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MODELING STABLE ONE-TYPES

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Abstract. Classification of homotopy n-types has focused on developing algebraic categories which are equivalent to categories of n-types. We expand this theory by providing algebraic models of homotopy-theoretic constructions for stable one-types. These include a model for the Postnikov one-truncation of the sphere spectrum, and for its action on the model of a stable one-type. We show that a bicategorical cokernel introduced by Vitale models the cofiber of a map between stable one-types, and apply this to develop an algebraic model for the Postnikov data of a stable one-type.

Introduction

The homotopy category of groupoids is equivalent to the homotopy category of unstable one-types, via the classifying space and fundamental groupoid functors. This is one of the well-known results from a large body of work around the "algebraic homotopy" outlined by J.H.C. Whitehead in his 1950 address to the International Congress of Mathematicians. Crossed modules classify unstable two-types, and Conduché gives a generalization to unstable three-types [Con84].

A related body of work focuses on stable homotopy type. Stable one-types are classified by Picard groupoids, i.e., group-like symmetric monoidal groupoids. This is a well-known result for which we give a new proof in Section 1. Picard groupoids were first introduced in the thesis of Sính Hoang Xuan [Sín75], where the author gives a thorough algebraic classification theorem. Since then, various results have further established the link between Picard groupoids and stable one types. Garzón and Miranda [GM97] develop a model structure for the categories of Picard groupoids, identifying the path and cylinder constructions therein. They use this setting to model homotopy classes of maps between spaces with nontrivial homotopy groups in degrees n and $n+1$ for $n \geq 1$. Garzón-Miranda-del Río [GMdR02] give categorical models for the n^{th} homotopy groupoid of a space for $n \geq 2$, showing that the resulting monoidal categories are braided for $n = 2$ and symmetric for $n \geq 3$. As a generalization of Eilenberg-Mac Lane cohomology, Bullejos-Carrasco-Cegarra [BCC93] define a cohomology of simplicial sets with coefficients in a Picard groupoid. Their work uses this to give alternate categorical models for spaces with homotopy groups in degrees n and $n+1$ for $n \geq 3$—the stable range.

Our proof that Picard groupoids classify stable one-types is given in Theorem 1.5 using the perspective of E_∞ action on the categorical and topological objects. Our main

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results go beyond the basic classification to describe the homotopical structure of stable one-types through corresponding structure of Picard groupoids.

Specifically, we study the decomposition of stable one-types by their Postnikov data. This consists of abelian groups π_0 and π_1, and a single k-invariant, which is a map of Eilenberg-Mac Lane spectra $H\pi_0 \to \Sigma^2 H\pi_1$. Using the isomorphism $[H\pi_0, \Sigma^2 H\pi_1] \cong Hom(\pi_0/2\pi_0, \pi_1)$ [EM54, (2.7)], this k-invariant can be identified with the quadratic map $\eta^* : \pi_0 \to \pi_1$, induced by precomposition with the Hopf map $\eta : S^3 \to S^2$ [BM08, §8]. Our main results model the Postnikov data and sphere action on a stable one-type directly in terms of Picard groupoid data. Note, in particular, that the target of the Postnikov invariant is a stable two-type (of a special kind). Hence our algebraic models lead naturally toward models for stable two-types.

Our main results are as follows. For Picard groupoids $\mathscr{C}$ and $\mathscr{D}$ and a symmetric monoidal functor $F : \mathscr{C} \to \mathscr{D}$, we describe a symmetric monoidal bicategory $\operatorname{Coker}(F)$ first introduced by Vitale [Vit02]. In Section 4 we apply a long exact sequence argument to prove the following result as Theorem 4.3 and Corollaries 4.5 and 4.8.

Theorem A. *Let $F : \mathscr{C} \to \mathscr{D}$ be a functor of Picard groupoids. Then there is a bigroupoid* $\operatorname{Coker}(F)$ *and a natural pseudofunctor*

$$C_F \colon \mathscr{D} \to \operatorname{Coker}(F)$$

which models the stable cofiber in the following sense:

i. $\operatorname{Coker}(F)$ *is symmetric monoidal and C_F is a symmetric monoidal pseudofunctor.*

ii. *Taking classifying spaces yields a cofibration sequence of grouplike E_∞ spaces:*

$$B\mathscr{C} \to B\mathscr{D} \to B\operatorname{Coker}(F).$$

iii. *When $\mathscr{D} = \mathscr{C}_0$ is the discrete category of isomorphism classes of objects in $\mathscr{C}$ and $F = \alpha_0$ is the induced monoidal functor, we have an equivalence with the Postnikov tower of $B\mathscr{C}$:*

$$\begin{tikzcd} B\mathscr{C} \arrow[r] \arrow[d, equal] & K(\pi_0,0) \arrow[r] \arrow[d, "\simeq"] & K(\pi_1,2) \arrow[d, "\simeq"] \\ B\mathscr{C} \arrow[r, "B\alpha_0"] & B\mathscr{C}_0 \arrow[r, "BC_{\alpha_0}"] & B\operatorname{Coker}(\alpha_0) \end{tikzcd}$$

Our work uses a strictification result which is somewhat stronger than one could expect for general symmetric monoidal categories, and it may be of independent interest. This is Theorem 2.2:

Theorem B. *Every Picard groupoid is equivalent as a symmetric monoidal category to one which is both skeletal and permutative.*

Our approach also reveals that the action of the truncated sphere spectrum on a stable one-type is present in the algebraic model. This is hinted at in the unstable literature [BCC93, GMdR02] but not described explicitly. We prove the following as Propositions 3.1, 3.3 and 3.4.

THEOREM C. *There is a Picard groupoid $\mathbb{S}$ which models the one-type of the sphere spectrum in the following sense:*

i. The Picard groupoid $\mathbb{S}$ is the free Picard groupoid on one object.

ii. The classifying space $B\mathbb{S}$ is the Postnikov 1-truncation of QS^0.

iii. Let $\mathscr{C}$ be a Picard groupoid. There is a natural action of $\mathbb{S}$ on $\mathscr{C}$ such that the induced action of $B\mathbb{S}$ on $B\mathscr{C}$ is equivalent to the action of the truncated sphere spectrum on $B\mathscr{C}$.

This work is a proving ground for a larger project joint with J.P. May which models stable two-types via symmetric monoidal bicategories. The top Postnikov invariant in that case lands in a stable 3-type, which should be modeled by a symmetric monoidal tricategory; one purpose of our program is to use this approach as leverage to understand symmetric monoidal structure on higher weak n-categories.

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1. Stable one-types

Let $\mathscr{S}$ denote any symmetric monoidal model category of spectra. We denote by $\mathscr{S}_0^1$ the full subcategory of $\mathscr{S}$ whose objects are spectra with all homotopy groups equal to zero except at levels 0 and 1. The objects of this category are called *stable one-types*. A map between stable one-types is a stable equivalence if it induces isomorphisms of homotopy groups.

1.1. DEFINITION. Let $(\mathscr{C}, \oplus, I)$ be a symmetric monoidal category. An object x is *invertible* if there exists an object y and an isomorphism

$$\varepsilon\colon y \oplus x \to I.$$

If such a y exists it is unique up to isomorphism. When one exists, we will sometimes use x^* to denote a specified inverse of x.

1.2. DEFINITION. A *Picard groupoid* $\mathscr{C}$ is a symmetric monoidal groupoid such that every object is invertible. The isomorphism classes of objects form an abelian group denoted $\pi_0\mathscr{C}$, and the endomorphisms of the identity object I form an abelian group denoted $\pi_1\mathscr{C}$.

1.3. Definition. The category $\mathcal{P}ic$ has as objects the Picard groupoids and as morphisms strong symmetric monoidal functors. For symmetric monoidal functors into groupoids, the notions of lax and strong monoidal coincide. Throughout the paper our monoidal functors are assumed to be strong monoidal. A symmetric monoidal functor is a *weak equivalence* if it is an equivalence of the underlying categories.

1.4. Remark. The term *Picard category* is used in some literature for what we call a Picard groupoid. Although some will interpret our terminology as redundant, we hope others will find it maximally comprehensible.

The classification of stable one-types by Picard groupoids appears explicitly and implicitly in various parts of the literature. For example, Patel [Pat12, §5] shows that there is an equivalence between the homotopy categories of stable one-types and Picard groupoids, making precise a "vauge idea" of Drinfeld [Dri06, §5.5]. This is also sketched by Hopkins-Singer in [HS05, §B] and Ganter-Kapranov in [GK11, §3]. An equivalent result of Bullejos-Carrasco-Cegarra appears in [BCC93, §5], where the authors prove that the homotopy category of spaces with nontrivial homotopy groups π_n and π_{n+1}, $n \geq 3$, is equivalent to the homotopy category of Picard groupoids. We give another proof of this result based on compatibility of the fundamental groupoid and classifying space functors with E_∞ actions.

1.5. Theorem. *There is an equivalence between the categories $\mathcal{H}o(\mathscr{S}_0^1)$ and $\mathcal{H}o(\mathcal{P}ic)$ induced by the fundamental groupoid and classifying space functors.*

Proof. We first recall that the homotopy category of connective spectra is equivalent to the homotopy category of group-like E_∞ spaces, and this equivalence descends to the category of stable one-types and the subcategory of group-like E_∞ spaces with no higher homotopy groups. Thus we can work in the context of E_∞ spaces.

It is a classical result that the classifying space and fundamental groupoid functors give an equivalence

$$\Pi_1 \colon \mathcal{H}o(\mathcal{T}op_0^1) \underset{\longleftarrow}{\overset{\simeq}{\longrightarrow}} \mathcal{H}o(\mathcal{G}pd) \colon B$$

where $\mathcal{T}op_0^1$ is the category of one-type spaces. Thus it suffices to show that Π_1 and B induce an equivalence between the homotopy categories of group-like E_∞ one-types and Picard groupoids.

Let $\mathscr{O}$ be the categorical Barrat-Eccles operad—its jth category $\mathscr{O}(j)$ is the translation groupoid of the action of Σ_j on itself and its algebras are permutative categories [May72]. Then $B\mathscr{O}$ is an E_∞ operad in $\mathcal{T}op$: If $\mathscr{C}$ is a symmetric monoidal category, then $B\mathscr{C}$ is an E_∞ space and if $\mathscr{C}$ is a Picard groupoid, then $B\mathscr{C}$ is a group-like E_∞ one-type. If $F \colon \mathscr{C} \to \mathscr{D}$ is a functor between Picard groupoids, then $BF \colon B\mathscr{C} \to B\mathscr{D}$ is an E_∞ map.

The operad $\Pi_1 B\mathscr{O}$ is an E_∞ operad in categories, and Π_1 preserves products. If X is an E_∞ one-type, then $\Pi_1 X$ is a symmetric monoidal groupoid. Moreover, $\pi_0 X \cong \pi_0 \Pi_1 X$, so $\Pi_1 X$ is a Picard category if X is group-like. If $f \colon X \to Y$ is a map of E_∞ spaces, then $\Pi_1 f$ is a symmetric monoidal functor.

Now consider the equivalence $\mathscr{C} \to \Pi_1 B\mathscr{C}$. Since this functor is part of a natural transformation of functors from $\mathcal{G}pd$ to itself, we have functors $\mathscr{O}(j) \times \mathscr{C}^j \to \Pi_1 B\mathscr{O}(j) \times (\Pi_1 B\mathscr{C})^j$ that commute with the structure maps of the algebras $\mathscr{C}$ and $\Pi_1 B\mathscr{C}$, thus showing that the equivalence $\mathscr{C} \to \Pi_1 B\mathscr{C}$ is a symmetric monoidal functor. A similar argument shows that for a stable one-type X, the weak equivalence $X \to B\Pi_1 X$ is an E_∞ map. ∎

1.6. Remark. One can actually show that the fundamental groupoid of an E_3 algebra is symmetric monoidal. Indeed, if $\mathscr{O}$ is an E_3 operad in $\mathcal{T}op$, then $\Pi_1\mathscr{O}$ is an E_∞ operad in $\mathcal{C}at$. This is because the fundamental groupoid depends only on the homotopy one-type of a space, and thus the obstructions to lifting an E_3 structure to an E_∞ structure on a groupoid vanish. Alternatively, one can provide an explicit argument using specific points of the little 3-cubes operad $\mathscr{C}_3$ to prove that if X is an algebra over $\mathscr{C}_3$ then $\Pi_1 X$ is a symmetric monoidal category. An example of this strategy can be found in [Gur11, Theorem 15], where the author proves that the fundamental 2-groupoid of an algebra over the little 2-cubes operad is braided monoidal.

2. Strictification

In this section we prove a strictification result for skeletal Picard groupoids. The result is an algebraic reflection of the fact that the first k-invariant of a connected double loop space is trivial [BC97, Theorem 5.8].

2.1. Definition. A Picard groupoid is *permutative* if it is strictly associative and strictly unital.

2.2. Theorem. *Every Picard groupoid is equivalent to one which is both skeletal and permutative.*

The proof appears after Proposition 2.5, which classifies Picard groupoids by symmetric 3-cocycles. Analogous results for stable crossed modules appear in [BC97].

2.3. Definition. [Symmetric 3-cocycle] Let G be an abelian group and M a trivial G-module. A *symmetric 3-cocycle* for G with coefficients in M is a pair (h, c) where h is a normalized 3-cocycle: for $x, y, z \in G$

$$h(x,0,z) = 0,$$
$$h(x,y,z) + h(u,x+y,z) + h(u,x,y) = h(u,x,y+z) + h(u+x,y,z),$$

and $c\colon G^2 \to M$ is a function satisfying

$$h(y,z,x) + c(x,y+z) + h(x,y,z) = c(x,z) + h(y,x,z) + c(x,y),$$
$$c(x,y) = -c(y,x).$$

We say two symmetric 3-cocycles (h, c) and (h', c') are cohomologous if there exists a function $k\colon G^2 \to M$ satisfying

$$\begin{aligned} k(x,0) &= k(0,y) = 0, \\ h(x,y,z) - h'(x,y,z) &= k(y,z) - k(x+y,z) + k(x,y+z) - k(x,y), \\ c(x,y) - c'(x,y) &= k(x,y) - k(y,x). \end{aligned}$$

We denote the group of cohomology classes of symmetric 3-cocycles by $H^3_{\mathrm{sym}}(G; M)$.

2.4. Definition. [Sín75, Chapter 2, §2],[JS93, §3] Let G be an abelian group, M a trivial G module, and (h, c) a symmetric 3-cocycle for G with coefficients in M. We define a skeletal Picard groupoid $\mathscr{T} = \mathscr{T}(G, M, (h, c))$ whose objects are the elements of G and whose morphisms are given by

$$\mathscr{T}(x,y) = \begin{cases} M & \text{if } x = y \\ \emptyset & \text{if } x \neq y. \end{cases}$$

Composition is defined by the addition in M and the monoidal structure is addition in G. The associativity is determined by

$$h(x,y,z)\colon (x+y)+z \to x+(y+z)$$

and the symmetry isomorphism is determined by

$$c(x,y)\colon x+y \to y+x.$$

The axioms of a symmetric 3-cocycle are precisely the axioms for compatibility of the symmetry and associativity in a skeletal symmetric monoidal groupoid.

2.5. Proposition. [Sín75, Chapter II, §2.1],[JS93, §3] *Every Picard groupoid $\mathscr{C}$ is equivalent to a skeletal one of the form $\mathscr{T}(G, M, (h, c))$ where $G = \pi_0\mathscr{C}$, $M = \pi_1\mathscr{C}$, and $(h, c) \in H^3_{sym}(G; M)$ is a symmetric 3-cocycle that represents the associativity and the symmetry.*

Two skeletal Picard groupoids $\mathscr{T}(G, M, (h, c))$ and $\mathscr{T}(G, M, (h', c'))$ are equivalent if and only if (h, c) and (h', c') are cohomologous.

Proof of Theorem 2.2. By Proposition 2.5, it suffices to consider a skeletal Picard groupoid $\mathscr{C} = \mathscr{T}(G, M, (h, c))$. Moreover, any abelian group G is a filtered colimit of finitely generated abelian groups, and any finitely generated abelian group is a direct sum of cyclic groups. Thus, by making use of the Künneth theorem and colimits over finitely generated abelian groups, it suffices to consider the case where G is cyclic. This strategy for studying Picard groupoids appears in [EM54, 26.4] and [JS93, 3.2].

Let G be a cyclic group. We now prove that there is some c' such that $[(h, c)] = [(0, c')]$ in $H^3_{\mathrm{sym}}(G; M)$. This is immediate in the infinite cyclic case since $H^3(\mathbb{Z}; M) = 0$. Now suppose $G = \mathbb{Z}/n$. The third cohomology group is

$$H^3(\mathbb{Z}/n; M) \cong \{\mu \in M | n\mu = 0\}.$$

Following Joyal and Street [JS93, §3] we have an explicit formula for cocycle representatives corresponding to μ:

$$h_\mu(x,y,z) = \begin{cases} 0 & \text{for } y+z<n \\ x\mu & \text{for } y+z \geq n \end{cases}$$

where x, y, z are taken to be integers in $\{0,\ldots,n-1\}$ and addition is performed over the integers to determine the values of h_μ.

A calculation shows

$$(h,c) \sim (h_{nc(1,1)}, \rho_{c(1,1)}), \tag{2.1}$$

where $\rho_{c(1,1)}$ denotes the symmetry given by

$$(x,y) \mapsto xy \cdot c(1,1).$$

This equivalence of cocycles determines a symmetric monoidal equivalence of the corresponding symmetric monoidal categories. Since the braiding on $\mathscr{C}$ is a symmetry, we have $c(x,y) = -c(y,x)$.

Now note that if (h,c) is a symmetric 3-cocycle, then $nc(1,1) = 0$ by Lemma 2.6, so $h_{nc(1,1)} = h_0 = 0$. Therefore Eq. (2.1) shows that $\mathscr{C}$ is equivalent (as a Picard groupoid) to one whose representing cocycle is $(0, \rho_c(1,1))$ and thus is both skeletal and permutative. ∎

2.6. Lemma. *If (h,c) is a symmetric 3-cocycle of $\mathbb{Z}/n$ with coefficients in M, then $nc(1,1) = 0$.*

Proof. Since c is symmetric, $c(1,1) = -c(1,1)$, and thus $2c(1,1) = 0$. If n is even, then the result follows; if n is odd, we make use of the identity $c(x,x) = x^2c(1,1)$ for all $x \in \mathbb{Z}/n$:

$$c(1,1) = (-1)^2c(1,1) = c(-1,-1) = c(n-1,n-1) = (n-1)^2c(1,1).$$

For n odd, $n-1$ is even and thus the last term is zero. ∎

The calculation of Eq. (2.1) shows that the symmetry c completely determines the cohomology class of the symmetric 3-cocycle (h,c) of a skeletal Picard groupoid. But [JS93, §3] shows that the symmetry determines and is determined by the quadratic map

$$q = c \circ \Delta \colon G \to M.$$

2.7. Definition. [Quadratic map] A map $q\colon G \to M$ is *quadratic* if

$$q(x) = q(-x),$$
$$q(x+y+z) + q(x) + q(y) + q(z) = q(y+z) + q(z+x) + q(x+y).$$

Eilenberg and Mac Lane [EM54], and Loday [Lod82] show that the set of quadratic maps $q : G \to M$ is isomorphic to the set of homotopy classes of maps $[K(G,n), K(M,n+2)]$ for $n \geq 3$, which is the set of stable homotopy classes of maps $[K(G,0), K(M,2)]_{\text{stable}}$. This is the set of possible Postnikov invariants of a stable one-type with $\pi_0 = G$ and $\pi_1 = M$. Thus we have the following refinement of Theorem 1.5:

2.8. COROLLARY. *The stable one-types with $\pi_0 = G$ and $\pi_1 = M$ are classified by the symmetric structures on a skeletal and permutative monoidal groupoid with objects G and each endomorphism group isomorphic to M.*

2.9. REMARK. The contrast between triviality of unstable k-invariants and non-triviality of stable k-invariants may be worth clarifying: When modeling connected spaces with nontrivial π_1 and π_2, it is the associativity of a monoidal groupoid (with invertible objects) which gives the first k-invariant of the corresponding space. However when modeling spectra with nontrivial π_0 and π_1 it is the *symmetry* of a Picard groupoid which gives the first (stable) k-invariant. A consequence of Theorem 1.5, Corollary 2.8, and [BC97, Theorem 5.8] is that the first k-invariant of a stable one-type is unstably trivial.

3. The truncated sphere spectrum

We now define a skeletal and permutative Picard groupoid $\mathbb{S}$ and explain how it is an algebraic model of the truncated sphere spectrum. The objects of $\mathbb{S}$ are the integers under addition, and the morphisms are given by

$$\mathbb{S}(m,n) = \begin{cases} \emptyset & \text{if } m \neq n \\ \mathbb{Z}/2 & \text{if } m = n. \end{cases}$$

We let η_n denote the nontrivial element of $\mathbb{S}(n,n)$ for each n. The monoidal structure is symmetric, with the symmetry isomorphism given by

$$c_{m,n} = \begin{cases} 0 & \text{if } mn \text{ is even} \\ \eta_{m+n} & \text{if } mn \text{ is odd.} \end{cases}$$

Note that this symmetry isomorphism gives rise to the stable quadratic map $q : \mathbb{Z} \to \mathbb{Z}/2$ given by the mod 2 map.

The Picard groupoid $\mathbb{S}$ is closely related to the category of finite sets, as we now describe. Let $\mathscr{E}$ be the skeletal category whose objects are the finite sets $\underline{0} = \emptyset$, $\underline{n} = \{1, 2, \ldots, n\}$ and whose morphism sets are given by the symmetric groups. This is a permutative category, with sum given by sum in $\mathbb{N}$ and with symmetry isomorphism $c^{\oplus}_{m,n}$ given by the permutation that sends $(1, 2, \ldots, m+n)$ to $(m+1, m+2, \ldots, m+n, 1, 2, \ldots, m)$. Note that $\mathscr{E}$ is skeletal and is equivalent to the category of finite sets.

There is a symmetric monoidal functor

$$\xi : \mathscr{E} \to \mathbb{S}$$

given on objects by the inclusion of $\mathbb{N}$ into $\mathbb{Z}$ and on morphisms by the sign homomorphism $\Sigma_n \to \mathbb{Z}/2$. This is the functor that first abelianizes the group of endomorphisms of each object $\underline{n}$, and then includes into $\mathbb{S}$.

The next three results justify our notation for $\mathbb{S}$ by showing that it is the free Picard groupoid on one object, its classifying space is the Postnikov 1-truncation of QS^0, and its natural action on a Picard groupoid $\mathscr{C}$ is a model for the action of the truncated sphere spectrum on $B\mathscr{C}$.

3.1. Proposition. *The Picard groupoid* $\mathbb{S}$ *is symmetric monoidally equivalent to the free Picard groupoid on one object,* $\mathscr{F}_{Pic}(*)$.

Proof. The free Picard groupoid functor $\mathscr{F}_{Pic}$ is equal to the composite of the free symmetric monoidal groupoid functor, $\mathscr{F}_{symMon}$, with the functor that freely adjoins inverses for objects, $\mathscr{F}_{inv}$. The free symmetric monoidal category on one object, $\mathscr{F}_{symMon}(*)$, is symmetric monoidally equivalent to the category $\mathscr{E}$ defined above.

We now show that $\mathbb{S}$ satisfies the universal property for $\mathscr{F}_{inv}(\mathscr{E})$: Let $\mathscr{C}$ be a Picard category and $G : \mathscr{E} \to \mathscr{C}$ a symmetric monoidal functor. We construct a symmetric monoidal functor H making the diagram commute:

$$\begin{tikzcd} \mathscr{E} \arrow[r, "G"] \arrow[d, "\xi"'] & \mathscr{C}. \\ \mathbb{S} \arrow[ur, dashed, "H"'] & \end{tikzcd}$$

For every object $x \in \mathscr{C}$ fix an inverse x^*. We define H on objects as

$$H(n) = \begin{cases} G(n) & \text{if } n \geq 0 \\ G(|n|)^* & \text{if } n < 0. \end{cases}$$

To define H on morphisms, note that $\mathscr{C}(x, x)$ is an abelian group for all $x \in \mathscr{C}$ and therefore G factors through the abelianization of $\mathscr{E}(n, n)$ and hence through ξ. This factorization determines H on the endomorphism group of n for $n \geq 0$, and the values of H on endomorphisms of negative n are determined by translation. It is easy to see that H is a symmetric monoidal functor.

Now let H' be another symmetric monoidal functor making the diagram commute. Note that for $n \geq 0$, we must have $H(n) = G(n) = H'(n)$. On the other hand we have natural isomorphisms

$$H(n) \oplus H(-n) \xrightarrow{\cong} I_{\mathscr{C}} \xleftarrow{\cong} H'(n) \oplus H'(-n)$$

and hence natural isomorphisms

$$H(-n) \xrightarrow{\cong} H(n)^* = H'(n)^* \xleftarrow{\cong} H'(-n).$$

These assemble to form a monoidal natural isomorphism between H and H'. ∎

3.2. Remark. Although a model for one-types of ring spectra is beyond the scope of this paper, we do note that $\mathbb{S}$ has a second symmetric monoidal structure, so that it is a bipermutative groupoid. This second monoidal structure is given by:

$$(m, n) \mapsto mn \in \mathbb{Z}$$
$$(f : m \to m, g : n \to n) \mapsto nf + mg \in \mathbb{Z}/2,$$

The symmetry isomorphism is given by

$$c^{\otimes}_{m,n} = \begin{cases} 0 & \text{if } \binom{m}{2}\binom{n}{2} \text{ is even} \\ \eta_{nm} & \text{if } \binom{m}{2}\binom{n}{2} \text{ is odd.} \end{cases}$$

By [May09], $B\mathbb{S}$ is an E_∞ ring space.

Furthermore, the category $\mathscr{E}$ described above is a bipermutative category, with second product given by multiplication in $\mathbb{N}$. This models the cartesian product of finite sets. The map $\xi : \mathscr{E} \to \mathbb{S}$ is a bipermutative functor, and $B\xi : B\mathscr{E} \to B\mathbb{S}$ is therefore an E_∞ ring map.

3.3. Proposition. *Let QS^0 be the zeroth space of the sphere spectrum. Then there is a map of E_∞ ring spaces*

$$\overline{B\xi} : QS^0 \longrightarrow B\mathbb{S}$$

which is the Postnikov 1-truncation of QS^0.

Proof. By Remark 3.2, we have a map of E_∞ ring spaces $B\xi : B\mathscr{E} \to B\mathbb{S}$. Since $B\mathbb{S}$ is group-like, this map factors through the group completion of $B\mathscr{E}$, which is equivalent to QS^0:

$$\begin{tikzcd} B\mathscr{E} \arrow[r, "B\xi"] \arrow[d] & B\mathbb{S}. \\ QS^0 \arrow[ur, "\overline{B\xi}"'] & \end{tikzcd}$$

The map $\overline{B\xi}$ is an isomorphism on $\pi_0 = \mathbb{Z}$, $\pi_1 = \mathbb{Z}/2$, and thus it is the Postnikov 1-truncation. ■

Let $(\mathscr{C}, \oplus, I)$ be any Picard groupoid. By Theorem 2.2, we can assume without loss of generality that $\mathscr{C}$ is both skeletal and permutative. Then each object x in $\mathscr{C}$ has a strict inverse, x^*, so that $x \oplus x^* = I = x^* \oplus x$. There is a natural action of $\mathbb{S}$ on $\mathscr{C}$

$$\mathbb{S} \times \mathscr{C} \xrightarrow{\cdot} \mathscr{C}$$

defined on objects as follows:

$$\begin{aligned} &0 \times x \mapsto I \\ &1 \times x \mapsto x \\ &n \times x \mapsto ((n-1)\cdot x) \oplus x && \text{for } n > 1 \\ &n \times x \mapsto |n| \cdot x^* && \text{for } n < 0. \end{aligned}$$

Let c denote the symmetry of $\mathscr{C}$. The action $\cdot$ on morphisms is defined by:

$$\begin{aligned} &\eta_2 \times 1_x \mapsto c(x,x), \\ &\eta_n \times 1_x \mapsto c(x,x) \oplus 1_{(n-2)\cdot x} && \text{for } n \neq 2. \end{aligned}$$

3.4. Proposition. *Let X be a stable one-type modeled by a Picard groupoid $\mathscr{C}$, so $B\mathscr{C} \simeq X$. Then the action of the truncated sphere spectrum on X is modeled by the action of $\mathbb{S}$ on $\mathscr{C}$.*

Proof. The action of $\mathbb{S}$ on $\mathscr{C}$ passes to an action of $B\mathbb{S}$ on $B\mathscr{C}$ which is homotopic to that of the group completion QS^0: The top triangle in the diagram below commutes because $B\mathscr{C}$ is group complete; the bottom commutes because the group completion abelianizes π_1 and hence the action of even permutations (the alternating group) is trivial.

$$\begin{tikzcd}
B\mathscr{E} \times B\mathscr{C} \arrow[rr] \arrow[d] & & B\mathscr{C} \\
QS^0 \times B\mathscr{C} \arrow[urr] \arrow[d] & & \\
B\mathbb{S} \times B\mathscr{C} \arrow[uurr] & &
\end{tikzcd}$$

■

3.5. Remark. An alternate argument for Proposition 3.4 notes that the action of the truncated sphere spectrum on $B\mathscr{C}$ determines and is determined by the unique nontrivial Postnikov invariant

$$K(\pi_0 B\mathscr{C}, 0) \xrightarrow{k_0} K(\pi_1 B\mathscr{C}, 2)$$

which is given by precomposition with η. The discussion preceding Corollary 2.8 shows that this Postnikov invariant is modeled by the stable quadratic map $q\colon \pi_0\mathscr{C} \to \pi_1\mathscr{C}$ given by $q(x) = c(x,x)$. This, in turn, determines and is determined by the action of $\mathbb{S}$ on $\mathscr{C}$ since η_2 acts by the symmetry c. In Section 4.6 we define the Postnikov invariant of a Picard groupoid and show that it models the Postnikov invariant of $B\mathscr{C}$ (Corollary 4.8).

4. Cokernels of Picard groupoid maps

Here we describe the cokernel of a map of Picard groupoids and the resulting exact sequence in homotopy groups.

4.1. Definition. [Bigroupoid] A *bigroupoid* is a bicategory $\mathscr{G}$ in which the 1-cells are invertible up to 2-isomorphism and the 2-cells are isomorphisms. The set $\pi_0\mathscr{G}$ is given by the equivalence classes of objects. For an object $x \in \mathscr{G}$, the group $\pi_1(\mathscr{G}, x)$ is given by the isomorphism classes of 1-endomorphisms of x. The group $\pi_2(\mathscr{G}, x)$ is given by the 2-endomorphisms of 1_x, the identity 1-cell of x.

4.2. Definition. [Cokernel [Vit02]] Let $F\colon \mathscr{C} \to \mathscr{D}$ be a map of Picard groupoids. The cokernel of F is a bigroupoid $\mathrm{Coker}(F)$ defined as follows: The objects of $\mathrm{Coker}(F)$ are the objects of $\mathscr{D}$. The 1-cells between objects x and y are pairs (f, n), where

$$x \xrightarrow{f} y \oplus F(n)$$

is a morphism of $\mathscr{D}$. The 2-cells between (f,n) and (f',n') are given by morphisms $\alpha\colon n \to n'$ of $\mathscr{C}$ such that the following diagram commutes:

$$\begin{tikzcd} & x \arrow[dl, "f"'] \arrow[dr, "f'"] & \\ y \oplus F(n) \arrow[rr, "1\oplus F(\alpha)"'] & & y \oplus F(n') \end{tikzcd}$$

The composite of two 1-cells

$$(f,n)\colon x \to y \qquad \text{and} \qquad (g,m)\colon y \to z$$

is given by the following composite morphism in $\mathscr{D}$:

$$x \xrightarrow{f} y \oplus F(n) \xrightarrow{g\oplus 1} (z \oplus F(m)) \oplus F(n) \to z \oplus F(m \oplus n).$$

Further details of the definition can be found in [Vit02, §2]; note that the cokernel is denoted $\mathrm{Cok}\,(F)$ there.

There is a natural pseudofunctor $C_F : \mathscr{D} \to \mathrm{Coker}(F)$ which is the identity on objects and which takes a morphism $f\colon x \to y$ to the 1-cell $\widehat{f} = (f, I_{\mathscr{C}})$ determined by the morphism

$$x \xrightarrow{f} y \to y \oplus I_{\mathscr{D}} \to y \oplus F(I_{\mathscr{C}}).$$

4.3. Theorem. *The symmetric monoidal structure on $\mathscr{D}$ induces a symmetric monoidal structure on the bicategory* $\mathrm{Coker}(F)$. *The pseudofunctor C_F is symmetric monoidal.*

We prove Theorem 4.3 in Section 5. In the remainder of this section we apply this cokernel to model stable cofibers and Postnikov invariants.

4.4. Theorem. *A map of Picard groupoids $F\colon \mathscr{C} \to \mathscr{D}$ gives rise to a long exact sequence of homotopy groups between $\mathscr{C}$, $\mathscr{D}$, and* $\mathrm{Coker}(F)$

$$0 \to \pi_2\,\mathrm{Coker}(F) \to \pi_1\mathscr{C} \to \pi_1\mathscr{D} \to \pi_1\,\mathrm{Coker}(F) \to \pi_0\mathscr{C} \to \pi_0\mathscr{D} \to \pi_0\,\mathrm{Coker}(F) \to 0.$$

Proof. Exactness at most positions is verified by [Vit02], noting that the $\mathrm{Ker}(F)$ used there has $\pi_0\,\mathrm{Ker}(F) \cong \pi_1\,\mathrm{Coker}(F)$ and $\pi_1\,\mathrm{Ker}(F) \cong \pi_2\,\mathrm{Coker}(F)$. Exactness at the remaining positions, $\pi_1\mathscr{D}$ and at $\pi_1\,\mathrm{Coker}(F)$, is straightforward from the definitions: An element in $\pi_1\mathscr{C}$ is represented by a morphism $f : I_{\mathscr{C}} \to I_{\mathscr{C}}$. The image of this element in $\pi_1\mathscr{D}$ is represented by the composite

$$I_{\mathscr{D}} \xrightarrow{\cong} F(I_{\mathscr{C}}) \xrightarrow{F(f)} F(I_{\mathscr{C}}) \xrightarrow{\cong} I_{\mathscr{D}}.$$

This composite morphism maps to the trivial element in $\pi_1\,\mathrm{Coker}(F)$ because it factors through a morphism in the image of F (namely, $F(f)$). Likewise, if $g : F(X) \to I_{\mathscr{D}}$ represents an element of $\pi_1\mathscr{D}$ whose image in $\pi_1\,\mathrm{Coker}(F)$ is trivial (factors through a morphism in the image of F), then the trivialization provides an element of $\pi_1\mathscr{C}$ whose image in $\pi_1\mathscr{D}$ is the element represented by g. Exactness at $\pi_1\,\mathrm{Coker}(F)$ is similar, and left to the reader. ∎

The two previous results, together with [Oso12, GO] show that the cokernel of Picard groupoids models the cofiber of stable one-types:

4.5. Corollary. *Let $F : \mathscr{C} \to \mathscr{D}$ be a map of Picard groupoids. Then the following is a cofibration sequence of group-like E_∞ spaces:*

$$B\mathscr{C} \to B\mathscr{D} \to B\,\mathrm{Coker}(F).$$

Proof. Note that since $\mathrm{Coker}(F)$ is a group-like symmetric monoidal bicategory, [Oso12, Theorem 2.1] and the improved results of [GO] imply that $B\,\mathrm{Coker}(F)$ is a group-like E_∞ space. Let C be the cofiber of the map on classifying spaces. Then the dashed arrow to $B\,\mathrm{Coker}(F)$ exists by the universal property of C, and it is an equivalence by Theorem 4.4.

$$\begin{tikzcd} B\mathscr{C} \arrow[r] & B\mathscr{D} \arrow[r] \arrow[dr] & C \arrow[d, dashed, "\simeq"] \\ & & B\,\mathrm{Coker}(F) \end{tikzcd}$$

∎

4.6. Modeling Postnikov invariants

4.7. Definition. Let $\mathscr{C}$ be a Picard groupoid, and let $\mathscr{C}_0$ be the category of isomorphism classes of objects of $\mathscr{C}$, with only identity morphisms. Let

$$\alpha_0 : \mathscr{C} \to \mathscr{C}_0$$

be the monoidal functor which takes each object to its isomorphism class and takes morphisms to identity morphisms. Let $k_0 = C_{\alpha_0}$ be the natural pseudofunctor from $\mathscr{C}_0$ to $\mathrm{Coker}(\alpha_0)$. We call the sequence

$$\mathscr{C} \xrightarrow{\alpha_0} \mathscr{C}_0 \xrightarrow{k_0} \mathrm{Coker}(\alpha_0)$$

the *Postnikov tower* of $\mathscr{C}$.

By Theorem 4.4, $\mathrm{Coker}(\alpha_0)$ has only one non-trivial homotopy group, which is $\pi_1\mathscr{C}$ in degree two. We refer to k_0 as the Postnikov invariant of $\mathscr{C}$. Our terminology is motivated by the following result.

4.8. Corollary. *The Postnikov tower of $\mathscr{C}$ models the Postnikov tower of $B\mathscr{C}$.*

Proof. This follows immediately from Corollary 4.5 and the fact that $B\mathscr{C}_0 \simeq K(\pi_0, 0)$.

$$\begin{tikzcd} B\mathscr{C} \arrow[r] \arrow[d, equal] & K(\pi_0,0) \arrow[r] \arrow[d, "\simeq"] & K(\pi_1,2) \arrow[d, dashed, "\simeq"] \\ B\mathscr{C} \arrow[r, "B\alpha_0"] & B\mathscr{C}_0 \arrow[r, "Bk_0"] & B\,\mathrm{Coker}(\alpha_0) \end{tikzcd}$$

∎

5. Proof of Theorem 4.3

5.1. Proposition. *The bicategory* Coker(F) *is symmetric monoidal.*

To prove this proposition we will construct a double category $\mathbb{C}\text{oker}(F)$ and use the results of [Shu10], which are analogous to those of [GG09, §6]. The idea behind this method is that it is usually easier to construct symmetric monoidal double categories than symmetric monoidal bicategories, and for certain double categories the symmetric monoidal structure lifts to a symmetric monoidal structure in a related bicategory.

The double category $\mathbb{C}\text{oker}(F)$ is constructed as follows. The category of objects, $\mathbb{C}\text{oker}(F)_0$, is $\mathscr{D}$. The category of morphisms, $\mathbb{C}\text{oker}(F)_1$, has as objects the quadruples (x, y, f, n), where x and y are objects of $\mathscr{D}$, n is an object of $\mathscr{C}$ and $f : x \to y \oplus F(n)$ is a morphism in $\mathscr{D}$.

A morphism in $\mathbb{C}\text{oker}(F)_1$ from (x, y, f, n) to (z, v, g, m) is given by a triple (a, b, α), where $a : x \to z$ and $b : y \to v$ are morphisms in $\mathscr{D}$, and $\alpha : n \to m$ is a morphism in $\mathscr{C}$, such that the following diagram commutes

$$\begin{tikzcd} x \arrow[r, "f"] \arrow[d, "a"'] & y \oplus F(n) \arrow[d, "b\oplus F(\alpha)"] \\ z \arrow[r, "g"'] & v \oplus F(m) \end{tikzcd}$$

Composition of morphisms in the category $\mathbb{C}\text{oker}(F)_1$ is given by composition componentwise. We follow the notation from [Shu10, Def. 2.1] to define the rest of the structure of the double category.

The unit functor U is defined as

$$\begin{aligned} U : \mathbb{C}\text{oker}(F)_0 &\longrightarrow \mathbb{C}\text{oker}(F)_1 \\ x &\longmapsto (x, x, \widehat{1}_x, I_{\mathscr{C}}) \\ x \xrightarrow{a} z &\longmapsto (a, a, 1_{I_{\mathscr{C}}}), \end{aligned}$$

where $\widehat{1}_x$ is given by the composition $x \to x \oplus I_{\mathscr{D}} \to x \oplus F(I_{\mathscr{C}})$. The functors for source and target, S and T, are given by:

$$\begin{aligned} S, T : \mathbb{C}\text{oker}(F)_1 &\longrightarrow \mathbb{C}\text{oker}(F)_0 \\ (x, y, f, n) &\longmapsto x, y \\ (a, b, \alpha) &\longmapsto a, b. \end{aligned}$$

Finally, the composition functor is given by

$$\begin{aligned} \odot : \mathbb{C}\text{oker}(F)_1 \times_{\mathbb{C}\text{oker}(F)_0} \mathbb{C}\text{oker}(F)_1 &\longrightarrow \mathbb{C}\text{oker}(F)_1 \\ [(x, y, f, n), (y, z, g, m)] &\longmapsto (x, z, f \bullet g, m \oplus n) \\ [(a, b, \alpha), (b, c, \beta)] &\longmapsto (a, c, \beta \oplus \alpha), \end{aligned}$$

where $f \bullet g$ denotes the composition

$$x \xrightarrow{f} y \oplus F(z) \xrightarrow{g \oplus 1} (z \oplus F(m)) \oplus F(n) \to z \oplus F(m \oplus n).$$

The associativity and unit constraints come from those in the monoidal structure of $\mathscr{C}$.

5.2. Proposition. *The double category* $\mathbb{C}\mathrm{oker}(F)$ *is symmetric monoidal.*

Proof. The category $\mathbb{C}\mathrm{oker}(F)_0 = \mathscr{D}$ is symmetric monoidal. We now give a symmetric monoidal structure to $\mathbb{C}\mathrm{oker}(F)_1$. On objects it is given by:

$$(x, y, f, n) \oplus (z, v, g, m) = (x \oplus z, y \oplus v, f \star g, n \oplus m),$$

where $f \star g$ is the composition

$$x \oplus z \xrightarrow{f \oplus g} (y \oplus F(n)) \oplus (v \oplus F(m)) \to (y \oplus v) \oplus (F(n) \oplus F(m)) \to (y \oplus v) \oplus F(n \oplus m).$$

On morphisms it is defined by applying the sum componentwise. The associativity, unit, and symmetry constraints are inherited from those in $\mathscr{C}$ and $\mathscr{D}$.

The globular isomorphism

$$\begin{array}{c} ((x, y, f, n) \oplus (y, z, g, m)) \odot ((x', y', f', n') \oplus (y', z', g', m')) \\ \Big\downarrow \mathfrak{x} \\ ((x, y, f, n) \odot (x', y', f', n')) \oplus ((y, z, g, m) \odot (y', z', g', m')) \end{array}$$

is given by the structural isomorphism in $\mathscr{C}$

$$(m \oplus m') \oplus (n \oplus n') \to (m \oplus n) \oplus (m' \oplus n').$$

The globular morphism $\mathfrak{u} : U_{x \oplus y} \to U_x \oplus U_y$ is given by the morphism $I \to I \oplus I$ in $\mathscr{C}$.

It is clear that all the necessary diagrams commute since they all involve compositions of morphisms of the symmetric monoidal structures on $\mathscr{C}$ and $\mathscr{D}$. ∎

We recall that a double category is *fibrant* in the terminology of [Shu10] if every vertical 1-morphism has a companion and a conjoint. These are horizontal 1-morphisms that allow transport of vertical structure to horizontal structure [Shu10, §3]:

5.3. Definition. Let $a : x \to z$ be a morphism in $\mathscr{D}$. A *companion* for a is given by $(x, z, \hat{a}, I_{\mathscr{C}})$, where $\hat{a}$ is the composite

$$x \xrightarrow{a} z \to z \oplus I_{\mathscr{D}} \to z \oplus F(I_{\mathscr{C}}).$$

The following diagrams commute and therefore the equations of [Shu10, 3.1] are trivially satisfied.

$$\begin{tikzcd} x \arrow[r, "\hat{a}"] \arrow[d, "a"'] & z \oplus F(I_{\mathscr{C}}) \arrow[d, equal] \\ z \arrow[r, "\widehat{1}_z"'] & z \oplus F(I_{\mathscr{C}}) \end{tikzcd} \qquad \begin{tikzcd} x \arrow[r, "\widehat{1}_x"] \arrow[d, equal] & x \oplus F(I_{\mathscr{C}}) \arrow[d, "a"] \\ x \arrow[r, "\hat{a}"] & z \oplus F(I_{\mathscr{C}}) \end{tikzcd}$$

A *conjoint* for a is given by $(z, x, \check{a}, I_{\mathscr{C}})$, where $\check{a}$ is the composite

$$z \xrightarrow{a^{-1}} x \to x \oplus I_{\mathscr{D}} \to x \oplus F(I_{\mathscr{D}}).$$

This is the companion of a in the double category obtained from $\mathbb{C}\mathrm{oker}(F)$ by taking the same category of objects and the opposite category of morphisms.

Proof of Proposition 5.1. The horizontal bicategory $\mathscr{H}(\mathbb{C}\mathrm{oker}(F))$ is precisely $\mathrm{Coker}(F)$. Since every morphism in $\mathscr{D}$ has a companion and a conjoint, $\mathbb{C}\mathrm{oker}(F)$ is a fibrant double category. Therefore by [Shu10, Thm 5.1] $\mathrm{Coker}(F)$ is symmetric monoidal. ∎

5.4. Remark. Vitale [Vit02] points out that $\mathrm{Coker}(F)$ is a bigroupoid. We note, moreover, that the objects are weakly invertible since they are the objects of $\mathscr{D}$ with the same monoidal structure. Thus $\mathrm{Coker}(F)$ is what one might call a *Picard bigroupoid.*

5.5. Proposition. *The pseudofunctor* $C_F : \mathscr{D} \to \mathrm{Coker}(F)$ *is symmetric monoidal.*

Proof. We need to specify transformations

$$(\chi_{x,y}, \chi_{f,g}) : C_F(x) \oplus C_F(y) \to C_F(x \oplus y)$$

and

$$\iota : I \to C_F(I).$$

We let $\chi_{x,y} : x \oplus y \to x \oplus y$ be the identity 1-cell in $\mathrm{Coker}(F)$, that is, $\widehat{1}_{x \oplus y}$. The 2-cell

$$\chi_{f,g} : \widehat{f \oplus g} \circ \widehat{1}_{x \oplus y} \Rightarrow \widehat{1}_{x' \oplus y'} \circ (\widehat{f} \oplus \widehat{g})$$

is given by the unique structural morphism in $\mathscr{C}$, $I \oplus I \to I \oplus (I \oplus I)$. It is an easy verification that this is a valid 2-cell in $\mathrm{Coker}(F)$, and that these data forms a transformation.

Similarly, we let $\iota : I \to I$ be the identity 1-cell. The rest of the data for a symmetric monoidal pseudofunctor consists of four modifications, which are collections of 2-cells. In the four cases, the source and target of the modifications have products of copies of the unit $I \in \mathscr{C}$ as their second component, and hence the modifications are given by the unique structural morphism connecting these two products in $\mathscr{C}$. The coherence of the symmetric monoidal structure on $\mathscr{C}$ therefore implies that these modifications satisfy all of the necessary equations. ∎

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