f.

Second, we must show that β_{1_x} is the composite

$$\beta_x \circ G1 \cong \beta_x 1 \cong \beta_x \cong 1\beta_x \cong F1 \circ \beta_x$$

where every isomorphism is given by a unique coherence isomorphism. To do this, we need only show that the composite above gives the equality of pasting diagrams we used to define β_{1_x}. This is trivial using the unit axiom for the transformation α and the fact that



is the unique coherence isomorphism

$$(\alpha_x \circ \beta_x) \circ G1 \cong G1 \circ (\alpha_x \circ \beta_x)$$

by the definition of β_{1_x}.

The third and final transformation axiom follows from a similar proof. ∎

1.8. Lemma. *Assume that every equivalence f in C is part of an adjoint equivalence $f \dashv_{eq} g$, and let $F : B \to C$ be a functor which is locally an equivalence. Then every equivalence r in B is part of an adjoint equivalence $r \dashv_{eq} s$.*

Proof. If $r : x \to y$ is an equivalence in B, then there is an $s : y \to x$ such that $rs \cong 1_y$ and $sr \cong 1_x$. Then Fr is an equivalence in C with Fs as a pseudoinverse. By hypothesis, we can find an adjoint equivalence $Fr \dashv_{eq} t$. By the uniqueness of pseudo-inverses, we must have $t \cong Fs$, so we have an adjoint equivalence $Fr \dashv_{eq} Fs$. Since F is locally full, this means we can find a 2-cell in B which maps to the following composite.

$$F1 \cong 1 \xrightarrow{\eta'} FsFr \cong F(sr)$$

(Here η' denotes the unit of the adjoint equivalence $Fr \dashv_{eq} Fs$.) This 2-cell will be the unit of our adjoint equivalence, and the counit is constructed similarly; the triangle identities follow from coherence for functors and the fact that F is locally faithful. ∎

1.9. Theorem. *Let B be a bicategory, and let f be an equivalence in B. Then f is part of an adjoint equivalence $f \dashv_{eq} g$.*

Proof. Since every equivalence in **Cat** is part of an adjoint equivalence, the same is true for $[B^{\mathrm{op}}, \mathbf{Cat}]$. Let $Y : B \to [B^{\mathrm{op}}, \mathbf{Cat}]$ be the Yoneda embedding. The functor Y satisfies the hypotheses of the above lemma, hence there is an adjoint equivalence $f \dashv_{eq} g$ in B. ∎

1.10. Remark. We have actually shown something stronger than the fact that every equivalence is part of an adjoint equivalence. We have actually shown that given any equivalence f, a pseudo-inverse g, and an isomorphism $\alpha : fg \cong 1$, there is a unique adjoint equivalence $f \dashv g$ with α as its counit. This is true in **Cat**, hence in any functor bicategory into **Cat**. Therefore any two adjoint equivalences $f \dashv_{eq} g$ with the same counit $\alpha : fg \Rightarrow 1$ will necessarily have the same unit after applying the Yoneda embedding, therefore must have the same unit before applying Y since it is a local equivalence.

2. Definitions

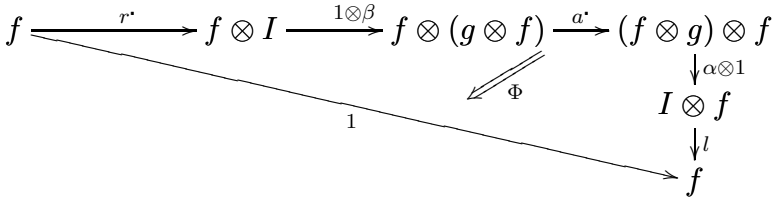
This section will provide the definition of a biadjoint biequivalence in an arbitrary tricategory T. This proceeds in two steps: first we define a biadjunction in a tricategory, and then equip it with extra structure to define a biadjoint biequivalence. There are two possible options for the definition of a biadjoint biequivalence. We provide the concise definition first (omitting the "horizontal cusp" axioms), and then explain how it is equivalent to a definition with additional data and axioms.

Before giving the definition of a biadjunction in a tricategory T, we note that our definition is merely the weakening of previous definitions of biadjunctions in **Gray**-categories. This weakening is done in the most straightforward manner, and is done because the most natural and concise definition of a biadjunction in a **Gray**-category uses the **Gray**-category axioms implicitly. A different approach to these structures might, for instance,

involve constructing the "free living biadjunction" - this would be a tricategory $\mathbb{B}$ with the property that biadjunctions in an arbitrary tricategory T would correspond to maps $\mathbb{B} \to T$. This is the approach taken by Lack in [L1] in the context of **Gray**-categories in order to discuss the relationship between biadjunctions and pseudomonads. Since our focus is on biadjoint biequivalences, and not the more general biadjunctions, we do not proceed in this fashion.

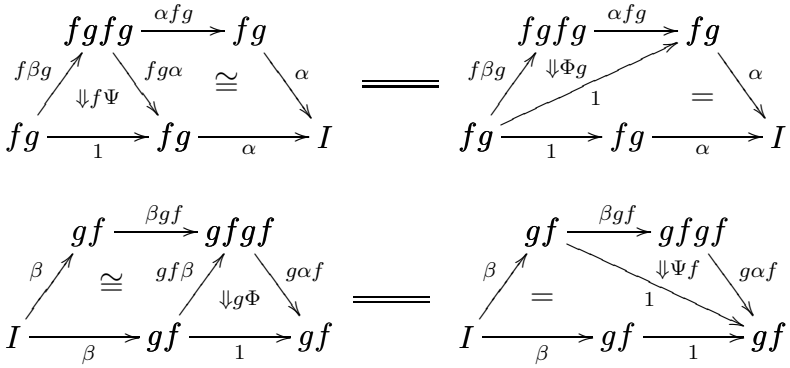
2.1. Definition. *Let T be a tricategory. Then a biadjunction $f \dashv_{bi} g$ consists of*

- *1-cells $f : x \to y, g : y \to x$,*
- *2-cells $\alpha : f \otimes g \Rightarrow I_y, \beta : I_x \Rightarrow g \otimes f$, and*
- *invertible 3-cells Φ, Ψ below,*



such that the pasting diagrams in Figures 1 and 2 are both the identity.

2.2. Remark. In the presence of the simplifying assumption that the tricategory T is actually a strict, cubical tricategory (i.e., a **Gray**-category), the axioms simplify to the equality of pasting diagrams below.



See [St], [Ver], or [L1] for earlier definitions.

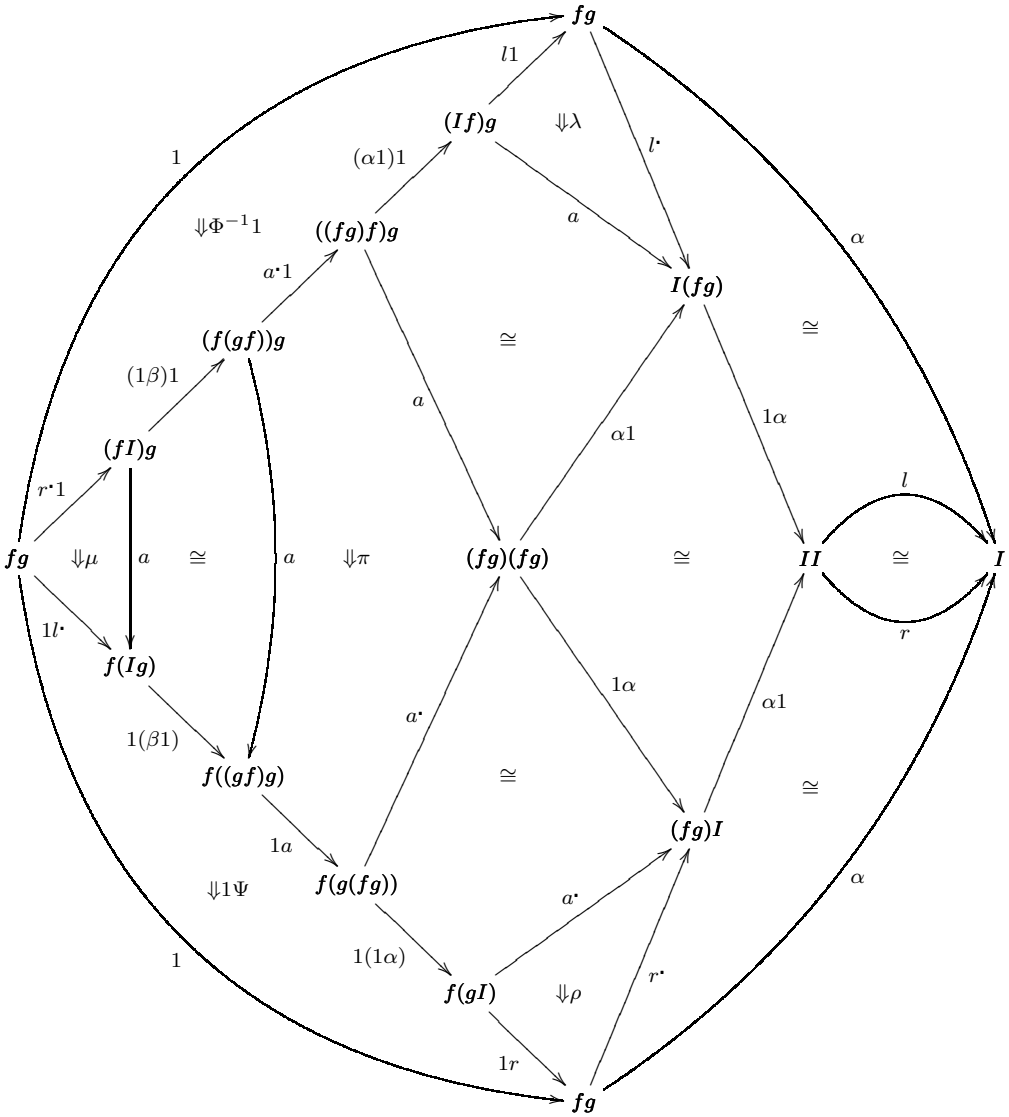


Figure 1: First pasting

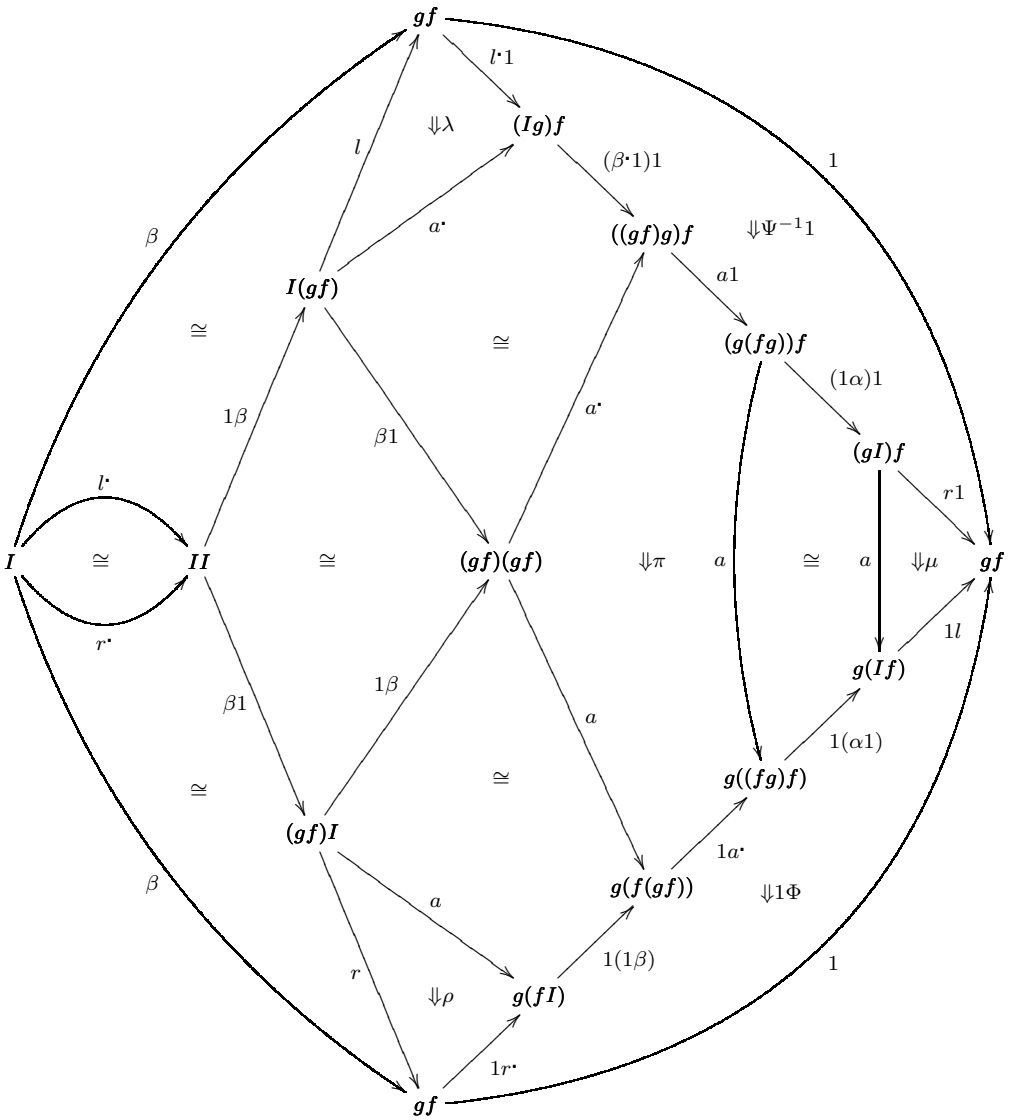


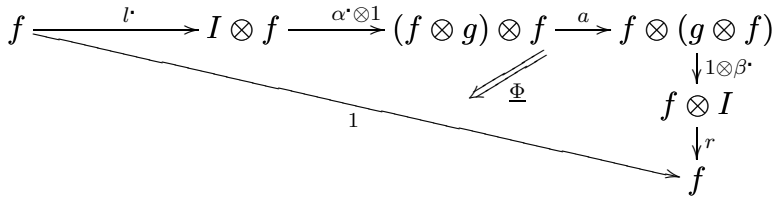
Figure 2: Second pasting

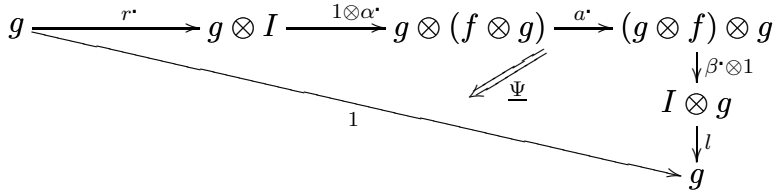
2.3. Definition. *Let T be a tricategory. Then a biadjoint biequivalence $f \dashv_{bieq} g$ consists of*

- *a biadjunction $f \dashv_{bi} g$ and*
- *adjoint equivalences $\alpha \dashv_{eq} \alpha^{\cdot}, \beta \dashv_{eq} \beta^{\cdot}$ in the respective hom-bicategories.*

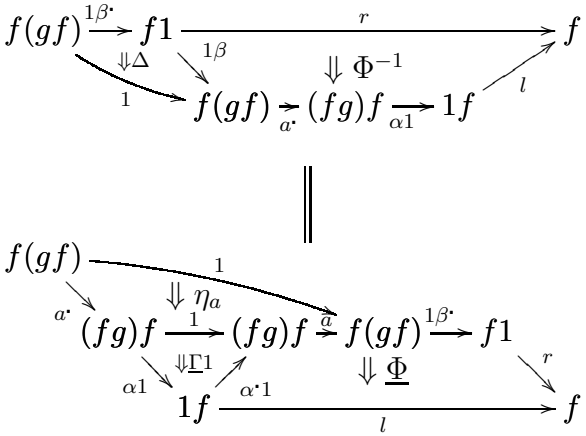
It is also possible to give a longer version of a biadjoint biequivalence which includes extra data satisfying the so-called horizontal cusp axioms. Such a definition is equivalent to the one given above by using the calculus of mates, as we explain below.

The extra data needed to express the horizontal cusp axioms are a pair of invertible 3-cells $\underline{\Phi}, \underline{\Psi}$.





These additional 3-cells are then required to satisfy the horizontal cusp axioms, named for their relationship with certain "braid movie moves" between braided surfaces in $\mathbb{R}^4$. One such axiom, written categorically, is given below.



It is then clear that this axiom merely says that $\underline{\Phi}$ is the mate of Φ, and similarly for $\underline{\Psi}$ and Ψ. All of the horizontal cusp-type axioms can be expressed in this fashion.

3. Biequivalences in **Bicat**

This section presents a computational proof that every biequivalence in the tricategory **Bicat** is part of a biadjoint biequivalence. The proof here will proceed much as the proof in **Cat** did, by relying on an alternate description of biequivalences. Thus we begin with a simple lemma.

3.1. Lemma. *Let $F : B \to C$ be a functor between bicategories. Then F is biessentially surjective and locally an equivalence of categories if and only if there is a functor $G : C \to B$ such that $FG \simeq 1_C$ and $GF \simeq 1_B$ in the respective functor bicategories.*

Proof. Since F is biessentially surjective, for every object c in C we can find an object b in B and an adjoint equivalence $f_c \dashv_{eq} g_c$ between Fb and c by Theorem 1.9; here we choose f_c to have source Fb and target c. We choose such an adjoint equivalence for every c, and define the functor G on objects by $Gc = b$. Now $F_{b,b'} : B(b, b') \to C(Fb, Fb')$ is an equivalence of categories for every pair b, b', and we choose an adjoint equivalence $F_{b,b'} \dashv_{eq} \tilde{G}_{b,b'}$. We define the functor G on hom-categories $G_{b,b'} : C(c, c') \to B(Gc, Gc')$ to be the composite

$$C(c, c') \overset{f_c^*}{\to} C(Fb, c') \overset{g_{c*}}{\to} C(Fb, Fb') \overset{\tilde{G}_{b,b'}}{\to} B(b, b').$$

We must now construct isomorphisms $1_{Gc} \cong G(1_c)$, $Gf \circ Gg \cong G(f \circ g)$ and check the axioms for a functor. For the first of these, we compute that

$$G(1_c) = \tilde{G}_{b,b}\Big(g_c \circ (1_c \circ f_c)\Big),$$

while $1_{Gc} = 1_b$. Now note that the adjoint equivalence $F_{b,b'} \dashv_{eq} \tilde{G}_{b,b'}$ has a unit isomorphism $1 \Rightarrow \tilde{G}_{b,b'} \circ F_{b,b'}$, and when specialized to the case $b = b'$ and then evaluated at 1_b yields

$$1_b \cong \tilde{G}_{b,b}(F_{b,b}(1_b)).$$

Since F is a functor, we have an isomorphism $\varphi_0^F : 1_{Fb} \cong F(1_b)$ which we can compose with the previous isomorphism to get

$$1_b \cong \tilde{G}_{b,b}(F_{b,b}(1_b)) \overset{\tilde{G}_{b,b}(\varphi_0^F)}{\longrightarrow} \tilde{G}_{b,b}(1_{Fb})$$

which we denote by $\overline{\varphi}_0^G$. Writing $\eta_c : 1_{Fb} \Rightarrow g_c \circ f_c$ for the unit of the adjoint equivalence $f_c \dashv_{eq} g_c$, we obtain the isomorphism

$$\varphi_0^G : 1_b \cong G(1_c)$$

as the following composite.

$$1_b \overset{\overline{\varphi}_0^G}{\longrightarrow} \tilde{G}_{b,b}(1_{Fb}) \overset{\tilde{G}_{b,b}(\eta_c)}{\longrightarrow} \tilde{G}_{b,b}(g_c \circ f_c) \overset{\tilde{G}_{b,b}(1 * l^{-1})}{\longrightarrow} \tilde{G}_{b,b}\Big(g_c \circ (1 \circ f_c)\Big)$$

The isomorphism $\varphi_2^G : Gf \circ Gg \cong G(f \circ g)$ is obtained in a similar fashion, and the functor axioms for G follow from those for F and the adjoint equivalence axioms for both $f_c \dashv_{eq} g_c$ and $F_{b,b'} \dashv_{eq} \tilde{G}_{b,b'}$.

We must finally check that $FG \simeq 1_C$ and $GF \simeq 1_B$ in the relevant functor bicategories. For the first of these, note that $FG(c) = Fb$ by construction. We already have equivalence 1-cells $f_c : FG(c) \to 1_C(c)$ that we now need to complete to a natural transformation. This requires that we give natural isomorphisms

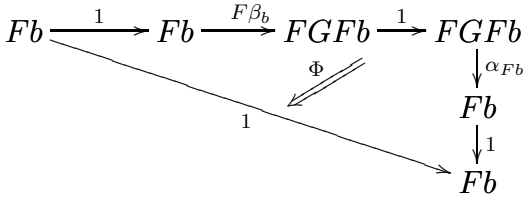
$$f_r : f_{c'} g_{c'} r f_c \cong r f_c,$$

one for each r, which we define to be the obvious whiskering of the counit isomorphism $f_{c'} g_{c'} \cong 1_{c'}$ composed with the left unit isomorphism. The adjoint equivalence axioms for $f_c \dashv_{eq} g_c$ and coherence for bicategories imply all of the transformation axioms. This shows that $FG \simeq 1_C$, and we leave it to the reader to prove $GF \simeq 1_B$. ∎

3.2. Theorem. *Let $F : B \to C$ be a biequivalence between bicategories. Then there is a biadjoint biequivalence $F \dashv_{bieq} G$.*

Proof. Since F is a biequivalence, choose a functor $G : C \to B$ such that FG is equivalent to 1_C in the bicategory $\mathbf{Bicat}(C, C)$ and GF is equivalent to 1_B in $\mathbf{Bicat}(B, B)$. Taking any equivalence $\alpha : FG \Rightarrow 1_C$ exhibiting this fact, we can construct an adjoint equivalence $\alpha \dashv_{eq} \alpha^{\cdot}$ in $\mathbf{Bicat}(C, C)$ by Theorem 1.9. We will write $\Gamma : \alpha\alpha^{\cdot} \Rrightarrow 1$ and $\underline{\Gamma} : 1 \Rrightarrow \alpha^{\cdot}\alpha$ for the counit and unit of this adjoint equivalence, respectively.

Now we construct the adjoint equivalence $\beta \dashv_{eq} \beta^{\cdot}$ (between 1_B and GF) and the invertible modification Φ simultaneously. The component of β at an object $b \in B$ is a 1-cell $\beta_b : b \to GFb$. The component of Φ at $b \in B$ is an invertible 2-cell in C



since the associativity and unit 2-cells in **Bicat** have identities as their components. By coherence, such an invertible 2-cells determines and is determined by an invertible 2-cell $\widetilde{\Phi} : \alpha_{Fb} \circ F(\beta_b) \Rightarrow 1_{Fb}$. Since α is an equivalence, giving such an isomorphism is equivalent to giving an isomorphism $\alpha^{\cdot}_{Fb} \cong F(\beta_b)$. Now F is locally an equivalence of categories, and in particular essentially surjective, so there exists a morphism $\beta_b : b \to GFb$ such that $F(\beta_b) \cong \alpha^{\cdot}_{Fb}$. For every object $b \in B$, choose such a β_b and a specified isomorphism $\delta_b : F(\beta_b) \cong \alpha^{\cdot}_{Fb}$.

For the component of β at $f : b \to c$, consider the following composite.

$$\begin{array}{rcccl} F(\beta_c \circ f) & \cong & F\beta_c \circ Ff & \overset{\delta_c * 1}{\to} & \alpha^{\cdot}_{Fc} \circ Ff \\ & \overset{\alpha^{\cdot}_f}{\to} & FGFf \circ \alpha^{\cdot}_{Fb} & \overset{1 * \delta_b^{-1}}{\to} & FGFf \circ F\beta_b \\ & & & \cong & F\Big(GFf \circ \beta_b\Big) \end{array}$$

Since F is locally an equivalence, there is a unique isomorphism

$$\beta_f : \beta_c \circ f \Rightarrow GFf \circ \beta_b$$

that maps to the composite above. It is then simple to check that β is a transformation $1 \Rightarrow GF$, and that it is an equivalence. This construction also immediately implies that δ is an invertible modification

$$\delta : 1_F \otimes \beta \Rrightarrow \alpha^{\cdot}.$$

We then define the adjoint equivalence $\beta \dashv_{eq} \beta^{\cdot}$ to be any adjoint equivalence containing β.

The 2-cell $\widetilde{\Phi} : \alpha_{Fb} \circ F(\beta_b) \Rightarrow 1_{Fb}$ is defined to be the following composite.

$$\alpha_{Fb} \circ F(\beta_b) \overset{1*\delta_b}{\Rightarrow} \alpha_{Fb} \circ \alpha^{\cdot}_{Fb} \overset{\Gamma}{\Rightarrow} 1_{Fb}$$

By coherence for bicategories, this determines the 2-cell Φ_b uniquely. These 2-cells Φ_b then give the data for an invertible modification Φ since all of the cells used to construct the Φ_b are either components of modifications or are appropriately natural.

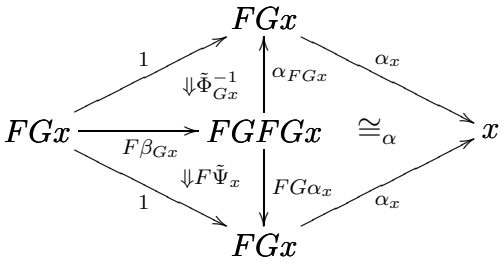
All that remains is to construct the invertible 3-cell Ψ and to check the two biadjunction axioms. Before doing so, we remind the reader that, for 3-cells in a tricategory, $\circ$ denotes the composition along 2-cell boundaries, $*$ denotes composition along 1-cell boundaries, and $\otimes$ denotes composition along 0-cell boundaries. Now the second axiom determines the 3-cell $(1_G \otimes \Psi) * 1_\beta$. Since the 2-cell β is an equivalence, the functor $- \circ \beta$ is an equivalence of categories, and in particular the cell $(1_G \otimes \Psi) * 1_\beta$ uniquely determines the cell $1_G \otimes \Psi$. Similarly, since the functor F is a biequivalence, G is also, so the functor $G \circ -$ is a biequivalence of bicategories; thus $1_G \otimes \Psi$ uniquely determines the invertible modification Ψ. By construction, the second biadjoint biequivalence axiom is satisfied.

Now we show that this choice of Ψ satisfies the first biadjoint biequivalence axiom. First, note that, while **Bicat** is not a **Gray**-category, it does have a strictly associative and unital composition law for 1-cells in the following sense. The composite $H(GF)$ equals the composite $(HG)F$, and similarly $F1 = F = 1F$, but we still have associativity and unit equivalences for this composition law. These are the 1-cells in the biadjoint biequivalence axioms labeled a, l, r, and they have components on objects given by identities and components on morphisms given by unique coherence 2-cells. For examples, the transformation $l : 1F \Rightarrow F$ has its component at an object x the identity $1_{Fx} : Fx \to Fx$ and its component at a 1-cell $f : x \to y$ the unique coherence cell

$$1_{Fy} \circ Ff \cong Ff \circ 1_{Fx}.$$

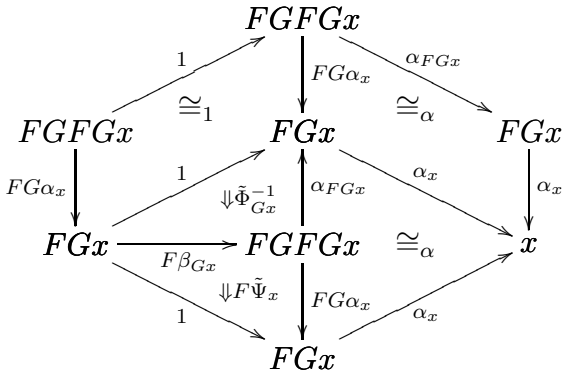
Similarly, the modifications π, μ, λ, and ρ in **Bicat** all have unique coherence 2-cells as their components. Thus coherence for bicategories reduces the first biadjoint biequivalence

to checking that the pasting

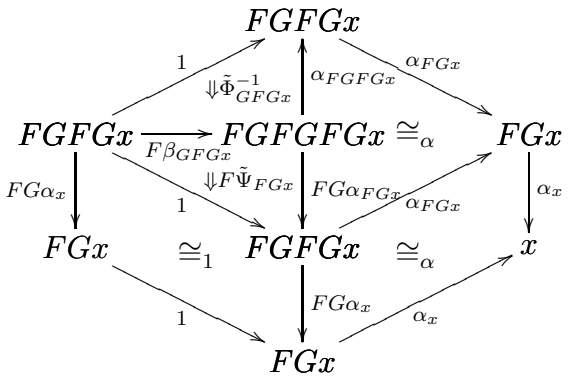


is equal to the identity on $\alpha \circ 1_{FG}$. (Here we use the same convention that $\widetilde{\Psi}$ is derived from Ψ via unique coherence isomorphisms.) From this point on, we mark our naturality isomorphisms with a subscript to indicate which transformation they are naturality isomorphisms for to avoid confusion, and we refer to all instances of the above pasting diagram as "Axiom 1", perhaps with some descriptor to indicate which object x is being used.

First, note that Axiom 1 is the identity if and only if it is the identity when x is of the form Fy for some y in B. Indeed, consider the following pasting diagram.

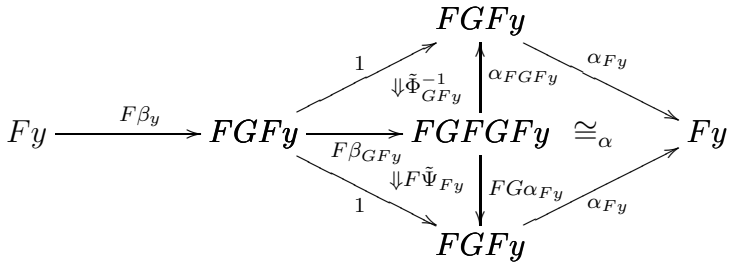


Using the modification and transformation axioms, it is equal to the pasting below.

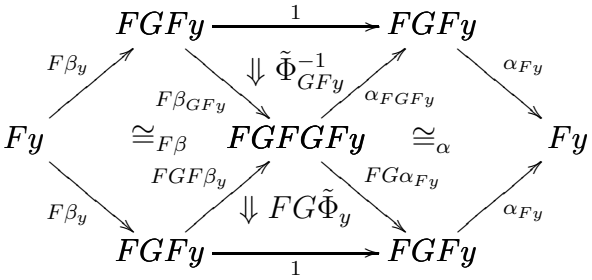


Thus the pasting Axiom 1 for x is the identity if and only if Axiom 1 for FGx is the identity, so taking $y = Gx$ proves the claim. From this point, we replace x with Fy.

Now Axiom 1 for Fy is the identity if and only if the following pasting diagram is the identity since $F\beta_y$ is an equivalence 1-cell.



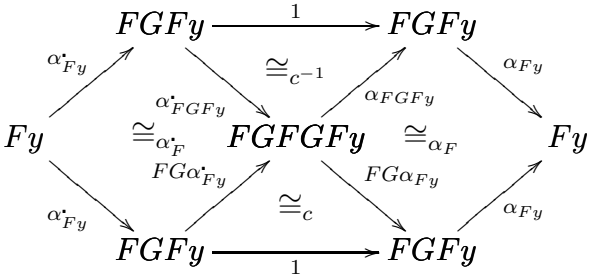
Applying F to the second biadjoint biequivalence axiom and rewriting, we see that the above pasting diagram is the identity if and only if the following one is.



Recall now that β was constructed together with an invertible modification δ with components $\delta_y : F\beta_y \cong \alpha^{\bullet}_{Fy}$ such that the composite

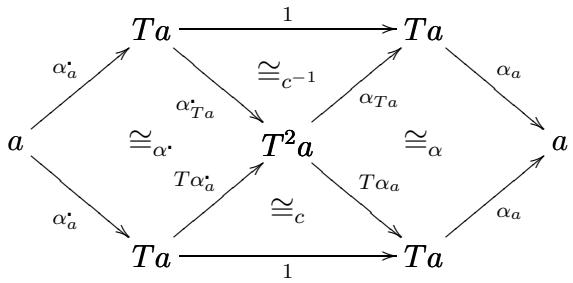
$$\alpha_{Fy} \circ \alpha^{\bullet}_{Fy} \overset{1*\delta_y^{-1}}{\Longrightarrow} \alpha_{Fy} \circ F\beta_y \overset{\Phi_y}{\Longrightarrow} 1_{Fy}$$

is the counit of the adjoint equivalence $\alpha \dashv_{eq} \alpha^{\bullet}$. The previous pasting diagram is then the identity if and only if we pre-compose it with δ_y^{-1} and post-compose it with δ_y. Using the naturality axiom for $\alpha^{\bullet}$, the modification axiom for δ, and the equality relating δ to the counit of the adjoint equivalence $\alpha \dashv_{eq} \alpha^{\bullet}$, the pre- and post-composed pasting is the identity if and only if the one displayed below is the identity.

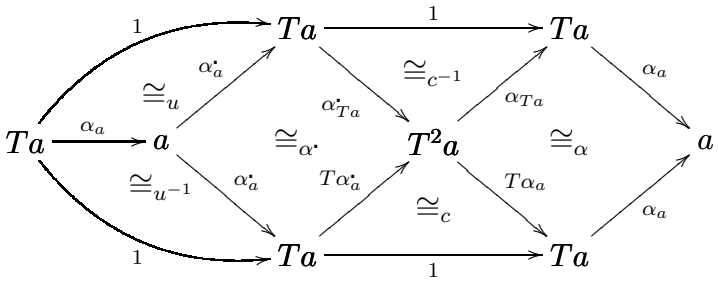


Here we have written $\cong_c$ for the counit isomorphism of the adjoint equivalence $\alpha \dashv \alpha^{\bullet}$, and $\cong_{c^{-1}}$ for the inverse of the counit. This diagram is the identity following from a general lemma on mates that we give below. ∎

3.3. Lemma. *Let B be a bicategory, and $T : B \to B$ a functor. Let $\alpha : T \Rightarrow 1$, $\alpha^{\cdot} : 1 \Rightarrow T$ be part of an adjoint equivalence $\alpha \dashv_{eq} \alpha^{\cdot}$. Then for any object a in B, the pasting diagram below is the identity.*

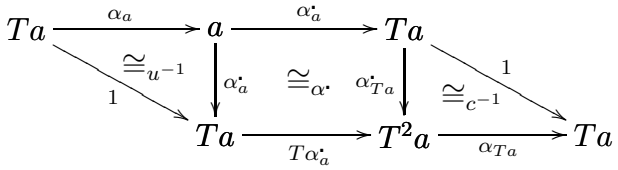


Proof. First, the above pasting diagram is the identity if and only if the one below is.



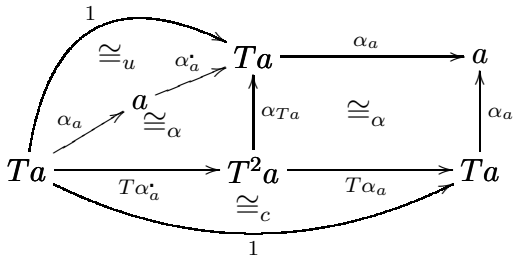
Here we have written $\cong_u$ to denote a unit isomorphism, and $\cong_{u^{-1}}$ the inverse of a unit isomorphism.

It is now a simple calculation using the adjoint equivalence axioms for transformations to show that the pasting below (modulo unit isomorphisms to alter the 1-cell source and target, which we ignore for now but record the presence of for later)

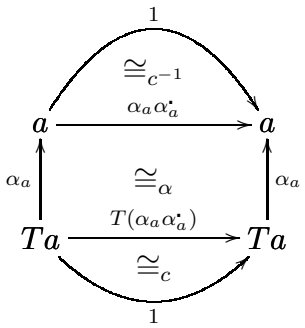


is equal to the naturality square below.

Thus we have shown that the pasting diagram in the previous paragraph is equal (modulo units) to the pasting diagram displayed below.



Using the triangle identities and the naturality axioms for α, this is now equal to the pasting below (once again modulo units).



By the naturality axioms for α and coherence for functors, this is equal to a composite of left and right unit isomorphisms so the original diagram is a composite of coherence isomorphisms, hence is the identity by coherence for bicategories. ∎

4. Biequivalences in general tricategories

This section will establish the general result that every biequivalence in a tricategory T is part of a biadjoint biequivalence in T. Our proof will proceed largely as did the general case for equivalences in bicategories by first examining the case of functor tricategories and then using a Yoneda embedding. Since a Yoneda embedding is only known for cubical tricategories rather than the general case, the proof for tricategories is slightly longer although essentially the same. We refer the reader to [GPS], [Gur], or the upcoming [Gur2] for the relevant tricategorical results.

Recall that if S is any tricategory and T is a **Gray**-category, then there is a **Gray**-category $\mathbf{Tricat}(S,T)$ with objects functors $S \to T$, 1-cells transformations, 2-cells modifications, and 3-cells perturbations.

4.1. Proposition. *Let S be any tricategory and T be a **Gray**-category. Assume that every biequivalence 1-cell f in T is part of a biadjoint biequivalence $f \dashv_{bieq} g$. Then every biequivalence 1-cell α in $\mathbf{Tricat}(S,T)$ is part of a biadjoint biequivalence $\alpha \dashv_{bieq} \beta$.*

4.2. Remark. We have written the proof of this proposition out so that it should be obvious to the reader that it remains true when T is merely a tricategory and not a **Gray**-category. By this we mean the following: the assumption that T is a **Gray**-category is only present to use the results of [Gur] to give a concrete construction of the tricategory $\mathbf{Tricat}(S,T)$. Using coherence for tricategories, it is possible to construct a tricategory $\mathbf{Tricat}(S,T)$ when T is any tricategory, and then the proof below applies verbatim to the analogous proposition.

Proof. Let $\alpha : F \Rightarrow G$ be a biequivalence in $\mathbf{Tricat}(S,T)$. Then $\alpha_a : Fa \to Ga$ is a biequivalence in T for every object a, so we choose biadjoint biequivalences $\alpha_a \dashv_{bieq} \beta_a$ for every object a of S. To complete the proof, we must do the following:

1. equip the components β_a with the structure of a transformation;
2. equip the componentwise adjoint equivalences

$$\boldsymbol{\varepsilon}_a : \alpha_a\beta_a \dashv_{eq} 1_{Ga}, \quad \boldsymbol{\eta}_b : 1_{Fb} \dashv_{eq} \beta_b\alpha_b$$

   with the structure of adjoint equivalences in the hom-bicategories $\mathbf{Tricat}(S,T)(G,G)$, $\mathbf{Tricat}(S,T)(F,F)$; and
3. check that Φ, Ψ are perturbations.

Since equations between perturbations are checked componentwise, the fact we have biadjoint biequivalences $\alpha_a \dashv_{bieq} \beta_a$ will then imply that there is a global biadjoint biequivalence $\alpha \dashv_{bieq} \beta$ in $\mathbf{Tricat}(S,T)$.

We begin by defining a transformation $\beta : G \Rightarrow F$ with components given by these β_a. Since we have already given the components on objects, there are three pieces of data left to define. The first is an adjoint equivalence

$$\begin{array}{ccc} S(a,b) & \xrightarrow{G} & T(Ga,Gb) \\ {\scriptstyle F}\downarrow & \swarrow\!\!\swarrow {\scriptstyle \boldsymbol{\beta}} & \downarrow{\scriptstyle T(1,\beta_b)} \\ T(Fa,Fb) & \xrightarrow[T(\beta_a,1)]{} & T(Ga,Fb) \end{array}$$

in the bicategory $\mathbf{Bicat}\Big(S(a,b), T(Ga,Fb)\Big)$. We write down the component β_f of β at an object f and leave it to the reader to construct the rest of the adjoint equivalence in the obvious fashion.

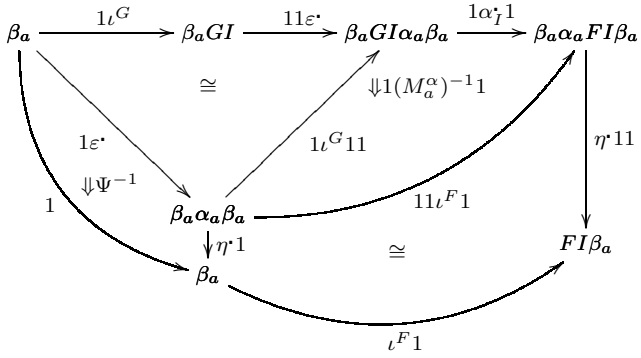
$$\begin{array}{rcl} \beta_b \otimes Gf & \xrightarrow{1\otimes r^{\bullet}} & \beta_b \otimes (Gf \otimes 1) \\ & \xrightarrow{1\otimes(1\otimes \varepsilon^{\bullet}_a)} & \beta_b \otimes (Gf \otimes (\alpha_a \otimes \beta_a)) \\ & \xrightarrow{1\otimes a^{\bullet}} & \beta_b \otimes ((Gf \otimes \alpha_a) \otimes \beta_a) \\ & \xrightarrow{1\otimes(\alpha^{\bullet}_f\otimes 1)} & \beta_b \otimes ((\alpha_b \otimes Ff) \otimes \beta_a) \\ & \xrightarrow{a^{\bullet}} & (\beta_b \otimes (\alpha_b \otimes Ff)) \otimes \beta_a \\ & \xrightarrow{a^{\bullet}\otimes 1} & ((\beta_b \otimes \alpha_b) \otimes Ff) \otimes \beta_a \\ & \xrightarrow{\eta^{\bullet}_b\otimes 1} & (1 \otimes Ff) \otimes \beta_a \\ & \xrightarrow{l\otimes 1} & Ff \otimes \beta_a \end{array}$$

We have written this out as if if were a 2-cell an arbitrary tricategory, not necessarily a **Gray**-category. In the case that T is **Gray**, this cell is as below.

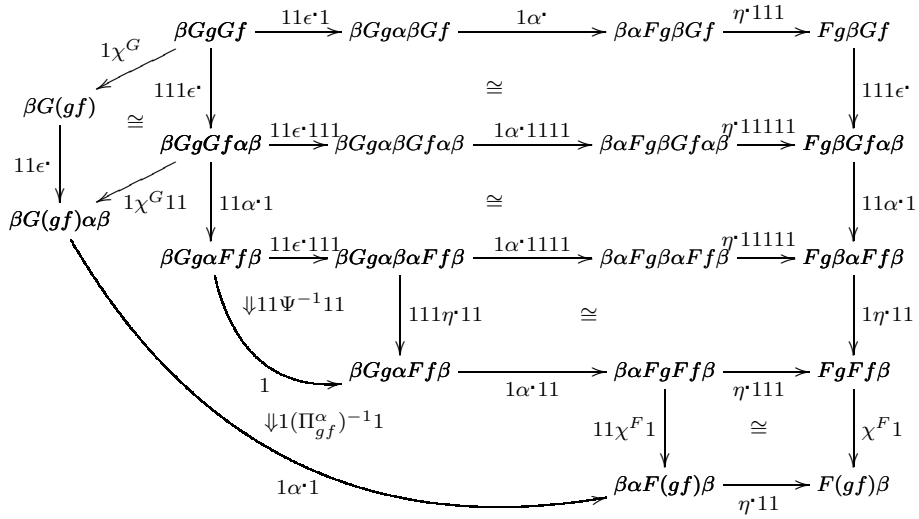
$$\beta_b Gf \xrightarrow{11\epsilon^{\bullet}_a} \beta_a Gf \alpha_a\beta_a \xrightarrow{1\alpha^{\bullet}_f 1} \beta_b\alpha_b Ff\beta_a \xrightarrow{\eta^{\bullet}_b 1} Ff\beta_a$$

We now must produce a pair of invertible modifications Π, M to complete the definition of the data for the transformation β. The component of the modification M^{β} at the object

a is given by the isomorphism shown below where the unmarked isomorphisms are unique by coherence and the two marked 3-cells are both appropriate mates.



The component of the modification Π^{β} at the composable pair (g, f) is given by the pasting below, once again following the same conventions. To conserve space, we omit the subscripts for the components of α and β given that they can be deduced from the other 1-cells in any given term.



All of the unmarked isomorphisms are isomorphisms of the form

$$(\beta \otimes 1) \circ (1 \otimes \alpha) \cong (1 \otimes \alpha) \circ (\beta \otimes 1)$$

arising from the functoriality of the horizontal composition

$$\otimes : T(y, z) \times T(x, y) \to T(x, z).$$

The transformation axioms for β then follow from those for α, thus completing the construction of a weak inverse for α.

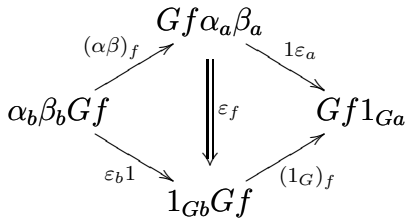
The next step is to construct adjoint equivalences

$$\boldsymbol{\varepsilon} : \alpha\beta \dashv_{eq} 1_G, \quad \boldsymbol{\eta} : 1_F \dashv_{eq} \beta\alpha.$$

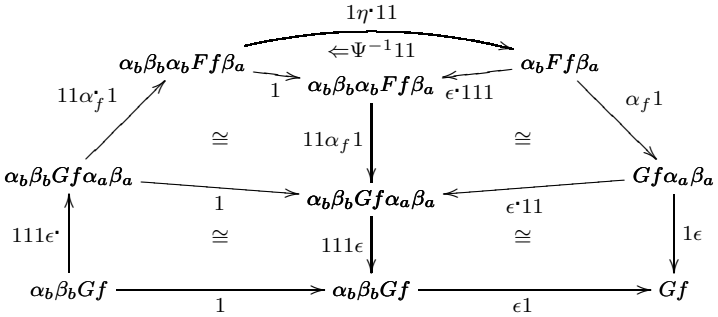
We already have the adjoint equivalences

$$\boldsymbol{\varepsilon}_a : \alpha_a\beta_a \dashv_{eq} 1_{Ga}, \quad \boldsymbol{\eta}_b : 1_{Fb} \dashv_{eq} \beta_b\alpha_b$$

on components in the tricategory T, we need only lift these to adjoint equivalences in the hom-bicategories $\mathbf{Tricat}(S,T)(G,G), \mathbf{Tricat}(S,T)(F,F)$, respectively. Thus we need to equip the collection $\varepsilon_a : \alpha_a\beta_a \Rightarrow I_{Ga}$ with the structure of a trimodification. The 2-cell ε_a is the 2-cell in T of the same name, so we must give the invertible 3-cell displayed below.



The component $(1_G)_f$ is given by $r^{\bullet} \otimes l$, so once again using coherence we write this as the identity and can define ε_f as shown below.



The unmarked isomorphisms in this pasting diagram are (up to coherence) counits for adjoint equivalences for the two left squares, functoriality of the tensor for the top right square, and a counit together with functoriality of the tensor for the bottom right square. There are now two trimodification axioms to check, one for composition and one for units. Both of these axioms follows from coherence, the definitions of Π^{β} and M^{β}, and the definitions of $\Pi^{\beta\alpha}$ and $M^{\beta\alpha}$; they do not require the biadjunction axioms for $\alpha_a \dashv_{bieq} \beta_a$.

To complete the proof that we have an adjoint equivalence

$$\varepsilon : \alpha\beta \dashv_{eq} 1_G$$

in $\mathbf{Tricat}(S,T)(G,G)$, we need only note that a modification $m : \theta \to \phi$ in this bicategory is an equivalence if and only if each component $m_x : \theta_x \Rightarrow \phi_x$ is an equivalence in the appropriate hom-bicategory of T. In this case, the modification m is ε, and each component ε_a is an equivalence in $T(Ga, Ga)$ by construction. We note in passing that the components of the specified pseudo-inverse for ε can be taken to be the 2-cells $\varepsilon_a^{\bullet}$ that appear in the pointwise adjoint equivalences

$$\varepsilon : \alpha_a\beta_a \dashv_{eq} 1_{Ga}.$$

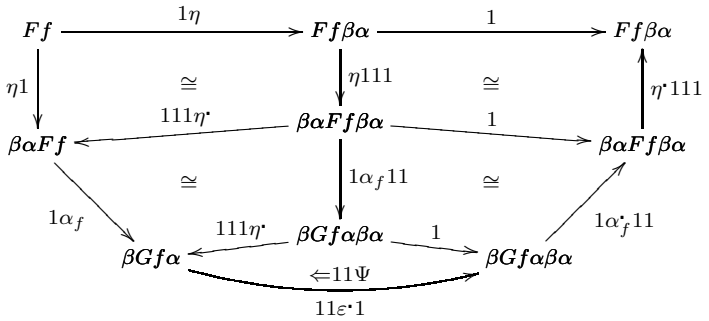
The construction of the adjoint equivalence

$$\eta : 1_F \dashv_{eq} \beta\alpha$$

follows exactly the same pattern as for ε, so we omit most of the details here. In order to check that Φ and Ψ are perturbations, we will need to know the components

$$\eta_f : 1\eta_a \circ (1_F)_f \Rrightarrow (\beta\alpha)_f \circ \eta_b 1$$

for each 1-cell $f : a \to b$. Using coherence, we write η_f as the pasting below.



The isomorphisms are all obtained from units of adjunctions and functoriality of the tensor.

Finally, we must check that the cells Φ_a, Ψ_a constitute a pair of invertible perturbations. This involves checking a single axiom for each 1-cell $f : a \to b$ which we leave to the reader, but it follows from coherence and the biadjunction axioms for both $\alpha_a \dashv_{bieq} \beta_a$ and $\alpha_b \dashv_{bieq} \beta_b$. ∎

To prove our main result, we require two lemmas.

4.3. Lemma. *Let $f : a \to b$ and $g, h : b \to a$ be 1-cells in a tricategory T. If both gf is equivalent to 1_a in $T(a,a)$ and fh is equivalent to 1_b in $T(b,b)$, then h is equivalent to g in $T(b,a)$. In particular, if f is a biequivalence in T and both g_1, g_2 are 1-cells such that $fg_i \simeq 1$ and $g_i f \simeq 1$ for $i = 1, 2$, then $g_1 \simeq g_2$.*

Proof. This follows in the standard way by considering the 1-cell $g_1 f g_2$ in $T(b,a)$. ∎

4.4. Lemma. *Let S, T be tricategories, and assume that every biequivalence f in T is part of a biadjoint biequivalence $f \dashv_{bieq} g$. Assume that $F : S \to T$ is 2-locally an equivalence, i.e., every functor*

$$F : S(a,b)(f,g) \to T(Fa,Fb)(Ff,Fg)$$

is an equivalence of categories. Then every biequivalence h in S is part of a biadjoint biequivalence.

Proof. This lemma follows in the same way that Lemma 1.7 did; the proof requires using the previous lemma in exactly the same way that Lemma 1.7 required the uniqueness of weak inverses in a bicategory. ∎

We now present our main result.

4.5. THEOREM. *Let T be a tricategory, and let f be a biequivalence in T. Then there is a 1-cell g in T and a biadjoint biequivalence $f \dashv_{bieq} g$.*

PROOF. First, recall that for every tricategory T there is a cubical tricategory (i.e., a tricategory in which the hom-bicategories are 2-categories and the unit and composition functors are cubical) $\mathrm{st}(T)$ and a triequivalence $T \to \mathrm{st}(T)$. Since triequivalences satisfy the hypotheses of the lemma above, we are left proving the theorem in the case that T is a cubical tricategory. If T is cubical, then it has a Yoneda embedding $T \hookrightarrow \mathbf{Tricat}(T^{\mathrm{op}}, \mathbf{Gray})$ which also satisfies the hypotheses of the above lemma. Thus if we show that every biequivalence in **Gray** is part of a biadjoint biequivalence, we will have proven the theorem for arbitrary T. But the inclusion $\mathbf{Gray} \hookrightarrow \mathbf{Bicat}$ satisfies the hypotheses of the lemma and we have already proven the claim directly for **Bicat**. ∎

5. Application: lifting monoidal structures

This section will show how to lift monoidal structures on bicategories using the results of Section 4. We refer the reader to [DS] for the definitions of the morphisms and higher cells between monoidal bicategories in the case of **Gray**-monoids, and [GPS] or [Gur] for the definitions of higher cells between tricategories from which these are derived. From this point forward, most calculations will only be described as they are generally straightforward but the pastings used can be very large.

5.1. THEOREM. *Let B be a monoidal bicategory and let C be any bicategory. If $F : C \to B$ is a biequivalence from C to the underlying bicategory of B, then C can be equipped with the structure of a monoidal bicategory and F can be compatibly equipped with the structure of a monoidal functor.*

PROOF. First choose a biadjoint biequivalence $F \dashv_{bieq} G$ in **Bicat**. We now define a monoidal structure on C as follows. The tensor $\boxtimes : C \times C \to C$ is the composite

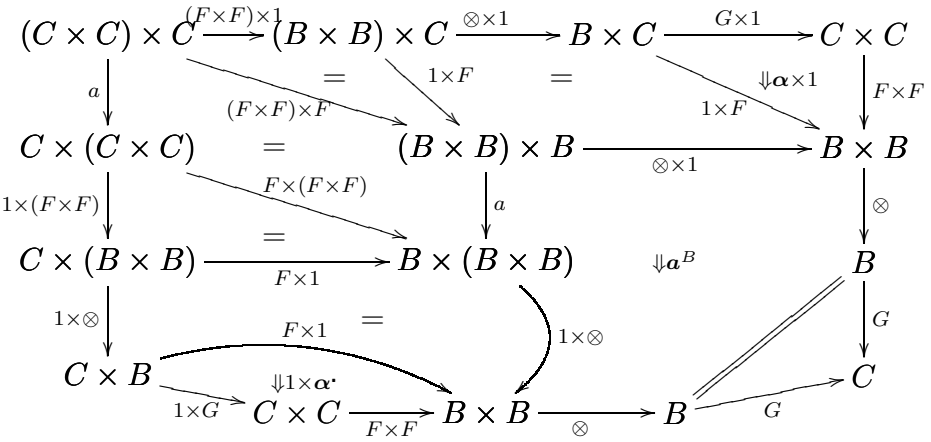
$$C \times C \xrightarrow{F \times F} B \times B \xrightarrow{\otimes} B \xrightarrow{G} C.$$

The unit $I^C : * \to C$ is the composite

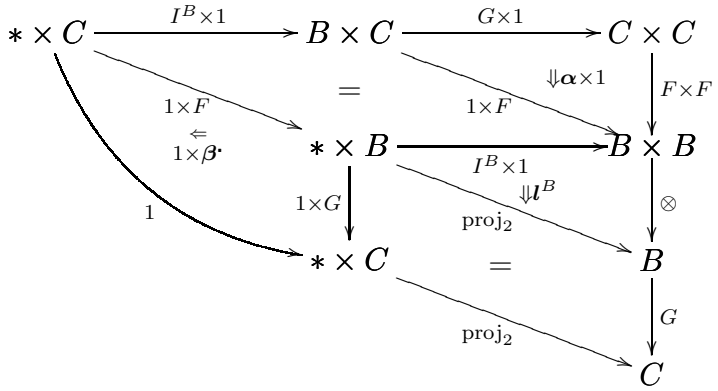
$$* \xrightarrow{I^B} B \xrightarrow{G} C.$$

The associativity adjoint equivalence is given by the pasting below. Here we have

marked identity adjoint equivalences with equal signs.



The left unit adjoint equivalence is given by the pasting below, following the same conventions as above.



The right unit adjoint equivalence is constructed in an analogous fashion.

In order to construct the remaining data, it is useful to compute the components of the above transformations. The component of a^C at (x, y, z) is given as the following composite which we write omitting all the identity components from adjoint equivalences marked with an equal sign.

$$\begin{array}{ll} G\big(FG(Fx \otimes Fy) \otimes Fz\big) & \xrightarrow{G\left(\alpha \otimes 1\right)} \quad G\big((Fx \otimes Fy) \otimes Fz\big) \\ & \xrightarrow{Ga^B} \quad G\big(Fx \otimes (Fy \otimes Fz)\big) \\ & \xrightarrow{G\left(1 \otimes \alpha^{\cdot}\right)} \quad G\big(Fx \otimes FG(Fy \otimes Fz)\big) \end{array}$$

The component of l^C at x is given by the following composite.

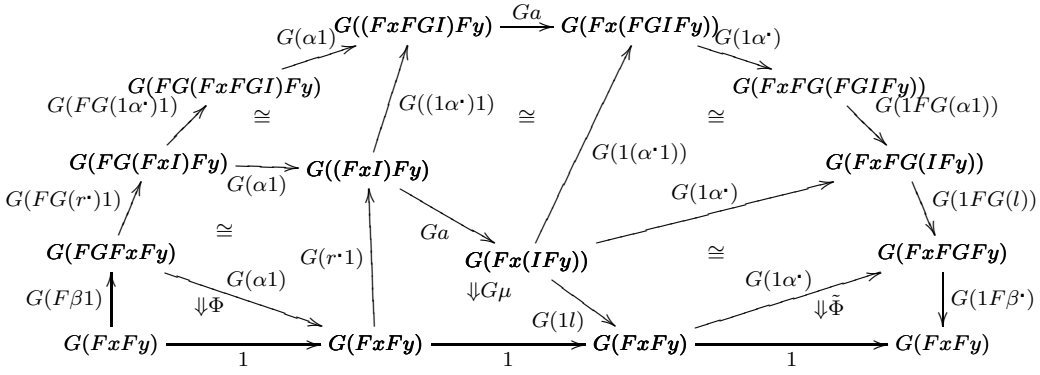
$$G(FGI \otimes Fx) \xrightarrow{G(\alpha \otimes 1)} G(I \otimes Fx) \xrightarrow{Gl} GFx \xrightarrow{\beta^{\cdot}} x$$

The component of r^C at x is given by the following composite.

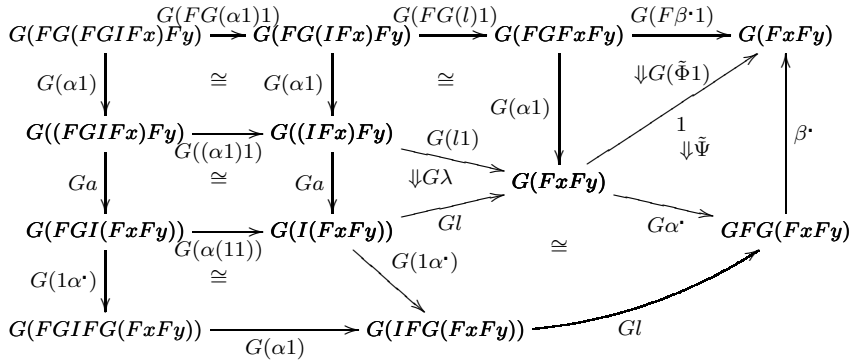
$$G(Fx \otimes FGI) \overset{G(1\otimes\alpha)}{\longrightarrow} G(Fx \otimes I) \overset{Gr}{\to} GFx \overset{\beta\cdot}{\to} x$$

Now we define the invertible modifications π, μ, λ, ρ. We explicitly define the unit modifications μ, λ, ρ, but only describe the construction of the modification π due to the size of the pasting diagram.

The unit modification μ is the pasting below (composed with the unique coherence isomorphism $1 \circ 1 \circ 1 \cong 1$), where all unmarked isomorphisms are naturality isomorphisms and $\widetilde{\Phi}$ is the mate of Φ.

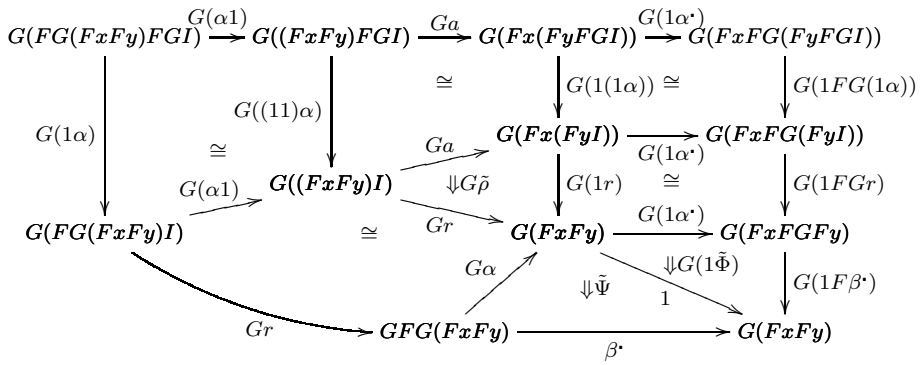


The unit modification λ is the pasting below.



For the unit modification $\rho : 1 \otimes r^{\bullet} \Rightarrow a \circ r^{\bullet}$, we give the following non-standard presentation. Let $\tilde{\rho}$ denote the mate of ρ^{-1} with source $(1 \otimes r) \circ a$ and target r. We thus define $\tilde{\rho}$ as the pasting below and leave it to the reader to construct the usual modification

ρ as necessary.



We now describe the construction of the modification π for the monoidal structure on C. This invertible modification is constructed like the unit modifications by pasting together

- naturality isomorphisms for $\alpha, \alpha^{\bullet}$, and the associator a;
- a single counit coming from $\alpha \dashv_{eq} \alpha^{\bullet}$; and
- the cell $G(\pi^B)$.

We leave it to the reader to construct the appropriate pasting from these cells.

There are now three monoidal bicategory axioms to check. We leave these to the reader as the diagrams are large but the computations simple – the associativity axiom follows by coherence, naturality, and the associativity axiom in B, while both unit axioms follow by coherence, naturality axioms, the corresponding unit axioms in B, and the biadjoint biequivalence axioms.

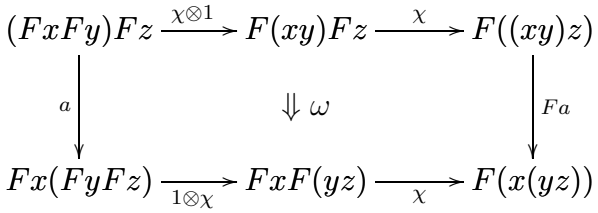
Now we show that F can be equipped with the structure of a monoidal functor. The adjoint equivalence

$$\boldsymbol{\chi} : \otimes_B \circ (F \times F) \to F \circ \otimes_C$$

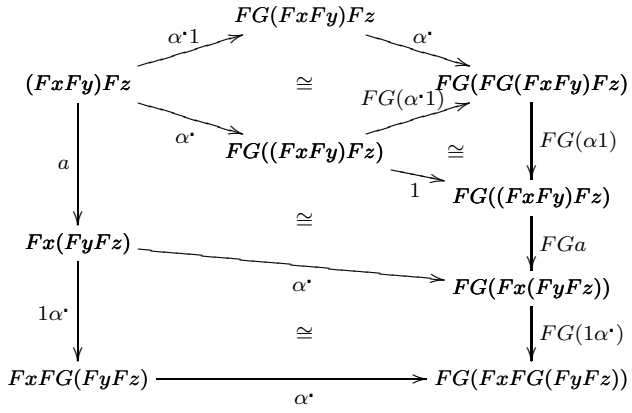
is $\boldsymbol{\alpha}^{\bullet}$, and the adjoint equivalence

$$\boldsymbol{\iota} : I_B \to F \circ I_C$$

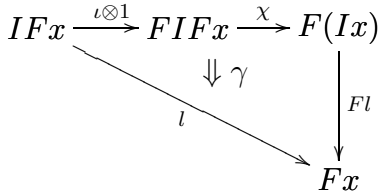
is $\boldsymbol{\alpha}^{\bullet}_I$. The invertible modification ω



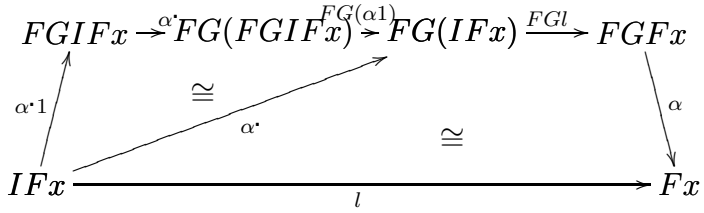
is given by the pasting diagram below in which all the cells are naturality isomorphisms or a counit for $FG(\alpha \otimes 1) \dashv FG(\alpha^{\bullet} \otimes 1)$.



The invertible modification γ



is the pasting diagram shown below where both cells are mates of naturality isomorphisms.



The invertible modification $\delta : \chi \circ (1 \otimes \iota) \circ r^{\bullet} \Rightarrow Fr^{\bullet}$ is defined similarly. There are now two axioms to check to show that these data give a monoidal functor between monoidal bicategories, and once again we leave these simple albeit long computations to the reader. The associativity axiom follows from the transformation axioms for α while the unit axiom requires using the biadjoint biequivalence axioms. ∎

5.2. Remark. It would be possible at this point to prove that the functor G chosen in the proof above can also be given the structure of a monoidal functor. For instance, the transformation χ is given by the transformation β used in constructing the biadjoint biequivalence $F \dashv_{bieq} G$. We could go further, and even show that the entire biadjoint biequivalence $F \dashv_{bieq} G$ can be lifted from **Bicat** to **MonBicat**, showing that the forgetful functor

$$\mathbf{MonBicat} \to \mathbf{Bicat}$$

is the tricategorical analogue of an isofibration. Put another way, the free monoidal bicategory construction is an example of a "flexible 3-monad." The proofs of all of these statements follow in exactly the same fashion as the construction of the monoidal structure on C and F given above.

5.3. Remark. We could also lift braided monoidal, sylleptic monoidal, or symmetric monoidal structures along biequivalences in a similar fashion. For instance, if B is braided with braiding $R_{x,y} : xy \to yx$ then C can be given a braided structure with braiding

$$F(GxGy) \overset{FR_{Gx,Gy}}{\longrightarrow} F(GyGx).$$

In these cases, as above, the entire biadjoint biequivalence could be lifted from **Bicat** to the relevant tricategory of monoidal bicategories of the kind considered.

6. Application: Picard 2-categories

This section will present an application of the main result to the study of Picard 2-categories. This is the analogue, for monoidal bicategories, of the result of Baez and Lauda that the 2-category of 2-groups (or Picard groupoids) is 2-equivalent to the 2-category of coherent 2-groups [BL]. It should be noted that all of the results of this section remain true when we add braided, sylleptic, or symmetric structures. This result could be considered the beginning of the coherence theory for Picard 2-categories, which itself is likely a crucial ingredient in understanding the relationship between Picard 2-categories and stable homotopy 2-types.

6.1. Definition. *A Picard 2-category X is a monoidal bicategory such that for every object x, there exists an object y such that*

$$x \otimes y \simeq I \simeq y \otimes x.$$

6.2. Remark. The reader should note that we call these Picard 2-categories even though the underlying object is a mere bicategory. We have also not assumed that all the 1- and 2-cells are invertible, nor that the monoidal structure is braided, sylleptic, or symmetric. All of these additional features (strictness, invertible higher cells, and symmetry) can be added as desired to produce the notion of Picard 2-category appropriate to a particular application. Analogous results to those we present below can then be proven.

6.3. Definition. *The tricategory* **Pic2Cat** *is the full sub-tricategory of* **MonBicat** *consisting of those monoidal bicategories which are Picard 2-categories.*

For the next definition, recall that every monoidal bicategory X gives rise to a tricategory ΣX with a single object $*$ and single hom-bicategory given by

$$\Sigma X(*, *) = X;$$

horizontal composition is then given by the tensor product, and all of the coherence constraints for the tricategory are given by those for the monoidal structure on X. Thus a biadjoint biequivalence $x \dashv_{bieq} y$ between objects of a monoidal bicategory is defined to be a biadjoint biequivalence $x \dashv_{bieq} y$ in ΣX where now x, y are treated as 1-cells of the tricategory ΣX.

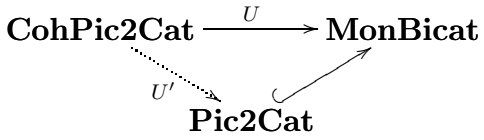
6.4. Definition. *A coherent Picard 2-category (X, inv) is a monoidal bicategory X, a function $inv : ob\ X \to ob\ X$, and for each object x a biadjoint biequivalence $x \dashv_{bieq} inv(x)$.*

6.5. Definition. *The tricategory* **CohPic2Cat** *has as its 0-cells coherent Picard 2-categories X, hom-bicategories defined as*

$$\mathbf{CohPic2Cat}\Big((X, inv_X), (Y, inv_Y)\Big) = \mathbf{MonBicat}(X, Y),$$

and all coherence constraints those inherited from the tricategory **MonBicat**.

6.6. Theorem. *The underlying monoidal bicategory functor U factors (as a strict functor between tricategories) through the inclusion of* **Pic2Cat** *into* **MonBicat**.



*The strict functor $U' :$ **CohPic2Cat** $\to$ **Pic2Cat** is a triequivalence.*

Proof. The first statement is clear, as the underlying monoidal bicategory of a coherent Picard 2-category (X, inv) is obviously a Picard 2-category, and all of the higher dimensional structure involved in the definitions of these two tricategories agrees. For the second statement, we must prove that U' is locally a biequivalence and triessentially surjective. Now U' is the identity functor on hom-bicategories, so is locally a biequivalence. To show that U' is triessentially surjective, note that Theorem 4.5 actually implies that U' is surjective on objects as we can always choose a biadjoint biequivalence $x \dashv_{bieq} y$ for any object x with the property that $x \otimes y \simeq I \simeq y \otimes x$. ∎

This theorem produces the most basic kind of equivalence between the theory of Picard 2-categories and its coherent version. For the rest of this paper, we will sketch an improvement to this equivalence by explaining how one might go about proving that not only can Picard 2-categories be replaced by coherent ones, but also that monoidal functors can be replaced by ones that preserve the choice of inverses up to equivalence.

6.7. Definition. *1. Let X be a bicategory. Define the bicategory X^{op} to be the one with*

- *the same objects as X,*
- *$X^{op}(a, b) = X(b, a)$,*
- *composition given by $g \circ^{op} f = f \circ g$, and*

- *constraints given by* $a^{op}_{h,g,f} = a^{-1}_{f,g,h}$, $l^{op}_f = r_f$, *and* $r^{op}_f = l_f$.

2. Let Y *be a monoidal bicategory. Define the monoidal bicategory* Y^{rev} *to be the one with*

- *underlying bicategory the same as* Y,
- $a \otimes^{rev} b = b \otimes a$,
- $I^{rev} = I$,
- *all adjoint equivalences given by the opposites of the appropriate adjoint equivalences for the monoidal structure on* Y, *and*
- *all invertible 2-cell data the same as that for* Y.

3. Let Z *be a monoidal bicategory. Define* Z^r *to be* $(Z^{op})^{rev} = (Z^{rev})^{op}$.

6.8. Proposition. *Let* (X, inv) *be a coherent Picard 2-category. The function on objects*

$$inv : ob\ X \to ob\ X$$

extends to a monoidal functor of the same name,

$$inv : X \to X^r.$$

Proof. We have already defined inv on objects, so now it is time to define it on 1- and 2-cells. To do this, we must fix notation. For an object x, the biadjoint biequivalence $x \dashv_{bieq} \mathrm{inv}(x)$ consists of

- an adjoint equivalence $\epsilon_x \dashv \epsilon_x^{\cdot}$ between the objects $x \otimes \mathrm{inv}(x)$ and I,
- an adjoint equivalence $\eta_x \dashv \eta_x^{\cdot}$ between the objects I and $\mathrm{inv}(x) \otimes x$,
- and invertible 2-cells Ψ_x, Φ_x, satisfying the necessary axioms.

Thus for a 1-cell $f : x \to y$, we define $\mathrm{inv}(f) : \mathrm{inv}(y) \to \mathrm{inv}(x)$ as the following composite (ignoring associativity and unit constraints by coherence).

$$\mathrm{inv}(y) \xrightarrow{\eta_x 1} \mathrm{inv}(x) \otimes x \otimes \mathrm{inv}(y) \xrightarrow{1f1} \mathrm{inv}(x) \otimes y \otimes \mathrm{inv}(y) \xrightarrow{1\epsilon_y} \mathrm{inv}(x)$$

We then define $\mathrm{inv}(\alpha)$ to be $1 * (1 \otimes \alpha \otimes 1) * 1$.

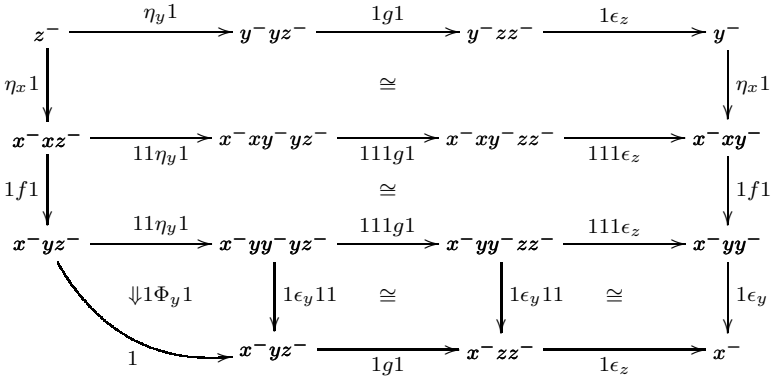
Next we must define structure constraints

$$\mathrm{inv}(g) \circ^r \mathrm{inv}(f) \cong \mathrm{inv}(g \circ f)$$
$$1_x \cong \mathrm{inv}(1_x)$$

and check that these give a functor of bicategories. Now $\mathrm{inv}(g) \circ^r \mathrm{inv}(f)$ in X^r is defined to be the composite $\mathrm{inv}(f) \circ \mathrm{inv}(g)$, so we in fact require an isomorphism of the form

$$\mathrm{inv}(f) \circ \mathrm{inv}(g) \cong \mathrm{inv}(g \circ f).$$

It is given by the pasting diagram below, in which we have written $\mathrm{inv}(x)$ as x^- and all the unmarked isomorphisms are functoriality of the tensor product.



The isomorphism $1_{\mathrm{inv}(x)} \cong \mathrm{inv}(1_x)$ is (modulo coherence) Φ_x^{-1}.

There are now three axioms to check, one for associativity and two for units. All three of these axioms follow by using the functoriality of the tensor product and then invoking coherence for monoidal bicategories (i.e., coherence for tricategories in the single-object case). The unit axioms then require the equation $\Phi \circ \Phi^{-1} = 1$, while the associativity axiom only uses naturality. Thus we have shown that inv is a functor of bicategories $X \to X^{op}$. Now we turn to showing that it is monoidal.

The first step in showing that inv is a monoidal functor $X \to X^r$ is to construct an adjoint equivalence

$$\chi : \otimes^r \circ (\mathrm{inv} \times \mathrm{inv}) \Rightarrow \mathrm{inv} \circ \otimes.$$

On components, this gives adjoint equivalences

$$\mathrm{inv}(x) \otimes^r \mathrm{inv}(y) = \mathrm{inv}(y) \otimes \mathrm{inv}(x) \to \mathrm{inv}(x \otimes y)$$

in X^r, hence adjoint equivalences in X with the source and target reversed. We define the component $\chi_{x,y}$ below, and the rest of the adjoint equivalence will be defined in the obvious fashion. We retain the same convention as above for writing $\mathrm{inv}(x)$ as x^-.

$$(xy)^- \xrightarrow{\eta_y 1} y^- y(xy)^- \xrightarrow{1\eta_x 11} y^- x^- xy(xy)^- \xrightarrow{11\epsilon_{xy}} y^- x^-$$

The second step in giving inv a monoidal structure is to construct an adjoint equivalence

$$\iota : I^r \Rightarrow \mathrm{inv}(I)$$

in X^r. Since $I^r = I$, this is the obvious adjoint equivalence with left adjoint shown below.

$$I^- \xrightarrow{l\cdot} II^- \xrightarrow{\epsilon_I} I$$

The third step in giving inv a monoidal structure is to define three invertible modifications ω, γ, δ. We leave it to the reader to write down the pasting diagrams as they are quite large, but we explain here which cells will be included in each. In each case, there will be a large number of coherence cells from the monoidal bicategory structure, most of which will arise from the functoriality of the tensor product. The other cells in each case are as follows:

- for ω, the remaining cells are $\Phi_{xy}, \Psi_{\mathrm{inv}(xyz)}$, and Φ_{yz}^{-1};
- for γ, the remaining cells are Φ_{Ix}, Φ_I, and $\Psi_{\mathrm{inv}(x)}$;
- for δ, the remaining cell is Φ_I.

(The apparent asymmetry in the definitions of γ and δ is due to the fact that γ has the cell Fl in the source while δ has the cell $Fr^{\cdot}$ in the target.)

Finally, there are two monoidal functor axioms to check. Both follow from coherence for monoidal functors (a special case of coherence for functors of tricategories) and the biadjoint biequivalence axioms. ∎

6.9. Remark. One can go further to prove that the monoidal functor inv is actually a monoidal biequivalence. First we can equip X^r with a canonical coherent Picard 2-category structure using the one given on X. Then it is easy to show that $\mathrm{inv}^2(x)$ is equivalent to x in X as both are weak inverses for $\mathrm{inv}(x)$, from which it follows that inv squares to the identity.

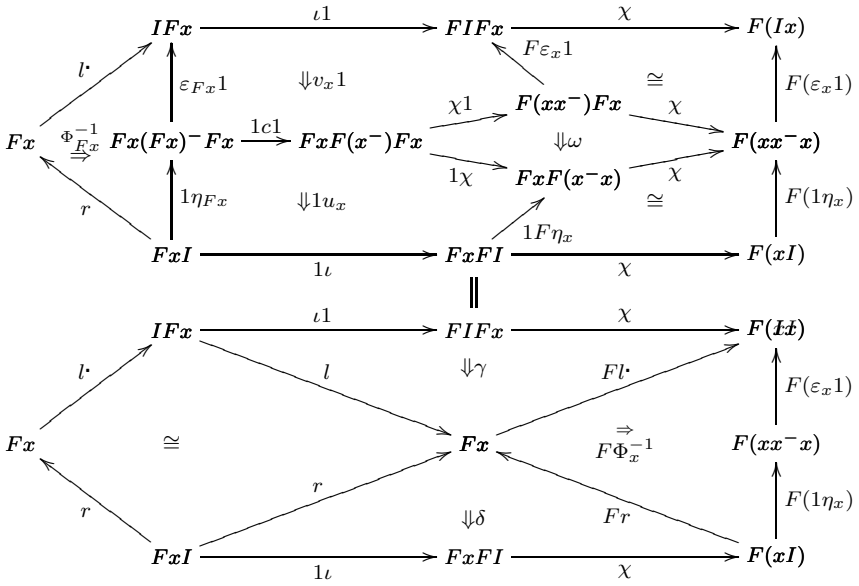
6.10. Definition. *A coherent functor $(F, c, u, v) : (X, inv_X) \to (Y, inv_Y)$ between coherent Picard 2-categories consists of the following data:*

- *a monoidal functor $F : X \to Y$,*
- *an equivalence 1-cell $c_x : (Fx)^- \to F(x^-)$ for each object $x \in X$, and*
- *a pair of invertible 2-cells u_x, v_x for each object $x \in X$ as displayed below.*

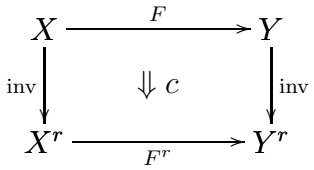
$$\begin{array}{ccccc}
(Fx)^- Fx & \xrightarrow{c_x 1} & F(x^-)Fx & \xrightarrow{\chi} & F(x^- x) \\
{\scriptstyle \eta_{Fx}}\uparrow & & \Downarrow u_x & & \uparrow{\scriptstyle F\eta_x} \\
I & & \xrightarrow[\iota]{} & & FI
\end{array}$$

$$\begin{array}{ccccc}
I & & \xrightarrow{\iota} & & FI \\
{\scriptstyle \varepsilon_{Fx}}\uparrow & & \Downarrow v_x & & \uparrow{\scriptstyle F\varepsilon_x} \\
Fx(Fx)^- & \xrightarrow[1c_x]{} & FxF(x^-) & \xrightarrow[\chi]{} & F(xx^-)
\end{array}$$

These are subject to the following axiom.



6.11. Remark. We could have structured this definition in a slightly different fashion in a variety of ways. First, the equivalence 1-cells c_x could have been the components of a transformation c as shown here.



Second, we could have asked that the invertible 2-cells u_x, v_x could have been the components of a pair of invertible modifications. To express the axioms above as diagrams of modifications would have required that the η_x, ε_x be the components of transformations, which in turn would require that the functor inv be covariant instead of contravariant. Thus we would have to restrict attention to those Picard 2-categories in which every 1-cell is an equivalence; in fact, we would need every 1-cell to come as part of a specified adjoint equivalence in order to prescribe inv as a covariant *functor*.

Third, we could have required a third axiom about how c, u, v interact with Ψ. This axiom follows from the first axiom by using the biadjoint biequivalence axioms. In addition, the pastings involved are larger than the one for the axiom above as they involve two different uses of c instead of just one, so it requires additional naturality squares.

6.12. Theorem. *Let $(X, inv_X), (Y, inv_Y)$ be coherent Picard 2-categories, and let $F : X \to Y$ be a monoidal functor. Then F underlies a coherent functor*

$$(F, c, u, v) : (X, inv_X) \to (Y, inv_Y).$$

PROOF. The 1-cell $c_x : F(x^-) \to Fx^-$ is given by the following composite where the last arrow is given by a composite of coherence cells and is thus unique up to unique isomorphism by coherence for functors.

$$F(x^-) \xrightarrow{\eta_{Fx}1} Fx^-FxF(x^-) \xrightarrow{1\chi} Fx^-F(xx^-) \xrightarrow{1F\varepsilon} Fx^-FI \longrightarrow Fx^-$$

It is immediate that c_x is an equivalence 1-cell.

We must now construct the invertible 2-cells u_x, v_x and check the two axioms. The cell u_x is a pasting of coherence cells from both Y and the functor F, together with Φ_{Fx}. The cell v_x is constructed similarly out of coherence cells and $F\Phi_x^{-1}$. The two axioms are straightforward diagram chases. ∎

6.13. COROLLARY. *Let X be a Picard 2-category. Then X has a coherent structure which is unique in the following sense: if (X, inv) and (X, inv') are two coherent structures on X, then the identity functor on X lifts to a coherent functor*

$$(1, c, u, v) : (X, inv) \to (X, inv').$$

6.14. REMARK. We leave it to the reader to define coherent transformations and modifications. A coherent transformation will involve additional data, while a coherent modification will only involve a new axiom. Defined correctly, it is then possible to prove that the forgetful functor from the tricategory in which all cells are coherent to the tricategory **Pic2Cat** is a triequivalence. This shows that a coherent structure on a given Picard 2-category is unique in the strongest possible sense.

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