Theory and Applications of Categories, Vol. 26, No. 2, 2012, pp. 30–59.

ON THE ITERATION OF WEAK WREATH PRODUCTS

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Abstract. Based on a study of the 2-category of weak distributive laws, we describe a method of iterating Street's weak wreath product construction. That is, for any 2-category $\mathcal{K}$ and for any non-negative integer n, we introduce 2-categories $\mathsf{Wdl}^{(n)}(\mathcal{K})$, of $(n+1)$-tuples of monads in $\mathcal{K}$ pairwise related by weak distributive laws obeying the Yang-Baxter equation. The first instance $\mathsf{Wdl}^{(0)}(\mathcal{K})$ coincides with $\mathsf{Mnd}(\mathcal{K})$, the usual 2-category of monads in $\mathcal{K}$, and for other values of n, $\mathsf{Wdl}^{(n)}(\mathcal{K})$ contains $\mathsf{Mnd}^{n+1}(\mathcal{K})$ as a full 2-subcategory. For the local idempotent closure $\overline{\mathcal{K}}$ of $\mathcal{K}$, extending the multiplication of the 2-monad Mnd, we equip these 2-categories with n possible 'weak wreath product' 2-functors $\mathsf{Wdl}^{(n)}(\overline{\mathcal{K}}) \to \mathsf{Wdl}^{(n-1)}(\overline{\mathcal{K}})$, such that all of their possible n-fold composites $\mathsf{Wdl}^{(n)}(\overline{\mathcal{K}}) \to \mathsf{Wdl}^{(0)}(\overline{\mathcal{K}})$ are equal; that is, such that the weak wreath product is 'associative'. Whenever idempotent 2-cells in $\mathcal{K}$ split, this leads to pseudofunctors $\mathsf{Wdl}^{(n)}(\mathcal{K}) \to \mathsf{Wdl}^{(n-1)}(\mathcal{K})$ obeying the associativity property up-to isomorphism. We present a practically important occurrence of an iterated weak wreath product: the algebra of observable quantities in an Ising type quantum spin chain where the spins take their values in a dual pair of finite weak Hopf algebras. We also construct a fully faithful embedding of $\mathsf{Wdl}^{(n)}(\overline{\mathcal{K}})$ into the 2-category of commutative $n+1$ dimensional cubes in $\mathsf{Mnd}(\overline{\mathcal{K}})$ (hence into the 2-category of commutative $n+1$ dimensional cubes in $\mathcal{K}$ whenever $\mathcal{K}$ has Eilenberg-Moore objects and its idempotent 2-cells split). Finally we give a sufficient and necessary condition on a monad in $\overline{\mathcal{K}}$ to be isomorphic to an n-ary weak wreath product.

Introduction

At the heart of the iteration of wreath products in the work [Cheng (2011)] of Eugenia Cheng, lies the 2-monad Mnd on the 2-category 2-Cat of 2-categories, 2-functors and 2-natural transformations, first discussed in [Street (1972)]. For any 2-category $\mathcal{K}$, the iteration of its associative multiplication $\mathsf{Mnd}^n(\mathcal{K}) \to \mathsf{Mnd}^{n-1}(\mathcal{K}) \to \ldots \to \mathsf{Mnd}(\mathcal{K})$ takes an $(n+1)$-tuple of monads, pairwise related by distributive laws obeying the Yang-Baxter equality, to a unique monad in $\mathcal{K}$. The resulting monad (that appeared already in [Jara Martínez et al. (2008)] in the case when $\mathcal{K}$ is the monoidal category of vector spaces) can be interpreted as an iterated wreath product.

The aim of this paper is to find a similar iteration process for weak wreath products introduced by Ross Street in [Street (2009)] and by Stefaan Caenepeel and Erwin De Groot in [Caenepeel, De Groot (2000)].

These weak wreath products are defined in 2-categories in which idempotent 2-cells split, see [Street (2009)]. They are induced by weak distributive laws; that is, certain

Received by the editors 2011-10-09 and, in revised form, 2012-01-25.

Transmitted by R.J. Wood. Published on 2012-01-30.

2000 Mathematics Subject Classification: 18C15, 18D05, 16W30.

Key words and phrases: monad, weak distributive law, n-ary weak wreath product, Yang-Baxter equation, quantum spin chain.

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2-cells relating two monads. They obey the usual compatibility conditions of distributive laws with the multiplications of the monads, but the compatibility conditions with the units are weakened [Caenepeel, De Groot (2000)], [Street (2009)]. Making weak distributive laws conceptually different from their non-weak counterparts, they are not known to be monads in any 2-category. (However, a weak distributive law can be characterized as a pair of monads in 2-categories extending $\mathsf{Mnd}(\mathcal{K})$ and its variant $\mathsf{Mnd}^*(\mathcal{K})$, respectively, see [Böhm (2010)]).

In Section 2, for any 2-category $\mathcal{K}$, we construct a 2-category $\mathsf{Wdl}^{(n)}(\mathcal{K})$ for every non-negative integer n. Its objects are $(n+1)$-tuples of monads in $\mathcal{K}$ pairwise related by weak distributive laws obeying the Yang-Baxter equation. The first one, $\mathsf{Wdl}^{(0)}(\mathcal{K})$ is isomorphic to $\mathsf{Mnd}(\mathcal{K})$, the 2-category of monads in $\mathcal{K}$ as defined in [Street (1972)]. The next one, $\mathsf{Wdl}^{(1)}(\mathcal{K})$ is the 2-category of weak distributive laws, obtained by dualizing the definition in [Böhm, Lack, Street (2011)]. For every n, $\mathsf{Wdl}^{(n)}(\mathcal{K})$ contains $\mathsf{Mnd}^{n+1}(\mathcal{K})$ as a full 2-subcategory. But, in contrast to the classical (that is, non-weak) case, $\mathsf{Wdl}^{(n)}(\mathcal{K})$ is not known to arise by the $(n+1)$-fold application of some 2-monad. Although in this way we can not interpret them as multiplications of some 2-monad, for each value of n we describe n different 2-functors $\mathsf{Wdl}^{(n)}(\overline{\mathcal{K}}) \to \mathsf{Wdl}^{(n-1)}(\overline{\mathcal{K}})$ (where $\overline{\mathcal{K}}$ denotes the local idempotent closure of $\mathcal{K}$). They extend the n possible multiplications $\mathsf{Mnd}^{n+1}(\overline{\mathcal{K}}) \to \mathsf{Mnd}^{n}(\overline{\mathcal{K}})$. They give rise to a unique composite $\mathsf{Wdl}^{(n)}(\overline{\mathcal{K}}) \to \mathsf{Wdl}^{(0)}(\overline{\mathcal{K}})$ whose value on an object of $\mathsf{Wdl}^{(n)}(\overline{\mathcal{K}})$ is regarded as the (associatively iterated) weak wreath product of the $n+1$ occurring monads in $\overline{\mathcal{K}}$. Whenever idempotent 2-cells in $\mathcal{K}$ split; that is, $\overline{\mathcal{K}}$ and $\mathcal{K}$ are biequivalent, our construction yields pseudofunctors $\mathsf{Wdl}^{(n)}(\mathcal{K}) \to \mathsf{Wdl}^{(n-1)}(\mathcal{K})$ giving rise to a composite $\mathsf{Wdl}^{(n)}(\mathcal{K}) \to \mathsf{Wdl}^{(0)}(\mathcal{K})$ which is unique up-to a pseudonatural equivalence in the choice of the biequivalence $\overline{\mathcal{K}} \to \mathcal{K}$.

Our motivation to study iterated weak wreath products comes from mathematical physics. The Ising model is a quantum spin chain in which the spins take their values in the sign group $\mathbb{Z}(2)$. In its various generalizations, the spins may take their values in arbitrary finite groups [Szlachányi, Vecsernyés (1993)], in finite dimensional Hopf algebras [Nill, Szlachányi (1997)] or in finite dimensional weak Hopf algebras [Nill, Szlachányi, Wiesbrock (preprint 1998)], [Böhm (1997)]. In all of these, except the last quoted family of models, the algebra of the observable quantities in any finite interval is given by an iterated wreath product. In quantum spin chains based on weak Hopf algebras, however, the algebras of observables are iterated weak wreath products. In Section 3 we present this example in some detail.

The definition of $\mathsf{Wdl}^{(n)}(\overline{\mathcal{K}})$ is further motivated in Section 4 by a fully faithful embedding of it into the 2-category of $n+1$ dimensional cubes in the 2-category of monads in $\overline{\mathcal{K}}$. Whenever idempotent 2-cells in $\mathcal{K}$ split, this yields a fully faithful embedding of $\mathsf{Wdl}^{(n)}(\mathcal{K})$ into the 2-category of $n+1$ dimensional cubes in the 2-category of monads in $\mathcal{K}$; and also into the 2-category of $n+1$ dimensional cubes in $\mathcal{K}$ whenever in addition $\mathcal{K}$ admits Eilenberg-Moore objects for monads.

In our final Section 5 we analyze the n-ary factorization problem associated to weak distributive laws. That is, we give a complete characterization of those monads in the local idempotent closure of a 2-category which arise as iterated weak wreath products of n monads.

Throughout, for a technical simplification, we work with 2-categories. There is no difficulty, however, to extend our considerations to bicategories.

Acknowledgement I would like to express my gratitude to Ross Street and Kornél Szlachányi for their helpful comments on this work. Partial support by the Hungarian Scientific Research Fund OTKA K68195 is gratefully acknowledged.

1. Preliminaries on weak distributive laws

In this section we revisit some recent 'weak' generalizations of the formal theory of monads that will be used in the sequel.

1.1. Local idempotent closure To any 2-category $\mathcal{K}$ we associate another 2-category $\overline{\mathcal{K}}$ by freely splitting idempotent 2-cells. In more detail, the 0-cells of $\overline{\mathcal{K}}$ are the same as those in $\mathcal{K}$. The 1-cells in $\overline{\mathcal{K}}$ are pairs consisting of a 1-cell v and a 2-cell $\overline{v} : v \to v$ in $\mathcal{K}$ such that $\overline{v}.\overline{v} = \overline{v}$; that is, $\overline{v}$ is idempotent. The 2-cells $(v, \overline{v}) \to (v', \overline{v}')$ in $\overline{\mathcal{K}}$ are 2-cells $\omega : v \to v'$ in $\mathcal{K}$ such that $\overline{v}'.\omega = \omega = \omega.\overline{v}$. Horizontal and vertical compositions in $\overline{\mathcal{K}}$ are induced by those in $\mathcal{K}$. The identity 2-cell is $\overline{v} : (v, \overline{v}) \to (v, \overline{v})$.

Throughout, we shall use the notation seen above: If it is not otherwise stated, in a 1-cell in $\overline{\mathcal{K}}$, for the idempotent 2-cell part we use the overlined version of the same symbol that denotes the 1-cell part.

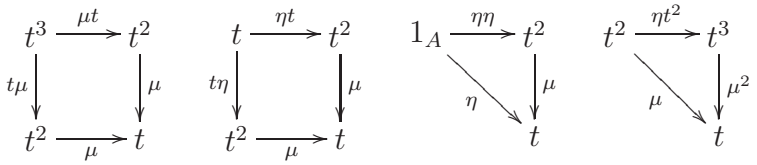
For any 2-category $\mathcal{K}$, there is an evident inclusion 2-functor $\mathcal{K} \to \overline{\mathcal{K}}$, acting on the 0-cells as the identity map, taking a 1-cell v to (v, v) – that is, the 1-cell with identity 2-cell part – and acting on the 2-cells again as the identity map.

We say that idempotent 2-cells in a 2-category $\mathcal{K}$ split if, for any idempotent 2-cell $\theta : v \to v$ there exist a 1-cell w and 2-cells $\iota : w \to v$ and $\pi : v \to w$ such that $\pi.\iota = w$ and $\iota.\pi = \theta$. If the splitting exists then it is unique up-to an isomorphism of w. Clearly, in $\overline{\mathcal{K}}$ idempotent 2-cells split for any 2-category $\mathcal{K}$.

Whenever in $\mathcal{K}$ idempotent 2-cells split, the inclusion $\mathcal{K} \to \overline{\mathcal{K}}$ becomes a biequivalence. (Since it acts on the 0-cells as the identity map, this simply means that it induces an equivalence of the hom categories.) Hence there is a *pseudofunctor* $\overline{\mathcal{K}} \to \mathcal{K}$ which is its inverse (in the sense of inverse biequivalences). On the 0-cells also this pseudofunctor acts as the identity map. On a 1-cell $(v, \overline{v})$ its action is constructed via a *chosen* splitting of the idempotent 2-cell $\overline{v}$. If $(\iota : w \to v, \pi : v \to w)$ is this chosen splitting, then the image of $(v, \overline{v})$ is w. A 2-cell $\omega : (v, \overline{v}) \to (v', \overline{v'})$ is taken to $w \xrightarrow{\iota} v \xrightarrow{\omega} v' \xrightarrow{\pi'} w'$. Let us stress that the biequivalence $\overline{\mathcal{K}} \to \mathcal{K}$ is not a 2-functor in general and it is unique only up-to a pseudonatural equivalence arising from the choice of the splitting of each idempotent 2-cell.

1.2. Demimonads In simplest terms, a demimonad in a 2-category is a monad $(A, (t, \overline{t}))$ in the local idempotent closure, cf. [Böhm, Lack, Street (2012)]. Explicitly, it is given by a 1-cell $t : A \to A$ and 2-cells $\mu : t^2 \to t$ and $\eta : 1_A \to t$ such that

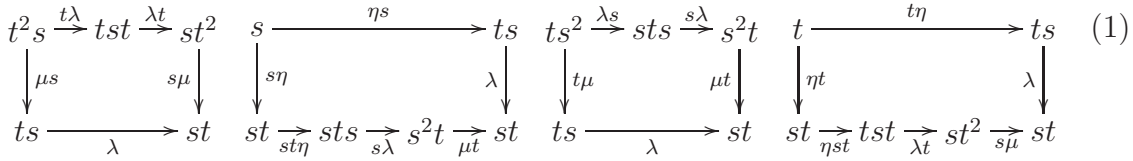
the following diagrams commute.



For a monad $(A,(t,\overline{t}))$ in the local idempotent closure, the 2-cell η being a two-sided unit for μ means that the idempotent 2-cell $\overline{t}$ must be equal to $\mu.t\eta = \mu.\eta t$ (hence it is a redundant information that will be often omitted in the sequel). This structure occurred in [Böhm (2010)] under the name 'pre-monad'.

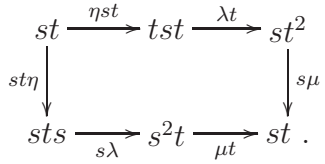
A demimonad $(A,(t,\overline{t}))$ is the image of a monad under the inclusion $\mathcal{K} \to \overline{\mathcal{K}}$ if and only if $\overline{t}$ is the identity 2-cell t.

1.3. Weak distributive laws Extending the notion of distributive law due to Jon Beck (see [Beck (1969)]), weak distributive laws in a 2-category were introduced by Ross Street in [Street (2009)] as follows. They consist of two monads (A,t) and (A,s) on the same object, and a 2-cell $\lambda : ts \to st$ such that the following diagrams commute.



(1)

The same set of axioms occurred also in [Caenepeel, De Groot (2000)]. By Proposition 2.2 in [Street (2009)], the second and fourth diagrams can be replaced by a single diagram



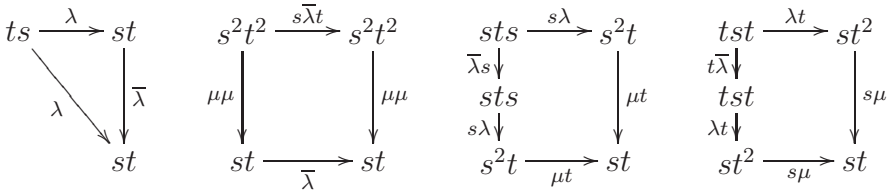
(2)

The equal paths around (2) give rise to an idempotent 2-cell $\overline{\lambda} : st \to st$ (which occurs also in the bottom rows of the second and fourth diagrams in (1)). It is an identity if and only if λ is a distributive law in the strict sense. Since (2) generalizes the 'unitality conditions' $\lambda.t\eta = \eta t$ and $\lambda.\eta s = s\eta$ on a distributive law λ, we refer to it as the 'weak unitality condition'.

Note that a weak distributive law in $\mathcal{K}$ is the same as a weak distributive law in $\mathcal{K}^{\mathrm{op}}$, the dual of $\mathcal{K}$ with respect to 1-cells.

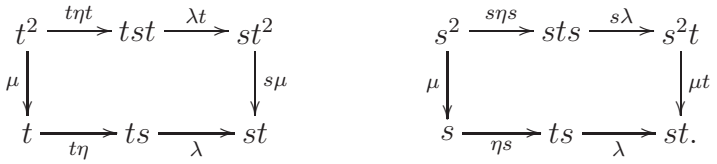
A weak distributive law in $\overline{\mathcal{K}}$ is then given by demimonads (A,t) and (A,s) and a 2-cell $\lambda : ts \to st$ rendering commutative the diagrams in (1) and obeying in addition the normalization conditions

In the sequel we shall need some identities on weak distributive laws (in $\overline{\mathcal{K}}$). The axioms imply commutativity of the following diagrams, see [Street (2009)].



(3)

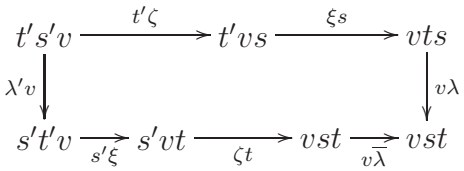
Moreover, by the associativity of μ, the left-bottom path in the last diagram in (1) commutes with the multiplication by t on the right; in the sense that $s\mu.\overline{\lambda}t.\eta t^2 = \overline{\lambda}.\eta t.\mu$. Hence also the top-right path in the last diagram in (1) commutes with the right multiplication by t, meaning the commutativity of the first diagram in



(4)

Commutativity of the second diagram follows symmetrically.

1.4. The 2-category of weak distributive laws Dualizing in the appropriate sense the definition of the 2-category of *mixed* weak distributive laws in the paper [Böhm, Lack, Street (2011)], the following 2-category $\mathsf{Wdl}(\mathcal{K})$ of weak distributive laws in $\mathcal{K}$ is obtained (see [Böhm, Gómez-Torrecillas (preprint 2011), Paragraph 1.9]). The 0-cells are the weak distributive laws $\lambda : ts \to st$. The 1-cells $\lambda \to \lambda'$ are triples consisting of a 1-cell $v : A \to A'$ and 2-cells $\xi : t'v \to vt$ and $\zeta : s'v \to vs$ in $\mathcal{K}$, such that $(v, \xi) : (A, t) \to (A', t')$ and $(v, \zeta) : (A, s) \to (A', s')$ are 1-cells in $\mathsf{Mnd}(\mathcal{K})$ (also called *monad morphisms* in [Street (1972)]) and the following diagram commutes.



(5)

The 2-cells $(v, \xi, \zeta) \to (v', \xi', \zeta')$ are 2-cells $\omega : v \to v'$ in $\mathcal{K}$ which are 2-cells in $\mathsf{Mnd}(\mathcal{K})$ (called *monad transformations* in [Street (1972)]); both $(v, \xi) \to (v', \xi')$ and $(v, \zeta) \to (v', \zeta')$. Horizontal and vertical compositions are induced by those in $\mathcal{K}$. As in [Böhm, Lack, Street (2011)], this definition can be interpreted in terms of (weak) liftings.

There is a fully faithful embedding $\mathsf{Mnd}^2(\mathcal{K}) \to \mathsf{Wdl}(\mathcal{K})$ as follows. It takes a 0-cell $((A, t), (s, \lambda))$ to the distributive law $\lambda : ts \to st$, regarded as a weak distributive law. It takes a 1-cell $((v, \xi), \zeta)$ to (v, ξ, ζ) and it acts on the 2-cells as the identity map.

1.5. Weak wreath product The weak wreath product induced by a weak distributive law in a 2-category in which idempotent 2-cells split, was discussed by Ross Street in [Street (2009), Theorem 2.4]. In the particular case of the monoidal category

(that is, one object bicategory) of modules over a commutative ring, it appeared in [Caenepeel, De Groot (2000), Theorem 3.2].

For an arbitrary 2-category $\mathcal{K}$, there is a *weak wreath product* 2-functor $\mathsf{Wdl}(\overline{\mathcal{K}}) \to \mathsf{Mnd}(\overline{\mathcal{K}})$, which sends a weak distributive law $\lambda : ts \to st$ to the monad $(st, \overline{\lambda})$ in $\overline{\mathcal{K}}$, with multiplication and unit

$$(st)^2 \xrightarrow{s\lambda t} s^2t^2 \xrightarrow{\mu\mu} st \quad \text{and} \quad 1 \xrightarrow{\eta\eta} ts \xrightarrow{\lambda} st \ .$$

It sends a 1-cell $((v, \overline{v}), \xi, \zeta) : \lambda \to \lambda'$ to the monad morphism with the same 1-cell part $(v, \overline{v})$ and the 2-cell part

$$s't'v \xrightarrow{s'\xi} s'vt \xrightarrow{\zeta t} vst \xrightarrow{v\overline{\lambda}} vst \ .$$

On the 2-cells it acts as the identity map.

Whenever idempotent 2-cells in $\mathcal{K}$ split, the biequivalence $\overline{\mathcal{K}} \simeq \mathcal{K}$ induces a pseudo-functor $\mathsf{Wdl}(\mathcal{K}) \xrightarrow{\simeq} \mathsf{Wdl}(\overline{\mathcal{K}}) \to \mathsf{Mnd}(\overline{\mathcal{K}}) \xrightarrow{\simeq} \mathsf{Mnd}(\mathcal{K})$. (It can be chosen, in fact, to be a 2-functor by choosing the biequivalence $\overline{\mathcal{K}} \to \mathcal{K}$ adopting the convention that we split identity 2-cells trivially; that is, via identity 2-cells.) Its object map yields Street's weak wreath product in $\mathcal{K}$.

1.6. Binary factorization Let $\mathcal{K}$ be any 2-category. As proved in the work [Böhm, Gómez-Torrecillas (preprint 2011)], a demimonad (A, r) is isomorphic to a weak wreath product induced by some weak distributive law $ts \to st$ in $\overline{\mathcal{K}}$ if and only if the following hold.

(a) There are 1-cells in $\mathsf{Mnd}(\overline{\mathcal{K}})$ with trivial 1-cell parts

$$(A, (t, \overline{t})) \xleftarrow{((A,A),\alpha)} (A, (r, \overline{r})) \xrightarrow{((A,A),\beta)} (A, (s, \overline{s})) \ ;$$

(b) The 2-cell

$$\pi := \big(\ (st, \overline{s}\overline{t}) \xrightarrow{\beta\alpha} (rr, \overline{r}\,\overline{r}) \xrightarrow{\mu} (r, \overline{r}) \ \big)$$

in $\overline{\mathcal{K}}$ possesses section ι (meaning $\pi.\iota = \overline{r} \equiv \mu.r\eta$) which is an s-t bimodule morphism with respect to the t- and s-actions induced on r by α and β, respectively.

Indeed, for the weak wreath product induced by a weak distributive law $\lambda : ts \to st$, we have 1-cells

$$(A, (t, \overline{t})) \xleftarrow{((A,A),\lambda.t\eta)} (A, (st, \overline{\lambda})) \xrightarrow{((A,A),\lambda.\eta s)} (A, (s, \overline{s}))$$

in $\mathsf{Mnd}(\overline{\mathcal{K}})$. Moreover, the 2-cell π in part (b) comes out as

$$\big(\ (st, \overline{s}\overline{t}) \xrightarrow{\lambda\lambda.\eta st\eta} (stst, \overline{\lambda}\,\overline{\lambda}) \xrightarrow{\mu\mu.s\lambda t} (st, \overline{\lambda}) \ \big) = \big(\ (st, \overline{s}\overline{t}) \xrightarrow{\overline{\lambda}} (st, \overline{\lambda}) \ \big) \ ,$$

which is split by the bimodule morphism $\overline{\lambda} : (st, \overline{\lambda}) \to (st, \overline{s}\overline{t})$.

Conversely, if properties (a) and (b) hold, then

$$ts \xrightarrow{\alpha\beta} rr \xrightarrow{\mu} r \xrightarrow{\iota} st$$

is a weak distributive law with corresponding idempotent equal to $\iota.\pi : st \to st$. The isomorphism between the induced weak wreath product and (A, r) is provided by $(st, \iota.\pi) \underset{\iota}{\overset{\pi}{\rightleftarrows}} (r, \overline{r})$ in $\overline{\mathcal{K}}$. For the details of the proof we refer to the article [Böhm, Gómez-Torrecillas (preprint 2011)].

2. 2-categories of weak distributive laws and the iterated weak wreath product

Throughout this section, $\mathcal{K}$ is an arbitrary 2-category and $\overline{\mathcal{K}}$ stands for its local idempotent closure. For any non-negative integer n, we define a 2-category $\mathsf{Wdl}^{(n)}(\mathcal{K})$. Its objects are $(n+1)$-tuples of monads pairwise related by weak distributive laws obeying the Yang-Baxter condition (see (6) below). For each value of n, we construct n different 2-functors $\mathsf{Wdl}^{(n)}(\overline{\mathcal{K}}) \to \mathsf{Wdl}^{(n-1)}(\overline{\mathcal{K}})$ corresponding to taking the weak wreath product of two consecutive monads of the $n+1$ occurring ones. We show that these 2-functors give rise to a unique composite $\mathsf{Wdl}^{(n)}(\overline{\mathcal{K}}) \to \mathsf{Wdl}^{(0)}(\overline{\mathcal{K}}) = \mathsf{Mnd}(\overline{\mathcal{K}})$. We regard its object map as the n-ary weak wreath product of the involved monads.

2.1. The 2-category $\mathsf{Wdl}^{(n)}(\mathcal{K})$ For any non-negative integer n, a 0-cell is given by $n+1$ monads $s_0, s_1, \ldots, s_n$ together with weak distributive laws $\lambda_{i,j} : s_j s_i \to s_i s_j$ for all $0 \le i < j \le n$, obeying for all $0 \le i < j < k \le n$ the Yang-Baxter relation

$$\begin{array}{ccccc} s_k s_j s_i & \xrightarrow{\lambda_{j,k} s_i} & s_j s_k s_i & \xrightarrow{s_j \lambda_{i,k}} & s_j s_i s_k \\ {\scriptstyle s_k\lambda_{i,j}}\downarrow & & & & \downarrow{\scriptstyle \lambda_{i,j} s_k} \\ s_k s_i s_j & \xrightarrow[\lambda_{i,k} s_j]{} & s_i s_k s_j & \xrightarrow[s_i \lambda_{j,k}]{} & s_i s_j s_k. \end{array} \tag{6}$$

The 1-cells consist of a 1-cell v and 2-cells $\xi_i : s_i' v \to v s_i$ in $\mathcal{K}$ for all $0 \le i \le n$, such that (v, ξ_i, ξ_j) is a 1-cell $\lambda_{i,j} \to \lambda'_{i,j}$ in $\mathsf{Wdl}(\mathcal{K})$ (see Paragraph 1.4), for all $0 \le i < j \le n$. The 2-cells are those 2-cells $\omega : v \to v'$ in $\mathcal{K}$ which are 2-cells $(v, \xi_i, \xi_j) \to (v', \xi_i', \xi_j')$ in $\mathsf{Wdl}(\mathcal{K})$ (in the sense of Paragraph 1.4), for all $0 \le i < j \le n$. Since $\mathsf{Wdl}(\mathcal{K})$ is closed under the horizontal and vertical compositions in $\mathcal{K}$, so is $\mathsf{Wdl}^{(n)}(\mathcal{K})$. Hence it is a 2-category with the horizontal and vertical compositions induced by those in $\mathcal{K}$.

Recall from [Cheng (2011)] that a 0-cell in $\mathsf{Mnd}^n(\mathcal{K})$ is given by n monads, pairwise related by distributive laws obeying the Yang-Baxter condition. The 1-cells consist of n monad morphisms for the n involved monads with a common underlying 1-cell, obeying (5) (though in that case the occurring idempotents are identities). The 2-cells are those 2-cells in $\mathcal{K}$ which are monad transformations for all of the n monad morphisms. With this description in mind, extending that in Paragraph 1.4, there is an evident fully faithful embedding $\mathsf{Mnd}^{n+1}(\mathcal{K}) \to \mathsf{Wdl}^{(n)}(\mathcal{K})$.

Taking any $m+1$-element subset of $\{0, 1, \ldots, n\}$ induces an evident 2-functor $\mathsf{Wdl}^{(n)}(\mathcal{K}) \to \mathsf{Wdl}^{(m)}(\mathcal{K})$.

2.2. Lemma. *Take any object* $\{\lambda_{i,j} : s_j s_i \to s_i s_j\}_{0\le i<j\le 2}$ *of* $\mathsf{Wdl}^{(2)}(\overline{\mathcal{K}})$. *In addition to* $\overline{s}_i : s_i \to s_i$ *for* $0 \le i \le 2$, *and* $\overline{\lambda}_{i,j} : s_i s_j \to s_i s_j$ *for* $0 \le i < j \le 2$, *let us introduce the the following idempotent 2-cells in* $\mathcal{K}$.

$$\overrightarrow{\lambda}_{0,p,q} := \big(s_0 s_p s_q \xrightarrow{s_0 s_p s_q \eta_0} s_0 s_p s_q s_0 \xrightarrow{s_0 s_p \lambda_{0,q}} s_0 s_p s_0 s_q \xrightarrow{s_0 \lambda_{0,p} s_q} s_0^2 s_p s_q \xrightarrow{\mu_0 s_p s_q} s_0 s_p s_q\big),$$

$$\overleftarrow{\lambda}_{k,l,2} := \big(s_k s_l s_2 \xrightarrow{\eta_2 s_k s_l s_2} s_2 s_k s_l s_2 \xrightarrow{\lambda_{k,2} s_l s_2} s_k s_2 s_l s_2 \xrightarrow{s_k \lambda_{l,2} s_2} s_k s_l s_2^2 \xrightarrow{s_k s_l \mu_2} s_k s_l s_2\big),$$

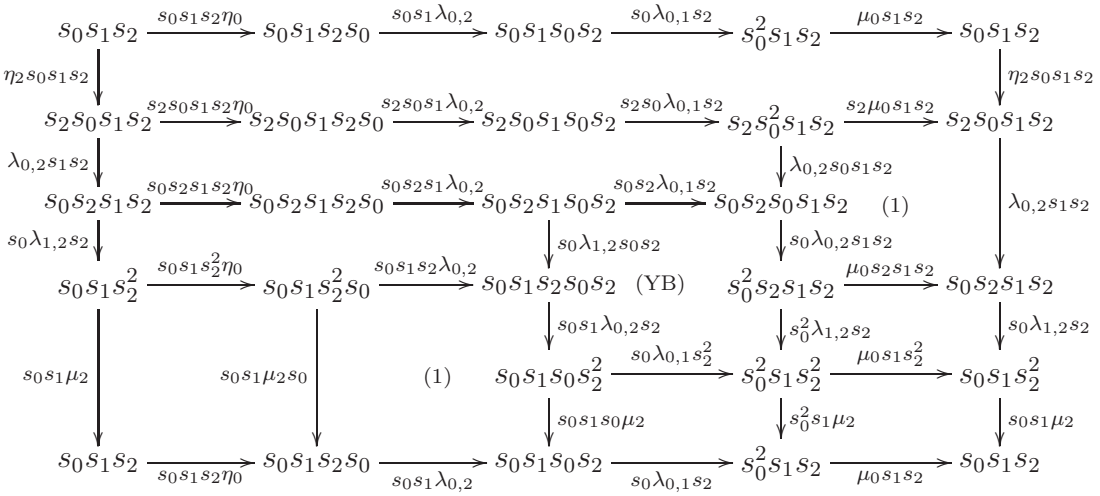
for $p = 1, q = 2$ *and* $p = 2, q = 1$; *and for* $k = 0, l = 1$ *and* $k = 1, l = 0$. *They obey the following equalities.*

$$\overrightarrow{\lambda}_{0,p,q}.\overline{\lambda}_{0,p}s_q = \overrightarrow{\lambda}_{0,p,q} = \overline{\lambda}_{0,p}s_q.\overrightarrow{\lambda}_{0,p,q} \tag{7}$$
$$\overrightarrow{\lambda}_{0,1,2}.s_0\lambda_{1,2} = s_0\lambda_{1,2}.\overrightarrow{\lambda}_{0,2,1} \tag{8}$$
$$\overrightarrow{\lambda}_{0,p,q}.\lambda_{0,p}s_q = \lambda_{0,p}s_q.s_p\overline{\lambda}_{0,q} \tag{9}$$
$$\overleftarrow{\lambda}_{k,l,2}.s_k\overline{\lambda}_{l,2} = \overleftarrow{\lambda}_{k,l,2} = s_k\overline{\lambda}_{l,2}.\overleftarrow{\lambda}_{k,l,2} \tag{10}$$
$$\overleftarrow{\lambda}_{0,1,2}.\lambda_{0,1}s_2 = \lambda_{0,1}s_2.\overleftarrow{\lambda}_{1,0,2} \tag{11}$$
$$\overleftarrow{\lambda}_{k,l,2}.s_k\lambda_{l,2} = s_k\lambda_{l,2}.\overline{\lambda}_{k,2}s_l \tag{12}$$
$$\overleftarrow{\lambda}_{0,1,2}.\overrightarrow{\lambda}_{0,1,2} = \overleftarrow{\lambda}_{0,1,2}.\overline{\lambda}_{0,1}s_2 = \overrightarrow{\lambda}_{0,1,2}.s_0\overline{\lambda}_{1,2} = \overrightarrow{\lambda}_{0,1,2}.\overleftarrow{\lambda}_{0,1,2}\ . \tag{13}$$

In what follows, we shall denote by $\overline{\lambda}_{012}$ the equal 2-cells in (13) (we shall see later the irrelevance of inserting any comma between the labels).

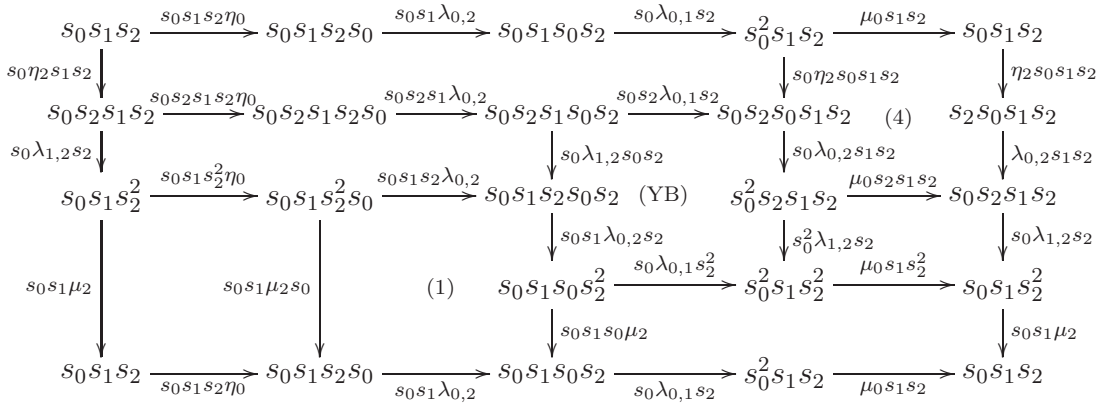
Proof. We only present a proof of (13), verification of the other equalities is left to the reader.

The first and the last expressions in (13) are equal by commutativity of the following diagram.



The first and the third expressions in (13) are equal by commutativity of the following

diagram.



Equality of the second and last expressions in (13) follows symmetrically. ∎

2.3. Lemma. *For any object* $\{\lambda_{i,j} : s_j s_i \to s_i s_j\}_{0\le i<j\le 2}$ *of* $\mathsf{Wdl}^{(2)}(\overline{\mathcal{K}})$*, consider the monads* $(s_0s_1, \overline{\lambda}_{0,1})$ *and* $(s_1s_2, \overline{\lambda}_{1,2})$ *in* $\overline{\mathcal{K}}$*, induced by the weak distributive laws* $\lambda_{0,1}$ *and* $\lambda_{1,2}$*, respectively. There are weak distributive laws*

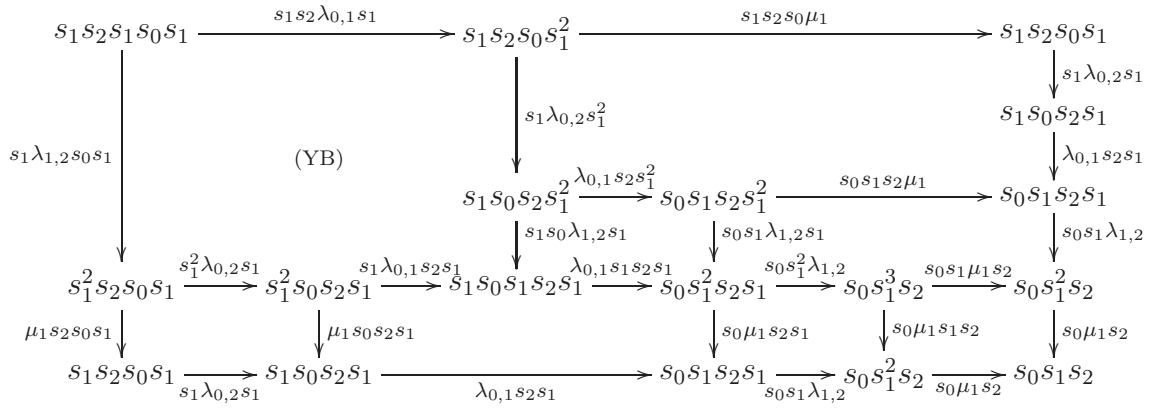
$$\begin{aligned}
\lambda_{01,2} &:= \big(s_2(s_0s_1) \xrightarrow{\lambda_{0,2}s_1} s_0s_2s_1 \xrightarrow{s_0\lambda_{1,2}} s_0s_1s_2 \xrightarrow{\overrightarrow{\lambda}_{0,1,2}} (s_0s_1)s_2\big) \\
&= \big(s_2(s_0s_1) \xrightarrow{s_2\overline{\lambda}_{01}} s_2s_0s_1 \xrightarrow{\lambda_{0,2}s_1} s_0s_2s_1 \xrightarrow{s_0\lambda_{1,2}} (s_0s_1)s_2\big) \qquad \textit{and} \\
\lambda_{0,12} &:= \big((s_1s_2)s_0 \xrightarrow{s_1\lambda_{0,2}} s_1s_0s_2 \xrightarrow{\lambda_{0,1}s_2} s_0s_1s_2 \xrightarrow{\overleftarrow{\lambda}_{0,1,2}} s_0(s_1s_2)\big) \\
&= \big((s_1s_2)s_0 \xrightarrow{\overline{\lambda}_{12}s_0} s_1s_2s_0 \xrightarrow{s_1\lambda_{0,2}} s_1s_0s_2 \xrightarrow{\lambda_{0,1}s_2} s_0(s_1s_2)\big)
\end{aligned}$$

in $\overline{\mathcal{K}}$*. Moreover, their induced monads* $(s_0s_1s_2, \overline{\lambda}_{01,2})$ *and* $(s_0s_1s_2, \overline{\lambda}_{0,12})$ *are equal.*

Proof. Both given forms of $\lambda_{01,2}$ are equal by (8) and (9). It is a 2-cell in $\overline{\mathcal{K}}$ by (7). Compatibility with the multiplication of s_2 holds since both $\lambda_{0,2}$ and $\lambda_{1,2}$ are compatible with it. With the normalization conditions $\overline{\lambda}_{01}.\mu_{01} = \mu_{01} = \mu_{01}.\overline{\lambda}_{01}s_0s_1 = \mu_{01}.s_0s_1\overline{\lambda}_{01}$ at hand, compatibility of $\lambda_{01,2}$ with the multiplication of s_0s_1 follows by the compatibilities of $\lambda_{0,2}$ with μ_0 and of $\lambda_{1,2}$ with μ_1 and the Yang-Baxter condition. The weak unitality condition (2) follows by the equality of the second and third expressions in (13) (so that $\overline{\lambda}_{01,2} = \overline{\lambda}_{012}$). This proves that $\lambda_{01,2}$ is a weak distributive law and $\lambda_{0,12}$ can be handled symmetrically (in particular, $\overline{\lambda}_{0,12} = \overline{\lambda}_{012}$).

Equality of the units in the induced monads follows immediately by the Yang-

Baxter condition. Concerning the multiplications, composing the equal paths around



by $s_1s_2\eta_1s_0s_1$ on the right, we obtain

$$s_0\mu_1s_2.\lambda_{0,1}\lambda_{1,2}.s_1\lambda_{0,2}s_1.\overline{\lambda}_{12}s_0s_1 = s_0\mu_1s_2.\lambda_{0,1}\lambda_{1,2}.s_1\lambda_{0,2}s_1.s_1s_2\overline{\lambda}_{01}. \tag{14}$$

Inserting these equal 2-cells $s_1s_2s_0s_1 \to s_0s_1s_2$ into $\mu_0s_1\mu_2.s_0(-)s_2$, we conclude the equality of the multiplications induced on $s_0s_1s_2$ by $\lambda_{01,2}$ and $\lambda_{0,12}$, respectively. ∎

2.4. On the Yang-Baxter condition Actually, also a sort of converse of Lemma 2.3 holds. Consider weak distributive laws $\{\lambda_{i,j} : s_js_i \to s_is_j\}_{0\le i<j\le 2}$ in $\overline{\mathcal{K}}$. Assume that

$$\lambda_{01,2} := \big(s_2(s_0s_1) \xrightarrow{s_2\overline{\lambda}_{01}} s_2s_0s_1 \xrightarrow{\lambda_{0,2}s_1} s_0s_2s_1 \xrightarrow{s_0\lambda_{1,2}} (s_0s_1)s_2\big) \qquad \text{and}$$

$$\lambda_{0,12} := \big((s_1s_2)s_0 \xrightarrow{\overline{\lambda}_{12}s_0} s_1s_2s_0 \xrightarrow{s_1\lambda_{0,2}} s_1s_0s_2 \xrightarrow{\lambda_{0,1}s_2} s_0(s_1s_2)\big)$$

are weak distributive laws inducing equal monads $(s_0s_1s_2, \overline{\lambda}_{01,2})$ and $(s_0s_1s_2, \overline{\lambda}_{0,12})$. Then the Yang-Baxter condition holds.

Indeed, equality of the multiplications $\mu_{0,12}$ and $\mu_{01,2}$ is equivalent to (14). With this identity at hand, from the compatibility of $\lambda_{0,12}$ with μ_{12} we obtain

$$\lambda_{0,12}.\mu_{12}s_0 = s_0\mu_1\mu_2.\lambda_{0,1}s_1s_2^2.s_1s_0\lambda_{1,2}s_2.s_1\lambda_{0,2}s_1s_2.s_1s_2\lambda_{0,12}\ .$$

Precomposing this equality with $\eta_1s_2s_1\eta_2s_0$, we conclude that

$$\lambda_{0,1}s_2.s_1\lambda_{0,2}.\lambda_{1,2}s_0 = s_0s_1\mu_2.s_0\lambda_{1,2}s_2.\lambda_{0,2}s_1s_2.s_2\lambda_{0,1}s_2.s_2s_1\lambda_{0,2}.s_2\lambda_{1,2}s_0.s_2\eta_2s_1s_0\ . \tag{15}$$

Symmetrically,

$$s_0\lambda_{1,2}.\lambda_{0,2}s_1.s_2\lambda_{0,1} = \mu_0s_1s_2.s_0\lambda_{0,1}s_2.s_0s_1\lambda_{0,2}.s_0\lambda_{1,2}s_0.\lambda_{0,2}s_1s_0.s_2\lambda_{0,1}s_0.s_2s_1\eta_0s_0\ . \tag{16}$$

It follows by the associativity of μ_1 that $s_0\overline{\lambda}_{1,2}.\overline{\lambda}_{0,1}s_2 = \overline{\lambda}_{0,1}s_2.s_0\overline{\lambda}_{1,2}$. Precomposing with $\eta_0s_1\eta_2$, we obtain from this $s_0\lambda_{1,2}.s_0\eta_2s_1.\lambda_{0,1}.s_1\eta_0 = \lambda_{0,1}s_2.s_1\eta_0s_2.\lambda_{1,2}.\eta_2s_1$. Inserting these latter equal expressions into

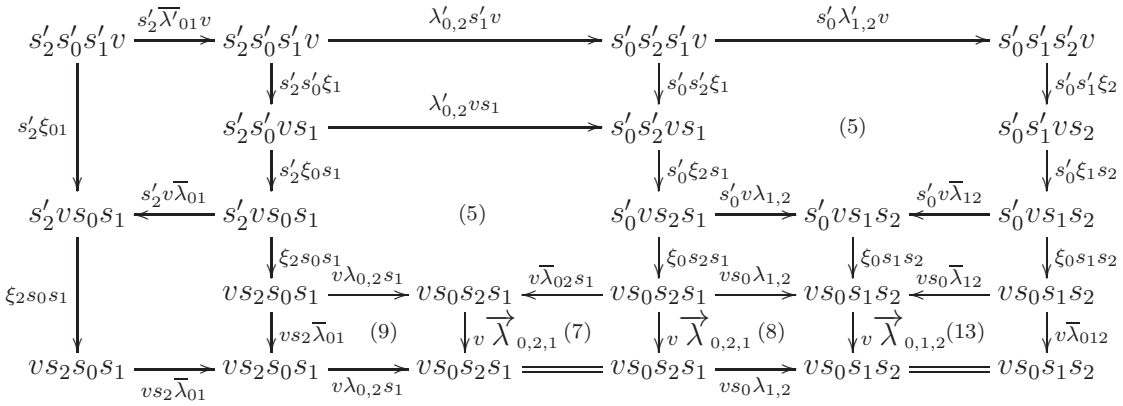
$$s_0s_1\mu_2.s_0\lambda_{1,2}s_2.\lambda_{0,2}s_1s_2.s_2\mu_0s_1s_2.s_2s_0\lambda_{0,1}s_2.s_2s_0s_1\lambda_{0,2}.s_2(-)s_0$$

and using (15) and (16) to simplify both sides of the resulting equality, we obtain the Yang-Baxter condition.

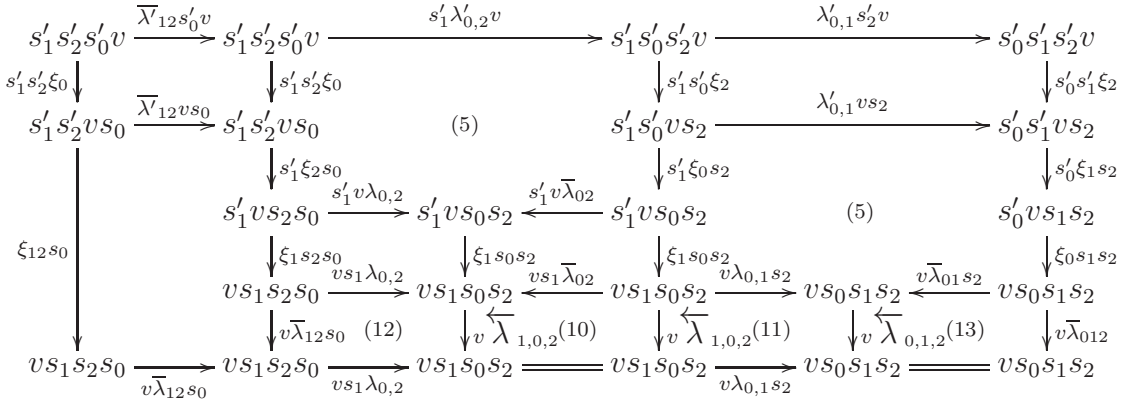
2.5. LEMMA. *For any 1-cell* $\{\xi_i : s'_i v \to v s_i\}_{0\le i\le 2}$ *in* $\mathsf{Wdl}^{(2)}(\overline{\mathcal{K}})$*, the following yield 1-cells in* $\mathsf{Wdl}(\overline{\mathcal{K}})$ *between the 0-cells described in Lemma 2.3.*

$$\{\xi_{01} := \big(\ s'_0s'_1v \xrightarrow{s'_0\xi_1} s'_0vs_1 \xrightarrow{\xi_0 s_1} vs_0s_1 \xrightarrow{v\overline{\lambda}_{01}} vs_0s_1\ \big),\qquad s'_2v \xrightarrow{\xi_2} vs_2\ \}\quad \textit{and}$$
$$\{\ s'_0v \xrightarrow{\xi_0} vs_0\ ,\qquad \xi_{12} := \big(\ s'_1s'_2v \xrightarrow{s'_1\xi_2} s'_1vs_2 \xrightarrow{\xi_1 s_2} vs_1s_2 \xrightarrow{v\overline{\lambda}_{12}} vs_1s_2\ \big)\}$$

PROOF. By Paragraph 1.5, ξ_{01} and ξ_{12} are 1-cells in $\mathsf{Mnd}(\overline{\mathcal{K}})$. Moreover, ξ_{01} and ξ_2 obey (5) by commutativity of the following diagram.



The top-left region commutes since ξ_{01} is a 2-cell in $\mathsf{Mnd}(\overline{\mathcal{K}})$ of domain $(s'_0s'_1v, \overline{\lambda'}_{0,1}\overline{v})$. Also ξ_0 and ξ_{12} obey (5), hence constitute a 1-cell in $\mathsf{Wdl}(\overline{\mathcal{K}})$, by commutativity of the similar diagram below. The bottom-left region commutes by the normalization of ξ_{12}.



■

2.6. THEOREM. *For any 2-category* $\mathcal{K}$*, and any positive integer* n*, there are* n *different 2-functors* $C_k : \mathsf{Wdl}^{(n)}(\overline{\mathcal{K}}) \to \mathsf{Wdl}^{(n-1)}(\overline{\mathcal{K}})$*, for* $1 \le k \le n$*, as follows. They take a*

0-cell $\{\lambda_{i,j} : s_j s_i \to s_i s_j\}_{0\le i<j\le n}$ *to*

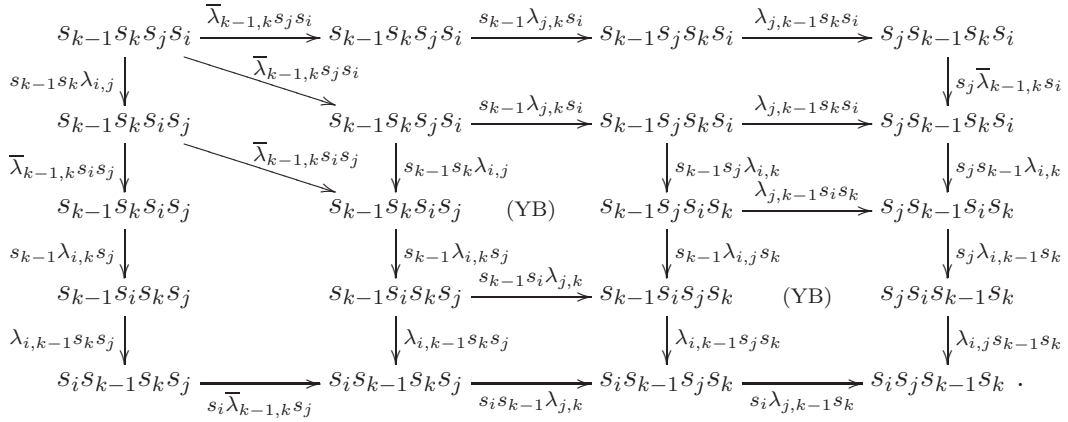
$$\left\{\begin{array}{ll} s_j s_i \xrightarrow{\lambda_{i,j}} s_i s_j & \text{if } i,j \notin \{k-1,k\} \\ s_j(s_{k-1}s_k) \xrightarrow{s_j\overline{\lambda}_{k-1,k}} s_j s_{k-1} s_k \xrightarrow{\lambda_{k-1,j}s_k} s_{k-1}s_j s_k \xrightarrow{s_{k-1}\lambda_{k,j}} (s_{k-1}s_k)s_j & \text{if } k<j \\ (s_{k-1}s_k)s_i \xrightarrow{\overline{\lambda}_{k-1,k}s_i} s_{k-1}s_k s_i \xrightarrow{s_{k-1}\lambda_{i,k}} s_{k-1}s_i s_k \xrightarrow{\lambda_{i,k-1}s_k} s_i(s_{k-1}s_k) & \text{if } i<k-1 \end{array}\right\} \tag{17}$$

where $(s_{k-1}s_k, \overline{\lambda}_{k-1,k})$ *is the monad in* $\overline{\mathcal{K}}$ *induced by the weak distributive law* $\lambda_{k-1,k}$. *They send a 1-cell* $\{\xi_i : s'_i v \to v s_i\}_{0\le i\le n}$ *to*

$$\left\{\begin{array}{ll} s'_i v \xrightarrow{\xi_i} v s_i & \text{if } 0\le i<k-1 \\ (s'_{k-1}s'_k)v \xrightarrow{s'_{k-1}\xi_k} s'_{k-1} v s_k \xrightarrow{\xi_{k-1}s_k} v s_{k-1}s_k \xrightarrow{v\overline{\lambda}_{k-1,k}} v(s_{k-1}s_k) & \\ s'_{i+1} v \xrightarrow{\xi_{i+1}} v s_{i+1} & \text{if } k-1<i<n \end{array}\right\} \tag{18}$$

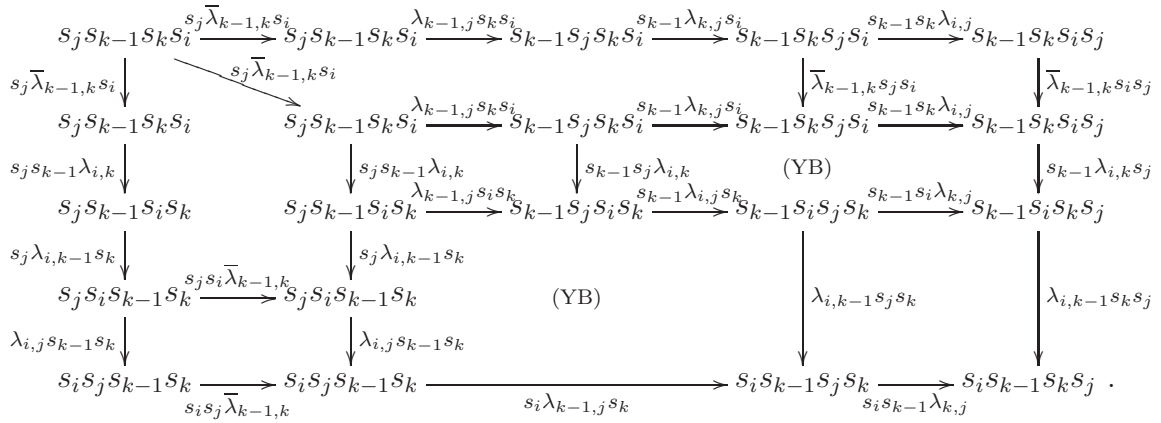
On the 2-cells they act as the identity map.

Proof. By Lemma 2.3, each line in (17) is a weak distributive law in $\overline{\mathcal{K}}$. We only need to check the Yang-Baxter conditions. For $0\le i<j<k-1<n$ the Yang-Baxter condition follows by commutativity of



The top-right region and the bottom-left region commute by Lemma 2.3. The $0<k<i<j\le n$ case is treated symmetrically. The Yang-Baxter condition in the last case, when $0\le i<k-1$ and $k<j\le n$, follows by commutativity of the similar

diagram



Both regions at the top-left commute by Lemma 2.3. This proves that (17) describes a 0-cell in $\mathsf{Wdl}^{(n-1)}(\overline{\mathcal{K}})$. By Lemma 2.5, (18) is a 1-cell in $\mathsf{Wdl}^{(n-1)}(\overline{\mathcal{K}})$. Evidently, 2-cells in $\mathsf{Wdl}^{(n)}(\overline{\mathcal{K}})$ are 2-cells in $\mathsf{Wdl}^{(n-1)}(\overline{\mathcal{K}})$ as well. Hence the stated maps define 2-functors C_k which clearly preserve the horizontal and vertical compositions. ∎

Via the fully faithful embedding $\mathsf{Mnd}^{n+1}(\overline{\mathcal{K}}) \to \mathsf{Wdl}^{(n)}(\overline{\mathcal{K}})$ in Paragraph 2.1, the 2-functors in Theorem 2.6 extend the multiplication C of the 2-monad Mnd. That is, the following diagram commutes, for all $1 \leq k \leq n$.

$$\begin{array}{ccc} \mathsf{Mnd}^{n+1}(\overline{\mathcal{K}}) & \xrightarrow{\mathsf{Mnd}^{k-1}C\mathsf{Mnd}^{n-k}(\overline{\mathcal{K}})} & \mathsf{Mnd}^{n}(\overline{\mathcal{K}}) \\ \downarrow & & \downarrow \\ \mathsf{Wdl}^{(n)}(\overline{\mathcal{K}}) & \xrightarrow[C_k]{} & \mathsf{Wdl}^{(n-1)} \end{array}$$

By associativity of the 2-monad Mnd, the n-fold iteration of the 2-functor in the top row; that is,

$$\mathsf{Mnd}^{n+1}(\overline{\mathcal{K}}) \xrightarrow{\mathsf{Mnd}^{k_n-1}C\mathsf{Mnd}^{n-k_n}(\overline{\mathcal{K}})} \quad \cdots \quad \xrightarrow{\mathsf{Mnd}^{k_2-1}C\mathsf{Mnd}^{2-k_2}(\overline{\mathcal{K}})} \mathsf{Mnd}^{2}(\overline{\mathcal{K}}) \xrightarrow{C} \mathsf{Mnd}(\overline{\mathcal{K}})$$

does not depend on the values of $k_i \in \{1, \ldots, i\}$, for $1 \leq i \leq n$. That is, its object map describes an 'associative' wreath product of monads. Although the 2-functor in the bottom row is not known to correspond to the multiplication in any 2-monad, in the rest of this section we show that it describes an associative weak wreath product in an appropriate sense.

2.7. Lemma. *For any integer $n > 1$, and for any 0-cell $\{\lambda_{i,j} : s_j s_i \to s_i s_j\}_{0 \leq i < j \leq n}$ of $\mathsf{Wdl}^{(n)}(\overline{\mathcal{K}})$, consider the idempotent 2-cell*

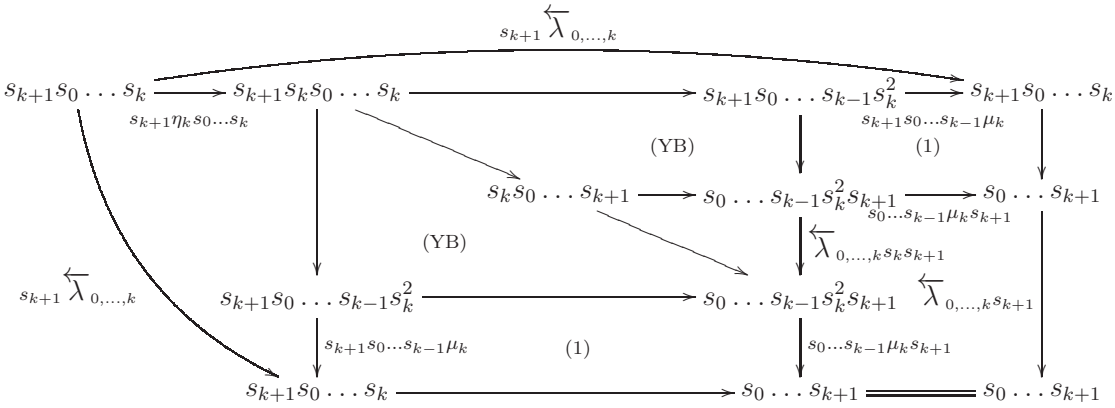
$$\overleftarrow{\lambda}_{0,1,\ldots,n} := \big(s_0 s_1 \ldots s_n \xrightarrow{\eta_n s_0 s_1 \ldots s_n} s_n s_0 s_1 \ldots s_n \longrightarrow s_0 s_1 \ldots s_{n-1} s_n^2 \xrightarrow{s_0 s_1 \ldots s_{n-1} \mu_n} s_0 s_1 \ldots s_n\big)$$

in $\mathcal{K}$ (where the unlabelled arrow denotes the unique composite of $\lambda_{i,j}$s of the given domain and codomain) and

$$\overline{\lambda}_{01\ldots n} := \overleftarrow{\lambda}_{0,1,\ldots,n} . \overleftarrow{\lambda}_{0,1,\ldots,n-1} s_n . \ \cdots \ . \overleftarrow{\lambda}_{0,1,2} s_3 s_4 \ldots s_n . \overline{\lambda}_{01} s_2 s_3 \ldots s_n \ . \tag{19}$$

This construction in (19) associates the same idempotent 2-cell to any 0-cell $\{\lambda_{i,j} : s_j s_i \to s_i s_j\}_{0\le i<j\le n}$ *of* $\mathsf{Wdl}^{(n)}(\overline{\mathcal{K}})$ *and to its image under any of the 2-functors* C_k *in Theorem 2.6. That is, for all* $1 \le k \le n$, $\overline{\lambda}_{01\dots n} = \overline{\lambda}_{01\dots k-2(k-1,k)k+1\dots n}$.

Proof. Using commutativity of

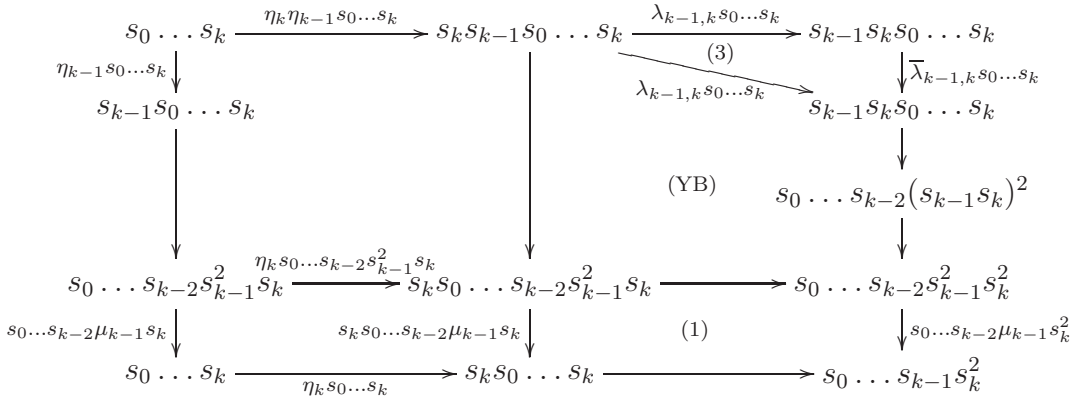


(20)

for any $1 \le k < n$, it follows easily that $\overline{\lambda}_{0\dots n}$ is idempotent. By (17) on one hand, and by (20) on the other,

$$\overleftarrow{\lambda}_{0,\dots,k-2,(k-1,k),k+1,\dots m} = \overleftarrow{\lambda}_{0,\dots,m}.s_0\dots s_{k-2}\overline{\lambda}_{k-1,k}s_{k+1}\dots s_m$$
$$= s_0\dots s_{k-2}\overline{\lambda}_{k-1,k}s_{k+1}\dots s_m.\overleftarrow{\lambda}_{0,\dots,m}.s_0\dots s_{k-2}\overline{\lambda}_{k-1,k}s_{k+1}\dots s_m$$

for all $k < m \le n$. Moreover, by commutativity of



we obtain $\overleftarrow{\lambda}_{0,\dots,k-2,(k-1,k)} = \overleftarrow{\lambda}_{0,\dots,k}.\overleftarrow{\lambda}_{0,\dots,k-1}s_k = s_0\dots s_{k-2}\overline{\lambda}_{k-,k}.\overleftarrow{\lambda}_{0,\dots,k}.\overleftarrow{\lambda}_{0,\dots,k-1}s_k$. Combining these identities we conclude the claim. ∎

2.8. Lemma. *In terms of the 2-functors in Theorem 2.6, for any integer* $n > 1$, *the composite*

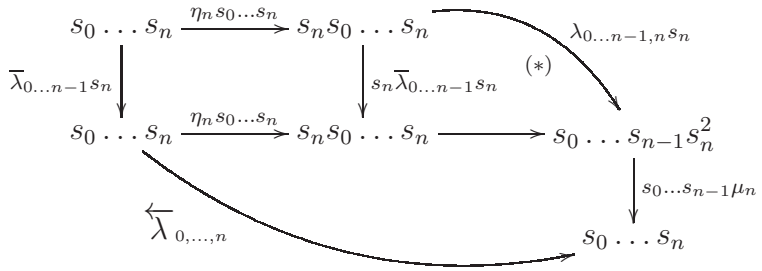
$$\mathsf{Wdl}^{(n)}(\overline{\mathcal{K}}) \xrightarrow{C_1} \mathsf{Wdl}^{(n-1)}(\overline{\mathcal{K}}) \xrightarrow{C_1} \cdots \xrightarrow{C_1} \mathsf{Wdl}(\overline{\mathcal{K}})$$

takes a 0-cell $\{\lambda_{i,j} : s_j s_i \to s_i s_j\}_{0\le i<j\le n}$ *to the weak distributive law*

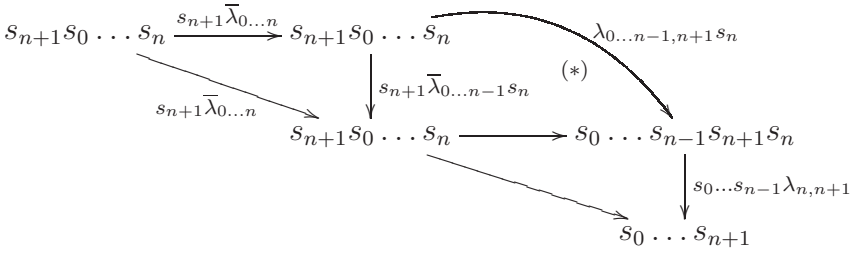
$$\lambda_{0\dots n-1,n} := \big(\ s_n(s_0\dots s_{n-1}) \xrightarrow{s_n\overline{\lambda}_{0\dots n-1}} s_n s_0\dots s_{n-1} \longrightarrow (s_0\dots s_{n-1})s_n\ \big), \qquad (21)$$

where the unlabelled arrow denotes the unique combination of $\lambda_{i,j}$*s with the given domain and codomain.*

PROOF. We proceed by induction in n. For $n = 2$ the claim follows by Theorem 2.6. Assume that it holds for some $n \geq 2$. Then the following diagram commutes.



The region marked by $(*)$ commutes by the induction hypothesis. The left bottom path is equal to $\overline{\lambda}_{0\dots n}$, see (19). Hence we conclude that $\overline{\lambda}_{0\dots n}$ is equal to $\overline{\lambda}_{0\dots n-1,n}$; that is, the idempotent associated to the weak distributive law $\lambda_{0\dots n-1,n}$. Using this observation and (17) (for the monads $s_0 \dots s_{n-1}$, s_n and s_{n+1}), in the top-right path of the following diagram we recognize $\lambda_{0\dots n,n+1}$.



The region marked by $(*)$ commutes by the induction hypothesis and the triangle on the left commutes by (20). ∎

Applying (8) and (9) to the monads $s_0 \dots s_{n-2}$, s_{n-1} and s_n, from (21) we obtain the equal expression

$$\lambda_{0\dots n-1,n} = \big(s_n(s_0 \dots s_{n-1}) \longrightarrow s_0 \dots s_n \xrightarrow{\overline{\lambda}_{0\dots n}} (s_0 \dots s_{n-1}) s_n \big). \tag{22}$$

2.9. LEMMA. *In terms of the 2-functors in Theorem 2.6, for any positive integer n, consider the composite*

$$\mathsf{Wdl}^{(n)}(\overline{\mathcal{K}}) \xrightarrow{C_1} \mathsf{Wdl}^{(n-1)}(\overline{\mathcal{K}}) \xrightarrow{C_1} \cdots \xrightarrow{C_1} \mathsf{Wdl}(\overline{\mathcal{K}}) \xrightarrow{C_1} \mathsf{Mnd}(\overline{\mathcal{K}}) \ . \tag{23}$$

(1) It takes a 0-cell $\{\lambda_{i,j} : s_j s_i \to s_i s_j\}_{0 \leq i < j \leq n}$ to the monad $(s_0 s_1 \dots s_n, \overline{\lambda}_{01\dots n})$ in $\overline{\mathcal{K}}$, with multiplication $\mu_{0\dots n}$ equal to

$$s_0 s_1 \dots s_n s_0 s_1 \dots s_n \longrightarrow s_0^2 s_1^2 \dots s_n^2 \xrightarrow{\mu_0 \mu_1 \dots \mu_n} s_0 s_1 \dots s_n \xrightarrow{\overline{\lambda}_{01\dots n}} s_0 s_1 \dots s_n \tag{24}$$

and unit $\eta_{0\dots n}$ equal to

$$1 \xrightarrow{\eta_n \eta_{n-1} \dots \eta_0} s_n s_{n-1} \dots s_0 \longrightarrow s_0 s_1 \dots s_n \xrightarrow{\overline{\lambda}_{01\dots n}} s_0 s_1 \dots s_n, \tag{25}$$

where the unlabelled arrows denote the unique (by the Yang-Baxter condition) combinations of $\lambda_{i,j}$s with the given domain and codomain.

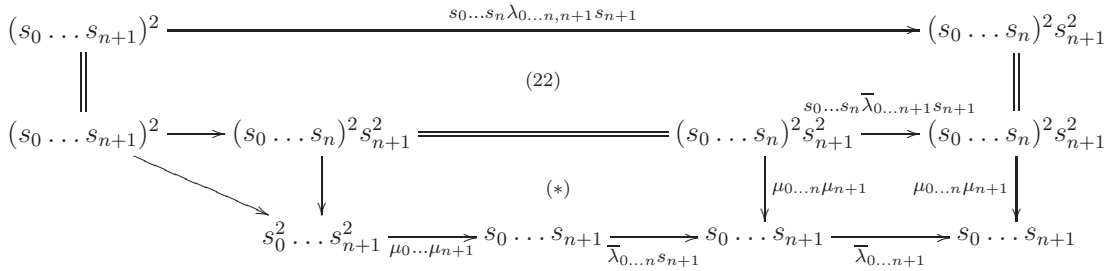
(2) It takes a 1-cell $\{\xi_i : s'_i v \to v s_i\}_{0\le i\le n}$ *to*

$$s'_0 s'_1 \dots s'_n v \xrightarrow{s'_0 s'_1 \dots s'_{n-1} \xi_n} s'_0 s'_1 \dots s'_{n-1} v s_n \xrightarrow{s'_0 s'_1 \dots s'_{n-2} \xi_{n-1} s_n} \cdots \xrightarrow{\xi_0 s_1 s_2 \dots s_n} v s_0 s_1 \dots s_n \xrightarrow{v\overline{\lambda}_{01\dots n}} v s_0 s_1 \dots s_n,$$

to be denoted by $\xi_{01\dots n}$.

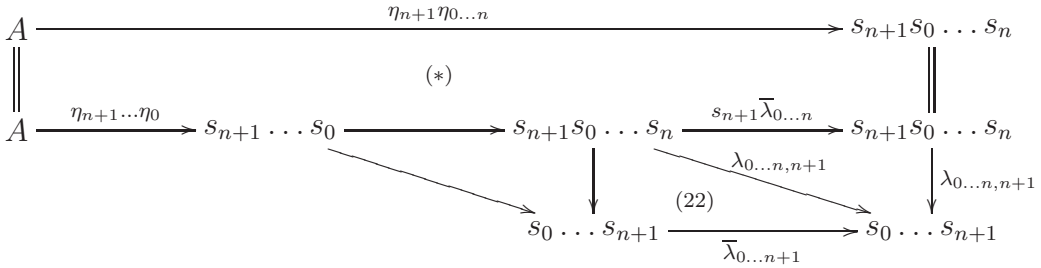
(3) On the 2-cells it acts as the identity map.

Proof. (1) We proceed by induction in n. For $n = 1$ the claim holds by Paragraph 1.5. Assume that it holds for some $n \ge 1$. Then $\mu_{0\dots n+1}$ occurs in the top-right path of the following diagram.



The region marked by $(*)$ commutes by the induction hypothesis and the bottom-right region commutes by the bilinearity of $\overline{\lambda}_{0\dots n+1} = \overline{\lambda}_{0\dots n,n+1}$, cf. (3). The composite of the last two arrows in the bottom row is equal to $\overline{\lambda}_{0\dots n+1}$ by the explicit form in (19).

Similarly, $\eta_{0\dots n+1}$ occurs in the top-right path of the following diagram.



The region marked by $(*)$ commutes by the induction hypothesis.

Part (2) is easily proved by induction in n and part (3) is trivial. ∎

2.10. Theorem. *For any 2-category* $\mathcal{K}$*, and any positive integer* n*, the 2-functors in Theorem 2.6 give rise to a unique composite*

$$\mathsf{Wdl}^{(n)}(\overline{\mathcal{K}}) \xrightarrow{C_{k_n}} \mathsf{Wdl}^{(n-1)}(\overline{\mathcal{K}}) \xrightarrow{C_{k_{n-1}}} \cdots \xrightarrow{C_{k_2}} \mathsf{Wdl}(\overline{\mathcal{K}}) \xrightarrow{C_{k_1}} \mathsf{Mnd}(\overline{\mathcal{K}})$$

which is independent of the choice of the index set $k_i \in \{1, 2, \dots, i\}$*, for* $1 \le i \le n$.

Proof. We proceed by induction in n. For $n = 1$ the claim is trivial: there is only one 2-functor $C_1 : \mathsf{Wdl}^{(1)}(\overline{\mathcal{K}}) \equiv \mathsf{Wdl}(\overline{\mathcal{K}}) \to \mathsf{Wdl}^{(0)}(\overline{\mathcal{K}}) \equiv \mathsf{Mnd}(\overline{\mathcal{K}})$, that recalled in Paragraph 1.5. Assume now that the claim holds for some positive integer n; that is, $C_1 C_{k_2} \dots C_{k_n}$ does not depend on $k_i \in \{1, 2, \dots, i\}$, for $1 \le i \le n$. With the explicit form of the 2-functor (23) in Lemma 2.9, and the explicit form of $C_k : \mathsf{Wdl}^{(n+1)}(\overline{\mathcal{K}}) \to \mathsf{Wdl}^{(n)}(\overline{\mathcal{K}})$ in (17) and (18) at hand, the equality $C_k C_1 C_1 \dots C_1 = C_1 C_1 C_1 \dots C_1$ of the $n+1$-fold composites follows by Lemma 2.7, for all $1 \le k \le n+1$. ∎

2.11. If idempotent 2-cells split Let us take a 2-category $\mathcal{K}$ which is locally idempotent complete; that is, $\mathcal{K} \simeq \overline{\mathcal{K}}$. Then we may consider the pseudofunctors

$$\mathsf{Wdl}^{(n)}(\mathcal{K}) \xrightarrow{\simeq} \mathsf{Wdl}^{(n)}(\overline{\mathcal{K}}) \xrightarrow{C_k} \mathsf{Wdl}^{(n-1)}(\overline{\mathcal{K}}) \xrightarrow{\simeq} \mathsf{Wdl}^{(n-1)}(\mathcal{K}),$$

for all values of $1 \leq k \leq n$. (Choosing the biequivalence $\overline{\mathcal{K}} \to \mathcal{K}$ by adopting the convention that we split identity 2-cells trivially, that is, via identity 2-cells; they become in fact 2-functors.) Their n-fold iteration is pseudonaturally equivalent to

$$\mathsf{Wdl}^{(n)}(\mathcal{K}) \xrightarrow{\simeq} \mathsf{Wdl}^{(n)}(\overline{\mathcal{K}}) \longrightarrow \mathsf{Mnd}(\overline{\mathcal{K}}) \xrightarrow{\simeq} \mathsf{Mnd}(\mathcal{K})$$

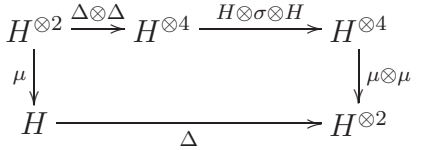
where the unlabelled arrow stands for *the* 2-functor in Theorem 2.10. This pseudofunctor (or in fact 2-functor with an appropriate choice) takes an object $\{\lambda_{i,j} : s_j s_i \to s_i s_j\}_{0 \leq i < j \leq n}$ of $\mathsf{Wdl}^{(n)}(\mathcal{K})$, considered as an object of $\mathsf{Wdl}^{(n)}(\overline{\mathcal{K}})$, to the image of the idempotent $\overline{\lambda}_{0\ldots n}$ in (19). This is regarded as the weak wreath product of the monads $s_0, s_1, \ldots, s_n$ in $\mathcal{K}$. It is unique – that is, the weak wreath product is associative – up-to an isomorphism arising from the chosen splittings of the occurring idempotents.

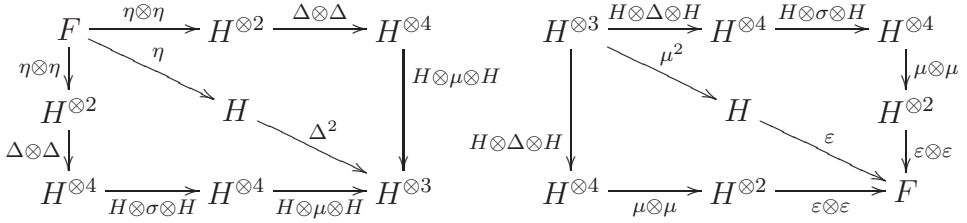
3. Examples from Ising type spin chains

In this section $\mathcal{K} := \mathsf{Vec}$ will be the one-object 2-category (in fact bicategory); that is, monoidal category of vector spaces over a given field F. Thus there is only one 0-cell F; the 1-cells are the F-vector spaces and the 2-cells are the linear maps. The horizontal composition (that is, monoidal product) is given by the tensor product $\otimes$ and the vertical composition is given by the composition of linear maps. Monads are just the F-algebras. We shall make use of the fact that the monoidal category of vector spaces is *symmetric*; the symmetry natural isomorphism (that is, the flip map) will be denoted by σ. Clearly, Vec is idempotent complete.

Our aim is to present an object of $\mathsf{Wdl}^{(n)}(\mathsf{Vec})$ (for any positive integer n) in terms of a finite dimensional weak bialgebra. We start with recalling the notion of *weak bialgebra* from [Nill (preprint 1998)], [Böhm, Nill, Szlachányi (1999)].

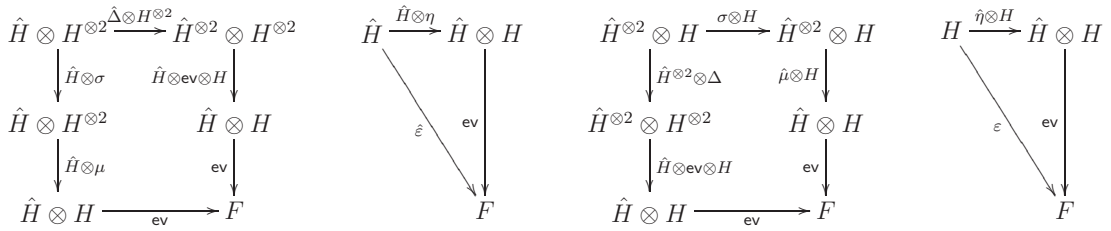
3.1. Definition. *A* weak bialgebra *is a vector space H equipped with an algebra (that is, monad) structure $\mu : H \otimes H \to H$, $\eta : F \to H$ and a coalgebra (that is, comonad) structure $\Delta : H \to H \otimes H$, $\varepsilon : H \to F$ such that the following diagrams commute.*





This definition can be generalized from Vec to any braided monoidal category, see [Alonso Álvarez et al. (2008)], [Pastro, Street (2009)]. Note that the axioms of a weak bialgebra are self-dual in the sense that they are closed under the reversing of the arrows in the diagrams depicting them.

3.2. Duals of weak bialgebras Whenever the 1-cell underlying a monad in a 2-category possesses a (left or right) adjoint, this adjoint comes equipped with the canonical structure of a comonad. Conversely, the adjoint of a comonad is a monad. In the monoidal category of vector spaces, a 1-cell H – that is, a vector space – possesses a (left and right) adjoint if and only if it is finite dimensional over F; in which case the adjoint is the linear dual $\hat{H} := \mathsf{Hom}(H, F)$; that is, the vector space of linear maps from H to F. In particular, the dual of a finite dimensional weak bialgebra is both an algebra – with multiplication $\hat{\mu} := \mathsf{Hom}(\Delta, F) : \hat{H} \otimes \hat{H} \cong \mathsf{Hom}(H \otimes H, F) \to \hat{H}$ and unit $\hat{\eta} := \mathsf{Hom}(\varepsilon, F) : F \to \hat{H}$ – and a coalgebra – with comultiplication $\hat{\Delta} := \mathsf{Hom}(\mu, F)$ and counit $\hat{\varepsilon} := \mathsf{Hom}(\eta, F)$. That is to say, the (co)algebra structure of $\hat{H}$ is defined by the following commutative diagrams



(26)

where $\mathsf{ev} : \hat{H} \otimes H \to F$ stands for the evaluation map (that is, the counit of the adjunction $\hat{H} \dashv H$). What is more, by self-duality of the weak bialgebra axioms, $\hat{H}$ is a weak bialgebra again with the above algebra and coalgebra structures.

3.3. The iterated weak wreath product of a finite weak bialgebra and its dual In terms of a finite dimensional weak bialgebra H, an object of $\mathsf{Wdl}^{(n)}(\mathsf{Vec})$ is given as follows. If $0 \leq i \leq n$ is even, then let s_i be the algebra underlying H and if i is odd then let s_i be the algebra underlying $\hat{H}$. If $j - i > 1$ then let $\lambda_{i,j}$ be given by the flip map σ. If i is odd, then let $\lambda_{i,i+1}$ be equal to λ defined as

$$H \otimes \hat{H} \xrightarrow{\sigma} \hat{H} \otimes H \xrightarrow{\hat{\Delta} \otimes \Delta} \hat{H} \otimes \hat{H} \otimes H \otimes H \xrightarrow{\hat{H} \otimes \mathsf{ev} \otimes H} \hat{H} \otimes H,$$

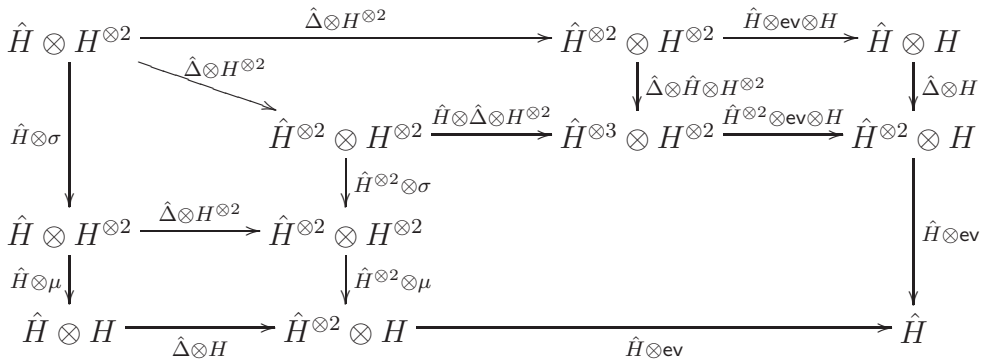
and if i is even then let $\lambda_{i,i+1}$ be equal to $\hat{\lambda}$ given by

$$\hat{H} \otimes H \xrightarrow{\sigma} H \otimes \hat{H} \xrightarrow{\Delta \otimes \hat{\Delta}} H \otimes H \otimes \hat{H} \otimes \hat{H} \xrightarrow{H \otimes \sigma \otimes \hat{H}} H \otimes \hat{H} \otimes H \otimes \hat{H} \xrightarrow{H \otimes \mathsf{ev} \otimes \hat{H}} H \otimes \hat{H}.$$

The symmetry $\sigma : X \otimes Y \to Y \otimes X$, for $X, Y \in \{H, \hat{H}\}$, is a distributive law hence a weak distributive law. We show that $\lambda : H \otimes \hat{H} \to \hat{H} \otimes H$ is a weak distributive law. The morphism

$$\xi := \left(\hat{H} \otimes H \xrightarrow{\hat{\Delta} \otimes H} \hat{H}^{\otimes 2} \otimes H \xrightarrow{\hat{H} \otimes \mathsf{ev}} \hat{H} \right)$$

is an associative (and evidently unital) action in the sense of commutativity of

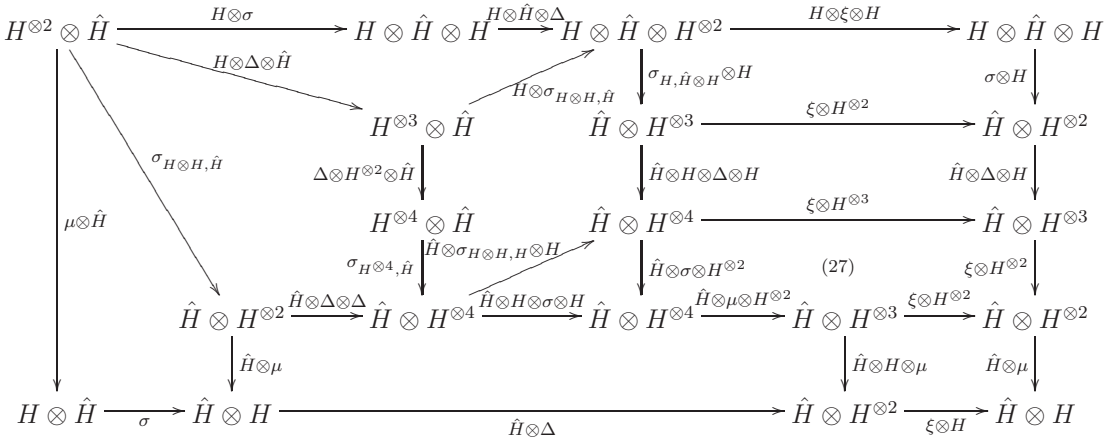


(27)

where the bottom-right region commutes by the first identity in (26). In terms of ξ,

$$\lambda = \big(\, H\otimes\hat{H} \xrightarrow{\sigma} \hat{H}\otimes H \xrightarrow{\hat{H}\otimes\Delta} \hat{H}\otimes H^{\otimes 2} \xrightarrow{\xi\otimes H} \hat{H}\otimes H \,\big).$$

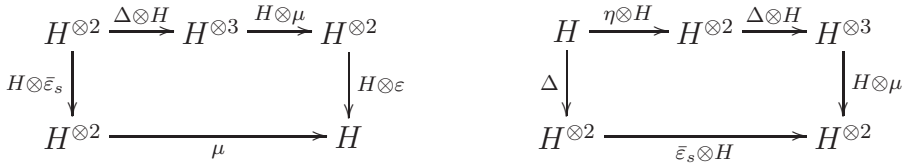
Using this form of λ, its compatibility with the multiplication of H follows by commutativity of the diagram below.



The region at the middle of the bottom row commutes by the first weak bialgebra axiom. Symmetrically, in terms of $\zeta := (\mathsf{ev}\otimes H).(\hat{H}\otimes\Delta)$, we can write $\lambda = (\hat{H}\otimes\zeta).(\hat{\Delta}\otimes H).\sigma$. With this form of λ at hand, its compatibility with the multiplication of $\hat{H}$ follows symmetrically. It remains to check the weak unitality condition (2). For that consider the (idempotent) morphism

$$\bar{\varepsilon}_s := \big(\, H \xrightarrow{\eta\otimes H} H^{\otimes 2} \xrightarrow{\Delta\otimes H} H^{\otimes 3} \xrightarrow{H\otimes\mu} H^{\otimes 2} \xrightarrow{H\otimes\varepsilon} H \,\big).$$

Recall from [Böhm, Caenepeel, Janssen (2011)] (equations (4) and (8), respectively) that the following diagrams involving $\bar{\varepsilon}_s$ commute.



(28)

Using the definitions in (26), the first identity in (28) is equivalent to $\xi.(\hat{H}\otimes\bar{\varepsilon}_s) = \hat{\mu}.(\hat{H}\otimes\xi).(\hat{H}\otimes\hat{\eta}\otimes H)$, implying commutativity of the bottom-right region in

$$\begin{array}{ccccccccc}
\hat{H}\otimes H & \xrightarrow{\eta\otimes\hat{H}\otimes H} & H\otimes\hat{H}\otimes H & \xrightarrow{\sigma\otimes H} & \hat{H}\otimes H^{\otimes 2} & \xrightarrow{\hat{H}\otimes\Delta\otimes H} & \hat{H}\otimes H^{\otimes 3} & \xrightarrow{\xi\otimes H^{\otimes 2}} & \hat{H}\otimes H^{\otimes 2} \\
\| & & & \searrow^{\hat{H}\otimes\eta\otimes H} & & (28) & \downarrow{\scriptstyle \hat{H}\otimes H\otimes\mu} & & \downarrow{\scriptstyle \hat{H}\otimes\mu} \\
\hat{H}\otimes H & & \xrightarrow{\hat{H}\otimes\Delta} & & \hat{H}\otimes H^{\otimes 2} & \xrightarrow{\hat{H}\otimes\bar{\varepsilon}_s\otimes H} & \hat{H}\otimes H^{\otimes 2} & \xrightarrow{\xi\otimes H} & \hat{H}\otimes H \\
\downarrow{\scriptstyle \hat{H}\otimes H\otimes\hat{\eta}} & \swarrow^{\hat{H}\otimes\hat{\eta}\otimes H} & & & \downarrow{\scriptstyle \hat{H}\otimes\hat{\eta}\otimes H^{\otimes 2}} & & & & \| \\
\hat{H}\otimes H\otimes\hat{H} & \xrightarrow[\hat{H}\otimes\sigma]{} & \hat{H}^{\otimes 2}\otimes H & \xrightarrow[\hat{H}^{\otimes 2}\otimes\Delta]{} & \hat{H}^{\otimes 2}\otimes H^{\otimes 2} & \xrightarrow[\hat{H}\otimes\xi\otimes H]{} & \hat{H}^{\otimes 2}\otimes H & \xrightarrow[\hat{\mu}\otimes H]{} & \hat{H}\otimes H\ .
\end{array}$$

Any path in this diagram yields an alternative expression of the idempotent $\overline{\lambda}:\hat{H}\otimes H\to\hat{H}\otimes H$, proving that λ is a weak distributive law. By symmetrical considerations so is $\hat{\lambda}$.

With some routine computations using the weak bialgebra axioms, one checks that $\overline{\lambda}$ is equal to the identity map – that is, λ is a distributive law in the strict sense – if and only if $\Delta.\eta = \eta\otimes\eta$; that is, H is a bialgebra in the strict sense.

Our next task is to check the Yang-Baxter conditions. The symmetry operators among themselves obey the Yang-Baxter condition, hence for $\{i,j,k\}$ such that $j-i>1$ and $k-j>1$ we are done. For $\{i-1,i,j\}$ and $\{i,j,j+1\}$, such that $j-i>1$, the Yang-Baxter conditions follow by naturality of the symmetry. So we are left with the case $\{i-1,i,i+1\}$. Assume first that i is odd. Then the Yang-Baxter condition follows by commutativity of

$$\begin{array}{ccccccc}
H\otimes\hat{H}\otimes H & \xrightarrow{\sigma\otimes H} & \hat{H}\otimes H^{\otimes 2} & \xrightarrow{\hat{\Delta}\otimes\Delta\otimes H} & \hat{H}^{\otimes 2}\otimes H^{\otimes 3} & \xrightarrow{\hat{H}\otimes\mathsf{ev}\otimes H^{\otimes 2}} & \hat{H}\otimes H^{\otimes 2} \\
\downarrow{\scriptstyle H\otimes\sigma} & & \downarrow{\scriptstyle \sigma_{\hat{H}\otimes H,H}} & & \downarrow{\scriptstyle \sigma_{\hat{H}^{\otimes 2}\otimes H^{\otimes 2},H}} & & \downarrow{\scriptstyle \hat{H}\otimes\sigma} \\
 & & & & & & \hat{H}\otimes H^{\otimes 2} \\
 & & & & & & \downarrow{\scriptstyle \sigma\otimes H} \\
H^{\otimes 2}\otimes\hat{H} & \xrightarrow{\sigma_{H,H\otimes\hat{H}}} & H\otimes\hat{H}\otimes H & \xrightarrow{H\otimes\hat{\Delta}\otimes\Delta} & H\otimes\hat{H}^{\otimes 2}\otimes H^{\otimes 2} & \xrightarrow{H\otimes\hat{H}\otimes\mathsf{ev}\otimes H} & H\otimes\hat{H}\otimes H \\
\downarrow{\scriptstyle H\otimes\Delta\otimes\hat{\Delta}} & & \downarrow{\scriptstyle \Delta\otimes\hat{\Delta}\otimes H} & & \downarrow{\scriptstyle \Delta\otimes\hat{\Delta}\otimes\hat{H}\otimes H^{\otimes 2}} & & \Delta\otimes\hat{\Delta}\otimes H\downarrow \\
H^{\otimes 3}\otimes\hat{H}^{\otimes 2} & & H^{\otimes 2}\otimes\hat{H}^{\otimes 2}\otimes H & \xrightarrow{H^{\otimes 2}\otimes\hat{H}\otimes\hat{\Delta}\otimes\Delta} & H^{\otimes 2}\otimes\hat{H}^{\otimes 3}\otimes H^{\otimes 2} & & H^{\otimes 2}\otimes\hat{H}^{\otimes 2}\otimes H \\
\downarrow{\scriptstyle H^{\otimes 2}\otimes\mathsf{ev}.\sigma\otimes\hat{H}} & & & & & & \\
H^{\otimes 2}\otimes\hat{H} & & \downarrow{\scriptstyle H\otimes\mathsf{ev}.\sigma\otimes\hat{H}\otimes H} & & \downarrow{\scriptstyle H\otimes\otimes\mathsf{ev}.\sigma\otimes\hat{H}^{\otimes 2}\otimes H^{\otimes 2}} & & \downarrow{\scriptstyle H\otimes\mathsf{ev}.\sigma\otimes\hat{H}\otimes H} \\
\downarrow{\scriptstyle \sigma\otimes\hat{H}} & \searrow^{\sigma_{H,H\otimes\hat{H}}} & & & & & \\
H^{\otimes 2}\otimes\hat{H} & \xrightarrow[H\otimes\sigma]{} & H\otimes\hat{H}\otimes H & \xrightarrow[H\otimes\hat{\Delta}\otimes\Delta]{} & H\otimes\hat{H}^{\otimes 2}\otimes H^{\otimes 2} & \xrightarrow[H\otimes\hat{H}\otimes\mathsf{ev}\otimes H]{} & H\otimes\hat{H}\otimes H\ .
\end{array}$$

The case when i is even is treated symmetrically. This proves that the construction in this paragraph yields an object in $\mathsf{Wdl}^{(n)}(\mathsf{Vec})$ (which is an object of $\mathsf{Mnd}^{n+1}(\mathsf{Vec})$ if and only if H is a bialgebra in the strict sense). Hence by Theorem 2.10 there is a corresponding weak wreath product monad (that is, F-algebra) given as the image of the idempotent (19). Since σ is unital, one obtains the following explicit forms of this idempotent. If n is odd, then it is

$$(H\otimes\hat{H})^{\otimes\frac{n+1}{2}} \xrightarrow{\overline{\lambda}^{\otimes\frac{n+1}{2}}} H\otimes(\hat{H}\otimes H)^{\otimes\frac{n-1}{2}}\otimes\hat{H} \xrightarrow{H\otimes\overline{\lambda}^{\otimes\frac{n-1}{2}}\otimes\hat{H}} (H\otimes\hat{H})^{\otimes\frac{n+1}{2}}$$

and if n is even, then it is

$$(H\otimes\hat{H})^{\otimes\frac{n}{2}}\otimes H\xrightarrow{\overline{\tilde{\lambda}}^{\otimes\frac{n}{2}}\otimes H}H\otimes(\hat{H}\otimes H)^{\otimes\frac{n}{2}}\xrightarrow{H\otimes\overline{\lambda}^{\otimes\frac{n}{2}}}(H\otimes\hat{H})^{\otimes\frac{n}{2}}\otimes H.$$

If H is a bialgebra in the strict sense (for instance it is the linear span of a finite group), then these idempotents become identity maps and so the above weak wreath products reduce to wreath products in the strict sense (see [Nill, Szlachányi (1997)] and [Jara Martínez et al. (2008)]).

In the quantum spin chains in [Nill, Szlachányi, Wiesbrock (preprint 1998)] and [Böhm (1997)], where the spins take their values in a dual pair of finite dimensional weak Hopf algebras, this (n+1)-ary weak wreath product is regarded as the algebra of observable quantities localized in the interval $[0,n]$ of the one dimensional lattice. In particular, in spin chains built on dual pairs of finite dimensional Hopf algebras (for instance on pairs of a finite group algebra and the algebra of linear functions on this group), the observable algebra is a proper (n+1)-ary wreath product. In the classical Ising model – where the spins only have 'up' and 'down' positions – these dual Hopf algebras are both isomorphic to the linear span of the sign group $\mathbb{Z}(2)$.

4. A fully faithful embedding

In this section we show that, for any 2-category $\mathcal{K}$, and any non-negative integer n, $\mathsf{Wdl}^{(n)}(\overline{\mathcal{K}})$ admits a fully faithful embedding into the power 2-category $\mathsf{Mnd}(\overline{\mathcal{K}})^{\mathbf{2}^{n+1}}$. Whenever idempotent 2-cells in $\mathcal{K}$ split, this gives rise to a fully faithful embedding $\mathsf{Wdl}^{(n)}(\mathcal{K})\to\mathsf{Mnd}(\mathcal{K})^{\mathbf{2}^{n+1}}$. If in addition $\mathcal{K}$ admits Eilenberg-Moore objects, this amounts to a fully faithful embedding $\mathsf{Wdl}^{(n)}(\mathcal{K})\to\mathcal{K}^{\mathbf{2}^{n+1}}$.

4.1. The 2-category $\mathcal{K}^{\mathbf{2}^n}$ The 2-category $\mathbf{2}$ has two 0-cells 0 and 1; an only non-identity 1-cell $1\to 0$; and all of its 2-cells are identities. For any 2-category $\mathcal{K}$, there is a 2-category $\mathcal{K}^{\mathbf{2}}$ of 2-functors $\mathbf{2}\to\mathcal{K}$, 2-natural transformations and modifications. Iteratively, for $n>1$ we define $\mathcal{K}^{\mathbf{2}^n}$ as $(\mathcal{K}^{\mathbf{2}^{n-1}})^{\mathbf{2}}$. That is, $\mathcal{K}^{\mathbf{2}^n}$ is isomorphic to the 2-category of 2-functors from the n-fold Cartesian product $\mathbf{2}\times\ldots\times\mathbf{2}$ to $\mathcal{K}$, 2-natural transformations and modifications. An explicit description is given as follows. The 0-cells are the n dimensional oriented cubes whose 2-faces are commutative squares of 1-cells in $\mathcal{K}$. A 1-cell from an n-cube of edges $\{v_{\underline{p},\underline{q}}:A_{\underline{p}}\to A_{\underline{q}}\}$ to $\{v'_{\underline{p},\underline{q}}:A'_{\underline{p}}\to A'_{\underline{q}}\}$ consists of 1-cells $\{u_{\underline{p}}:A_{\underline{p}}\to A'_{\underline{p}}\}$ in $\mathcal{K}$ such that the $n+1$-cube $\{v_{\underline{p},\underline{q}}:A_{\underline{p}}\to A_{\underline{q}},u_{\underline{p}}:A_{\underline{p}}\to A'_{\underline{p}},v'_{\underline{p},\underline{q}}:A'_{\underline{p}}\to A'_{\underline{q}}\}$ is commutative. That is, for all values of $\underline{p}$ and $\underline{q}$, $v'_{\underline{p},\underline{q}}.u_{\underline{p}}=u_{\underline{q}}.v_{\underline{p},\underline{q}}$. Finally, 2-cells consist of 2-cells $\omega_{\underline{p}}:u_{\underline{p}}\to\tilde{u}_{\underline{p}}$ in $\mathcal{K}$ such that $v'_{\underline{p},\underline{q}}.\omega_{\underline{p}}=\omega_{\underline{q}}.v_{\underline{p},\underline{q}}$.

In Cartesian coordinates, the vertices of an n-cube can be labelled by the elements $\underline{p}=(p_1,\ldots,p_n)$ of the set $\{0,1\}^n$. Sometimes we represent $\underline{p}\in\{0,1\}^n$ by listing those values of i for which $p_i=1$. For example, $12=(1,1,0,\ldots,0)$, $3=(0,0,1,0,\ldots,0)$, etc.. The n-cube has an edge $\underline{p}\to\underline{q}$ if and only if there is some integer $1\leq i\leq n$ such that $p_j=q_j$ for all $j\neq i$, $p_i=0$ and $q_i=1$. We denote this situation by $\underline{q}=\underline{p}+i$. For $\underline{p},\underline{q}\in\{0,1\}^n$, we say that $\underline{p}<\underline{q}$ if, for any $1\leq i,j\leq n$, the equality $p_iq_j=1$

implies $i < j$. For $\underline{p} < \underline{q}$ we define $\underline{p} + \underline{q} \in \{0,1\}^n$ putting $(\underline{p} + \underline{q})_i := p_i + q_i$. We denote by $\underline{0} := (0, 0, \ldots, 0)$ and $\underline{1} := (1, 1, \ldots, 1)$ the constant elements of $\{0,1\}^n$.

The construction of the promised 2-functor $\mathsf{Wdl}^{(n)}(\overline{\mathcal{K}}) \to \mathsf{Mnd}(\overline{\mathcal{K}})^{\mathbf{2}^{n+1}}$ relies on a few lemmas below. A routine computation proves the first one:

4.2. Lemma. *For any object $\{\lambda_{i,j} : s_j s_i \to s_i s_j\}_{0 \le i < j \le 2}$ of $\mathsf{Wdl}^{(2)}(\overline{\mathcal{K}})$, there is a homomorphism of monads in $\overline{\mathcal{K}}$,*

$$\varphi^1_{0,2} := \overline{\lambda}_{012}.s_0\eta_1 s_2 : (s_0 s_2, \overline{\lambda}_{02}) \to (s_0 s_1 s_2, \overline{\lambda}_{012}).$$

This means that there is a 1-cell $((A,A), \varphi^1_{0,2}) : (A, (s_0 s_1 s_2, \overline{\lambda}_{012})) \to (A, (s_0 s_2, \overline{\lambda}_{02}))$ in $\mathsf{Mnd}(\overline{\mathcal{K}})$ (where A is the object underlying the monads (A, s_i)).

4.3. Lemma. *For any object $\{\lambda_{i,j} : s_j s_i \to s_i s_j\}_{0 \le i < j \le 4}$ of $\mathsf{Wdl}^{(4)}(\overline{\mathcal{K}})$, the morphisms as in Lemma 4.2 constitute a commutative diagram in $\overline{\mathcal{K}}$:*

$$\begin{array}{ccc}
(s_0 s_2 s_4, \overline{\lambda}_{024}) & \xrightarrow{\varphi^1_{0,24}} & (s_0 s_1 s_2 s_4, \overline{\lambda}_{0124}) \\
\downarrow{\scriptstyle \varphi^3_{02,4}} & & \downarrow{\scriptstyle \varphi^3_{012,4}} \\
(s_0 s_2 s_3 s_4, \overline{\lambda}_{0234}) & \xrightarrow[\varphi^1_{0,234}]{} & (s_0 s_1 s_2 s_3 s_4, \overline{\lambda}_{01234})\ .
\end{array}$$

Proof. In view of Lemma 4.2, both paths around the diagram are equal to $\overline{\lambda}_{01234}.\, s_0\eta_1 s_2\eta_3 s_4$. ∎

4.4. Lemma. *For any 1-cell $\{\xi_i : s'_i v \to v s_i\}_{0 \le i \le 2}$ in $\mathsf{Wdl}^{(2)}(\overline{\mathcal{K}})$, the morphisms in Lemma 4.2 induce a commutative square*

$$\begin{array}{ccc}
(A, (s_0 s_1 s_2, \overline{\lambda}_{012})) & \xrightarrow{((v,\overline{v}),\xi_{012})} & (A', (s'_0 s'_1 s'_2, \overline{\lambda'}_{012})) \\
\downarrow{\scriptstyle ((A,A),\varphi^1_{0,2})} & & \downarrow{\scriptstyle ((A',A'),\varphi'^1_{0,2})} \\
(A, (s_0 s_2, \overline{\lambda}_{02})) & \xrightarrow[((v,\overline{v}),\xi_{02})]{} & (A', (s'_0 s'_2, \overline{\lambda'}_{02}))
\end{array}$$

in $\mathsf{Mnd}(\overline{\mathcal{K}})$.

Proof. Both paths around the diagram are computed to be equal to

$$v\overline{\lambda}_{012}.vs_0\eta_1 s_2.\xi_0 s_2.s'_0\xi_2 = v\overline{\lambda}_{012}.\xi_0 s_1 s_2.s'_0\xi_1 s_2.s'_0 s'_1\xi_2.\overline{\lambda'}_{012}v.s'_0\eta'_1 s'_2 v,$$

where the last equality follows by the unitality of ξ_i and the normalization property $\xi_{012}.\overline{\lambda'}_{012}v = \xi_{012}$. ∎

4.5. A 2-FUNCTOR $F : \mathsf{Wdl}^{(n)}(\overline{\mathcal{K}}) \to \mathsf{Mnd}(\overline{\mathcal{K}})^{\mathbf{2}^{n+1}}$ For any non-negative integer n, and any $\underline{0} \neq \underline{p} = (p_0, \dots, p_n) \in \{0,1\}^{n+1}$, taking those values of $0 \leq i \leq n$ for that $p_i = 1$, defines a 2-functor $\mathsf{Wdl}^{(n)}(\overline{\mathcal{K}}) \to \mathsf{Wdl}^{(-1+\sum_i p_i)}(\overline{\mathcal{K}})$. Composing it with the unique iterated weak wreath product 2-functor $\mathsf{Wdl}^{(-1+\sum_i p_i)}(\overline{\mathcal{K}}) \to \mathsf{Mnd}(\overline{\mathcal{K}})$ in Section 2, yields a 2-functor $\mathsf{Wdl}^{(n)}(\overline{\mathcal{K}}) \to \mathsf{Mnd}(\overline{\mathcal{K}})$. Denote the image of

$$\{\lambda_{i,j} : s_j s_i \to s_i s_j\}_{0\leq i<j\leq n} \quad \overset{\{\xi_i : s'_i v \to v s_i\}_{0\leq i\leq n}}{\underset{\{\xi'_i : s'_i v' \to v' s_i\}_{0\leq i\leq n}}{\Downarrow \omega}} \quad \{\lambda'_{i,j} : s'_j s'_i \to s'_i s'_j\}_{0\leq i<j\leq n}$$

under it by

$$(A, (s_{\underline{p}}, \overline{\lambda}_{\underline{p}})) \quad \overset{\xi_{\underline{p}} : s'_{\underline{p}} v \to v s_{\underline{p}}}{\underset{\xi'_{\underline{p}} : s'_{\underline{p}} v' \to v' s_{\underline{p}}}{\Downarrow \omega}} \quad (A', (s'_{\underline{p}}, \overline{\lambda'}_{\underline{p}})) \ .$$

Then $(A, s_{\underline{p}})$ is the weak wreath product of those demimonads (A, s_i) for which $p_i = 1$. For $\underline{0} \in \{0,1\}^{n+1}$, put $(A, s_{\underline{0}}) := (A, A)$ and $\xi_{\underline{0}} := v$.

By Lemmas 4.2 and 4.3, for any object $\{\lambda_{i,j} : s_j s_i \to s_i s_j\}_{0\leq i<j\leq n}$ of $\mathsf{Wdl}^{(n)}(\overline{\mathcal{K}})$, for any $\underline{p} \in \{0,1\}^{n+1}$, and for any $0 \leq i, j \leq n$ such that $p_i = p_j = 0$, there is a commutative square

$$\begin{array}{ccc} (A, (s_{\underline{p}+i+j}, \overline{\lambda}_{\underline{p}+i+j})) & \xrightarrow{((A,A),\varphi^i_{\underline{p}+j})} & (A, (s_{\underline{p}+j}, \overline{\lambda}_{\underline{p}+j})) \\ {\scriptstyle ((A,A),\varphi^j_{\underline{p}+i})}\downarrow & & \downarrow{\scriptstyle ((A,A),\varphi^j_{\underline{p}})} \\ (A, (s_{\underline{p}+i}, \overline{\lambda}_{\underline{p}+i})) & \xrightarrow[((A,A),\varphi^i_{\underline{p}})]{} & (A, (s_{\underline{p}}, \overline{\lambda}_{\underline{p}})) \end{array}$$

in $\mathsf{Mnd}(\overline{\mathcal{K}})$. Such squares constitute a commutative $n+1$-cube in $\mathsf{Mnd}(\overline{\mathcal{K}})$.

By Lemma 4.4, for any 1-cell $\{\xi_i : s'_i v \to v s_i\}_{0\leq i\leq n}$ of $\mathsf{Wdl}^{(n)}(\overline{\mathcal{K}})$, there is a commutative square

$$\begin{array}{ccc} (A, (s_{\underline{p}+i}, \overline{\lambda}_{\underline{p}+i})) & \xrightarrow{((v,\overline{v}),\xi_{\underline{p}+i})} & (A', (s'_{\underline{p}+i}, \overline{\lambda'}_{\underline{p}+i})) \\ {\scriptstyle ((A,A),\varphi^i_{\underline{p}})}\downarrow & & \downarrow{\scriptstyle ((A',A'),\varphi'^i_{\underline{p}})} \\ (A, (s_{\underline{p}}, \overline{\lambda}_{\underline{p}})) & \xrightarrow[((v,\overline{v}),\xi_{\underline{p}})]{} & (A', (s'_{\underline{p}}, \overline{\lambda'}_{\underline{p}})) \end{array}$$

in $\mathsf{Mnd}(\overline{\mathcal{K}})$. Hence the 1-cells $((v,\overline{v}),\xi_{\underline{p}}) : (A, (s_{\underline{p}}, \overline{\lambda}_{\underline{p}})) \to (A', (s'_{\underline{p}}, \overline{\lambda'}_{\underline{p}}))$ constitute a 1-cell in $\mathsf{Mnd}(\overline{\mathcal{K}})^{\mathbf{2}^{n+1}}$.

Finally, for a 2-cell $\omega : \{\xi_i : s'_i v \to v s_i\}_{0\leq i\leq n} \to \{\xi'_i : s'_i v' \to v' s_i\}_{0\leq i\leq n}$ in $\mathsf{Wdl}^{(n)}(\overline{\mathcal{K}})$, ω is a 2-cell $((v,\overline{v}),\xi_{\underline{p}}) \to ((v',\overline{v'}),\xi'_{\underline{p}})$ in $\mathsf{Mnd}(\overline{\mathcal{K}})$, for any $\underline{p} \in \{0,1\}^{n+1}$, which constitutes a 2-cell in $\mathsf{Mnd}(\overline{\mathcal{K}})^{\mathbf{2}^{n+1}}$.

The above maps define the stated 2-functor $F : \mathsf{Wdl}^{(n)}(\overline{\mathcal{K}}) \to \mathsf{Mnd}(\overline{\mathcal{K}})^{\mathbf{2}^{n+1}}$.

4.6. THEOREM. *For any 2-category $\mathcal{K}$, and any non-negative integer n, the 2-functor $\mathsf{Wdl}^{(n)}(\overline{\mathcal{K}}) \to \mathsf{Mnd}(\overline{\mathcal{K}})^{\mathbf{2}^{n+1}}$ in Paragraph 4.5 is fully faithful.*

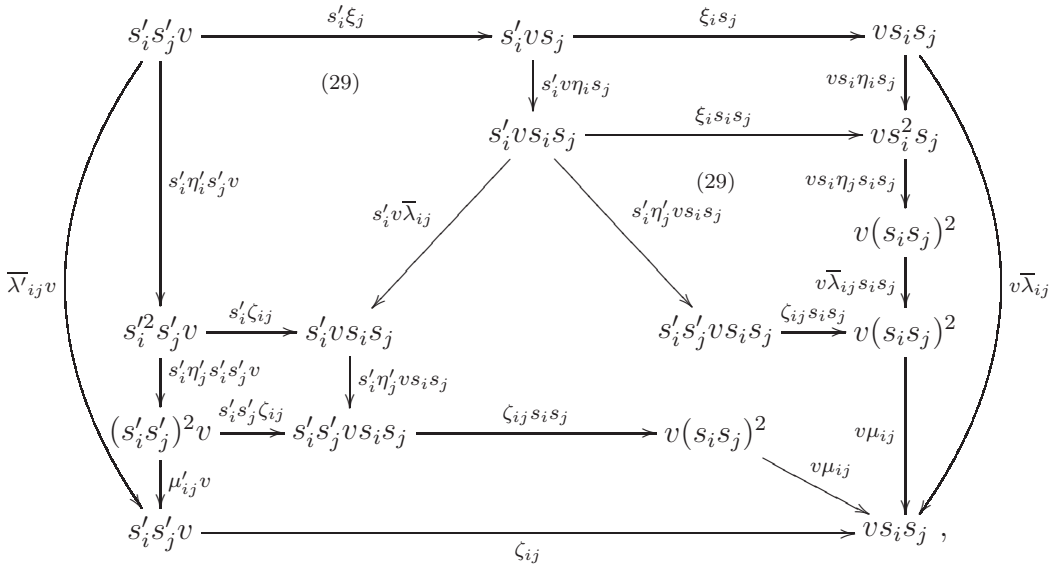
PROOF. Faithfulness is obvious. In order to prove fullness on the 1-cells, take a 1-cell $\{\zeta_{\underline{p}} : s'_{\underline{p}}v \to vs_{\underline{p}}\}_{\underline{p}\in\{0,1\}^{n+1}}$ in $\mathsf{Mnd}(\overline{\mathcal{K}})^{\mathbf{2}^{n+1}}$ between objects arising from 0-cells $\{\lambda_{i,j} : s_js_i \to s_is_j\}_{0\le i<j\le n}$ and $\{\lambda'_{i,j} : s'_js'_i \to s'_is'_j\}_{0\le i<j\le n}$ of $\mathsf{Wdl}^{(n)}(\overline{\mathcal{K}})$. This includes in particular 1-cells $\xi_i := \zeta_i : s'_iv \to vs_i$ in $\mathsf{Mnd}(\overline{\mathcal{K}})$. We claim that for $i \in \{0,\dots,n\}$ they constitute a 1-cell in $\mathsf{Wdl}^{(n)}(\overline{\mathcal{K}})$ and each $\zeta_{\underline{p}}$ is equal to their weak wreath product. By commutativity of the squares

$$\begin{array}{ccc} (A,(s_is_j,\overline{\lambda}_{ij})) & \xrightarrow{((v,\overline{v}),\zeta_{ij})} & (A',(s'_is'_j,\overline{\lambda'}_{ij})) \\ {\scriptstyle((A,A),\varphi^j_i)}\downarrow & & \downarrow{\scriptstyle((A',A'),\varphi'^j_i)} \\ (A,(s_i,\overline{s}_i)) & \xrightarrow[((v,\overline{v}),\xi_i)]{} & (A',(s'_i,\overline{s}'_i)) \end{array} \qquad \begin{array}{ccc} (A,(s_is_j,\overline{\lambda}_{ij})) & \xrightarrow{((v,\overline{v}),\zeta_{ij})} & (A',(s'_is'_j,\overline{\lambda'}_{ij})) \\ {\scriptstyle((A,A),\varphi^i_j)}\downarrow & & \downarrow{\scriptstyle((A',A'),\varphi'^i_j)} \\ (A,(s_j,\overline{s}_j)) & \xrightarrow[((v,\overline{v}),\xi_j)]{} & (A',(s'_j,s'_j)) \end{array}$$

in $\mathsf{Mnd}(\overline{\mathcal{K}})$, we conclude the commutativity of the diagrams

$$\begin{array}{ccccc} s'_iv & \xrightarrow{s'_i\eta'_jv} & s'_is'_jv & \xrightarrow{\overline{\lambda'}_{ij}v} & s'_is'_jv \\ {\scriptstyle\xi_i}\downarrow & & & \searrow^{\zeta_{ij}} & \downarrow{\scriptstyle\zeta_{ij}} \\ vs_i & \xrightarrow[vs_i\eta_j]{} & vs_is_j & \xrightarrow[v\overline{\lambda}_{ij}]{} & vs_is_j \end{array} \qquad \begin{array}{ccccc} s'_jv & \xrightarrow{\eta'_is'_jv} & s'_is'_jv & \xrightarrow{\overline{\lambda'}_{ij}v} & s'_is'_jv \\ {\scriptstyle\xi_j}\downarrow & & & \searrow^{\zeta_{ij}} & \downarrow{\scriptstyle\zeta_{ij}} \\ vs_j & \xrightarrow[vs_i\eta_j]{} & vs_is_j & \xrightarrow[v\overline{\lambda}_{ij}]{} & vs_is_j \end{array} \tag{29}$$

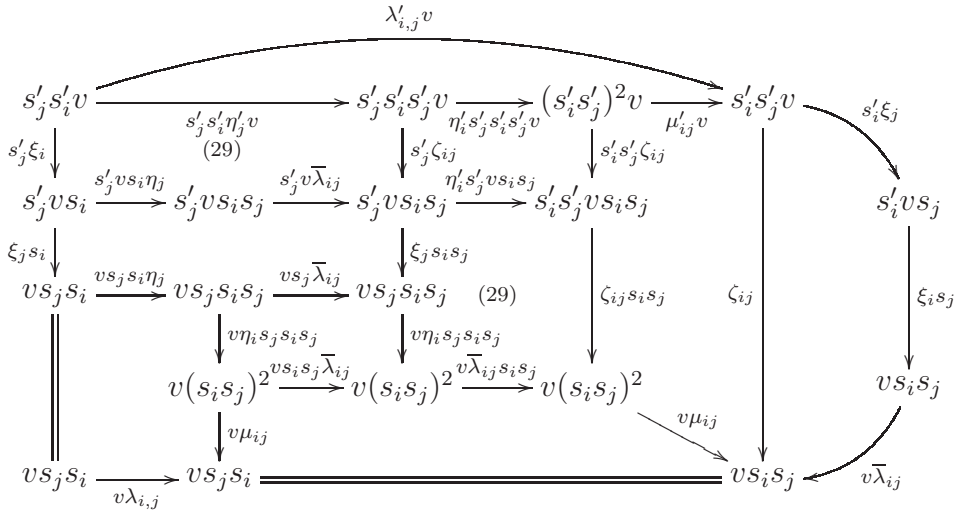
in $\mathcal{K}$. Since $((v,\overline{v}),\zeta_{ij})$ is a 1-cell in $\mathsf{Mnd}(\overline{\mathcal{K}})$, the following diagram commutes



(30)

where we denoted $\mu_{ij} = \mu_i\mu_j.s_i\lambda_{i,j}s_j$. Since $\zeta_{ij}.\overline{\lambda'}_{ij}v = \zeta_{ij}$, this says that ζ_{ij} is equal to $v\overline{\lambda}_{ij}.\xi_is_j.s'_i\xi_j$; that is, ζ_{ij} is the weak wreath product of ξ_i and ξ_j. With this expression

of ζ_{ij} at hand, also the following diagram commutes.



That is, (v, ξ_i, ξ_j) is a 1-cell in $\mathsf{Wdl}(\overline{\mathcal{K}})$. Thus the collection $\{\xi_i : s'_iv \to vs_i\}_{0\leq i\leq n}$ is a 1-cell in $\mathsf{Wdl}^{(n)}(\overline{\mathcal{K}})$. The same reasoning as in (30) shows that its image under the 2-functor in the claim is the 1-cell $\{\zeta_{\underline{p}} : s_{\underline{p}}v \to vs_{\underline{p}}\}_{\underline{p}\in\{0,1\}^{n+1}}$ in $\mathsf{Mnd}(\overline{\mathcal{K}})^{\mathbf{2}^{n+1}}$ that we started with. Fullness on the 2-cells is evident. ∎

4.7. If idempotent 2-cells split Let $\mathcal{K}$ be a 2-category in which idempotent 2-cells split; that is, biequivalent to $\overline{\mathcal{K}}$, and consider the 2-functor

$$\mathsf{Wdl}^{(n)}(\mathcal{K}) \xrightarrow{\simeq} \mathsf{Wdl}^{(n)}(\overline{\mathcal{K}}) \xrightarrow{F} \mathsf{Mnd}(\overline{\mathcal{K}})^{\mathbf{2}^{n+1}} , \tag{31}$$

where F is the 2-functor introduced in Paragraph 4.5. By Paragraph 4.5, it takes a 1-cell $\{\xi_i : s'_iv \to vs_i\}_{0\leq i\leq n}$ to an $n+2$-cube in $\mathsf{Mnd}(\overline{\mathcal{K}})$, with faces of the form in the first diagram in

$$\begin{array}{ccc}
(A,(s_{\underline{p}+i},\overline{\lambda}_{\underline{p}+i})) & \xrightarrow{((v,v),\xi_{\underline{p}+i})} & (A',(s'_{\underline{p}+i},\overline{\lambda'}_{\underline{p}+i})) \\
{\scriptstyle((A,A),\varphi^i_{\underline{p}})}\downarrow & & \downarrow{\scriptstyle((A',A'),\varphi'^i_{\underline{p}})} \\
(A,(s_{\underline{p}},\overline{\lambda}_{\underline{p}})) & \xrightarrow[((v,v),\xi_{\underline{p}})]{} & (A',(s'_{\underline{p}},\overline{\lambda'}_{\underline{p}}))
\end{array}
\qquad\qquad
\begin{array}{ccc}
(A,z_{\underline{p}+i}) & \xrightarrow{(v,\zeta_{\underline{p}+i})} & (A',z'_{\underline{p}+i}) \\
{\scriptstyle(A,\psi^i_{\underline{p}})}\downarrow & & \downarrow{\scriptstyle(A',\psi'^i_{\underline{p}})} \\
(A,z_{\underline{p}}) & \xrightarrow[(v,\zeta_{\underline{p}})]{} & (A',z'_{\underline{p}}).
\end{array}
\tag{32}$$

Let us choose a biequivalence pseudofunctor $\overline{\mathcal{K}} \to \mathcal{K}$ adopting the convention that we split any identity 2-cell trivially; that is, via identity 2-cells. Then the induced biequivalence $\mathsf{Mnd}(\overline{\mathcal{K}}) \to \mathsf{Mnd}(\mathcal{K})$ takes the first square in (32) to a *commutative* square in $\mathsf{Mnd}(\mathcal{K})$ of the form in the second diagram. Mapping $\{\xi_i : s'_iv \to vs_i\}_{0\leq i\leq n}$ to the $n+2$-cube in $\mathsf{Mnd}(\mathcal{K})$ formed by these faces; and mapping a 2-cell ω to the 2-cell in $\mathsf{Mnd}(\mathcal{K})^{\mathbf{2}^{n+1}}$ whose value at each $\underline{p} \in \{0,1\}^{n+1}$ is given by ω; we obtain a fully faithful 2-functor $\mathsf{Wdl}^{(n)}(\mathcal{K}) \to \mathsf{Mnd}(\mathcal{K})^{\mathbf{2}^{n+1}}$. Its composition with the biequivalence $\mathsf{Mnd}(\mathcal{K})^{\mathbf{2}^{n+1}} \xrightarrow{\simeq} \mathsf{Mnd}(\overline{\mathcal{K}})^{\mathbf{2}^{n+1}}$ is 2-naturally isomorphic to (31).

4.8. If Eilenberg-Moore objects exist Recall (from [Street (1972)]) that a 2-category $\mathcal{K}$ is said to admit Eilenberg-Moore objects provided that the evident inclusion $I : \mathcal{K} \to \mathsf{Mnd}(\mathcal{K})$ possesses a right 2-adjoint J. Whenever J exists, it induces a fully faithful 2-functor $\mathsf{Mnd}(\mathcal{K}) \to \mathcal{K}^{\mathbf{2}}$ as follows. It takes a monad (A, t) to the 1-cell part of the counit of the 2-adjunction $I \dashv J$ evaluated at (A, t); that is, the so-called "forgetful morphism" $J(A, t) \to A$ in $\mathcal{K}$. (The terminology certainly comes from its form in $\mathcal{K} = \mathsf{Cat}$.) It takes a 1-cell (v, ψ) to the pair $(v, J(v, \psi))$ and it takes a 2-cell ω to the pair $(\omega, J\omega)$.

4.9. Corollary. *Let $\mathcal{K}$ be a 2-category in which idempotent 2-cells split and which admits Eilenberg-Moore objects. Then composing the fully faithful 2-functor $\mathsf{Wdl}^{(n)}(\mathcal{K}) \to \mathsf{Mnd}(\mathcal{K})^{\mathbf{2}^{n+1}}$ in Paragraph 4.7 with $J^{\mathbf{2}^{n+1}} : \mathsf{Mnd}(\mathcal{K})^{\mathbf{2}^{n+1}} \to \mathcal{K}^{\mathbf{2}^{n+1}}$, we obtain a fully faithful embedding.*

5. The n-ary factorization problem

The aim of this section is to find sufficient and necessary conditions on a demimonad (that is, a monad in the local idempotent closure of a 2-category) to be isomorphic to a weak wreath product of n demimonads. Some facts about the $n = 2$ case are recalled in Paragraph 1.6.

In the next theorem we shall use the notation introduced after Paragraph 4.1.

5.1. Theorem. *For any demimonad (A, s) in an arbitrary 2-category $\mathcal{K}$, the following assertions are equivalent.*

(i) There is an object $\{\lambda_{i,j} : s_j s_i \to s_i s_j\}_{1 \le i < j \le n}$ of $\mathsf{Wdl}^{(n-1)}(\overline{\mathcal{K}})$ such that the corresponding n-ary weak wreath product (that is, its image under the 2-functor in Theorem 2.10) is isomorphic to (A, s).

(ii) There is an n dimensional cube whose 2-faces are commutative squares of monad morphisms in $\overline{\mathcal{K}}$ of the form $(A, \varphi^i_{\underline{p}}) : (A, s_{\underline{p}+i}) \to (A, s_{\underline{p}})$, such that the following hold. For $\underline{p} < \underline{q} \in \{0,1\}^n$, denote by $\varphi_{\underline{p}}{}^{\underline{q}}$ and by $\varphi^{\underline{p}}{}_{\underline{q}}$ the (unique) morphisms composed along any path to $\underline{p} + \underline{q}$ from $\underline{p}$ and from $\underline{q}$, respectively. Then

(a) $(A, s_{\underline{0}})$ is the trivial monad (A, A) and $(A, s_{\underline{1}})$ is isomorphic to (A, s).

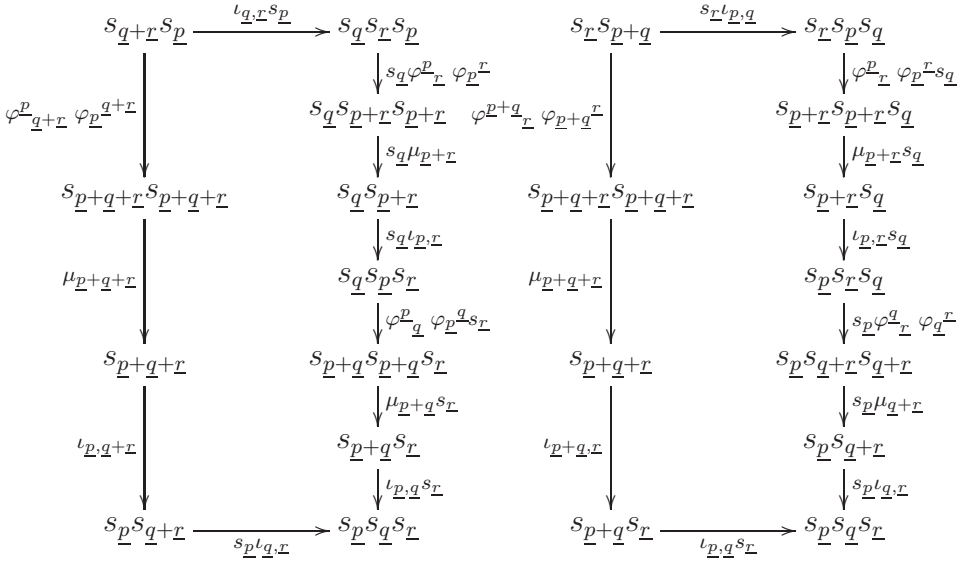
(b) For all $\underline{p} < \underline{q} \in \{0,1\}^n$, the 2-cell

$$\pi_{\underline{p},\underline{q}} := \Big(s_{\underline{p}} s_{\underline{q}} \xrightarrow{\varphi_{\underline{p}}{}^{\underline{q}}\, \varphi^{\underline{p}}{}_{\underline{q}}} s_{\underline{p}+\underline{q}} s_{\underline{p}+\underline{q}} \xrightarrow{\mu_{\underline{p}+\underline{q}}} s_{\underline{p}+\underline{q}} \Big)$$

possesses an $s_{\underline{p}}$-$s_{\underline{q}}$ bimodule section $\iota_{\underline{p},\underline{q}}$ in $\overline{\mathcal{K}}$.

(c) For all $\underline{p} < \underline{q} < \underline{r} \in \{0,1\}^n$, the morphisms ι in part (b) render commutative the following diagrams.

$$\begin{array}{ccc} s_{\underline{p}+\underline{q}+\underline{r}} & \xrightarrow{\iota_{\underline{p},\underline{q}+\underline{r}}} & s_{\underline{p}} s_{\underline{q}+\underline{r}} \\ {\scriptstyle \iota_{\underline{p}+\underline{q},\underline{r}}}\downarrow & & \downarrow{\scriptstyle s_{\underline{p}} \iota_{\underline{q},\underline{r}}} \\ s_{\underline{p}+\underline{q}} s_{\underline{r}} & \xrightarrow[\iota_{\underline{p},\underline{q}} s_{\underline{r}}]{} & s_{\underline{p}} s_{\underline{q}} s_{\underline{r}} \end{array} \tag{33}$$



(34)

PROOF. (i)⇒(ii). The 2-functor in Paragraph 4.5 takes $\{\lambda_{i,j} : s_j s_i \to s_i s_j\}_{1\leq i<j\leq n}$ to a commutative n-cube in $\mathsf{Mnd}(\overline{\mathcal{K}})$ with edges $(A, \varphi^i_{\underline{p}}) : (A, s_{\underline{p}+i}) \to (A, s_{\underline{p}})$ of the form in Lemma 4.2. In this cube $(A, s_{\underline{0}})$ is the trivial monad (A, A) and $(A, s_{\underline{1}})$ is the n-ary weak wreath product which is isomorphic to (A, s) by assumption. Thus property (a) holds. By construction of the 2-functor in Paragraph 4.5, $(A, s_{\underline{p}+\underline{q}})$ is the weak wreath product of $(A, s_{\underline{p}})$ and $(A, s_{\underline{q}})$, for all $\underline{p} < \underline{q} \in \{0, 1\}^n$. Hence the 2-cell $\pi_{\underline{p},\underline{q}}$ in part (b) possesses a bilinear section $\iota_{\underline{p},\underline{q}}$ by Paragraph 1.6. It remains to show that the diagrams in part (c) commute.

The monic 2-cell $\iota_{\underline{p},\underline{q}}$ is given by $\overline{\lambda}_{\underline{p}+\underline{q}} : (s_{\underline{p}} s_{\underline{q}}, \overline{\lambda}_{\underline{p}+\underline{q}}) \to (s_{\underline{p}} s_{\underline{q}}, \overline{s}_{\underline{p}} \overline{s}_{\underline{q}})$; $\iota_{\underline{p},\underline{q}+\underline{r}}$ is equal to $\overline{\lambda}_{\underline{p}+\underline{q}+\underline{r}} : (s_{\underline{p}} s_{\underline{q}} s_{\underline{r}}, \overline{\lambda}_{\underline{p}+\underline{q}+\underline{r}}) \to (s_{\underline{p}} s_{\underline{q}} s_{\underline{r}}, \overline{s}_{\underline{p}} \overline{\lambda}_{\underline{q}+\underline{r}})$ and so on. Hence (33) takes the form

$$\begin{array}{ccc}
(s_{\underline{p}} s_{\underline{q}} s_{\underline{r}}, \overline{\lambda}_{\underline{p}+\underline{q}+\underline{r}}) & \xrightarrow{\overline{\lambda}_{\underline{p}+\underline{q}+\underline{r}}} & (s_{\underline{p}} s_{\underline{q}} s_{\underline{r}}, \overline{s}_{\underline{p}} \overline{\lambda}_{\underline{q}+\underline{r}}) \\
{\scriptstyle \overline{\lambda}_{\underline{p}+\underline{q}+\underline{r}}}\downarrow & & \downarrow{\scriptstyle s_{\underline{p}} \overline{\lambda}_{\underline{q}+\underline{r}}} \\
(s_{\underline{p}} s_{\underline{q}} s_{\underline{r}}, \overline{\lambda}_{\underline{p}+\underline{q}} s_{\underline{r}}) & \xrightarrow[\overline{\lambda}_{\underline{p}+\underline{q}} \overline{s}_{\underline{r}}]{} & (s_{\underline{p}} s_{\underline{q}} s_{\underline{r}}, s_{\underline{p}} s_{\underline{q}} s_{\underline{r}})
\end{array}$$

which commutes by (7) and (10). In the vertical paths of the diagrams in (34), note the occurrence of the weak distributive laws $\lambda_{\underline{p},\underline{q}}$, $\lambda_{\underline{p},\underline{q}+\underline{r}}$, etc. Thus the first diagram in (34) takes the form

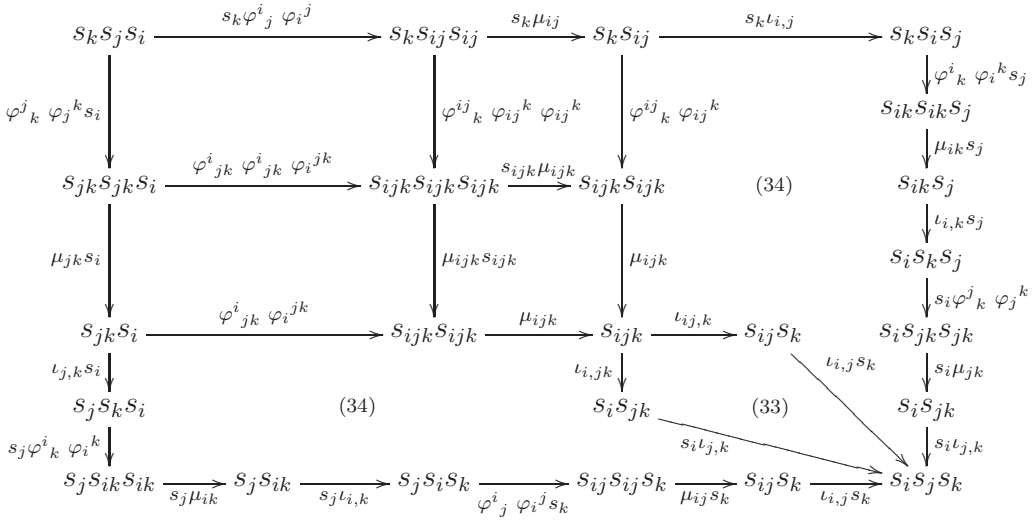
$$\begin{array}{ccccc}
(s_{\underline{q}} s_{\underline{r}} s_{\underline{p}}, \overline{\lambda}_{\underline{q}+\underline{r}} \overline{s}_{\underline{p}}) & & \xrightarrow{\lambda_{\underline{p},\underline{q}+\underline{r}}} & & (s_{\underline{p}} s_{\underline{q}} s_{\underline{r}}, \overline{s}_{\underline{p}} \overline{\lambda}_{\underline{q}+\underline{r}}) \\
{\scriptstyle \overline{\lambda}_{\underline{q}+\underline{r}} s_{\underline{p}}}\downarrow & & & & \downarrow{\scriptstyle s_{\underline{p}} \overline{\lambda}_{\underline{q}+\underline{r}}} \\
(s_{\underline{q}} s_{\underline{r}} s_{\underline{p}}, \overline{s}_{\underline{q}} \overline{s}_{\underline{r}} \overline{s}_{\underline{p}}) & \xrightarrow[s_{\underline{q}} \lambda_{\underline{p},\underline{r}}]{} & (s_{\underline{q}} s_{\underline{p}} s_{\underline{r}}, \overline{s}_{\underline{q}} \overline{s}_{\underline{p}} \overline{s}_{\underline{r}}) & \xrightarrow[\lambda_{\underline{p},\underline{q}} s_{\underline{r}}]{} & (s_{\underline{p}} s_{\underline{q}} s_{\underline{r}}, \overline{s}_{\underline{p}} \overline{s}_{\underline{q}} \overline{s}_{\underline{r}})
\end{array}$$

which is evidently commutative in view of (17). Commutativity of the second diagram in (34) follows symmetrically.

(ii)⇒(i). By assumption, for all $1 \leq i < j \leq n$, the 2-cell

$$\pi_{i,j} := \left(s_i s_j \xrightarrow{{\varphi_i}^j \, {\varphi^i}_j} s_{ij} s_{ij} \xrightarrow{\mu_{ij}} s_{ij} \right)$$

possesses an s_i-s_j bimodule section $\iota_{i,j}$. Hence by Paragraph 1.6, there is a weak distributive law $\lambda_{i,j} := \iota_{i,j}.\mu_{ij}.{\varphi^i}_j \, {\varphi_i}^j : s_j s_i \to s_i s_j$ in $\overline{\mathcal{K}}$ such that the corresponding weak wreath product is isomorphic to s_{ij}. Let us prove that the collection $\{\lambda_{i,j} : s_j s_i \to s_i s_j\}_{1 \leq i < j \leq n}$ obeys the Yang-Baxter conditions, for all $1 \leq i < j < k \leq n$. This follows by commutativity of the following diagram.



Thus $\{\lambda_{i,j} : s_j s_i \to s_i s_j\}_{1 \leq i < j \leq n}$ is an object of $\mathsf{Wdl}^{(n-1)}(\overline{\mathcal{K}})$. It remains to show that the corresponding n-ary weak wreath product is isomorphic to $(A, s_{\underline{1}}) \cong (A, s)$.

As observed above, (A, s_{12}) is isomorphic to the weak wreath product of (A, s_1) and (A, s_2) with respect to the weak distributive law $\lambda_{1,2} := \iota_{1,2}.\mu_{12}.{\varphi^1}_2 \, {\varphi_1}^2 : s_2 s_1 \to s_1 s_2$. Similarly, (A, s_{123}) is isomorphic to the weak wreath product of (A, s_{12}) and (A, s_3) with respect to $\lambda_{12,3} := \iota_{12,3}.\mu_{123}.{\varphi^{12}}_3 \, {\varphi_{12}}^3 : s_3 s_{12} \to s_{12} s_3$. Moreover, precomposing both paths around the second diagram in (34) (for $\underline{p} = 1$, $\underline{q} = 2$ and $\underline{r} = 3$) by $s_3 \pi_{1,2} = s_3 \mu_{12}.s_3 {\varphi_1}^2 {\varphi^1}_2$, we obtain that the weak distributive law $\lambda_{12,3} : (s_3 s_{12}, \overline{s}_3 \overline{s}_{12}) \to (s_{12} s_3, \overline{s}_{12} \overline{s}_3)$ differs by the isomorphisms $\iota_{1,2} : (s_{12}, \overline{s}_{12}) \rightleftarrows (s_1 s_2, \overline{\lambda}_{12}) : \pi_{1,2}$ from the image $s_1 \lambda_{2,3}.\lambda_{1,3} s_2.s_3 \overline{\lambda}_{12} : (s_3 s_1 s_2, s_3 \overline{\lambda}_{12}) \to (s_1 s_2 s_3, \overline{\lambda}_{12} s_3)$ of $\{\lambda_{i,j} : s_j s_i \to s_i s_j\}_{1 \leq i < j \leq 3}$ under the 2-functor $C_1 : \mathsf{Wdl}^{(2)}(\overline{\mathcal{K}}) \to \mathsf{Wdl}^{(1)}(\overline{\mathcal{K}})$ in Theorem 2.6. Hence (A, s_{123}) is isomorphic to the ternary weak wreath product of $\{(A, s_i)\}_{1 \leq i \leq 3}$. Iterating this reasoning we conclude that $(A, s_{\underline{1}})$ is isomorphic to the n-ary weak wreath product of $\{(A, s_i)\}_{1 \leq i \leq n}$. ∎

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