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# RANGE CATEGORIES I: GENERAL THEORY

J.R.B. COCKETT, XIUZHAN GUO AND PIETER HOFSTRA

ABSTRACT. In this two-part paper, we undertake a systematic study of abstract partial map categories in which every map has both a restriction (domain of definition) and a range (image). In this first part, we explore connections with related structures such as inverse categories and allegories, and establish two representational results. The first of these explains how every range category can be fully and faithfully embedded into a category of partial maps equipped with a suitable factorization system. The second is a generalization of a result from semigroup theory by Boris Schein, and says that every small range category satisfying the additional condition that every map is an epimorphism onto its range can be faithfully embedded into the category of sets and partial functions with the usual notion of range. Finally, we give an explicit construction of the free range category on a partial map category in terms of certain types of labeled trees.

## 1. Introduction

1.1. BACKGROUND AND MOTIVATION The pervasiveness of partiality both in mathematics and in theoretical computer science has led several researchers to develop categorical approaches to partial maps. For the present purposes, the work by Di Paola and Heller on dominical categories [Di Paola & Heller 1987], which was aimed at a categorical treatment of classical computability theory and Gödel's incompleteness theorems, is particularly relevant, as it was the first approach in which the partiality of a map was captured by an idempotent rather than a subobject. This prompted Robinson and Rosolini to define a class of abstract partial map categories called P-categories [Robinson & Rosolini 1988]; furthermore, they proved representational results for those categories in terms of traditional partial map categories (by which we mean categories where the morphisms are spans $(m, f)$ where $m$ belongs to a suitable class of monics; see Section 4 for details.) For a more detailed account of the history of categories of partial maps, as well as the relations to semigroup theory, we refer the reader to the introduction of the paper [Cockett & Manes 2009].

In 2002, Cockett and Lack proposed a more general axiomatization of partial map categories, called restriction categories. The characteristic feature of restriction categories is that the partiality of a map $f : A \to B$ is captured by an idempotent $\overline{f} : A \to A$ on $A$;

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in the case of sets and partial functions, this map $\overline{f}$ would be defined as

$$\overline{f}(x) = \begin{cases} x & \text{if } x \in dom(f) \\ \text{undefined} & \text{otherwise.} \end{cases}$$

The behaviour of the operator $f \mapsto \overline{f}$ can then be axiomatized by means of four equations (see Section 2). What sets apart restriction categories from P-categories, and hence in particular from the more special dominical categories, is that the definition doesn't require products; and unlike traditional partial map categories no pullbacks are required to define composition. As such, restriction categories are both economical and general.

Further developments include the study of partial map categories with additional categorical structure or properties. For example, extensive categories, and in particular the extensive completion of a category appear in [Cockett 1993], while partial cartesian closed categories were considered by in [Curien & Obtulowitz 1989]. An improved setting for the study of abstract computability theory was put forward in [Cockett & Hofstra 2007]; furthermore, a term logic for partial cartesian categories was presented in [Cockett & Hofstra 2010].

In this paper we continue the development of the theory of partial map categories with additional categorical structure by studying the class of *range categories*: these are categories of partial maps in which every morphism has a range, or image. The category of sets and partial functions is a typical such category, and in the category of spaces and partial continuous functions with open domain the maps with range are precisely[1] the open maps. Another motivating example is the category of partial recursive functions; both the domain and range of such a function are recursively enumerable. However, other similar categories, obtained by replacing the standard recursion theoretic model by a more general model of partial combinatory logic, need not have enumerable ranges, and hence this is a non-trivial extra property of a model of computation. Finally, an important class of examples comes from semigroup theory: every inverse monoid is a range category, as is the idempotent splitting of an inverse semigroup.

Our interest in range categories and our motivation for developing the basic theory as reported on in these papers is at least threefold. First, we believe the theory is of intrinsic interest, much in the same way as the theory of allegories is interesting in its own right. It has a simple algebraic axiomatization, is more general than other classes of categories which aim at modeling partial maps with images, and yet captures all of the motivating examples.

Second, the theory of restriction categories in general and of more specific classes such as range categories can play a bridging role between (inverse) semigroup theory and category theory. Various concepts and constructions in semigroup theory, for example certain kinds of (co)completions, can be carried out on the level of restriction categories (we shall see several examples in the present papers, but the reader may consider [Cockett

[1]Assuming a mild separation axiom; see Section 3 for details.

& Manes 2009] for others); this means that restriction categories play an organizational role, bringing out the categorical content of semigroup-theoretic results. Since restriction categories are much closer in spirit to (inverse) semigroups than partial map categories are, understanding the categorical contents of semigroup theory is much easier when working with restriction categories.

And third, in the development of computability theory within the setting of [Cockett & Hofstra 2007] it rapidly became apparent that it would be highly beneficial to have available a logic which would allow for standard computability-theoretic definitions and arguments to be interpretable in a suitable class of categories. This class of categories is that of discrete cartesian range categories, and the term logic, which will be described in Part II, is precisely the internal language of this class of categories. Consequently, (and this was in fact the immediate catalyst for the work) a clear understanding of this class (which is slightly larger than that of regular categories) was required.

1.2. Related work Various aspects of ranges and related concepts in the categorical context have already been considered in varying levels of detail in the literature. We mention here some of those, while throughout the text we elaborate on the connections with the present work more precisely.

The work which is closest (both in spirit and in terms of actual results) to ours is that of Rosolini, who introduces ranges in the setting of P-categories, notes a connection with factorization systems, and proves a representational result [Rosolini 1988]. The first difference with our approach is that we develop ranges without assuming the presence of finite products (which P-categories have by definition). The second difference is the approach in loc. cit. imposes a non-algebraic axiom which we do not take as part of our basic axiomatization. Therefore, the basic theory of ranges which we develop in this paper is more general; we recover the concepts from P-categories by adding subsequent layers of structure/properties to our setting. In particular, a restriction category-theoretic analogue of Rosolini's representation theorem is presented in part II of this paper.

In [Di Paola & Heller 1987] the authors consider *recursion categories* with images. These can be seen as a special class of P-categories with ranges, and thus are again more special than the range categories we consider here. Their goal, namely to give a categorical treatment of recursion theory, was also one of our motivations; however, Di Paolo and Heller did not develop the theory of ranges in much detail, instead they moved straight to recursion-theoretic applications.

Another related piece of work which should be mentioned here is the paper [Hughes & Jacobs 2002] on the connection between factorization systems and fibrations; the authors show that every factorization system gives rise to a bicomplete fibration, and that this fibration satisfies the Beck-Chevalley condition whenever the factorization system is pullback-stable. In the present paper, we prove a representational result for range categories in terms of factorization systems; the close relationship between this result and the one by Hughes and Jacobs is that every range category comes naturally equipped with a bifibration (in an appropriate restriction-categorical sense). Furthermore, in part 2 of this paper the Beck-Chevalley condition for range categories with products is examined.

Since range categories are related to factorization systems, fibrations and regular categories, there are close connections between results in these areas and some of the results about range categories developed in this paper. In fact, once one understands the details concerning the connections, one could in principle translate parts of the theory of, say, factorization systems into the language of range categories. While we will attempt throughout the paper to point out when a particular result could have been obtained in this manner, we believe there are reasons to develop the theory from the perspective of range categories.

The primary reason is that our aim is to give a coherent and self-contained presentation which does not heavily rely on other results. While it would indeed have been possible in certain cases to take a shortcut by referring to facts found in the literature, we usually prefer to give direct proofs which exemplify and illustrate reasoning about range categories. For example, one can deduce from the fact that one may freely add equalizers to a category that, as long as one is willing to split idempotents, each restriction category has a meet completion. However, this does not give a concrete description of what the resulting category then looks like, and doesn't explain how the construction relates to the meet-completion in semigroup theory. Our direct construction (Part II, Section 2.5) is elementary and immediately makes the connection with the semigroup-theoretic variant clear.

The second reason is that in some cases definitions or results about ranges do not correspond nicely to well-known results in neighbouring areas. For example, the term logic for discrete range categories which we present in Part II is quite natural from a proof-theoretic point of view, but it turns out to be strictly weaker than regular logic. As such, it doesn't really correspond to a well-studied fragment of categorical logic in the total world. Moreover, the inference rule which would have to be added to the logic in order to bridge the gap makes the system much less transparent. Closely related is the fact that while the regular completion of a category with finite limits is a simple and elegant construction, its partial analogue is a lot more complicated and is best viewed as a two step process, the first of which freely adds ranges, making the corresponding factorization system proper and pullback-stable, and the second of which forces the this factorization system to be the regular epi-mono factorization by means of a category of fractions construction. Because of such tensions, we prefer to give our primary attention to the natural unfolding of the theory on the level of restriction categories, only to focus on connections with related theory after that.

1.3. Contributions The main focus of the first part of this paper is on describing the fundamental structures and concepts involved, on developing the material in a systematic and reasonably self-contained fashion, and on generalizing and unifying various strands of research in this direction. The second part will deal with the interaction of ranges with other types of structure (in particular, the partial analogue of finite limits) as well as a term logic for reasoning about such categories.

The main contributions of the present paper are the following:

1. A thorough and systematic development of the basic theory of categories with

ranges, some examples, and their positioning with respect to other classes of categories such as allegories, categories with factorization systems, and fibrations with existential quantification.

2. A representation theorem for range categories, which says that every range category can be fully embedded into the partial map category associated to a category with a suitable factorization system, and moreover that every range category in which idempotents split is of that form.

3. A generalization of a result from semigroup theory by Boris Schein, which says that every small range category satisfying the additional condition that every morphism is an epimorphism onto its range can be embedded faithfully (but generally not fully) in the category of sets and partial functions.

4. A direct description of the free range category on a category or on a restriction category.

1.4. Outline The paper is organized as follows. First, we consider categories equipped with a support operator; such an operator captures the idea that to every map we can associate a domain, in the form of an idempotent. This setting is precisely sufficient to specify the notion of an open map, and we may then consider categories where all maps are open. Such categories then turn out to have a cosupport, or range, operator which is compatible with the support operator. We then develop the elementary theory of range categories and open maps, and establish various characterizations of ranges, both equationally and in terms of fibrations. We also prove a technical result needed for the representation theorem in Section 4, stating that range categories are stable under idempotent splitting.

Section 3 explores some examples of range categories and connections with more familiar concepts. Every allegory (abstract category of relations) admits a support and cosupport operator, and we show how there is a maximal subcategory which is a range category, namely by selecting the "simple maps" (single-valued relations). Next, we consider categories satisfying the axiom of choice, which in this setting means that every morphism admits a partial section. Such a category always admits a range operator. As a special case, we consider inverse categories; these are a generalization of inverse semigroups, and have the defining property that every map has a partial inverse. Finally, we consider ranges and open maps in the context of topological spaces and locales.

Next, we turn to the basic representational results. Sections 4.1– 4.8 show how splitting of idempotents of a range category gives a category whose underlying category of total maps admits a factorization system, and conversely how every suitable factorization system gives rise to a split range category. Section 4.11 is concerned with a generalization of Boris Schein's result about semigroups with domains and ranges (as explained in Jackson and Stokes [Jackson & Stokes 2009]); we prove that every range category satisfying an additional condition can be embedded into sets and partial functions via a faithful functor which preserves domain and range.

Finally, Section 5 is devoted to a construction of the free range category on a restriction category. The key technical idea is that suitable equivalence classes of labeled trees can be used to represent the domains and ranges in the free category. The main result is then that the construction is a left adjoint to the forgetful functor from range categories to restriction categories.

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## 2. Basic Theory

This section develops the basic theory of range categories and open maps. While the presentation is largely self-contained, we only briefly rehearse the main elements of the theory of restriction categories; the reader is referred to [Cockett & Lack 2002] for a detailed treatment.

2.1. SUPPORT AND RESTRICTION The starting point for our investigations is the concept of a category equipped with a *support operator*; such an operator captures the idea that morphisms in such a category have a domain of definition, represented by an idempotent on the source of the morphism.

Formally, a support (operator) on a category $\mathsf{C}$ is an operation on morphisms

$$\frac{f : X \to Y}{\overline{f} : X \to X}$$

which satisfies the following four equational axioms:

$$\begin{array}{rll}
[\mathbf{R.1}] & f\overline{f} = f & \\
[\mathbf{R.2}] & \overline{g}\,\overline{f} = \overline{f}\,\overline{g} & \text{whenever } \mathit{dom}(f) = \mathit{dom}(g) \\
[\mathbf{R.3}] & \overline{g\overline{f}} = \overline{g}\,\overline{f} & \text{whenever } \mathit{dom}(f) = \mathit{dom}(g) \\
[\mathbf{wR.4}] & \overline{gf} = \overline{\overline{g}f} & \text{whenever } \mathit{cod}(f) = \mathit{dom}(g)
\end{array}$$

The fourth axiom is actually a weakening of the following axiom (which, for reasons to become clear later, is called the *axiom of determinism*):

$$[\mathbf{R.4}] \quad f\overline{gf} = \overline{g}f \quad \text{whenever } \mathit{cod}(f) = \mathit{dom}(g)$$

If a support operators satisfies [**R.4**] then we shall call it a *restriction operator*, and a category equipped with a restriction operator is called a *restriction category*. The general theory of such abstract categories of partial maps was developed in detail in [Cockett & Lack 2002]; here we shall content ourselves with a brief review of those aspects pertinent to the present objectives and with some observations concerning the differences between restrictions and the more general supports.

The map $\overline{f}$ represents the domain of $f$; hence maps for which $\overline{f} = 1$ are called *total*. Total maps are closed under composition, and identities are total; hence we have a subcategory $\mathsf{Tot}(\mathsf{C})$ of $\mathsf{C}$ on the total maps.

We call morphisms $f$ which satisfy $f = \overline{f}$ *restriction idempotents*. (This terminology will be used both in general support categories and in restriction categories.) For a fixed object $A$, the collection of all restriction idempotents on $A$ is denoted by $\mathcal{O}(A)$. This set (assuming the ambient category is locally small) is in fact a meet-semilattice with top element. This structure is given by (for restriction idempotents $e, e'$):

$$\begin{array}{rcl} \top & = & 1_A \\ e \leq e' & \Leftrightarrow & ee' = e \\ e \wedge e' & = & ee' \end{array}$$

Given a morphism $f : B \to A$, there is an induced function

$$f^* : \mathcal{O}(A) \to \mathcal{O}(B); \qquad e \mapsto f^*(e) := \overline{ef}.$$

which we refer to as *pullback along $f$*. This function always preserves the ordering; it preserves the top element if and only if $f$ is total. If the support operator is a restriction, then $f^*$ also preserves binary meets.

Categories with support are automatically order-enriched: given parallel maps $f, g : A \to B$ set

$$f \leq g \Leftrightarrow f = g\overline{f}.$$

The ordering on the idempotent lattice $\mathcal{O}(A)$ arises by restriction of the ordering on $\mathsf{C}(A, A)$. Importantly, however, the orderings $\mathsf{C}(A, B)$ generally do not have a top element, nor do they have binary meets.

2.2. Examples.

1. The paradigmatic example of a restriction category is $\mathsf{Par}$, the category of sets and partial functions. For an object $A$, we have $\mathcal{O}(A) \cong \mathcal{P}(A)$, the full powerset of $A$.

2. Topological spaces and partial continuous functions with open domain form a restriction category, with $\mathcal{O}(A)$ the lattice of opens of the space $A$.

3. Partial recursive functions $\mathbb{N} \to \mathbb{N}$ form a restriction category (monoid). The restriction idempotents correspond to recursively enumerable sets.

4. Every category can be regarded as a restriction category in a trivial way via $\overline{f} = 1$ for all $f$. This shows that a category can be a restriction category in more than one way, and hence that a restriction is additional structure on a category, not a property.

5. The category $\mathsf{Rel}$ of sets and relations is supported, but not a restriction category. One verifies easily that a relation $f$ satisfies $f\overline{gf} = \overline{g}f$ for all $g$ if and only if $f$ is (the graph of) a partial function. The precise connection between categories of relations and support categories is explored in Section 3.1.

When $\mathsf{C}$ is a category with support, then a map $f$ is called *deterministic* if $f\overline{gf} = \overline{g}f$ for all $g$. Deterministic maps form a subcategory and include all restriction idempotents. Therefore:

2.3. Proposition. *Given a support category* $\mathsf{C}$*, the subcategory on the deterministic maps is a restriction category.*

For example, in the category $\mathsf{Rel}$, the deterministic maps are precisely the partial functions. Note also that the order-enrichment of $\mathsf{Rel}$ *qua support category* is not the same as that of $\mathsf{Rel}$ *qua allegory*: according to the latter, we have $f \subseteq g$ whenever the graph of $f$ is contained in the graph of $g$, but according to the former, we have $f \leq g$ whenever $f$ can be obtained from $g$ by restricting it to a subset. Thus we have $f \leq g$ implies $f \subseteq g$, but not vice versa. For deterministic maps $f, g$, the two orderings coincide.

We conclude this section by organizing restriction categories into a 2-category.

A functor $F : \mathsf{C} \to \mathsf{D}$ between two restriction categories is said to be a *restriction functor* if $F(\overline{f}) = \overline{F(f)}$ for every $f$. Restriction categories and restriction functors form a category, denoted by $\mathfrak{Rcat}_0$. There is an obvious forgetful functor $U_r : \mathfrak{Rcat}_0 \to \mathfrak{Cat}_0$ to the category of categories, which forgets the restriction structure. In [Cockett & Lack 2002] a left adjoint was explicitly given.

Given two restriction functors $F, G : \mathsf{C} \to \mathsf{D}$, a family of morphisms $\alpha_A : FA \to GA$ is called a *strict natural transformation* if it is a natural transformation in the usual sense and all of its components are total. It is a *lax* natural transformation if the naturality squares commute up to inequality:

$$\begin{tikzcd} FA \arrow[r, "\alpha_A"] \arrow[d, "Ff"'] \arrow[dr, phantom, "\leq"] & GA \arrow[d, "Gf"] \\ FB \arrow[r, "\alpha_B"'] & GB \end{tikzcd}$$

Restriction categories, restriction functors, and strict natural transformations thus form a 2-category, denoted $\mathfrak{Rcat}$. Similarly, restriction categories, restriction functors and lax natural transformations form a 2-category denoted $\mathfrak{Rcat}_l$.

2.4. Open Maps An important example of a restriction category is the category of topological spaces and partial continuous functions with open domains. Our notation $\mathcal{O}(A)$ for the poset of restriction idempotents is directly motivated by this example, and we often wish to think of this poset as the poset of open subobjects of $A$. It is natural to investigate to which extent topological notions can be given a sensible interpretation in the context of restriction categories. In this section we carry this out for the notion of open map. Before doing so, however, we remark that the idea of axiomatizing open maps in a category goes back to work by Joyal and others (see for example [Joyal & Moerdijk 1994]); however, the ambient setting in loc. sit. was one of much more richly structured categories, such as pretoposes.

In topology, a continuous function $f : X \to Y$ is called an open map if the direct image function which sends a subset $U \subseteq X$ to $f[U] \subseteq Y$ restricts to a map $f_! : \mathcal{O}(X) \to \mathcal{O}(Y)$, i.e., when the direct image under $f$ of an open set is again open. In this situation, $f_!$ is left adjoint to $f^{-1}$.

In order to generalize this, consider a category $\mathsf{C}$ with support, and a morphism $f : A \to B$. The associated "inverse image" function $f^* : \mathcal{O}(B) \to \mathcal{O}(A)$ has the property that it actually lands in $\mathcal{O}(A)/\overline{f}$: given any $e \in \mathcal{O}(B)$ we have $f^*(e) = \overline{ef} \leq \overline{f}$. Of course, the meet-semilattice $\mathcal{O}(A)/\overline{f}$ is the principal downset $\downarrow(\overline{f}) \subseteq \mathcal{O}(A)$; categorically, this is the same thing as the slice over $\overline{f}$.

2.5. Definition. [Open map] *A morphism $f : A \to B$ in a category with supports is called* open *when the poset map $f^* : \mathcal{O}(B) \to \mathcal{O}(A)/\overline{f}$ has a left adjoint $f_! \dashv f^*$ satisfying the Frobenius identity*

$$f_!(e \wedge f^*(e')) = f_!(e) \wedge e'$$

*for any two restriction idempotents $e \in \mathcal{O}(A)$ and $e' \in \mathcal{O}(B)$.*

We note that the inequality

$$f_!(e \wedge f^*(e')) \leq f_!(e) \wedge f_!(f^*(e')) \leq f_!(e) \wedge e'$$

holds regardless.

There is a slight reformulation of this definition which is sometimes convenient:

2.6. Lemma. *In a support category a map $f : A \to B$ is open if and only if there is a poset morphism $\exists_f : \mathcal{O}(A) \to \mathcal{O}(B)$ such that*

| | | |
|---|---|---|
| **[Open. 1]** | $\exists_f f^*(e') \leq e'$ | *for all* $e' \in \mathcal{O}(B)$ |
| **[Open. 2]** | $e \wedge f^*(e') \leq f^*(\exists_f(e) \wedge e')$ | *for all* $e \in \mathcal{O}(A)$ *and* $e' \in \mathcal{O}(B)$ |
| **[Open. 3]** | $e' \wedge \exists_f(e) \leq \exists_f(f^*(e') \wedge e)$ | *for all* $e \in \mathcal{O}(A)$ *and* $e' \in \mathcal{O}(B)$ |

Note that the last inequality gives Frobenius reciprocity as the inequality running in the other direction is always present.

Proof.

($\Rightarrow$) Suppose $f$ is open so that $f^*$ has a Frobenius left adjoint $f_! : \mathcal{O}(A)/\overline{f} \to \mathcal{O}(B)$. Then define the new $\exists_f(e) := f_!(\overline{f} \wedge e)$, so that

$$\begin{aligned}
\exists_f(f^*(e')) &= f_!(\overline{f} \wedge f^*(e')) = f_!(\overline{f}) \wedge e' \leq e' \\
e \wedge f^*(e') &= e \wedge \overline{f} \wedge f^*(e') \leq f^*(f_!(\overline{f} \wedge e)) \wedge f^*(e') \\
&= f^*(f_!(\overline{f} \wedge e) \wedge e') = f^*(\exists_f(e) \wedge e') \\
e' \wedge \exists_f(e) &= e' \wedge f_!(\overline{f} \wedge e) = f_!(f^*(e') \wedge \overline{f} \wedge e) \\
&= \exists_f(f^*(e') \wedge e).
\end{aligned}$$

($\Leftarrow$) Note that $\exists_f(\overline{f} \wedge e) = \exists_f(e)$ as

$$\exists_f(\overline{ef} \wedge e) \leq \exists_f(e) \quad \text{as} \quad \overline{f} \wedge e \leq e$$

and

$$\exists_f(e) = 1_B \wedge f_!(e) \leq \exists_f(f^*(1_B) \wedge e) = \exists_f(\overline{f} \wedge e)$$

Therefore we may view $\exists_f$ as acting on the slice $\mathcal{O}(A)/\overline{f}$. Moreover, we have

$$e \wedge \overline{f} = e \wedge f^*(1_B) \leq f^*(\exists_f(e) \wedge 1_B) = f^*(\exists_f(e)).$$

The Frobenius condition is immediate.

■

Given a support category $\mathsf{C}$, we are now interested in the subcategory on the open maps. which we will denote by $\mathsf{C}_{\text{open}}$. First, we verify that this is indeed a category with support.

2.7. Lemma. *In any support category all restriction idempotents are open maps and open maps are closed under composition.*

Proof. We must establish the three conditions from the previous lemma for a support idempotent $e$. This is easy: set $\exists_e(e') = ee'$. Then:

$$\begin{aligned} \exists_e(e^*(e')) &= eee' \leq e' \\ e' \wedge e^*(e'') &= e'ee'' \leq e^*(\exists_e(e') \wedge e'') \\ e' \wedge \exists_e(e'') &= e'ee'' = \exists_e(e^*(e') \wedge e'') \end{aligned}$$

It remains to show that open maps compose: if $f$ and $g$ are open maps which compose set $\exists_{gf} = \exists_g \exists_f$. We then have:

$$\begin{aligned} \exists_g(\exists_f(f^*(g^*(e)))) &\leq \exists_g(g^*(e)) \leq e \\ e \wedge f^*(g^*(e')) &\leq f^*(\exists_f(e) \wedge g^*(e')) \leq f^*(g^*(\exists_g(\exists_f(e)) \wedge e') \\ e' \wedge \exists_g(\exists_f(e)) &= \exists_g(g^*(e') \wedge \exists_f(e)) = \exists_g(\exists_f(f^*(g^*(e')) \wedge e)) \end{aligned}$$

(where $e$ is an idempotent on $dom(f)$ and $e'$ an idempotent on $cod(g)$). ■

2.8. Remark. As an aside we note that restriction functors do not, in general, preserve open maps. Therefore taking the subcategory on the open maps is not a functorial construction.

The open maps in a support category $\mathsf{C}$ are those which have a well-defined *range*. We put

$$\hat{f} := \exists_f(1),$$

and refer to $\hat{f}$ as the range, or *cosupport* of the map $f$. Thus we have an operator of the following type:

$$\frac{f : X \to Y}{\hat{f} : Y \to Y}$$

This operator satisfies the axioms dual to those of a support operator.

2.9. LEMMA. *The operator $f \mapsto \hat{f}$ is a support operator on the category $(\mathsf{C}_{\text{open}})^{op}$.*

PROOF.

$$
\begin{array}{rcl}
\hat{f}f & = & \exists_f(1_A)f = f\overline{\exists_f(1_A)f} = ff^*(\exists_f(1_A) \wedge 1_B) \geq 1_A \wedge f^*(1_A) = f\overline{f} = f \\
\hat{f}\hat{g} & = & \exists_f(1_A) \wedge \exists_g(1_{A'}) = \exists_g(1_{A'}) \wedge \exists_f(1_A) = \hat{g}\hat{f} \\
\widehat{\hat{f}g} & = & \exists_f(\hat{1}_A)g = \exists_{\exists_f(1_A)g}(1_{A'}) = \exists_{\exists_f(1_A)}(\exists_g(1_{A'})) = \exists_f(1_A) \wedge \exists_g(1_{A'}) = \hat{f}\hat{g} \\
\widehat{g\hat{f}} & = & \exists_{g\hat{f}}(1_B) = \exists_g(\exists_{\hat{f}}(1_B)) = \exists_g(\hat{f}) = \exists_g(\exists_f(1_A)) = \exists_{gf}(1_A) = \widehat{gf}.
\end{array}
$$

■

What is more, this cosupport operator is compatible with the support, in the sense that

$$\overline{\hat{f}} = \hat{f}; \qquad \hat{\overline{f}} = \overline{f}.$$

A category $\mathsf{C}$ with a support and a cosupport which are compatible in this sense will be called a *bisupport category.* We now have the main observation of this section:

2.10. THEOREM. *When $\mathsf{C}$ is a support category, then the subcategory on the open maps $\mathsf{C}_{\text{open}}$ is a bisupport category whose support agrees with that of $\mathsf{C}$, and which contains all restriction idempotents. Moreover, when the support on $\mathsf{C}$ is actually a restriction, then $\mathsf{C}_{open}$ is the largest bisupport subcategory of $\mathsf{C}$.*

In particular, this says that a bisupport category in which the support is a restriction is the same thing as a restriction category where all maps are open. See also Proposition 2.13.

PROOF. We have already established that the open maps form a subcategory with all the stated properties. It remains to be shown that it is the largest such category when the support is a restriction. For this note that if a map $f$ is included in a bisupport subcategory then it must be an open map as one may define $\exists_f(e) = \widehat{fe}$. This has the required properties:

$$
\begin{array}{rcl}
\exists_f(f^*(e')) & = & \widehat{f\overline{e'f}} = \widehat{e'f} = e'\hat{f} \leq e' \\
e \wedge f^*(e') & = & e'\overline{e'f} = \overline{e'fe} = \overline{e'\widehat{fe}fe} \leq \overline{e'\widehat{fe}f} = f^*(\exists_f(e) \wedge e') \\
e' \wedge \exists_f(e) & = & e'\widehat{fe} = \widehat{e'fe} = \widehat{f\overline{e'f}e} = \exists_f(f^*(e') \wedge e)
\end{array}
$$

Hence $f$ is an open map. ■

It should be noted that the cosupport is hardly ever a corestriction; see Section 3.9 for the exception. We also point out the the condition that the support is a restriction (which was used in the first step of the proof) cannot be omitted: in the bisupport category $\mathsf{Rel}$, a map is open if and only if it is deterministic.

2.11. Ranges We now turn to our main objects of study, namely range categories. A range category may be defined as a bisupport category in which the support is a restriction. Before we turn to an alternative characterization, we observe that given a support operator on a category there exists at most one compatible cosupport. Indeed, suppose that $\widetilde{(-)}$ is a second cosupport combinator; then we compute

$$\hat{f} = \widehat{\widetilde{f}\, f} = \widetilde{f}\, \hat{f} = \hat{f}\, \widetilde{f} = \widetilde{\hat{f}\, f} = \widetilde{f}.$$

Therefore, while a category may have many different support operators, having a compatible cosupport is a property of a support category. (Of course, one may also deduce this from the unicity of adjoints to pullback functors.) Consequently, being a range category is a property of a restriction category.

We now give the alternative definition of a range category:

2.12. Definition. [Range Category] *A restriction category* $\mathsf{C}$ *is a* range category *if it has an operator*

$$\frac{f : X \to Y}{\hat{f} : Y \to Y}$$

*satisfying the following four axioms:*

- **[RR.1]** $\overline{\widehat{f}} = \widehat{f}$
- **[RR.2]** $\widehat{f} f = f$
- **[RR.3]** $\widehat{\overline{g} f} = \overline{g}\, \widehat{f}$ *for all maps* $f, g$ *with* $\mathrm{codom}(f) = \mathrm{dom}(g)$
- **[RR.4]** $\widehat{g \widehat{f}} = \widehat{g f}$ *for all maps* $f, g$ *with* $\mathrm{codom}(f) = \mathrm{dom}(g)$

So far, we thus have three different ways of describing range categories:

2.13. Proposition. *For a restriction category* $\mathsf{C}$*, the following are equivalent:*

(i) $\mathsf{C}$ *is a range category*

(ii) $\mathsf{C}$ *has a compatible cosupport*

(iii) *all maps in* $\mathsf{C}$ *are open*

Proof. We have already shown that the second two conditions are equivalent (Theorem 2.10). The straightforward calculation that a combinator $f \mapsto \hat{f}$ a compatible cosupport if and only if it is a range combinator is omitted. ∎

The following lemma collects some useful basic facts about ranges. The proof is a straightforward exercise in the use of the axioms and is left to the reader.

2.14. LEMMA. *In a range category,*

(i) $\widehat{\hat{f}\, g} = \hat{f}\ \hat{g}$ *if* $\mathrm{codom}(f) = \mathrm{codom}(g)$

(ii) $\hat{f} = 1$ *if* $f$ *is epi. In particular,* $\hat{1} = 1$

(iii) $\hat{\overline{f}} = \overline{f}$ *for all* $f$

(iv) $\widehat{g\, f}\ \hat{g} = \widehat{g\, f}$ *if* $\mathrm{codom}(f) = \mathrm{dom}(g)$

A restriction functor $F : \mathsf{C} \to \mathsf{D}$ between two restriction categories is a *range functor* if it preserves not only the restriction but also the range:

$$F(\overline{f}) = \overline{F(f)}; \qquad F(\hat{f}) = \widehat{F(f)}.$$

The latter requirement is not automatic. For example, consider the category of sets, regarded with trivial restriction (and hence also range) structure. The inclusion into the category of sets and partial maps preserves the restriction but not the range.

Range restriction categories and range restriction functors form a category, denoted by $\mathfrak{RRcat}_0$, which can be enriched with either strict or lax transformations. There are evident forgetful functors

$$\mathfrak{RRcat}_0 \to \mathfrak{Rcat}_0 \to \mathfrak{Cat}_0,$$

which forget the restriction and range structures. Since range restriction categories are defined equationally, we know that $U : \mathfrak{RRcat}_0 \to \mathfrak{Cat}_0$ is monadic via a finitary monad so that $\mathfrak{RRcat}_0$ is locally finitely presentable. In Section 5 we give an explicit description of the free functor.

We note that there is another functor $\mathfrak{RRcat}_0 \to \mathfrak{Cat}_0$ which sends a range category to the subcategory on those maps which are total and cototal, i.e. satisfy $\overline{f} = 1, \hat{f} = 1$. The left adjoint to this functor equips a category with the trivial restriction and range.

2.15. IDEMPOTENT SPLITTING We now prove a technical result which will be used in Section 4, namely that the idempotent splitting of a range category is again a range category. In [Cockett & Lack 2002] it was already shown that the idempotent splitting of a restriction category is again a restriction category, and that split restriction categories are precisely partial map categories arising from systems of monics. Thus here we only focus on making sure the range structure is also well-behaved with respect to this construction.

2.16. DEFINITION. *A restriction idempotent* $\overline{f}$ *is* split *if it can be written as* $\overline{f} = mr$ *with* $rm = 1$*. In this case* $m$*, the monic part, is called a* restriction monic*. A restriction category is said to be* split *if all of its restriction idempotents split.*

If $\overline{f}$ splits as $\overline{f} = mr$ then $\overline{f} = \overline{r}$, because we have $\overline{f} = \overline{mr} = \overline{\overline{m}r} = \overline{r}$ since $m$ is monic, hence total.

We now consider a range category $\mathsf{C}$ and construct its splitting $\mathsf{Split}(\mathsf{C})$ by splitting the restriction idempotents. Explicitly:

**Objects:** restriction idempotents of $\mathsf{C}$,

**Maps:** a map $f$ from $(e_1 : A \to A)$ to $(e_2 : B \to B)$ is given by a map $f : A \to B$ such that $e_1 f e_2 = f$,

**Composition:** as in $\mathsf{C}$,

**Identities:** $1_e = e$ for any object $e$ of $\mathsf{Split}(\mathsf{C})$.

We observe that $\mathsf{Split}(\mathsf{C})$ inherits restriction and ranges from $\mathsf{C}$. To show this, it suffices to show that the restriction and range of a map are maps of $\mathsf{Split}(\mathsf{C})$.

2.17. Lemma. *If $f : e_1 \to e_2$ is a map of $\mathsf{Split}(\mathsf{C})$, then so are $\overline{f} : e_1 \to e_1$ and $\hat{f} : e_2 \to e_2$.*

Proof. The equalities

$$\overline{f}e_1 = \overline{f}\overline{e_1} = \overline{f\overline{e_1}} = \overline{fe_1} = \overline{f},$$
$$e_1\overline{f} = \overline{f}e_1 = \overline{f},$$
$$e_2\hat{f} = \overline{e_2}\hat{f} = \widehat{\overline{e_2}f} = \widehat{e_2 f} = \hat{f}$$

and

$$\hat{f}e_2 = e_2\hat{f} = f$$

show that $\overline{f} : e_1 \to e_1$ and $\hat{f} : e_2 \to e_2$ are maps in $\mathsf{Split}(\mathsf{C})$, as desired. ∎

This shows that $\mathsf{Split}(\mathsf{C})$ is a range category. We record:

2.18. Proposition. *If $\mathsf{C}$ is a range category, then so is $\mathsf{Split}(\mathsf{C})$ with the split restriction structure given by the restriction in $\mathsf{C}$.*

By the same process in the proof of Proposition 2.27 [Cockett & Lack 2002], we may regard the 2-category of split range categories $\mathfrak{RRcat}_s$ as a full reflexive sub-2-category of $\mathfrak{RRcat}$: the 2-functor $\mathsf{Split} : \mathfrak{RRcat} \to \mathfrak{RRcat}_s$, taking $\mathsf{C}$ to $\mathsf{Split}(\mathsf{C})$, is a left adjoint to the inclusion $\mathfrak{RRcat}_s \hookrightarrow \mathfrak{RRcat}$.

We point out that the above proposition also holds when we split all idempotents in $\mathsf{C}$, not just the restriction idempotents.

## 3. Examples and Connections

The time is ripe for some examples of range categories. We have already seen that given a restriction category $\mathsf{C}$, the subcategory of open maps is a range category. In $\mathsf{Par}$, every map is open, so this is a range category. In the category of spaces and partial continuous functions with open domain, the open maps which are open in the usual topological sense are open in our sense. The converse is true provided we impose the $T_1$ separation axiom (points are closed). To see why this is needed, consider the category of Alexandroff spaces; such spaces are characterized by the fact that for each subset of $X$ there exists a smallest open set containing it. Given a continuous map between such spaces, we may define its range to be the smallest open set containing its set-theoretic image. This does define a range category, but clearly not every map is open in the topological sense.

In the category of partial recursive functions (of one variable, say), every map is open as well: this may be seen as a consequence of the fact that this category satisfies the axiom of choice (see Section 3.5). Explicitly, given $f$, the range of $f$ is the domain of the function $y \mapsto \mu x.f(x) = y$. (In some texts, the definition of r.e. set is taken to be the image of a computable function; while this does stress the idea of enumerating a subset, it does not agree well with the axiomatic categorical approach, according to which one needs to have domains before one can have ranges.)

In the following sections, we construct more examples by investigating the connections between range categories and neighbouring areas.

3.1. Allegories Just as restriction categories are abstract categories of partial maps, allegories ([Freyd & Scedrov 1990]) are abstract categories of relations. For us, two relevant connections between allegories and range categories are that every allegory is a bisupport category, and that from every allegory one can extract a range category by taking deterministic maps.

Let us first rehearse the relevant definitions. An *involution* on a category $\mathsf{C}$ is a functor $(-)^\circ : \mathsf{C} \to \mathsf{C}^{op}$ which is the identity on objects, and which satisfies $f^{\circ\circ} = f$ for all morphisms $f$.

3.2. Definition. *An* allegory *is an order-enriched category equipped with an order-preserving involution. (We denote the ordering on the homsets by $\subseteq$.) Moreover, each homset is required to have binary meets, and these are to satisfy the modular law*

$$gf \cap h \subseteq g(f \cap g^\circ h)$$

Note that this does not imply that the binary meets in the homsets are preserved by composition. The statement that the involution preserves the ordering means that $f \subseteq g$ if (and only if) $f^\circ \subseteq g^\circ$.

The motivating example of an allegory is the category $\mathsf{Rel}$ of sets and relations; more generally, for any regular category $\mathsf{C}$ we may construct the allegory $\mathsf{Rel}(\mathsf{C})$ of relations in $\mathsf{C}$.

Each map $f$ in an allegory has a support defined by

$$\frac{A \xrightarrow{f} B}{A \xrightarrow[\overline{f} = f^\circ f \cap 1_A]{} A}$$

and dually each map has a cosupport:

$$\frac{A \xrightarrow{f} B}{B \xrightarrow[\hat{f} = ff^\circ \cap 1_B]{} B}$$

Towards the verification of the axioms of a (co)support, it is useful to observe the following:

3.3. LEMMA. *In an allegory, if $h, h' \subseteq 1$ then $hh' = h'h = h \cap h'$, $hh = h$, and $h = h^\circ$.*

PROOF. We show that $hh' = h \cap h'$. To this end, note that $hh' \subseteq h1 = h$ and $hh' \subseteq 1h' = h'$, so certainly $hh' \subseteq h \cap h'$. Conversely, $h \cap h' \subseteq h(1 \cap h^\circ h') \subseteq h(1 \cap h') \subseteq hh'$. All of the desired equalities follow from this. ∎

We now show:

3.4. PROPOSITION. *Every allegory is a bisupport category, with support and cosupport defined as above.*

PROOF. We verify the four support axioms.

$$\begin{aligned} f\overline{f} &= f(f^\circ f \cap 1) \subseteq f1 = f \\ f &= f \cap f \subseteq f(f^\circ f \cap 1) = f\overline{f} \end{aligned}$$

so $f\overline{f} = f$.

To establish that $\overline{f}\ \overline{g} = \overline{g}\ \overline{f}$ it suffices to show $\overline{f}\ \overline{g} = \overline{f} \cap \overline{g}$. However, note that $\overline{f} \subseteq 1$ and $\overline{g} \subseteq 1$, meaning that $\overline{f}, \overline{g}$ are subidentities, so that the result follows from the previous Lemma.

To establish that $\overline{g}\ \overline{f} = \overline{g\overline{f}}$ we show that for any subidentity $h \subseteq 1$ we have $\overline{g}\ h = \overline{gh}$.

$$\begin{aligned} \overline{g}\ h &= \overline{g} \cap h = 1 \cap g^\circ g \cap h = 1 \cap h \cap g^\circ g \\ &= 1 \cap h^\circ h \cap g^\circ g \subseteq 1 \cap h^\circ h(1 \cap hh^\circ g^\circ g) \\ &\subseteq 1 \cap h^\circ g^\circ gh = \overline{gh} \\ \overline{gh} &= 1 \cap h^\circ g^\circ gh \subseteq 1 \cap g^\circ g = \overline{g} \\ \overline{gh} &= 1 \cap h^\circ g^\circ gh \subseteq h(g^\circ gh \cap h^\circ(1 \cap h^\circ g^\circ gh)) \subseteq h(h^\circ g^\circ gh) \subseteq h \end{aligned}$$

So $\overline{gh} \subseteq \overline{g} \cap h = \overline{g}h$ whence the identity holds.

To show that $\overline{gf} = \overline{\overline{g}f}$ we first note that, using the first three restriction identities, $\overline{gf} \subseteq \overline{f}$ as $\overline{f} \cap \overline{gf} = \overline{f}\ \overline{gf} = \overline{gf\overline{f}} = \overline{gf}$. We then have:

$$\begin{aligned} \overline{\overline{g}f} &= 1 \cap \overline{f}\overline{g}f = 1 \cap f^\circ(1 \cap g^\circ g)f \\ &\subseteq 1 \cap f^\circ g^\circ gf = \overline{gf} \\ \overline{gf} &= \overline{g\overline{g}f} \subseteq \overline{\overline{g}f} \end{aligned}$$

The cosupport axioms follow by duality. Both structures agree on their subidentities and thus the support and cosupport are compatible. ∎

In general, the support in an allegory is not a restriction: the axiom [**R**.**4**] fails, because not all morphisms need be deterministic. However, we may take the subcategory on the deterministic maps, and this is a range category.

Another relevant subcategory of an allegory is given by the *simple morphisms*: a morphism $f : A \to B$ is simple if $ff^\circ \subseteq 1_B$. The simple maps are also closed under composition and contain all restriction idempotents. Every simple map is deterministic, so that the simple maps form a range category as well. Moreover, morphisms between allegories preserve simple maps, so this process is functorial.

3.5. Choice Assuming the axiom of choice, the category $\mathsf{Par}$ of sets and partial functions has as one of its special features that every morphism has a *partial section*: given a partial function $f : A \to B$ we may define $m : B \to A$ by

$$m(b) = \begin{cases} a & \text{where } a \text{ is such that } f(a) = b \\ \uparrow & \text{if no such } a \text{ exists.} \end{cases}$$

In the category of partial recursive functions, a similar construction works (making use of the well-ordering structure on the natural numbers).

3.6. Definition. *Let* $\mathsf{C}$ *be a restriction category.*

(i) *A map* $f$ *is called a* partial retraction *when there exists* $m$ *such that* $fm = \overline{m}$ *and* $fmf = f$. *In this case* $m$ *is called a* partial section *of* $f$.

(ii) *A map* $f$ *is called a* partial isomorphism *when there exists* $f^{-1}$ *such that* $f^{-1}f = \overline{f}$ *and* $ff^{-1} = \overline{f^{-1}}$.

(iii) *A restriction category satisfies the* axiom of choice *if every map has a partial section.*

Our aim in this section is to show that such categories always have a range: given $f$, choose a partial section $m$ and set $\hat{f} = \overline{m}$. In order to prove that this satisfies the axioms of a range, we need to establish some basic properties of partial retractions and partial sections.

The first thing to note is that partial retractions do not always compose. However, we have the following observations:

3.7. Lemma. *In any restriction category:*

(i) *A partial isomorphism is a partial retraction, and a partial retraction which is a partial section is a partial isomorphism.*

(ii) *Partial isomorphisms compose and have unique partial inverses.*

(iii) *If* $f$ *and* $gf$ *are partial retractions with partial sections* $m$ *and* $k$ *respectively then* $fk$ *is a partial section of* $g\overline{m}$.

(iv) *If* $f$ *is a partial retraction and* $g$ *is a partial isomorphism then* $gf$ *is a partial retraction.*

(v) *If $f$ is a partial retraction with partial section $m$ and $e$ is a restriction idempotent then $ef$ is a partial retraction with partial section $me$.*

PROOF.

*(i)* The first part of the statement is easy. For the second part, suppose $f$ is both a partial retraction and a partial section, say with $\overline{m} = fm$, $fmf = f$, $\overline{f} = gf$ and $gfg = g$ (so that $m$ is a partial section of $f$, and $f$ is a partial section of $g$). We show that $mf = \overline{f}$, which implies that $f$ is a partial isomorphism with partial inverse $m$. Note first that

$$m = m\overline{m} = m\overline{fm} = \overline{f}m.$$

Then

$$mf = \overline{f}mf = gfmf = gf = \overline{f}$$

and we're done.

*(ii)* If $f, g$ are partial isomorphisms then $f^{-1}g^{-1}gf = f^{-1}\overline{g}f = f^{-1}f\overline{gf} = \overline{fgf} = \overline{gf}$, so that indeed $f^{-1}g^{-1}$ is a partial inverse of $gf$. The second claim is left to the reader.

*(iii)* Note that from $gfk = \overline{k}$ it follows that $\overline{gfk} = \overline{fk} = \overline{k}$. To show that $fk$ is a partial section of $g\overline{m}$ we compute

$$(g\overline{m})(fk) = gfmfk = gfk = \overline{k} = \overline{fk}$$

and

$$(g\overline{m})(fk)(g\overline{m}) = gfmfkgfm = gfkgfm = gfm = g\overline{m}$$

where in the first identity we use $\overline{m} = fm$ twice, in the second we use $fmf = f$, and in the third $(gf)k(gf) = gf$.

*(iv)* Suppose that $m$ is a partial section of $f$. Then $mg^{-1}$ is a partial section of $gf$:

$$(gf)(mg^{-1}) = g\overline{m}g^{-1} = gg^{-1}\overline{mg^{-1}} = \overline{g^{-1}mg^{-1}} = \overline{mg^{-1}}$$

and

$$(gf)(mg^{-1})(gf) = g\overline{mg}f = g\overline{g}\overline{m}f = g\overline{m}f = gfmf = gf.$$

*(v)* This follows from the previous item, since any restriction idempotent is a partial isomorphism.

∎

Having these elementary facts at our disposal, we now state and prove:

3.8. PROPOSITION. *For any restriction category* $\mathsf{C}$ *with choice, the assignment* $\hat{f} = \overline{m}$, *where* $m$ *is any partial section of* $f$, *defines a range on* $\mathsf{C}$. *Moreover, any restriction functor out of* $\mathsf{C}$ *preserves this range.*

The first step in the proof is to observe that this definition of range is independent of the chosen partial section. So let $m, m'$ be two partial sections of $f$, and compute

$$\overline{m} = fm = fm'fm = \overline{m'}\overline{m}$$

so that $\overline{m} \leq \overline{m'}$. By symmetry $\overline{m} = \overline{m'}$.

Next, we verify the range axioms.

For [**RR.1**], consider a partial section $m$ of $f$, and calculate $\overline{\hat{f}} = \overline{\overline{m}} = \overline{m} = \hat{f}$.

For [**RR.2**], let $m$ be a partial section of $f$ again, so that $\hat{f}f = \overline{m}f = fmf = f$.

For [**RR.3**] we first use part *(v)* of Lemma 3.7 with $e = \overline{g}$: this gives that $m\overline{g}$ is a partial section of $\overline{g}f$. Then

$$\widehat{\overline{g}f} = \overline{m\overline{g}} = \overline{g}\ \overline{m} = \overline{g}\ \hat{f}.$$

For [**RR.4**] suppose two partial retractions $f, g$ are given; let $m$ be a partial section of $f$, and $k$ be a partial section of $gf$. By item *(ii)* of the lemma we know that $fk$ is a partial section of $g\overline{m}$. Then

$$\widehat{g\hat{f}} = \widehat{g\overline{m}} = \overline{fk} = \overline{k} = \widehat{gf}$$

as needed.

For the last claim, we simply observe that any restriction functor preserves partial retractions and partial sections.

Looking ahead to the representation of range categories in terms of categories with factorization systems (Section 4), we point out that range categories with choice can be embedded into partial map categories of categories with a factorization system in which the $\mathcal{E}$-maps are split epis.

3.9. INVERSE CATEGORIES Inverse categories generalize inverse monoids, which in turn are inverse semigroups with unit (see the textbook [Lawson 1998] for an exposition of these notions). We briefly describe now how inverse categories arise, and discuss some well-known alternative definitions.

3.10. DEFINITION. *A restriction category is said to be an* inverse category *when each arrow is a partial isomorphism.*

For an easy example, note that any groupoid can be regarded as an inverse category in which the restriction is trivial (i.e., $\overline{f} = 1$ for all $f$).

Another key example comes from inverse semigroups: when $S$ is an inverse semigroup, we may form its idempotent splitting: the objects of the resulting category are the idempotents $e$ of $S$, while a morphism $e \to f$ is an element $s \in S$ for which $fs = s = se$. As is well-known (see for example [Lawson & Steinberg 2004]) this is an inverse category with $\overline{s} = s^*s, \hat{s} = ss^*$. The partial inverse of $s$ is of course $s^*$.

From the results in the previous section (Proposition 3.8) we immediately get that an inverse category is a range category. The restriction is then $\overline{x} = x^{-1}x$ and the range is $\hat{x} = xx^{-1}$. In fact, the range is also a corestriction (meaning that the axiom $g\hat{f} = \widehat{gf}g$ holds). However, note that the converse does not hold: there exist codeterministic range categories which are not inverse. For the simplest counterexample, consider a category with two objects and only one non-trivial map $A \to B$.

From the fact that partial isomorphisms compose and that every restriction idempotent is a partial isomorphism we see that given any restriction category $\mathsf{C}$, the subcategory on the partial isomorphisms is an inverse category containing all restriction idempotents of $\mathsf{C}$.

Importantly an inverse category may be described in a number of different ways. A common approach is to define an inverse category as a category equipped with an involution $(-)^{-1}$ satisfying the following equations:

$$(x^{-1})^{-1} = x, \quad (xy)^{-1} = y^{-1}x^{-1}, \quad xx^{-1}x = x, \qquad \text{and} \quad xx^{-1}yy^{-1} = yy^{-1}xx^{-1}.$$

This forces the inverse to be unique in the sense that if $xyx = x$ and $yxy = y$ then $y = x^{-1}$. The calculation is well-known but quite tricky. First we note $y = yxx^{-1}$ and by symmetry $y = x^{-1}xy$ because

$$y = yxy = yy^{-1}yxx^{-1}xyy^{-1}y = yxx^{-1}y^{-1}yxyy^{-1}y = yxx^{-1}y^{-1}yy^{-1}y = yxx^{-1}.$$

Then we have $y = yxx^{-1} = x^{-1}xyxx^{-1} = x^{-1}xx^{-1} = x^{-1}$.

Alternatively, recall that semigroup theorists call a map $f$ *regular* (category theorists might prefer – to avoid confusion with regular monics and regular epics – to call $f$ *retractive* as in the idempotent splitting it is a retraction) if there is a $g$ such that $fgf = f$. A *regular inverse* (or retractive inverse) of a map $f$ is a $g$ such that $fgf = f$ and $gfg = g$. An inverse category may then be described as a category in which each map $f$ has a unique regular (or retractive) inverse $f^{-1}$.

Finally, just as for categories with choice, it is worth observing that any restriction functor automatically preserves inverses, so that the image of an inverse category is always an inverse category.

3.11. Indexed meet-semilattices and frames We have already mentioned the category of topological spaces and partial continuous maps with open domain, as well as the fact that we tend to think of the meet-semilattices of restriction idempotents as lattices of open subobjects. This section will explain how each restriction category has associated to it an indexed meet-semilattice, which generalizes the usual subobject fibration of a category (see e.g. [Jacobs 1999] for an exposition of fibrations and related notions). We then investigate this fibred structure in the case of range categories, and also consider the special case where the restriction idempotents actually form frames.

Our starting point is the category $\wedge\mathsf{SLat}_p$ whose objects are meet-semilattices with top element and whose morphisms are meet-preserving functions. (It is not required that the top element is preserved.) Then $\wedge\mathsf{SLat}_p^{\mathrm{op}}$ becomes a restriction category when we define, for a meet-semilattice homomorphism $f : X \to Y$, its restriction to be the function $\overline{f} : Y \to Y$ given by $\overline{f}(y) = f(\top) \wedge y$. Clearly the total maps in this category correspond to the homomorphisms which do preserve the top element. We write $\wedge\mathsf{SLat}^{op}$ for $\mathsf{Total}(\wedge\mathsf{SLat}_p^{op})$.

For a restriction category $\mathsf{C}$, we may now regard the so-called *fundamental functor*, which takes the form

$$\mathcal{O} : \mathsf{C} \to \wedge\mathsf{SLat}_p^{\mathrm{op}};$$

it sends an object to its meet-semilattice of restriction idempotents, and a morphism $f : A \to B$ to $f^* : \mathcal{O}(B) \to \mathcal{O}(A)$. Note that in the special case where $\mathsf{C} = \wedge\mathsf{SLat}_p^{\mathrm{op}}$ this functor is the identity. The fundamental functor is in fact a restriction functor; this leads to the following lemma (whose proof is immediate):

3.12. Lemma. *For any restriction category* $\mathsf{C}$*, we have a pullback diagram:*

$$\begin{tikzcd} \mathsf{Total}(\mathsf{C}) \arrow[r] \arrow[d] & \mathsf{C} \arrow[d, "\mathcal{O}"] \\ \wedge\mathsf{SLat}^{op} \arrow[r] & \wedge\mathsf{SLat}_p^{op} \end{tikzcd}$$

The left-hand vertical map may now be regarded as an ordinary subobject fibration, with the only difference that the subobjects are restriction idempotents. Observe in particular that this fibration admits comprehension (in the form of a right adjoint to the functor which sends an object $C$ to the terminal idempotent on $C$, namely the identity) precisely when the restriction idempotents of $\mathsf{C}$ split.

Naturally, we consider the subcategory of $\wedge\mathsf{SLat}_p^{\mathrm{op}}$ on the open maps which by definition is now a range category. Explicitly, the open morphisms of $\wedge\mathsf{SLat}_p^{\mathrm{op}}$ are those which, when regarded in the opposite category, have a Frobenius left adjoint.

Denoting the subcategory of $\mathsf{C}$ on the open maps by $\mathsf{C}_{\mathrm{open}}$, we find that the following diagram is a pullback.

$$\begin{tikzcd} \mathsf{C}_{\mathrm{open}} \arrow[r, tail] \arrow[d, "\mathcal{O}"'] & \mathsf{C} \arrow[d, "\mathcal{O}"] \\ \wedge\mathsf{SLat}_{\mathrm{open},p}^{\mathrm{op}} \arrow[r, tail] & \wedge\mathsf{SLat}_p^{\mathrm{op}} \end{tikzcd}$$

Putting this together with the previous lemma, we find that for a range category $\mathsf{C}$, we have an associated fibration on $\mathsf{Total}(\mathsf{C})$ of restriction idempotents which is actually a bifibration, i.e., which has existential quantification. This opens the way to the connection between range categories

and factorization systems, see next section for details. Whenever one has a $\mathsf{C}$-indexed meet-semilattice $\mathcal{P}$, one may form a restriction category whose objects are those of $\mathsf{C}$, but whose restriction idempotents at an object $A$ are given by the meet-semilattice $\mathcal{P}(A)$ (see [Cockett & Guo 2006]). This process is part of an adjunction between the category of restriction categories and that of indexed meet-semilattices (the right adjoint being the assignment of the fundamental functor to a category), and is to be thought of as freely adding partiality (as specified by the indexed semilattice) to a category. The categories so obtained have interesting structural properties; for example, they are unitary (in the sense of inverse semigroups).

The above construction can be adapted to produce range categories: suppose that we are given a category $\mathsf{D}$, and a functor $\mathcal{P} : \mathsf{D} \to \wedge\mathsf{SLat}^{\mathrm{op}}_{\mathrm{open}}$, i.e., an indexed meet-semilattice in which each reindexing functor $f^* : \mathcal{P}(B)/\overline{f} \to \mathcal{P}(A)$ has a Frobenius left adjoint. Then from this data we can construct a range category $\mathsf{D}[\mathcal{P}]$ whose objects are those of $\mathsf{D}$ and whose idempotent lattice at $A$ is $\mathcal{P}(A)$. (The proof follows the same lines as in [Cockett & Guo 2006].)

The category of meet-semilattices has an important subcategory, namely the category of frames $\mathsf{Frm}_p$ of frames and $\wedge, \bigvee$-preserving functions, but where the morphisms are not required to preserve the top element. Morphisms in this category always have a right adjoint, and hence are Frobenius left adjoints. This means that the category $\mathsf{Loc}_p$ of locales, the opposite of $\mathsf{Frm}_p$, is a range category. There is a slightly different description of this category: it is isomorphic to the category $\mathsf{Par}(\mathsf{Loc}, \mathrm{Open})$ of locales and partial maps with open domain. Indeed, any map $f : A \to B$ in $\mathsf{Loc}_p$ factors uniquely through the open sublocale of $A$ determined by $f^*(1_B)$.

## 4. Representation Theorems

The main representational result for restriction categories, proved in [Cockett & Lack 2002], states that every restriction category is embeddable in a category of partial maps. More precisely, every category of partial maps is a *split* restriction category, and every split restriction category is of this form.

In this section we first extend this result to range categories. The partial map categories corresponding to range categories will be seen to have a factorization system on their total map category satisfying a certain stability condition.

After that, we prove a different representational result, which states that every small range category which satisfies an additional axiom can be represented as a (non-full) subcategory of $\mathsf{Par}$, the category of sets and partial functions. For semigroups with ranges, this result is due to Boris Schein (see the exposition in [Jackson & Stokes 2009]); the result presented here extends his surprising proof.

4.1. Factorization systems and partial maps In a category, a collection $\mathcal{M}$ of monics which includes all isomorphisms and is closed under composition, is called *a system of monics.* A system of monics $\mathcal{M}$ is said to be *stable* if for any $m \in \mathcal{M}$ and any $f : A \to B$ the pullback of $m$ along $f$ exists and belongs to $\mathcal{M}$. In practice, all systems of monics we discuss here will be stable, so in order to reduce clutter we drop this adjective.

Given a category $\mathsf{C}$ with a system of monics $\mathcal{M}$, one may form the category of partial maps $\mathsf{Par}(\mathsf{C}, \mathcal{M})$ with:

**Objects:** $A \in \mathsf{C}$,

**Maps:** a map from $A$ to $B$ is a pair $(m, f)$, where $m : A' \to A$ is in $\mathcal{M}$ and $f : A' \to B$ is an arbitrary map in $\mathsf{C}$:

$$\begin{tikzcd} & A' \arrow[dl, "m"'] \arrow[dr, "f"] & \\ A & & B \end{tikzcd}$$

factored out by the equivalence relation: $(m, f) \approx (m', f')$ whenever there exists an isomorphism $\alpha$ in $\mathsf{C}$ such that $m'\alpha = m$ and $f'\alpha = f$,

**Identities:** $(1_A, 1_A) : A \to A$,

**Composition:** $(m', g)(m, f) = (mm'', gf')$, where $f'$ and $m''$ are given by the pullback diagram $(*)$:

$$\begin{tikzcd} & & A'' \arrow[dl, "m''"'] \arrow[dr, "f'"] & & \\ & A' \arrow[dl, "m"'] \arrow[dr, "f"'] & (*) & B' \arrow[dl, "m'"] \arrow[dr, "g"] & \\ A & & B & & C \end{tikzcd}$$

The original maps in $\mathsf{C}$ can be embedded into $\mathsf{Par}(\mathsf{C}, \mathcal{M})$ by $f \mapsto (1, f)$.

4.2. Theorem. [Cockett & Lack 2002], Proposition 3.1 *Let* $\mathsf{C}$ *be a category equipped with a system of monics* $\mathcal{M}$. *Then* $\mathsf{Par}(\mathsf{C}, \mathcal{M})$ *has a split restriction given by* $\overline{(m, f)} = (m, m)$. *Furthermore, the total maps in* $\mathsf{Par}(\mathsf{C}, \mathcal{M})$ *are precisely those in the image of* $\mathsf{C}$.

We now consider factorization systems $(\mathcal{E}, \mathcal{M})$ in a category $\mathsf{C}$. The factorization systems we consider are strong, in the sense that diagonal fillers are unique. (See [Adamek et. al 1990] for an exposition of the general theory of factorization systems.) We say that $(\mathcal{E}, \mathcal{M})$ is $\mathcal{M}$-*stable* when for every map $f = m_f e_f$, with $m_f \in \mathcal{M}$ and $e_f \in \mathcal{E}$, and $a \in \mathcal{M}$ we have that

$$\begin{tikzcd} & \bullet \arrow[dd, "a''"] \arrow[dr, "m'_f"] & \\ \bullet \arrow[ur, dashed, "e'_f"] \arrow[rr, "f'"] \arrow[ddd, "a'"'] & & \bullet \arrow[ddd, "a"] \\ & & \\ & \bullet \arrow[dr, "m_f"] & \\ \bullet \arrow[ur, "e_f"] \arrow[rr, "f"] & & \bullet \end{tikzcd}$$

where $f'$ and $m'_f$ are pullbacks of $f$ and $m_f$ along $a$, and $e'_f$ is the pullback of $e_f$ along $a''$, respectively, then $f' = m'_f e'_f$ is the $(\mathcal{E}, \mathcal{M})$-factorization of $f'$.

It is clear that a factorization system in a category where pullbacks along $\mathcal{M}$-maps exist is $\mathcal{M}$-stable when the $\mathcal{E}$-maps are stable under pullback along $\mathcal{M}$.

4.3. Example. It is well-known that the category $\mathsf{Top}$ of spaces and continuous maps is not regular because the regular epi-monic factorization is not pullback-stable. However, when we let $\mathcal{M}$ be the regular monics and $\mathcal{E}$ be the class of epimorphisms, then $\mathcal{M}$ is a system of monics, and we get a factorization system which is pullback stable (along all maps, not just $\mathcal{M}$-maps).

In [Hughes & Jacobs 2002] it is explained how an $\mathcal{M}$-stable factorization system $(\mathcal{E}, \mathcal{M})$ on a category $\mathsf{C}$ induces a fibration whose total category is $\mathcal{M}$, regarded as a subcategory of $\mathsf{C}^{\to}$. This fibration then has some distinguishing features: it has existential quantification, and it has full subset types. Conversely, it is proved in loc. cit. that such fibrations always arise from factorization systems. In the next two subsections we shall prove directly (i.e., without invoking fibrations) that split range categories correspond to stable factorization systems. In principle, we could have deduced this result from the correspondence exhibited in loc. cit., by using the fact that, as explained in Section 3.11, the total map category of a range category comes equipped with a fibration in which reindexing functors have left adjoints. However, we opt to complete the triangle by directly showing how the range structure gives rise to a factorization system and vice versa.

4.4. Range categories from factorization systems Next, we extend Theorem 4.2 by showing that if a category with a system of monics $\mathcal{M}$ has the additional feature of having an $\mathcal{M}$-stable $(\mathcal{E}, \mathcal{M})$ factorization system, then the resulting partial map category has ranges. Formally:

4.5. Theorem. *Let $\mathsf{C}$ be a category equipped with an $\mathcal{M}$-stable factorization system $(\mathcal{E}, \mathcal{M})$. Then the partial map category $\mathsf{Par}(\mathsf{C}, \mathcal{M})$ is a split range category.*

Proof. From Theorem 4.2 we already know that $\mathsf{Par}(\mathsf{C}, \mathcal{M})$ is a split restriction category with restriction $\overline{(m, f)} = (m, m)$. Therefore it suffices to define the range operator and to show that the axioms **[RR.1]**-**[RR.4]** are satisfied.

To define the range of a morphism $(m, f)$, consider the $(\mathcal{E}, \mathcal{M})$-factorization of $f = m_f e_f$. Then define

$$\widehat{(m, f)} := (m_f, m_f).$$

We now have:

**[RR.1]** $\overline{\widehat{(m, f)}} = \overline{(m_f, m_f)} = (m_f, m_f) = \widehat{(m, f)}$.

**[RR.2]** $\widehat{(m, f)}(m, f) = (m_f, m_f)(m, f) = (m, f)$ since the following (*) is a pullback diagram:

$$\begin{tikzcd}
 & & A' \arrow[dl, "1_{A'}"'] \arrow[drr, "e_f"] & & & & \\
 & A' \arrow[dl, "m"'] \arrow[drr, "f"] \arrow[rrr, phantom, "(*)"] & & & D \arrow[dl, "m_f"] \arrow[drr, "m_f"] & & \\
A & & & B & & & B
\end{tikzcd}$$

**[RR.3]** Let $(n', f')$ be the pullback of $(n, f)$ and $(n'', m'_f)$ the pullback of $(n, m_f)$. Then there is a unique map $e'_f : D \to D'$ in $\mathsf{C}$ such that $m'_f e'_f = f'$ and $n'' e'_f = e_f n'$ and so $(n', e'_f)$ is a pullback of $(n'', e_f)$:

$$\begin{tikzcd}
 & & E \arrow[rr, dashed, "e'_f"] \arrow[dl, "n'"'] \arrow[drr, "f'"] & & F \arrow[dl, "n''"'] \arrow[d, "m'_f"] & \\
 & A' \arrow[dl, "m"'] \arrow[rr, "e_f"] \arrow[drr, "f"'] & & D \arrow[d, "m_f"'] & B' \arrow[dl, "n"] \arrow[dr, "n"] & \\
A & & & B & & B
\end{tikzcd}$$

Hence, by hypothesis, $f' = m'_f e'_f$ is the $(\mathcal{E}, \mathcal{M})$-factorization of $f'$ and therefore $nf' = (nm'_f)e'_f$ is the factorization of $nf'$ since $nm'_f \in \mathcal{M}$ and $e'_f \in \mathcal{E}$. Thus,

$$\begin{aligned}
\overline{(n,g)\widehat{(m,f)}} &= \widehat{(n,n)(m,f)} \\
&= \widehat{(mn', nf')} \\
&= (m_{nf'}, m_{nf'}) \\
&= (nm'_f, nm'_f) \text{ (since } nf' = (nm'_f)e'_f) \\
&= (n,n)(m_f, m_f) \text{ (since } (n'', m'_f) \text{ is the pullback of } (n, m_f)) \\
&= \overline{(n,g)}\widehat{(m,f)}.
\end{aligned}$$

**[RR.4]** Suppose that $(f', n')$ and $(m'_f, n'')$ are pullbacks of $(f, n)$ and $(m_f, n)$, respectively. Again, there is a unique map $e'_f : E \to F$ such that $m'_f e'_f = f'$ and $n'' e'_f = e_f n'$ and so $(n', e'_f)$ is a pullback of $(n'', e_f)$:

$$\begin{tikzcd}
 & & E \arrow[rr, dashed, "e'_f"] \arrow[dl, "n'"'] \arrow[drr, "f'"] & & F \arrow[dl, "n''"'] \arrow[d, "m'_f"] \arrow[dr, "e_1"] & \\
 & A' \arrow[dl, "m"'] \arrow[rr, "e_f"] \arrow[drr, "f"'] & & D \arrow[d, "m_f"'] & B' \arrow[dl, "n"] \arrow[dr, "g"'] & G \arrow[d, "m_1"] \\
A & & & B & & C
\end{tikzcd}$$

If $gm'_f = m_1e_1$ is the factorization of $gm'_f$, then $gf' = g(m'_f e'_f) = m_1(e_1e'_f)$ is the factorization of $gf'$. Thus,

$$\begin{aligned}
\widehat{(n,g)\widehat{(m,f)}} &= (n,g)\widehat{(m_f,m_f)} \\
&= \widehat{(m_f n'', gm'_f)} \\
&= (m_{gm'_f}, m_1) \\
&= (m_{gf'}, m_{gf'}) \\
&= \widehat{(m_{gf'}, m_{gf'})} \\
&= \widehat{(n,g)(m,f)},
\end{aligned}$$

as desired. ■

4.6. Factorization systems from range categories We now embark on the proof of the converse of Theorem 4.5, namely that any split range category gives rise to a stable factorization system on its category of total maps.

4.7. Theorem. *Let* $\mathsf{C}$ *be a split range category. Then* $\mathsf{Total}(\mathsf{C})$*, the subcategory on the total maps, admits an* $\mathcal{M}$*-stable* $(\mathcal{E},\mathcal{M})$*-factorization system with*

$$\mathcal{E} = \{f|\overline{f} = 1, \hat{f} = 1\}; \qquad \mathcal{M} = \{m|\overline{m} = 1, m \text{ is a partial isomorphism}\}$$

Proof. We first show that $\mathcal{E}$-maps are stable under pullback along $\mathcal{M}$-maps. Let $e \in \mathcal{E}$ and $m \in \mathcal{M}$. Since $m$ is a partial section there exists $r$ such that $\overline{m} = rm, rmr = r$. By the fact that $\mathcal{M}$ is a stable system of monics, we know that the pullback of $e$ along $m$ exists:

$$\begin{tikzcd}
A \arrow[r, "e'=rem'"] \arrow[d, "m'"'] & B \arrow[d, "m"'] \\
C \arrow[r, "e"'] & D \arrow[u, bend right, "r"']
\end{tikzcd}$$

Moreover, we may take it to be $e' = rem'$, where we denote the pullback of $m$ along $e$ by $m'$; the latter is again a partial section, say with partial retraction $r'$. Now

$$\widehat{e'} = \widehat{rem} = \widehat{rem'\widehat{r}} = \widehat{rem'r'} = \widehat{re\overline{re}} = \widehat{re} = \widehat{r} = 1$$

This shows that $\widehat{e'} = 1$; since we already know that $\overline{e'} = 1$, it follows that $e' \in \mathcal{E}$.

Next, we show that $(\mathcal{E},\mathcal{M})$ is a factorization system. It is clear that $\mathcal{E}$ is closed under composition, since if $e, e' \in \mathcal{E}$ then $\widehat{e'e} = \widehat{e'\widehat{e}} = \widehat{e'} = 1$. Suppose that $f : X \to Y$ is a total map; since $\hat{f}$ is a split restriction idempotent, we may factor it as $\hat{f} = m_f r_f$ for some maps $r_f : Y \to Z$ and $m_f : Z \to Y$. with $r_f m_f = 1_Z$. We claim that the $(\mathcal{E},\mathcal{M})$-factorization of $f$ is $f = m_f(r_f f)$. First, note that $f = \hat{f}f = m_f r_f f$, so that $f$ indeed factors as such. Next, note that $m_f \in \mathcal{M}$ since it is a restriction monic. Next, we prove that $r_f f \in \mathcal{E}$: we have $\overline{r_f f} = \overline{\overline{m_f} r_f f} = \overline{m_f r_f f} = \overline{f} = 1$, and $\widehat{r_f f} = \widehat{r_f \hat{f}} = \widehat{r_f m_f r_f} = \widehat{r_f} = 1$.

To prove that $\mathcal{E}$-maps are orthogonal to $\mathcal{M}$-maps, consider a commutative square

$$\begin{tikzcd} A \arrow[r, "e"] \arrow[d, "p"'] & B \arrow[d, "q"] \arrow[dl, bend right, "r"'] \\ C \arrow[r, "m"'] & D \end{tikzcd}$$

where $e \in \mathcal{E}, m \in \mathcal{M}$ and where $r$ is a partial retraction of $m$. We claim that $rq : B \to C$ is a diagonal filler. That this map is total follows right away from the fact that $q = mrq$ is total. It is also easy to see that $rqe = rmp = p$. To show that $mrq = q$, we derive first that

$$\overline{r}\widehat{mp} = \widehat{\overline{mr}mp} = \widehat{mp}$$

whence $\widehat{mp} \leq \overline{r}$. Using this, we find

$$mr = \overline{r} \geq \widehat{mp} = \widehat{q}e = \widehat{q}$$

and therefore $mr\widehat{q} = \widehat{q}$. Now the desired equality

$$mrq = mr\widehat{q}q = \widehat{q}q = q$$

follows.

Finally, from the fact that $m$ is monic it follows that the filler $rq$ is unique.

∎

4.8. Main Result We now make precise in which sense categories with factorization systems are the same thing as range categories.

Let $\mathcal{M}\mathsf{Fac}$ be the 2-category with

**0-cells:** categories $\mathsf{C}$ equipped with a factorization systems $(\mathcal{E}, \mathcal{M})$, which is stable under pullbacks of $\mathcal{M}$-maps, and where all $\mathcal{M}$-maps are monic.

**1-cells:** a 1-cell from $(\mathsf{C}, \mathcal{E}, \mathcal{M})$ to $(\mathsf{D}, \mathcal{F}, \mathcal{N})$ is a functor $F : \mathsf{C} \to \mathsf{D}$ which sends $\mathcal{E}$-maps to $\mathcal{F}$-maps and $\mathcal{M}$-maps to $\mathcal{N}$-maps, and which preserves pullback squares along $\mathcal{M}$-maps.

**2-cells:** $\mathcal{M}$-cartesian natural transformations, i.e., natural transformations whose naturality squares involving $\mathcal{M}$-maps are pullback squares.

We are now in a position to state the main results. Recall that $\mathfrak{RRcat}_s$ is the 2-category whose 0-cells are split range categories, whose 1-cells are range-preserving restriction functors and whose 2-cells are total natural transformations.

4.9. Theorem. *The 2-categories $\mathcal{M}\mathsf{Fac}$ and $\mathfrak{RRcat}_s$ are 2-equivalent.*

Proof. We have shown that the assignments

$$\mathsf{Total} : \mathfrak{RRcat}_s \to \mathcal{M}\mathsf{Fac}, \qquad \mathsf{Par} : \mathcal{M}\mathsf{Fac} \to \mathfrak{RRcat}_s$$

are well-defined on objects. We know that these constructions form a 2-equivalence when regarded as 2-functors between the 2-category of split restriction categories and the 2-category of categories equipped with systems of monics. Hence we only need to show that they are well-defined on 1-cells.

Concretely, this means that we have to show that a range functor $F$ between split range categories gives a functor $\mathsf{Total}(F)$ which preserves $\mathcal{E}$- and $\mathcal{M}$-maps. Conversely, we need that the restriction functor $\mathsf{Par}(G)$ associated to a 1-cell $G$ in $\mathcal{M}\mathsf{Fac}$ preserves ranges.

For the first of these claims, recall that the $\mathcal{E}$-maps are defined to be those $f$ for which $\overline{f} = 1$ and $\widehat{f} = 1$; it is clear that any range functor preserves these. Similarly, the $\mathcal{M}$-maps are preserved because any restriction functor preserves restriction monics.

For the second part, consider a morphism $(m, f)$ in $\mathsf{Par}(\mathsf{C}, \mathcal{M}, \mathcal{E})$, and its range $(m_f, m_f)$ (where $m_f$ is the $\mathcal{M}$-part of $f$). Since the morphisms in $\mathcal{M}\mathsf{Fac}$ preserve factorizations, any such morphism $F$ will send $f = m_f e_f$ to $Ff = m_{Ff}e_{Ff}$. This shows that $(Fm_f, Fm_f) = (m_{Ff}, m_{Ff})$ is the range of $F(m, f)$, as needed. ∎

4.10. Theorem. *Any range category embeds via a full and faithful range preserving functor into a range category of the form* $\mathsf{Par}(\mathsf{C}, \mathcal{E}, \mathcal{M})$.

Proof. Given a range category $\mathsf{C}$, consider $\mathsf{Split}(\mathsf{C})$, which is a split range category. There is a full and faithful range preserving functor $\mathsf{C} \to \mathsf{Split}(\mathsf{C})$. Now apply the previous theorem, which implies that $\mathsf{Split}(\mathsf{C}) \cong \mathsf{Par}(\mathsf{Total}(\mathsf{Split}(\mathsf{C})), \mathcal{E}, \mathcal{M})$, for the induced factorization system $(\mathcal{E}, \mathcal{M})$. ∎

4.11. A Generalization of Schein's Theorem While the representational results in the previous section hold for all range categories, this section is concerned with those which satisfy the additional axiom

**[RR.5]** $\quad gf = hf \Rightarrow g\widehat{f} = h\widehat{f}.$

In particular, this axiom says that maps for which $\widehat{f} = 1$ are epimorphisms, and hence it excludes pathological examples of range categories such as the category of Alexandroff spaces, or the trivial restriction and range on a category. It is interesting to note that earlier authors [Di Paola & Heller 1987, Rosolini 1988], took this axiom together with **[RR.2]** as definition of range. The connection then is:

4.12. Lemma. *A combinator satisfying* **[RR.1]**, **[RR.2]** *and* **[RR.5]** *also satisfies the other range axioms.*

Proof. We first prove [**RR.4**]: we have

$$\begin{aligned} & \widehat{gf}gf = gf && \text{by } [\mathbf{RR.2}] \\ \Leftrightarrow\ & \widehat{gf}g\hat{f} = g\hat{f} && \text{by } [\mathbf{RR.5}] \\ \Leftrightarrow\ & \widehat{gf}\widehat{g\hat{f}} = \widehat{g\hat{f}} && \text{by } [\mathbf{RR.5}] \end{aligned}$$

so that $\widehat{g\hat{f}} \leq \widehat{gf}$, and similarly

$$\begin{aligned} & \widehat{g\hat{f}}g\hat{f} = g\hat{f} && \text{by } [\mathbf{RR.2}] \\ \Leftrightarrow\ & \widehat{g\hat{f}}gf = gf && \text{by } [\mathbf{RR.5}] \\ \Leftrightarrow\ & \widehat{g\hat{f}}\widehat{gf} = \widehat{gf} && \text{by } [\mathbf{RR.5}] \end{aligned}$$

so that $\widehat{gf} \leq \widehat{g\hat{f}}$.

Next, observe that we have $\overline{k} = \overline{k}\,\overline{k}$ and hence by [**RR.5**] $\hat{\overline{k}} = \overline{k}\,\hat{\overline{k}} = \overline{k}$. Now [**RR.3**] follows, since

$$\begin{aligned} \overline{g}\hat{f} &= \widehat{\overline{g}\hat{f}} && \text{as this is a restriction idempotent} \\ &= \widehat{\overline{g}f} && \text{by } [\mathbf{RR.4}] \end{aligned}$$

■

In terms of the representation of range categories using factorization systems, the range categories satisfying this extra axiom are precisely those corresponding to factorization systems of which the $\mathcal{E}$-maps are epimorphisms.

The representation to which we now turn is much more specific: it says that we can faithfully represent a small range category satisfying [**RR.5**] into the category of sets and partial functions. The price we pay for this gain in concreteness is the fact that this representation is generally not full.

Our reworking of Schein's original result (which was concerned with the special case of semigroups with ranges) follows the exposition in Jackson and Stokes [Jackson & Stokes 2009].

The aim is, for a range category $\mathsf{C}$ satisfying [**RR.5**], to define a faithful range functor

$$\mathsf{S} : \mathsf{C} \to \mathsf{Par}.$$

Towards the definition of this functor, consider an object $X$ of $\mathsf{C}$. A *stickleback* $\gamma$ on $X$ is a zigzag of maps of the following form:

$$\begin{tikzcd}[column sep=small] & Y_0 \arrow[dl, "\gamma_0"'] \arrow[dr, "\gamma_0'"] & & Y_1 \arrow[dl, "\gamma_1"'] \arrow[dr, "\gamma_1'"] & & \cdots & & Y_n \arrow[dl, "\gamma_n"'] \\ X = Z_0 & & Z_1 & & Z_2 & \cdots & Z_n & \end{tikzcd}$$

where $\overline{\gamma_0} = \overline{\gamma_0'}$, $\widehat{\gamma_0'} = \widehat{\gamma_1}$, $\overline{\gamma_1} = \overline{\gamma_1'}$ , ..., $\widehat{\gamma_{n-1}'} = \widehat{\gamma_n}$. This stickleback has length $n$: the shortest stickleback has one arrow.

Suppose that, in the above diagram, there exists a map $x$ making

$$\begin{tikzcd}
 & Y_i \arrow[dl, "\gamma_i"'] \arrow[dr, "\gamma_i'"] & \\
Z_i & & Z_{i+1} \arrow[ll, dotted, bend left, "x"]
\end{tikzcd}$$

commute. Then it follows that $\overline{x} \geq \widehat{\gamma_i'}$, since

$$\overline{x}\widehat{\gamma_i'} = \widehat{\overline{x}\gamma_i'} = \widehat{\gamma_i'\overline{\gamma_i}} = \widehat{\gamma_i'\overline{\gamma_i'}} = \widehat{\gamma_i'}.$$

This allows us to define a new stickleback, which is the same as $\gamma$ except for that we replace the fragment

$$\begin{tikzcd}
 & Y_i \arrow[dl, "\gamma_i"'] \arrow[dr, "\gamma_i'"] & & Y_{i+1} \arrow[dl, "\gamma_{i+1}"'] \\
Z_i & & Z_{i+1} &
\end{tikzcd}
\quad \text{by} \quad
\begin{tikzcd}
 & Y_{i+1} \arrow[dl, "x\gamma_{i+1}"'] \\
Z_i &
\end{tikzcd}$$

The stickleback so obtained is called a *shortening* of $\gamma$ *via* $x$, and the new map $x\gamma_{i+1}$ is called the *contracted map.* We say that a stickleback is *short* when it does not admit any shortenings. The key fact about shortenings is the following lemma:

4.13. Lemma. *Any stickleback on $X$ can be shortened to a short stickleback, which does not depend on the choice or order of shortenings.*

In effect, this says that shortening, considered as a reduction relation on sticklebacks, is strongly normalizing. It is clear of course that it is terminating, since shortening strictly decreases the length of a stickleback.

Proof. Consider a shortening as above, induced by a morphism $x : Z_{i+1} \to Z_i$. Suppose that $x' : Z_{i+1} \to Z_i$ also induces a shortening. Then we have:

$$\begin{aligned}
x\gamma_i' = \gamma_i = x'\gamma_i' \quad &\Rightarrow \quad x\widehat{\gamma_i} = x'\widehat{\gamma_i'} \\
&\Rightarrow \quad x\widehat{\gamma_{i+1}} = x'\widehat{\gamma_{i+1}} \\
&\Rightarrow \quad x\widehat{\gamma_{i+1}}\gamma_{i+1} = x'\widehat{\gamma_{i+1}}\gamma_{i+1} \\
&\Rightarrow \quad x\gamma_{i+1} = x'\gamma_{i+1}
\end{aligned}$$

so that $x$ and $x'$ give rise to the same shortening. The first implication is where the axiom [**RR.5**] is used.

Finally, we have to show that shortenings can be applied in any order. This is clear when the shortenings are not adjacent. When they are (say via $x : Z_{i+1} \to Z_i, y : Z_{i+2} \to Z_{i+1}$) then first shortening via $x$ leaves a new stickleback which we shorten via $yx$; alternatively, first shortening via $y$ leaves a stickleback which we shorten via $x$. Either way, the resulting stickleback has $xy\gamma_{i+2}$ as contracted map. ∎

The short stickleback associated to $\gamma$ will be denoted by $sh(\gamma)$. We now define $\mathsf{S}(X)$ to be the set of all short sticklebacks on $X$. For a morphism $f : X \to Y$, let $\mathsf{S}(f)$ be the partial function which sends the stickleback $\gamma$ to the shortening of

$$\begin{tikzcd}
& Y_0 \arrow[dl, "f\gamma_0"'] \arrow[dr, "\gamma_0'"] & & Y_1 \arrow[dl, "\gamma_1"'] \arrow[dr, "\gamma_1'"] & & \cdots & & Y_n \arrow[dl, "\gamma_n"'] \\
Y = Z_0 & & Z_1 & & Z_2 & \cdots & Z_n &
\end{tikzcd}$$

provided $\overline{f\gamma_0} = \overline{\gamma_0}$, and which is undefined otherwise.

The restriction of $\mathsf{S}(f)$ corresponds to the set of sticklebacks $\gamma$ for which $\overline{f\gamma_0} = \overline{\gamma_0}$. Because $\overline{f\gamma_0} = \overline{\overline{f}\gamma_0}$, this is the same as $\mathsf{S}(\overline{f})$.

The range of $\mathsf{S}(f)$ corresponds to all sticklebacks with first leg $f\gamma_0$. Clearly such sticklebacks are in the domain of $\mathsf{S}(\widehat{f})$; conversely if a stickleback $\delta$ on $Y$ is in the domain of $\mathsf{S}(\widehat{f})$, that means that $\overline{\widehat{f}\delta_0} = \overline{\delta_0}$, i.e. that $\widehat{f}\delta_0 = \delta_0$. Then consider the following stickleback $\gamma$ on $X$:

$$\begin{tikzcd}
& X \arrow[dl, "\overline{\widehat{\delta_0}f}"'] \arrow[dr, "\widehat{\delta_0}f"] & & Y_0 \arrow[dl, "\delta_0"'] \arrow[dr, "\delta_0'"] & & Y_1 \arrow[dl, "\delta_1"'] \arrow[dr, "\delta_1'"] & & \cdots & & Y_n \arrow[dl, "\delta_n"'] \\
X & & Y & & Y = Z_0 & & Z_1 & \cdots & Z_n &
\end{tikzcd}$$

It is now readily verified that this is indeed a stickleback and that $\mathsf{S}(f)(\gamma) = \delta$. Remaining details are left to the reader.

The result is now:

4.14. THEOREM. [Schein's Theorem for Range Categories] *Every small range category in which* [**RR.5**] *holds admits a faithful (but generally not full) embedding into the partial map category of a regular category, namely sets and partial functions.*

We remark that the condition [**RR.5**] is not only sufficient but also necessary, because in a regular category, every map is an epimorphism onto its range.

## 5. Free Range Categories

Since range categories are defined equationally, the category $\mathfrak{RRcat}_0$ of range categories and functors is monadic over the category of directed graphs. In this section we provide some insight into the explicit construction of the free range category. Consider first the following chain of forgetful functors:

$$\mathsf{DirGraph} \longleftarrow \mathfrak{Cat}_0 \longleftarrow \mathfrak{Rcat}_0 \longleftarrow \mathfrak{RRcat}_0.$$

It is well-known that categories are monadic over directed graphs, and in [Cockett & Lack 2002] an explicit construction of the left adjoint to the forgetful functor from restriction categories to categories was given. Here we extend this picture by constructing the left adjoint to the forgetful functor from range categories to restriction categories. In the MSc. thesis of the second author the left adjoint to the forgetful functor from range categories to graphs was constructed; the construction presented here is an adaptation of the one presented there.

5.1. Labeled Trees The main technical ingredient for the construction of the free range category on a restriction category $\mathsf{C}$ is a specific type of tree, suitably labeled by arrows from $\mathsf{C}$. These trees will represent the freely added range idempotents. Throughout this section, $\mathsf{C}$ is a fixed small restriction category.

First, define the category $\mathsf{Tree}_*$ as follows. The objects are finite trees with designated base point $*$, that is, finite directed graphs in which for every pair $a, b$ of distinct vertices there is a unique sequence

$$[a = v_0, e_0, v_1, e_1, \dots, e_{k-1}, v_k = b]$$

where the $v_i$ are pairwise distinct vertices and $e_i$ is an edge from $v_i$ to $v_{i-1}$ or from $v_{i-1}$ to $v_i$. We usually suppress the base point from the notation, unless there is risk of confusion. The morphisms between such based trees are graph homomorphisms which preserve the base point.

Given a tree $T$ we shall call a graph homomorphism $\alpha : T \to \mathsf{C}$ a $\mathsf{C}$-labeling of $T$. The object $\alpha(*)$ is referred to as the base point of $\alpha$; as we shall see, the labeling $\alpha$ specifies a formal idempotent on the object $\alpha(*)$. A typical labeling might look something like

$$\begin{tikzcd}
A_4 & A_1 \arrow[d, dashed, "f_4"'] & A_2 & A_3 & \\
A_3 & A_0 \arrow[ul, "f_6"] \arrow[dr, dashed, "f_5"'] & A_0 \arrow[u, "f_2"'] \arrow[ur, "f_3"'] \arrow[dr, dashed, "f_7"] & A_1 \arrow[d, dashed, "f_4"] & \\
A_2 & A_0 \arrow[l, "f_2"] \arrow[ul, "f_3"'] \arrow[r, dashed, "f_1"'] & \underline{A} \arrow[r, "f_0"'] & A_0 & A_1 \arrow[l, dashed, "f_4"]
\end{tikzcd}$$

The base point of this tree is the underlined object. Note that certain subtrees occur multiple times. This tree would represent the intersection of the following three idempotents on $A$:

$$\widehat{f_1\overline{f_2}\ \overline{f_3}} \qquad \widehat{f_5\widehat{f_4}\overline{f_6}} \qquad \overline{f_0\ \widehat{f_4}\widehat{f_4}f_7\widehat{\overline{f_2}\ \overline{f_3}}}$$

We make a typographical distinction between arrows which point towards the basepoint and those pointing away from it. The former are drawn dashed, and represent ranges; the latter are drawn solid and represent domains.

We refer to a tree with labeling $\alpha$ as above as a $\mathsf{C}$-*tree*. We denote such a structure by $(T, \alpha)$, or often simply by $T$ (we will not often encounter multiple labelings of the same tree). We also call a tree with $\alpha(*) = A$ a tree based at $A$. We have a category $\mathsf{C}\text{-}\mathsf{Tree}$ of $\mathsf{C}$-labeled trees and graph homomorphisms which respect the base point and the labeling.

5.2. Operations on Trees Since trees based at $A$ are to represent idempotents at $A$, we need to identify trees which give rise to the same idempotent. For example, the trees

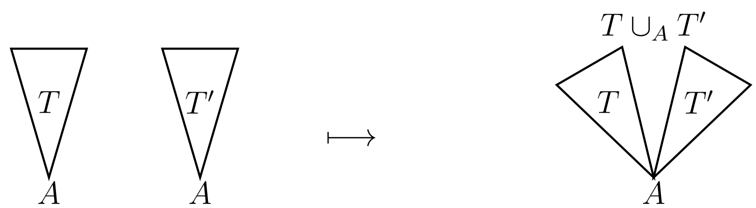
$$\begin{tikzcd} A_0 \arrow[dr, dashed, "f"] & \\ A_0 \arrow[r, dashed, "f"'] & \underline{A} \end{tikzcd} \qquad\qquad \begin{tikzcd} A_0 \arrow[r, dashed, "f"'] & \underline{A} \end{tikzcd}$$

both represent the idempotent $\hat{f} = \hat{f} \cap \hat{f}$ on $A$. We now develop several operations on trees which capture this invariance; together, these generate an equivalence relation on trees which we denote by $\sim$.

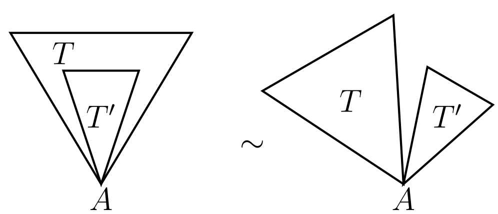
Most of these operations depend on tree grafting. Given a tree $T$, a chosen vertex $A \in T$, and another tree $T'$ with a chosen vertex also labeled $A$, we define

$$T \cup_A T'$$

to be the result of identifying the two copies of $A$ in the disjoint union (coproduct) of the trees $T, T'$. We depict this as follows:




Duplication Given a tree $T$, a vertex $x$ labeled $A$ and a subtree $T'$ of $T$ containing $A$, we may form a new tree $T \cup_A T'$ by first taking the disjoint union of $T$ and $T'$ and then identifying the two copies of $A$. Graphically:




Here the base point may be in $T$, it may be equal to $A$, or it may be in $T'$, in which case it may occur in either copy (but of course not in both).

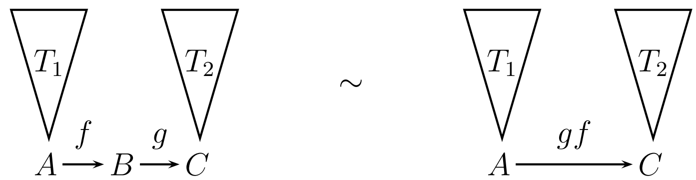
Identities The tree consisting of a single vertex $A$ (which then necessarily is the base point) is equal to the tree consisting of one edge labeled by $1_A$. The base point may be on either side.

$$\underline{A} \xrightarrow{1_A} A \qquad \sim \qquad \underline{A} \qquad \sim \qquad A \overset{1_A}{\dashrightarrow} \underline{A}$$

Composition Suppose $T$ can be decomposed as

$$T = T_1 \cup_A ( \; A \xrightarrow{f} B \xrightarrow{g} C \; ) \cup_C T_2.$$

Assume moreover that $B$ is not the base point of the tree. Then this tree is equivalent to the "contracted" tree $T_1 \cup_A ( A \xrightarrow{gf} C ) \cup_C T_2$. The base point may occur on either side. Graphically this may be depicted as




Restriction Given a tree $T$ and vertex labeled by $A$, and an outgoing edge $A \to^h B$ where $B$ is a leaf, we may replace $h : A \to B$ by an edge labeled $\overline{h} : A \to A$.

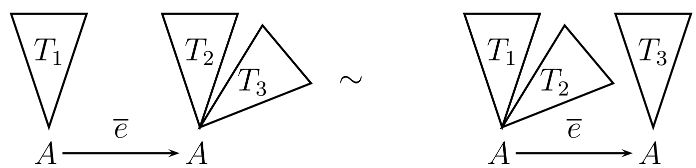
$$T \cup_A ( A \xrightarrow{h} B ) \qquad \sim \qquad T \cup_A ( A \xrightarrow{\overline{h}} A )$$

Here, the base point must be in $T$ (possibly $A$) but not $B$.

Sliding Subtrees may be moved across idempotents, using the identity

$$T_1 \cup_A ( A \xrightarrow{e} A ) \cup_A T_2 \cup_A T_3 \qquad \sim \qquad T_1 \cup_A T_2 \cup_A ( A \xrightarrow{e} A ) \cup_A T_3$$

Here, the base point may occur on either side, and it may also be slid across. Graphically:



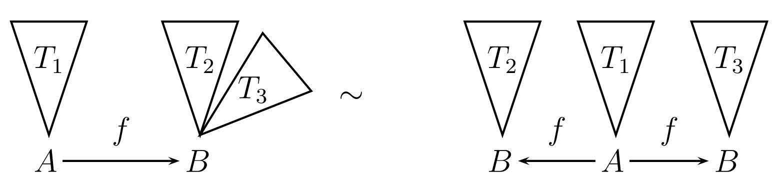

Determinism Given a tree at $A$ or at $B$ which decomposes as

$$T = T_1 \cup_A ( A \xrightarrow{f} B ) \cup_B T_2 \cup_B T_3$$

we may split at $f$ to form

$$T_1 \cup_A (( A \xrightarrow{f} B ) \cup_B T_2) \cup_A (( A \xrightarrow{f} B ) \cup_B T_3)$$

where again the base point may occur anywhere (if the base point was $B$, then we may choose either of the two copies).




We write $(T, \alpha) \sim (T', \alpha')$ to say that these two trees can be transformed into each other using the above manipulations.

5.3. CONSTRUCTION We now turn to the definition of the category $\mathcal{R}(\mathsf{C})$, the free range category on the restriction category $\mathsf{C}$. The objects will be those of $\mathsf{C}$, and the morphisms will be equivalence classes of pairs $(f, T)$, where $f$ is a morphism of $\mathsf{C}$ subsumed by the tree $T$. Formally:

5.4. DEFINITION. *A map $f : A \to B$ is* subsumed *by a tree $T$ if*

- *$T$ is based at $dom(f) = A$*
- *Adding $f$ as a new outgoing edge to $T$ results in an equivalent tree.*

*An equivalence relation on pairs $(f, T)$ is generated by:*

- *$(f, T) \sim (f, T')$ when $T \sim T'$*
- *$(f, T) \sim (fe, T)$ when $e = \overline{e}$ is subsumed by $T$.*

*The equivalence class of $(f, T)$ will be denoted $[f, T]$.*

Given a morphism $f : A \to B$ and a tree $S$ based at $B$, we define a new tree $f^*S$ at $A$ by

$$f^*S = (\underline{A} \to^f B) \cup_B S$$

i.e. by grafting in the edge labeled by $f$ at $B$ and letting $A$ be the new base point.

5.5. LEMMA. *The operation $f^*$ is well-defined w.r.t. the equivalence relation on trees, i.e. if $S \sim S'$ then $f^*S \sim f^*S'$. Moreover, it has the property that $f^*(S \cup_B S') \sim f^*S \cup_A f^*S'$.*

PROOF. The first claim is clear by inspection of each of the manipulations on trees. The second claim follows from the determinism rule. ∎

Next, we define an operation $\exists_f$, for $f : A \to B$ sending trees on $A$ to trees on $B$ by letting

$$\exists_f T = T \cup_A (A \to^f \underline{B})$$

Again it is easily proved that this is well-defined.

5.6. DEFINITION. *Let $(f, T)$ and $(g, S)$ be two suitable pairs, where $f : A \to B, g : B \to C$.*

1. *The composite $[g, S][f, T]$ is defined to be $[gf, T \cup_A f^*S]$.*
2. *The restriction of $[f, T]$ is defined to be $[1_A, T]$*
3. *The range of $[f, T]$ is defined to be $[1_B, \exists_f T]$.*

5.7. LEMMA. *The composition law on $\mathcal{R}(\mathsf{C})$ is well-defined.*

PROOF. We already know that $S \sim S'$ implies $f^*S \sim f^*S'$, and $T \cup_A f^*S \sim T \cup_A f^*S'$. But we also have $(f, T) \sim (fe, T)$, so that $T \cup_A (fe)^*S \sim (A \to^e A) \cup_A T \cup_A f^*S \sim T \cup_A f^*S$, where we used sliding for the first equivalence. The result now follows easily. ∎

5.8. Theorem. *Under the above definitions of composition, restriction and range, $\mathcal{R}(\mathsf{C})$ is a range category.*

Proof. First, we check that the composition is unital and associative. The identity at $A$ is the map $[1_A, \{1_A\}]$, where it doesn't matter because of sliding whether we take this to be an ingoing or outgoing edge. Then

$$[f,T][1_A,\{1_A\}] = [f,\{1_A\} \cup_A 1_A^* T \cup_A \{1_A\}] = [f,T].$$

Similarly

$$[1_B,\{1_B\}][f,T] = [f, T \cup_A f^*\{1_B\}] = [f, T \cup_A \{f\}] = [f,T].$$

For associativity, consider

$$\begin{aligned} ([h,R][g,S])[f,T] &= [hg, S \cup_B g^* R][f,T] \\ &= [hgf, T \cup_A f^*(S \cup_B g^* R)] \\ &= [hgf, T \cup_A f^* S \cup_A f^* g^* R] \\ &= [hgf, T \cup_A f^* S \cup_A (gf)^* R] \\ &= [h,R][gf, T \cup_A f^* S] \\ &= [h,R]([g,S][f,T]) \end{aligned}$$

Next, we verify the restriction identities. It is clear that the restriction operation is well-defined on equivalence classes.

**[R.1]** Given $[f,T] : A \to B$ we have

$$\begin{aligned} [f,T]\overline{[f,T]} &= [f,T][1,T] \\ &= [f, T \cup_A 1^* T] \\ &= [f, T \cup_A T] \\ &= [f,T] \end{aligned}$$

**[R.2]** Given $[f,T],[g,S] : A \to B$ we have

$$\begin{aligned} \overline{[f,T]}\ \overline{[g,S]} &= [1,T][1,S] \\ &= [1, T \cup_A 1^* S] \\ &= [1, T \cup_A S] \\ &= [1,S][1,T] \\ &= \overline{[g,S]}\ \overline{[f,T]} \end{aligned}$$

**[R.3]** Given $[f,T],[g,S] : A \to B$ we have

$$\begin{aligned} \overline{[g,S]\overline{[f,T]}} &= \overline{[g,S][1,T]} \\ &= \overline{[g, S \cup_A T]} \\ &= [1, S \cup_A T] \\ &= \overline{[g,T]}\ \overline{[f,S]} \end{aligned}$$

**[R.4]** Given $[f,T]: A \to B, g: B \to C$ we have

$$
\begin{aligned}
[f,T]\overline{[g,S][f,T]} &= [f,T]\overline{[gf, T \cup_A f^*S]} \\
&= [f,T][1, T \cup_A f^*S] \\
&= [f, T \cup_A T \cup_A f^*S] \\
&= [f, T \cup_A f^*S] \\
&= [1,S][f,T] \\
&= \overline{[g,S]}[f,T]
\end{aligned}
$$

**[RR.1]** Given $[f,T]: A \to B$, note that $\widehat{[f,T]} = [1, \exists_f T]$ is a restriction idempotent.

**[RR.2]** Given $[f,T]: A \to B$, consider

$$\widehat{[f,T]}[f,T] = [1,\exists_f T][f,T] = [f, f^*\exists_f T] = [f,T]$$

where the last equality uses determinism.

**[RR.3]** Given $[f,T]: A \to B$ and $[g,S]: B \to C$, we have

$$\widehat{\overline{[g,S]}[f,T]} = \widehat{[1,S][f,T]} = [f, \widehat{T \cup_A f^*S}] = [1, \exists_f (T \cup_A f^*S)]$$

Using determinism again, the tree $\exists_f(T \cup_A f^*S)$ is equivalent to $S \cup_B \exists_f T$ as needed.

**[RR.4]** Given $[f,T]: A \to B$ and $[g,S]: B \to C$, we have

$$\widehat{[g,S]\widehat{[f,T]}} = \widehat{[g,S][1,\exists_F T]} = [g, \widehat{\exists_f T \cup_B S}] = [1, \exists_g (S \cup_B \exists_f T)]$$

while

$$\widehat{[g,S][f,T]} = [gf, \widehat{T \cup_A f^*S}] = [1, \exists_{gf}(T \cup_A f^*S)].$$

These are equivalent trees using determinism once again. ∎

5.9. Universal Property The categories $\mathsf{C}$ and $\mathcal{R}(\mathsf{C})$ have the same objects, and given a map $f: A \to B$ in $\mathsf{C}$, there is an induced morphism $[f, \{f\}]: A \to B$ in $\mathcal{R}(\mathsf{C})$.

5.10. Lemma. *The above assignment is a restriction functor $\iota: \mathsf{C} \to \mathcal{R}(\mathsf{C})$.*

Proof. Given $f: A \to B, g: B \to C$ we have

$$[g,\{g\}][f,\{f\}] = [gf, \{gf, f\}] = [gf, \{gf\}]$$

using duplication and composition.

To see that the restriction is preserved, consider $f: A \to B$ and calculate

$$\overline{[f,\{f\}]} = [1,\{f\}] = [1,\{\overline{f}\}] = [\overline{f},\{\overline{f}\}]$$

using restriction of $f$ and the equivalence relation on the first component of maps.

∎

Next, we aim to show that this functor is universal amongst restriction functors into range categories.

Now, consider a restriction functor $F : \mathsf{C} \to \mathsf{D}$ where $\mathsf{D}$ is a range category. Given a tree $(T, \alpha)$ on $A$, we define an idempotent $F(T, \alpha)$ on $FA$ as follows. Denote the incoming edges at $A$ by $f_1 : A_1 \to A, \dots, f_k : A_k \to A$ and the outgoing edges by $g_1 : A \to B_1, \dots g_l : A \to B_l$, with residual trees $T_1, \dots, T_k, S_1, \dots, S_l$ respectively. Then by induction the idempotents $F(T_1, A_1), \dots F(T_k, A_k)$ and $F(S_1, B_1), \dots F(S_l, B_l)$ are defined, and so we may define

$$F(T, A) := \bigcap_{i=1}^{k} \exists_{f_i}(F(T_i, A_i)) \cap \bigcap_{i=1}^{l} g_i^*(F(S_i, B_i)).$$

The value at the empty tree is defined to be the identity. Note that $F(T, A)$ is order-reversing in the sense that if $T'$ is a subtree of $T$ based at $A$, then $F(T, A) \leq F(T', A)$. Also note that for trees $T, S$ both based at $A$, we have

$$F(T \cup_A S, A) = F(T, A) \cap F(S, A).$$

Here, the operation $\cap$ is the meet in the lattice of idempotents.

5.11. Lemma. *For trees $T, T'$ based at $A$ for which $T \sim T'$, we have $F(T, A) = F(T', A)$.*

Proof. Inspection of all the manipulation steps on trees. Invariance under duplication easily follows from the remarks preceding the statement of the lemma. The identities clause is also obvious.

Next, consider the operation of composition. One case is as follows: we have a decomposition of $T$ as

$$T = T_1 \cup_A (\, A \xrightarrow{f} B \xrightarrow{g} C \,) \cup_C T_2,$$

where $A$ is the base. Then we get

$$F(T, A) = F(T_1, A) \cap f^*(g^* F(T_2, C)) = F(T_1, A) \cap (gf)^* F(T_2, C).$$

Similarly, if the base is $C$ we would use the fact that $\exists_g \exists_f = \exists_{gf}$. Other possible cases are handled inductively.

For restriction, suppose that $f : A \to B$ is a branch of $T$ with $B$ a leaf. We assume that $A$ is the base point. Then $F(T, A) = F(T, A) \cap F\overline{f}$, and the result follows from the identities

$$(g\overline{f})^*(e) = \overline{f} \cap g^*(e)$$

and

$$\exists_{\overline{f}g}(e) = \exists_g(e) \cap \overline{f}$$

For sliding, there are two cases, depending on the location of the base point. In the first, we have

$$\begin{aligned} F(T_1, A) \cap e^*(F(T_2, A) \cap F(T_3, A)) &= F(T_1, A) \cap e^* F(T_2, A) \cap e^* F(T_3, A) \\ &= F(T_1, A) \cap F(T_2, A) \cap e^* F(T_3, A) \end{aligned}$$

using the fact that for idempotents $e, e'$ we have $e^*(e') = ee'$. In the other case we calculate

$$\exists_e F(T_1, A) \cap F(T_2, A) \cap F(T_3, A) = e \cap F(T_1, A) \cap F(T_2, A) \cap F(T_3, A)$$

using the identity $\exists_e(e') = ee'$.

Finally, for determinism we first consider the case where the base point is $A$. Then observe that $f^*(e \cap e') = f^*e \cap f^*e'$. Thus if $F(T_2, B) = e, F(T_3, B) = e'$ then

$$F(T, A) = F(T_1, A) \cap Ff^*(e \cap e') = F(T_1, A) \cap Ff^*e \cap Ff^*e'$$

and the latter is the value associated to the tree in which $f$ has been duplicated according to the determinism rule. When the base point is $B$, we use the Frobenius property. ∎

We now define the extension of the functor $F : \mathsf{C} \to \mathsf{D}$ to $\tilde{F} : \mathcal{R}(\mathsf{C}) \to \mathsf{D}$ to be

$$\tilde{F}A = FA \qquad \tilde{F}[f, T] = (Ff)F(T, A).$$

It is readily seen that this is well-defined: we already have shown invariance under reduction of $T$, and if $[f, T] = [fe, T]$ then we have $F(T, A) \leq Fe$, so that

$$\tilde{F}[f, T] = (Ff)F(T, A) = (Ff)FeF(T, A) = F(Fe)F(T, A) = \tilde{F}[Fe, T]$$

as needed.

For composition, observe that

$$\begin{aligned} (Fg)F(S, B)(Ff)F(T, A) &= (Fg)(Ff)(Ff)^*(F(S, B))F(T, A) \\ &= F(gf)F(f^*S, A)F(T, A) \\ &= F(gf)F(f^*S \cup_A T) \end{aligned}$$

as needed.

It is clear from the definition of restriction and range that these are preserved by $\tilde{F}$. It is also clear that $\tilde{F}$ is an extension of $F$. Finally, we show uniqueness of this extension: given another extension $G$ of $F$ (where $G$ is a range functor). Consider a map $[f, T] : A \to B$. This map may be factored as $[f, \{f\}][1, T]$, so that $G[f, T] = (Ff)G[1, T]$, and hence it suffices to show that $\tilde{F}$ and $G$ agree on restriction idempotents.

Given $[T]$ at $A$, we show by induction on $T$ that $G[T] = F[T, A]$. If $T$ consists of the identity only, then this is clear. Denote the incoming edges of $T$ at $A$ by $f_1 : A_1 \to A, \ldots, f_k : A_k \to A$ and the outgoing edges by $g_1 : A \to B_1, \ldots g_l : A \to B_l$, with residual trees $T_1, \ldots, T_k, S_1, \ldots, S_l$ respectively. Then by induction we have $F(T_1, A_1) = G[T_1], \ldots, F(T_k, A_k) = G[T_k]$ and $F(S_1, B_1) = G[S_1], \ldots, F(S_l, B_l) = G[S_l]$

$$\begin{aligned} F(T, A) &= \bigcap_{i=1}^{k} \exists_{f_i}(F(T_i, A_i)) \cap \bigcap_{i=1}^{l} g_i^*(F(S_i, B_i)) \\ &= \bigcap_{i=1}^{k} \exists_{f_i} G[T_i] \cap \bigcap_{i=1}^{l} g_i^* G[S_i] \\ &= G[T] \end{aligned}$$

since $G$ commutes with intersections of restriction idempotents, with pullback functors $f_i^*$ and with direct image maps $\exists_{g_i}$.

We conclude:

5.12. Theorem. *The assignment* $\mathsf{C} \mapsto \mathcal{R}(\mathsf{C})$ *underlies a functor* $\mathfrak{Rcat}_0 \to \mathfrak{RRcat}_0$ *which is left adjoint to the forgetful functor from range categories to restriction categories.*

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*Department of Computer Science, University of Calgary*
*2500 University Dr. NW, Calgary, Alberta Canada T2N 1N4, Canada*

*Department of Mathematics and Statistics, University of Ottawa*
*585 King Edward Avenue, Ottawa, ON, K1N 6N5, Canada*

Email: robin@ucalgary.ca
guox@cpsc.ucalgary.ca
phofstra@uottawa.ca

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