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A PRESHEAF INTERPRETATION OF THE GENERALIZED FREYD CONJECTURE

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ABSTRACT. We give a generalized version of the Freyd conjecture and a way to think about a possible proof. The essential point is to describe an elementary formal reduction of the question that holds in any triangulated category. There are no new results, but at least one known example drops out very easily.

In algebraic topology, the generating hypothesis, or Freyd conjecture, is a long-standing conjecture about the structure of the stable homotopy category. It was initially formulated in 1965 [Freyd, 1966] and remains open to this day. Because the original conjecture has proved difficult to analyze, recent work has turned to studying similar conjectures in categories that share many properties with the stable homotopy category in hopes of further understanding the types of categories in which such a conclusion holds. In this note, we state a version of the generating hypothesis for an arbitrary triangulated category and analyze conditions under which this hypothesis holds. We emphasize that we impose no additional conditions on our triangulated categories, so that our results show which *formal* properties of a category imply the Freyd conjecture. Therefore our results give conceptual insight into the kind of category in which the generating hypothesis can be expected to hold.

1. The generalized Freyd conjecture

Let $\mathscr{T}$ be a triangulated category and write $[X,Y]$ for the abelian group of maps $X \to Y$ in $\mathscr{T}$. Let $\mathscr{B}$ be a (small) full subcategory of $\mathscr{T}$ closed under its translation (or shift) functor Σ and let $\mathscr{C}$ be the thick full subcategory of $\mathscr{T}$ that is generated by $\mathscr{B}$; write $\iota\colon \mathscr{B} \to \mathscr{C}$ for the inclusion. For emphasis, we often write $\mathscr{B}(X,Y) = [X,Y]$ when $X, Y \in \mathscr{B}$ and $\mathscr{C}(X,Y) = [X,Y]$ when $X, Y \in \mathscr{C}$. The category $\mathscr{B}$ is pre-additive (enriched over $\mathscr{A}b$), and $\mathscr{C}$ is additive (has biproducts). Let $\mathscr{P}\mathscr{B}$ and $\mathscr{P}\mathscr{C}$ denote the categories of abelian presheaves defined on $\mathscr{B}$ and $\mathscr{C}$. They consist of the additive functors from $\mathscr{B}^{op}$ or $\mathscr{C}^{op}$ to $\mathscr{A}b$ and the additive natural transformations between them.

1.1. DEFINITION. Define the Freyd functor $\mathbb{F}\colon \mathscr{T} \to \mathscr{P}\mathscr{B}$ by sending an object X to the functor $\mathbb{F}X$ specified on objects and morphisms of $\mathscr{B}$ by $\mathbb{F}X(-) = [-,X]$ and sending a

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map $f\colon X \to Y$ to the natural transformation $f_* = [-, f]$. Define $\mathbb{Y}\colon \mathscr{T} \to \mathscr{P}\mathscr{C}$ similarly. We are only interested in the restrictions of $\mathbb{F}$ and $\mathbb{Y}$ to $\mathscr{C}$, and then $\mathbb{Y}\colon \mathscr{C} \to \mathscr{P}\mathscr{C}$ is the standard Yoneda embedding.

1.2. Conjecture. [The generalized Freyd conjecture] *The functor $\mathbb{F}\colon \mathscr{C} \to \mathscr{P}\mathscr{B}$ is faithful. Equivalently, $\mathbb{F}f = 0$ if and only if $f = 0$. We then say that the Freyd conjecture holds for the pair $(\mathscr{C}, \mathscr{B})$.*

The conjecture is also called the "generating hypothesis" since it is equivalent to the assertion that the set of objects of $\mathscr{B}$ forms a generator for the category $\mathscr{C}$. This means that if there are distinct maps $f, g\colon X \to Y$ in $\mathscr{C}$, then there is a map $e\colon W \to X$ in $\mathscr{C}$ such that $W \in \mathscr{B}$ and $f \circ e \neq g \circ e$.

We hasten to add that the conjecture is false in this generality. Additional hypotheses are needed, but none will be relevant to our formal analysis. For example, we might as well assume that a map $f\colon X \to Y$ in $\mathscr{C}$ is an isomorphism if and only $f_*\colon \mathbb{F}X \to \mathbb{F}Y$ is an isomorphism, as holds in the stable homotopy category. This condition is necessary but not sufficient for the Freyd conjecture to hold. Indeed, if Z is the third term in an exact triangle with one map f, then f is an isomorphism if and only if $Z = 0$ and $\mathbb{F}f$ is an isomorphism if and only if $\mathbb{F}Z = 0$. If $\mathbb{F}Z = 0$ and the Freyd conjecture is true, then $Z = 0$ since its identity map is the zero map.

1.3. Example. Take $\mathscr{T} = \mathrm{Ho}\mathscr{S}$ to be the stable homotopy category. Let $\mathscr{B}$ consist of the sphere spectra S^n for integers n. Then $\mathscr{C}$ is the homotopy category of finite CW spectra. Freyd [Freyd, 1966] conjectured that a map f in $\mathscr{C}$ is zero if and only if it induces the zero homomorphism $f_*\colon \pi_*(X) \to \pi_*(Y)$. By the following observation, this is a special case of our Conjecture 1.2.

1.4. Lemma. *In Example 1.3, the category $\mathscr{P}\mathscr{B}$ is isomorphic to the category $\mathscr{M}$ of right modules over the ring π_* of stable homotopy groups of spheres. Under this isomorphism, the Freyd functor $\mathbb{F}$ coincides with the stable homotopy group functor $\pi_*\colon Ho\mathscr{S} \to \mathscr{M}$.*

Proof. We have $\mathscr{B}(S^m, S^n) = \pi_m(S^n) \cong \pi_{m-n}$. For $T \in \mathscr{P}\mathscr{B}$, let $T_n = T(S^n)$. The additive contravariant functor T gives homomorphisms

$$T\colon \pi_{m-n} \cong \mathscr{B}(S^m, S^n) \to \mathscr{A}b(T_n, T_m).$$

By adjunction, these give homomorphisms $T_n \otimes \pi_{m-n} \to T_m$. The functoriality gives the formula $(tx)y = t(xy)$. No signs appear since we are taking right modules, as is dictated categorically by contravariance. Conversely, given a right π_*-module M, define $T(S^n) = M_n$ and, for $t \in M_n$ and $x \in \pi_{m-n} \cong \mathscr{B}(S^m, S^n)$ define $T(x)(t) = tx \in M_m$. The module axioms ensure that T is a functor. This specifies the isomorphism of categories, and the consistency of $\mathbb{F}$ and π_* is clear. ∎

The following example originally prompted us to take a presheaf perspective on the Freyd conjecture.

1.5. EXAMPLE. Let G be a compact Lie group and take $\mathscr{T} = \mathrm{Ho}G\mathscr{S}$ to be the equivariant stable homotopy category. Let $G\mathscr{B}$ consist of the orbit G-spectra

$$S^n[G/H] \equiv \Sigma^n\Sigma^\infty(G/H)_+$$

for integers n and closed subgroups H of G. The thick subcategory $G\mathscr{C}$ generated by $G\mathscr{B}$ is the category of retracts of finite G-CW spectra. The equivariant version of the Freyd conjecture asserts that a map f in $G\mathscr{C}$ is zero if and only if it induces the zero homomorphism $f_*\colon \pi_*^H(X) \to \pi_*^H(Y)$ for all H, where

$$\pi_n^H(X) \equiv \pi_n(X^H) \cong [S^n[G/H], X]_G.$$

Again, this is a special case of our Conjecture 1.2. This example is the focus of [Bohmann, 2010].

1.6. REMARK. The full subcategory $\mathscr{B}_0$ of $\mathscr{B}$ whose objects are the $S[G/H] \equiv S^0[G/H]$ is called the Burnside category. A Mackey functor (or G-Mackey functor) M is by definition an object of $\mathscr{P}G\mathscr{B}_0$. When G is finite, this agrees with the more usual algebraic definition ([Lewis et al, 1986, V§9] or [May, 1996, IX§4, XIX§3]). A map of Mackey functors is a natural transformation, that is, a map of presheaves.

1.7. DEFINITION. The graded Burnside category $G\pi_*$ has objects the $S[G/H]$. Its abelian group of maps of degree n from $S[G/H]$ to $S[G/J]$ is

$$\pi_n^H(S[G/J]) = [S^n[G/H], S[G/J]]_G.$$

Composition is induced by suspension and composition in $G\mathscr{B}$ in the evident fashion. Define a right $G\pi_*$-module M to be a graded presheaf, that is, a contravariant functor $G\pi_* \to \mathscr{A}b_*$, where $\mathscr{A}b_*$ is the category of graded abelian groups.

1.8. LEMMA. *In Example 1.5, the category $\mathscr{P}G\mathscr{B}$ is isomorphic to the category $G\mathscr{M}$ of right $G\pi_*$-modules. Under this isomorphism, the Freyd functor $\mathbb{F}$ coincides with the equivariant stable homotopy group functor $\pi_*^{(-)}\colon HoG\mathscr{S} \to G\mathscr{M}$.*

The proof is the same as that of Example 1.3. In this case, the presheaf formulation of the Freyd conjecture appears more natural to us than the equivalent homotopy group reformulation.

1.9. EXAMPLE. In the rational equivariant stable homotopy category, the Freyd conjecture is true if G is finite [Greenlees and May, 1995] and is false if $G = S^1$ [Bohmann, 2010].

Here is another example where the presheaf formulation, and the use of many objects and their shifts rather than a single object and its shifts in $\mathscr{B}$, may be more natural than the formulation of the Freyd conjecture studied so far.

1.10. Example. In several recent papers, Benson, Carlson, Chebolu, Christensen and Mináč [Benson et al, 2007, Carlson et al, 2009, Chebolu et al, 2007] study the Freyd conjecture in the stable module category of a finite group G over a field k whose characteristic divides the order of G. This is a triangulated category StMod_{kG} obtained from the category of kG–modules by modding out by maps that factor through a projective module. They restrict to the thick subcategory generated by the trivial representation k and ask whether the Tate cohomology functor is faithful on this subcategory. Since the Tate cohomology of a module M is given by maps out of k in the stable module category,

$$\hat{H}^i(G, M) = \mathrm{Stmod}_{kG}(\Omega^i k, M),$$

their formulation is equivalent to our Conjecture 1.2 with $\mathscr{B} = \{\Omega^i k\}$. In this context, they prove that the Freyd conjecture holds in the stable module category of kG–modules if and only if the p–Sylow subgroup of G is either C_2 or C_3, where p is the characteristic of k [Carlson et al, 2009]. Their proof in fact shows that this variant of the Freyd conjecture holds if and only if the thick subcategory generated by k consists of finite direct sums of suspensions of k.

For p-groups, the trivial module is the only irreducible module over kG, and the thick subcategory generated by k is the subcategory of compact objects in the stable module category. For non-p-groups, other irreducible modules exist. It is thus natural to generalize the Freyd conjecture to our presheaf context by letting $\mathscr{B}$ be the category whose objects are the suspensions of the irreducible kG–modules. The thick subcategory $\mathscr{C}$ generated by this $\mathscr{B}$ is then the subcategory of compact objects in the stable module category and one can ask when Conjecture 1.2 holds. This presheaf version of the generating hypothesis in StMod_{kG} takes into account all the generating objects in the stable module category.

2. A general line of argument

The presheaf perspective suggests a method of attack on the generalized Freyd conjecture. We have the forgetful functor

$$\mathbb{U} = \iota^* \colon \mathscr{P}\mathscr{C} \to \mathscr{P}\mathscr{B}$$

given by restricting presheaves defined on $\mathscr{C}^{op}$ to the full subcategory $\mathscr{B}^{op}$. The functor $\mathbb{U}$ has a left adjoint prolongation functor

$$\mathbb{P} = \iota_! \colon \mathscr{P}\mathscr{B} \to \mathscr{P}\mathscr{C}.$$

For $T \in \mathscr{P}\mathscr{B}$ and $K \in \mathscr{C}$, $\mathbb{P}T(K)$ is the categorical tensor product

$$\mathbb{P}T(K) = T \otimes_{\mathscr{B}} \mathscr{C}(K, -).$$

This is an example of a left Kan extension, and it is one used extensively in recent work on spectra (e.g. [Mandell and May, 2002, Mandell et al, 2001]). Since $\mathscr{B}$ is a full subcategory of $\mathscr{C}$, the unit $\eta\colon \mathrm{Id} \to \mathbb{UP}$ of the adjunction is a natural isomorphism [Mandell et al, 2001, I.3.2].

We focus attention on the counit $\varepsilon\colon \mathbb{PU} \to \mathrm{Id}$ of the adjunction. By its definition, we see that the Freyd functor $\mathbb{F}\colon \mathscr{C} \to \mathscr{P}\mathscr{B}$ coincides with the composite $\mathbb{UY}$. We therefore have the map

$$\varepsilon\colon (\mathbb{PF}X)(K) = (\mathbb{PUY}X)(K) \to (\mathbb{Y}X)(K) = [K, X] \tag{1}$$

for $K, X \in \mathscr{C}$. Note in particular that if $K \in \mathscr{B}$, then this map is an isomorphism. Indeed, we then have $(\mathbb{PF}X)(K) = (\mathbb{UPF}X)(K)$ and $\varepsilon = \mathbb{U}\varepsilon$. By one of the triangle identities, $\mathbb{U}\varepsilon\colon \mathbb{UPU} \to \mathbb{U}$ is an isomorphism since $\eta\colon \mathrm{Id} \to \mathbb{UP}$ is an isomorphism.

Consider a map $f\colon X \to Y$ in $\mathscr{C}$. Since $\mathbb{F} \cong \mathbb{UPF}$, $\mathbb{F}f = 0$ if and only if $\mathbb{PF}f = 0$. If the map (1) is an epimorphism for all $K, X \in \mathscr{C}$, then $\mathbb{PF}f = 0$ implies $\mathbb{Y}f = 0$. By the Yoneda lemma, $\mathbb{Y}f = 0$ if and only if $f = 0$. Therefore this epimorphism condition gives a sufficient condition for the Freyd conjecture for $(\mathscr{C}, \mathscr{B})$ to hold. Moreover, for any map $g\colon K \to X$ in $\mathscr{C}$, we have the commutative diagram

$$\begin{tikzcd}
\mathbb{PF}X(X) \arrow[r, "\varepsilon"] \arrow[d, "g^*"'] & \mathbb{Y}X(X) = [X, X] \arrow[d, "g^*"] \\
\mathbb{PF}X(K) \arrow[r, "\varepsilon"'] & \mathbb{Y}X(K) = [K, X].
\end{tikzcd}$$

Since g^* on the right takes the identity map of X to g, ε is an epimorphism in general if the identity map of X is in the image of ε. These elementary formal considerations already give the following conclusion.

2.1. Proposition. *The Freyd conjecture holds for $(\mathscr{C}, \mathscr{B})$ if for all $X \in \mathscr{C}$ the identity map of X is in the image of $\varepsilon\colon (\mathbb{PF}X)(X) \to [X, X]$.*

To see what is involved in verifying the hypothesis, consider an exact triangle

$$K \longrightarrow L \longrightarrow M \longrightarrow \Sigma K \tag{2}$$

in $\mathscr{T}$, where K, L, and M are in $\mathscr{C}$. We have the commutative diagram

$$\begin{tikzcd}
\cdots \arrow[r] & \mathbb{PF}X(\Sigma K) \arrow[r] \arrow[d, "\varepsilon"'] & \mathbb{PF}X(M) \arrow[r] \arrow[d, "\varepsilon"] & \mathbb{PF}X(L) \arrow[r] \arrow[d, "\varepsilon"] & \mathbb{PF}X(K) \arrow[r] \arrow[d, "\varepsilon"] & \cdots \\
\cdots \arrow[r] & \mathbb{Y}X(\Sigma K) \arrow[r] & \mathbb{Y}X(M) \arrow[r] & \mathbb{Y}X(L) \arrow[r] & \mathbb{Y}X(K) \arrow[r] & \cdots .
\end{tikzcd} \tag{3}$$

Since $\mathbb{Y}X(K) = [K, X]$, the lower row is exact. By definition, $\mathscr{C}$ is the smallest subcategory of $\mathscr{T}$ that contains $\mathscr{B}$, is closed under retracts, and has the property that if two terms of an exact triangle are in $\mathscr{C}$ then so is the third. By an easy retract argument and the five lemma, this gives the following conclusion.

2.2. Proposition. *If the top row of Diagram (3) is exact for every exact triangle (2) and every $X \in \mathscr{C}$, then $\varepsilon\colon \mathbb{PF}X \to \mathbb{Y}X$ is an isomorphism for every $X \in \mathscr{C}$ and the Freyd conjecture holds for $(\mathscr{C}, \mathscr{B})$.*

There is a reinterpretation of the exactness hypothesis that makes it reminiscent of the standard result that the adjoint (if it exists) of an exact functor between triangulated categories is exact. For K and X in $\mathscr{C}$, the abelian group $\mathbb{PF}X(K)$ is the coequalizer in $\mathscr{A}b$ displayed in the diagram

$$\begin{array}{c} \sum_{I,J\in\mathscr{B}} \mathscr{C}(J,X) \otimes \mathscr{B}(I,J) \otimes \mathscr{C}(K,I) \\ \downarrow\downarrow \\ \sum_{J\in\mathscr{B}} \mathscr{C}(J,X) \otimes \mathscr{C}(K,J) \\ \downarrow \\ \mathscr{C}(-,X) \otimes_{\mathscr{B}} \mathscr{C}(K,-), \end{array} \tag{4}$$

where the parallel arrows are given by composition in $\mathscr{C}$.

Let us write $\mathscr{P}_d\mathscr{B}$ and $\mathscr{P}_d\mathscr{C}$ for the categories of covariant additive functors on $\mathscr{B}$ and $\mathscr{C}$, and similarly write $\mathbb{U}_d$, $\mathbb{P}_d$, $\mathbb{F}_d$, and $\mathbb{Y}_d$ for the corresponding functors. (The d stands for dual.) We are just interchanging $\mathscr{B}$ and $\mathscr{C}$ with their opposite categories. Visibly, we again have $\mathbb{U}_d\mathbb{Y}_d = \mathbb{F}_d$ and again have an adjunction $(\mathbb{P}_d, \mathbb{U}_d)$ with $\mathbb{U}_d\mathbb{P}_d \cong \mathrm{Id}$. By symmetry, we have

$$\mathbb{PF}X(K) = \mathbb{P}_d\mathbb{F}_dK(X). \tag{5}$$

But in this dual reformulation, the exactness hypothesis on K for fixed X is now a levelwise exactness statement about the composite functor $\mathbb{P}_d\mathbb{F}_d\colon \mathscr{C}^{op} \to \mathscr{P}_d\mathscr{C}$. Since $\mathbb{F}_dK(J) = \mathscr{C}(K,J)$ for $J \in \mathscr{B}$, $\mathbb{F}_d$ clearly takes exact triangles in the variable K to exact sequences for each fixed J. Thus a more general question to ask is whether or not $\mathbb{P}_d\colon \mathscr{P}_d\mathscr{B} \to \mathscr{P}_d\mathscr{C}$ preserves levelwise exactness. That is, is it true that if $T' \to T \to T''$ is a sequence of diagrams $\mathscr{B} \to \mathscr{A}b$ such that the sequence $T'(J) \to T(J) \to T''(J)$ is exact for each $J \in \mathscr{B}$, then the sequence $\mathbb{P}T'(X) \to \mathbb{P}T(X) \to \mathbb{P}T''(X)$ is exact for all $X \in \mathscr{C}$?

Observe that we have not yet used any hypothesis on $\mathscr{B}$, other than that it generates the thick subcategory $\mathscr{C}$ of the triangulated category $\mathscr{T}$. Thus all that we have done is to give a purely formal reduction of the general problem.

3. The derived category of ring

Our framework for understanding the Freyd conjecture leads to a transparent proof of the result of Lockridge [Lockridge, 2007, 3.9] that the generalized Freyd conjecture holds in the derived category of a von Neumann regular ring. We simply observe that the hypotheses of Proposition 2.2 hold in this case by one of the equivalent definitions of a von Neumann regular ring. However, our methods do not prove the converse, which is proved in [Hovey et al, 2007, Lockridge, 2007].

Using right R-modules for definiteness, let $\mathscr{D}(R)$ be the derived category of a ring R and let $\mathscr{B}$ be the full subcategory of $\mathscr{D}(R)$ whose objects are the shifts $\Sigma^i R$ of the chain complex that is R concentrated in degree 0. Then $\mathscr{C}$ is the category of perfect chain complexes, namely those isomorphic in $\mathscr{D}(R)$ to bounded chain complexes of finitely generated projective R-modules. The Freyd functor assigns the homology groups $\mathscr{B}(\Sigma^i R, X) = H_i X$ to a chain complex X, and Conjecture 1.2 is the assertion that a map between perfect chain complexes is 0 in $\mathscr{D}(R)$ if it induces the zero map on homology. Defining $H^i(K) = \mathscr{C}(K, \Sigma^i R)$, as usual, we have the following observation in this case.

3.1. Lemma. *For perfect chain complexes K and X, $\mathbb{PF}X(K)$ is isomorphic to the abelian group $\sum_i H_i(X) \otimes_R H^i(K)$.*

Proof. By definition, $\mathbb{PF}X(K)$ is the coequalizer displayed in the diagram

$$\begin{array}{c}
\sum_{i,j\in\mathbb{Z}} \mathscr{C}(\Sigma^j R, X) \otimes \mathscr{B}(\Sigma^i R, \Sigma^j R) \otimes \mathscr{C}(K, \Sigma^i R) \\
\downarrow\downarrow \\
\sum_{i\in\mathbb{Z}} \mathscr{C}(\Sigma^i R, X) \otimes \mathscr{C}(K, \Sigma^i R) \\
\downarrow \\
\mathscr{C}(-, X) \otimes_{\mathscr{B}} \mathscr{C}(K, -).
\end{array}$$

There are no maps $\Sigma^i R \to \Sigma^j R$ unless $i = j$, when $\mathscr{B}(\Sigma^i R, \Sigma^i R) \cong R$. The composition maps

$$\mathscr{C}(\Sigma^i R, X) \otimes \mathscr{B}(\Sigma^i R, \Sigma^i R) \to \mathscr{C}(\Sigma^i R, X)$$

specify the right action of R on $H_i(X)$. Therefore our coequalizer diagram can be rewritten as

$$\begin{array}{c}
\sum_{i\in\mathbb{Z}} H_i(X) \otimes R \otimes H^i(K) \\
\downarrow\downarrow \\
\sum_{i\in\mathbb{Z}} H_i(X) \otimes H^i(K) \\
\downarrow \\
\sum_{i\in\mathbb{Z}} H_i(X) \otimes_R H^i(K).
\end{array}$$

The conclusion follows. ■

3.2. Proposition. *If the homology R-modules $H_i(X)$ of any perfect chain complex X are R-flat, then the Freyd conjecture holds for the pair $(\mathscr{C}, \mathscr{B})$ in $\mathscr{D}(R)$.*

Proof. Let $K \to L \to M \to \Sigma K$ be an exact triangle in $\mathscr{D}(R)$. Since the functor $\mathscr{C}(-, \Sigma^i R)$ takes exact triangles to exact sequences, the sequence

$$H^{i-1}(K) \to H^i(M) \to H^i(L) \to H^i(K)$$

is exact. By our flatness hypothesis, this sequence remains exact on tensoring with each $H_i(X)$. By Lemma 3.1, when we take the direct sum over i of these sequences, we obtain the exact sequence

$$\cdots \to \mathbb{PF}X(\Sigma K) \to \mathbb{PF}X(M) \to \mathbb{PF}X(L) \to \mathbb{PF}X(K) \to \cdots$$

The conclusion follows from Proposition 2.2. ∎

By one definition, the ring R is von Neumann regular if every R-module is flat.

3.3. Corollary. *The Freyd conjecture holds for the derived category $\mathscr{D}(R)$ of a von Neumann regular ring R.*

By another characterization, R is von Neumann regular if every finitely generated ideal is generated by an idempotent [Stenström, 1975, 12.1]. As Grigory Garkusha [Garkusha, 2011] pointed out to us, this holds if all finitely generated ideals I of R are FP-injective, meaning that $\mathrm{Ext}^1_R(F, I) = 0$ for all finitely presented R-modules F. Indeed, R/I is finitely presented, and from the short exact sequence $0 \to I \to R \to R/I \to 0$ we obtain the short exact sequence

$$0 \to \mathrm{Hom}_R(R/I, I) \to \mathrm{Hom}_R(R/I, R) \to \mathrm{Hom}_R(R/I, R/I) \to \mathrm{Ext}^1_R(R/I, R/I) = 0.$$

Let $\pi\colon R \to R/I$ be the quotient map and choose $\iota\colon R/I \to R$ such that $\pi\iota = \mathrm{id}$. Then R is the internal direct sum $I \oplus \iota(R/I)$, $e = \iota\pi(1)$ is an idempotent, and $1 - e$ is an idempotent that generates I. Therefore the elementary result of Corollary 3.3 comes reasonably close to one direction of the best possible result about $\mathscr{D}(R)$, namely [Hovey et al, 2007, Theorem 2.1], which states that the Freyd conjecture holds for $\mathscr{D}(R)$ if and only if all submodules of flat R-modules are flat and all finitely presented R-modules are FP-injective.

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