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CLOSED CATEGORIES VS. CLOSED MULTICATEGORIES

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ABSTRACT. We prove that the 2-category of closed categories of Eilenberg and Kelly is equivalent to a suitable full 2-subcategory of the 2-category of closed multicategories.

1. Introduction

The notion of closed category was introduced by Eilenberg and Kelly [2]. It is an axiomatization of the notion of category with internal function spaces. More precisely, a closed category is a category $\mathcal{C}$ equipped with a functor $\underline{\mathcal{C}}(-,-) : \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathcal{C}$, called the *internal* Hom-*functor*; an object $\mathbb{1}$ of $\mathcal{C}$, called the *unit object*; a natural isomorphism $i_X : X \xrightarrow{\sim} \underline{\mathcal{C}}(\mathbb{1}, X)$, and natural transformations $j_X : \mathbb{1} \to \underline{\mathcal{C}}(X, X)$ and $L^X_{YZ} : \underline{\mathcal{C}}(Y,Z) \to \underline{\mathcal{C}}(\underline{\mathcal{C}}(X,Y), \underline{\mathcal{C}}(X,Z))$. These data are to satisfy five axioms; see Definition 2.1 for details.

A wide class of examples is provided by closed monoidal categories. We recall that a monoidal category $\mathcal{C}$ is called *closed* if for each object X of $\mathcal{C}$ the functor $X \otimes -$ admits a right adjoint $\underline{\mathcal{C}}(X,-)$; i.e, there exists a bijection $\mathcal{C}(X \otimes Y, Z) \cong \mathcal{C}(Y, \underline{\mathcal{C}}(X,Z))$ that is natural in both Y and Z. Equivalently, a monoidal category $\mathcal{C}$ is closed if and only if for each pair of objects X and Z of $\mathcal{C}$ there exist an *internal* Hom-*object* $\underline{\mathcal{C}}(X,Z)$ and an *evaluation* morphism $\mathrm{ev}^{\mathcal{C}}_{X,Z} : X \otimes \underline{\mathcal{C}}(X,Z) \to Z$ satisfying the following universal property: for each morphism $f : X \otimes Y \to Z$ there exists a unique morphism $g : Y \to \underline{\mathcal{C}}(X,Z)$ such that $f = \mathrm{ev}^{\mathcal{C}}_{X,Z} \circ (1_X \otimes g)$. One can check that the map $(X,Z) \mapsto \underline{\mathcal{C}}(X,Z)$ extends uniquely to a functor $\underline{\mathcal{C}}(-,-) : \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathcal{C}$, which together with certain canonically chosen transformations i_X, j_X, and L^X_{YZ} turns $\mathcal{C}$ into a closed category.

While closed monoidal categories are in prevalent use in mathematics, arising in category theory, algebra, topology, analysis, logic, and theoretical computer science, there are also important examples of closed categories that are not monoidal. The author's motivation stemmed from the theory of A_∞-categories.

The notion of A_∞-category appeared at the beginning of the nineties in the work of Fukaya on Floer homology [3]. However its precursor, the notion of A_∞-algebra, was introduced in the early sixties by Stasheff [13]. It as a linearization of the notion of A_∞-space, a topological space equipped with a product operation which is associative up to homotopy, and the homotopy which makes the product associative can be chosen so that it satisfies a collection of higher coherence conditions. Loosely speaking, A_∞-categories

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are to A_∞-algebras what linear categories are to algebras. On the other hand, A_∞-categories generalize differential graded categories. Unlike in differential graded categories, in A_∞-categories composition need not be associative on the nose; it is only required to be associative up to a homotopy that satisfies a certain equation up to another homotopy, and so on.

Many properties of A_∞-categories follow from the discovery, attributed to Kontsevich, that for each pair of A_∞-categories $\mathcal{A}$ and $\mathcal{B}$ there is a natural A_∞-category $A_\infty(\mathcal{A}, \mathcal{B})$ with A_∞-functors from $\mathcal{A}$ to $\mathcal{B}$ as its objects. These A_∞-categories of A_∞-functors were also investigated by many other authors, e.g. Fukaya [4], Lefèvre-Hasegawa [10], and Lyubashenko [12]; they allow us to equip the category of A_∞-categories with the structure of a closed category.

In the recent monograph by Bespalov, Lyubashenko, and the author [1] the theory of A_∞-categories is developed from a slightly different perspective. Our approach is based on the observation that although the category of A_∞-categories is not monoidal, there is a natural notion of A_∞-functor of many arguments, and thus A_∞-categories form a *multicategory*.

The notion of multicategory (known also as colored operad or pseudo-tensor category) was introduced by Lambek [7, 8]. It is a many-object version of the notion of operad. If morphisms in a category are considered as analogous to functions, morphisms in a multicategory are analogous to functions in several variables. An arrow in a multicategory looks like $X_1, X_2, \dots, X_n \to Y$, with a finite sequence of objects as the domain and one object as the codomain. The most familiar example of multicategory is the multicategory of vector spaces and multilinear maps.

Multicategories generalize monoidal categories: a monoidal category $\mathcal{C}$ gives rise to a multicategory $\widehat{\mathcal{C}}$ whose objects are those of $\mathcal{C}$ and whose morphisms $X_1, X_2, \dots, X_n \to Y$ are morphisms $X_1 \otimes X_2 \otimes \dots \otimes X_n \to Y$ of $\mathcal{C}$. Multicategories arising from monoidal categories can be described by a simple axiom, which leads to the notion of representable multicategory [5]. The essence of the axiom is the existence, for each finite sequence $X_1, \dots, X_n$ of objects, of an arrow $X_1, \dots, X_n \to X$ that enjoys a universal property resembling that of tensor product of modules. Hermida proved [5] that the 2-category of monoidal categories, strong monoidal functors, and monoidal transformations is 2-equivalent to the 2-category of representable multicategories, multifunctors that preserve universal arrows, and multinatural transformations. This result was later extended by Bespalov, Lyubashenko, and the author [1] to a 2-equivalence (in fact, a **Cat**-equivalence) between the 2-category of lax monoidal categories, lax monoidal functors, and monoidal transformations, and the 2-category of lax representable multicategories, multifunctors, and multinatural transformations. Together with these works, the present papers finishes the program of giving a complete multicategorical expression of Eilenberg and Kelly's seminal work [2] by making explicit a precise relation between closed categories and closed multicategories.

Lambek defined closed multicategories in [7]. They generalize closed monoidal categories in the obvious way. Lambek's definition of a closed multicategory is equivalent to the following one. A multicategory C is *closed* if for every sequence $X_1, \dots, X_m, Z$ of objects of C there exists an internal Hom-object $\underline{\mathsf{C}}(X_1, \dots, X_m; Z)$ together with an evaluation morphism $\mathrm{ev}^{\mathsf{C}}_{X_1,\dots,X_m;Z} : X_1, \dots, X_m, \underline{\mathsf{C}}(X_1, \dots, X_m; Z) \to Z$ satisfying the following universal property: for each morphism $f : X_1, \dots, X_m, Y_1, \dots, Y_n \to Z$ there is a unique morphism $g : Y_1, \dots, Y_n \to \underline{\mathsf{C}}(X_1, \dots, X_m; Z)$ such that $f = \mathrm{ev}^{\mathsf{C}}_{X_1,\dots,X_m;Z} \circ (1_{X_1}, \dots, 1_{X_m}, g)$.

Bespalov, Lyubashenko, and the author proved [1] that the multicategory of A_∞-categories is closed, thus obtaining a conceptual explanation of the origin of the A_∞-categories of A_∞-functors.

This paper arose as an attempt to understand in general the relation between closed categories and closed multicategories. It turned out that these notions are essentially equivalent in a very strong sense. Namely, on the one hand, there is a 2-category of closed categories, closed functors, and closed natural transformations. On the other hand, there is a 2-category of closed multicategories with unit objects, multifunctors, and multinatural transformations. Because a 2-category is the same thing as a category enriched in **Cat**, it makes sense to speak about **Cat**-functors between 2-categories; these can be called strict 2-functors because they preserve composition of 1-morphisms and identity 1-morphisms strictly. We construct a **Cat**-functor from the 2-category of closed multicategories with unit objects to the 2-category of closed categories, and prove that it is a **Cat**-equivalence; see Proposition 4.6 and Theorem 5.1.

Both closed categories and multicategories can bear symmetries. With some additional work it can be proven that the 2-category of symmetric closed categories is **Cat**-equivalent to the 2-category of symmetric closed multicategories with unit objects. We are not going to explore this subject here.

Although we have not done so in this paper, the notion of closedness can be generalized to multicategories enriched in monoidal categories or even multicategories. The usefulness of such a generalization is indicated by the paper of Hyland and Power on pseudo-closed 2-categories [6], in which the notion of closed **Cat**-multicategory (i.e., multicategory enriched in the category **Cat** of categories) is implicitly present, although not spelled out. Martin Hyland told the author that he had known about the equivalence discussed in this paper and even made it a base for his considerations in computer science.

We should mention that the definition of closed category we adopt in this paper does not quite agree with the definition appearing in [2]. Closed categories have been generalized by Street [14] to extension systems; a closed category in our sense is an extension system with precisely one object. We discuss carefully the relation between these definitions because it is crucial for our proof of Theorem 5.1; see Remark 2.3 and Proposition 2.19. Our definition of closed category also coincides with the definition appearing in Laplaza's paper [9], to which we would like to pay special tribute because it allowed us to give an elegant construction of a closed multicategory with a given underlying closed category.

1.1. Notation We use interchangeably the notations $g \circ f$ and $f \cdot g$ for the composition of morphisms $f : X \to Y$ and $g : Y \to Z$ in a category, giving preference to the latter notation, which is more readable. Throughout the paper the set of nonnegative integers is denoted by $\mathbb{N}$, the category of sets is denoted by $\mathcal{S}$, and the category of categories is denoted by **Cat**.

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2. Closed categories

In this section we give preliminaries on closed categories. We begin by recalling the definition of closed category appearing in [14, Section 4] and [9].

2.1. Definition. *A* closed category $(\mathcal{C}, \underline{\mathcal{C}}(-,-), \mathbb{1}, i, j, L)$ *consists of the following data:*

- *a category* $\mathcal{C}$*;*
- *a functor* $\underline{\mathcal{C}}(-,-) : \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathcal{C}$*;*
- *an object* $\mathbb{1}$ *of* $\mathcal{C}$*;*
- *a natural isomorphism* $i : \mathrm{Id}_{\mathcal{C}} \xrightarrow{\sim} \underline{\mathcal{C}}(\mathbb{1}, -) : \mathcal{C} \to \mathcal{C}$*;*
- *a transformation* $j_X : \mathbb{1} \to \underline{\mathcal{C}}(X, X)$*, dinatural in* $X \in \mathrm{Ob}\,\mathcal{C}$*;*
- *a transformation* $L^X_{YZ} : \underline{\mathcal{C}}(Y, Z) \to \underline{\mathcal{C}}(\underline{\mathcal{C}}(X, Y), \underline{\mathcal{C}}(X, Z))$*, natural in* $Y, Z \in \mathrm{Ob}\,\mathcal{C}$ *and dinatural in* $X \in \mathrm{Ob}\,\mathcal{C}$*.*

These data are subject to the following axioms.

CC1. The following equation holds true:

$$\big[\mathbb{1} \xrightarrow{j_Y} \underline{\mathcal{C}}(Y, Y) \xrightarrow{L^X_{YY}} \underline{\mathcal{C}}(\underline{\mathcal{C}}(X, Y), \underline{\mathcal{C}}(X, Y))\big] = j_{\underline{\mathcal{C}}(X,Y)}.$$

CC2. The following equation holds true:

$$\big[\underline{\mathcal{C}}(X, Y) \xrightarrow{L^X_{XY}} \underline{\mathcal{C}}(\underline{\mathcal{C}}(X, X), \underline{\mathcal{C}}(X, Y)) \xrightarrow{\underline{\mathcal{C}}(j_X, 1)} \underline{\mathcal{C}}(\mathbb{1}, \underline{\mathcal{C}}(X, Y))\big] = i_{\underline{\mathcal{C}}(X,Y)}.$$

CC3. The following diagram commutes:

$$\begin{tikzcd}
\underline{\mathcal{C}}(U,V) \arrow[rr, "L^Y_{UV}"] \arrow[d, "L^X_{UV}"'] & & \underline{\mathcal{C}}(\underline{\mathcal{C}}(Y,U),\underline{\mathcal{C}}(Y,V)) \arrow[dd, "\underline{\mathcal{C}}(1,L^X_{YV})"] \\
\underline{\mathcal{C}}(\underline{\mathcal{C}}(X,U),\underline{\mathcal{C}}(X,V)) \arrow[d, "L^{\underline{\mathcal{C}}(X,Y)}_{\underline{\mathcal{C}}(X,U),\underline{\mathcal{C}}(X,V)}"'] & & \\
\underline{\mathcal{C}}(\underline{\mathcal{C}}(\underline{\mathcal{C}}(X,Y),\underline{\mathcal{C}}(X,U)),\underline{\mathcal{C}}(\underline{\mathcal{C}}(X,Y),\underline{\mathcal{C}}(X,V))) \arrow[rr, "\underline{\mathcal{C}}(L^X_{YU},1)"] & & \underline{\mathcal{C}}(\underline{\mathcal{C}}(Y,U),\underline{\mathcal{C}}(\underline{\mathcal{C}}(X,Y),\underline{\mathcal{C}}(X,V)))
\end{tikzcd}$$

CC4. The following equation holds true:

$$\bigl[\underline{\mathcal{C}}(Y,Z) \xrightarrow{L^{\mathbb{1}}_{YZ}} \underline{\mathcal{C}}(\underline{\mathcal{C}}(\mathbb{1},Y),\underline{\mathcal{C}}(\mathbb{1},Z)) \xrightarrow{\underline{\mathcal{C}}(i_Y,1)} \underline{\mathcal{C}}(Y,\underline{\mathcal{C}}(\mathbb{1},Z))\bigr] = \underline{\mathcal{C}}(1,i_Z).$$

CC5. The map $\gamma : \mathcal{C}(X,Y) \to \mathcal{C}(\mathbb{1},\underline{\mathcal{C}}(X,Y))$ that sends a morphism $f : X \to Y$ to the composite

$$\mathbb{1} \xrightarrow{j_X} \underline{\mathcal{C}}(X,X) \xrightarrow{\underline{\mathcal{C}}(1,f)} \underline{\mathcal{C}}(X,Y)$$

is a bijection.

We shall call $\underline{\mathcal{C}}(-,-)$ the internal Hom-functor *and $\mathbb{1}$ the* unit object.

2.2. Example. The category $\mathcal{S}$ of sets becomes a closed category if we set $\underline{\mathcal{S}}(-,-) = \mathcal{S}(-,-)$; take for $\mathbb{1}$ a set $\{*\}$, chosen once and for all, consisting of a single point $*$; and define i, j, L by:

$$\begin{aligned}
i_X(x)(*) &= x, && x \in X; \\
j_X(*) &= 1_X; && \\
L^X_{YZ}(g)(f) &= f \cdot g, && f \in \mathcal{S}(X,Y), \quad g \in \mathcal{S}(Y,Z).
\end{aligned}$$

2.3. Remark. Definition 2.1 is slightly different from the original definition by Eilenberg and Kelly [2, Section 2]. They require that a closed category $\mathcal{C}$ be equipped with a functor $C : \mathcal{C} \to \mathcal{S}$ such that the following axioms are satisfied in addition to CC1–CC4.

CC0. The following diagram of functors commutes:

$$\begin{tikzcd}
\mathcal{C}^{\mathrm{op}} \times \mathcal{C} \arrow[r, "\underline{\mathcal{C}}(-,-)"] \arrow[dr, "\mathcal{C}(-,-)"'] & \mathcal{C} \arrow[d, "C"] \\
& \mathcal{S}
\end{tikzcd}$$

CC5'. The map

$$Ci_{\underline{\mathcal{C}}(X,X)} : \mathcal{C}(X,X) = C\underline{\mathcal{C}}(X,X) \to C\underline{\mathcal{C}}(\mathbb{1}, \underline{\mathcal{C}}(X,X)) = \mathcal{C}(\mathbb{1}, \underline{\mathcal{C}}(X,X))$$

sends $1_X \in \mathcal{C}(X,X)$ to $j_X \in \mathcal{C}(\mathbb{1}, \underline{\mathcal{C}}(X,X))$.

[2, Lemma 2.2] implies that

$$\gamma = Ci_{\underline{\mathcal{C}}(X,Y)} : \mathcal{C}(X,Y) = C\underline{\mathcal{C}}(X,Y) \to C\underline{\mathcal{C}}(\mathbb{1}, \underline{\mathcal{C}}(X,Y)) = \mathcal{C}(\mathbb{1}, \underline{\mathcal{C}}(X,Y)),$$

so that a closed category in the sense of Eilenberg and Kelly is also a closed category in our sense. Furthermore, as we shall see later, an arbitrary closed category in our sense is isomorphic to a closed category in the sense of Eilenberg and Kelly.

2.4. Proposition. [2, Proposition 2.5] $i_{\underline{\mathcal{C}}(\mathbb{1},X)} = \underline{\mathcal{C}}(1, i_X) : \underline{\mathcal{C}}(\mathbb{1}, X) \to \underline{\mathcal{C}}(\mathbb{1}, \underline{\mathcal{C}}(\mathbb{1}, X))$.

Proof. The proof given in [2, Proposition 2.5] translates word by word to our setting. ■

2.5. Proposition. [2, Proposition 2.7] $j_{\mathbb{1}} = i_{\mathbb{1}} : \mathbb{1} \to \underline{\mathcal{C}}(\mathbb{1}, \mathbb{1})$.

Proof. The proof given in [2, Proposition 2.7] relies on the axiom CC5', and thus is not applicable here; we provide an independent proof for the sake of completeness. The map $\gamma : \mathcal{C}(\mathbb{1}, \underline{\mathcal{C}}(\mathbb{1}, \mathbb{1})) \to \mathcal{C}(\mathbb{1}, \underline{\mathcal{C}}(\mathbb{1}, \underline{\mathcal{C}}(\mathbb{1}, \mathbb{1})))$ is a bijection by the axiom CC5, so it suffices to prove that $\gamma(j_{\mathbb{1}}) = \gamma(i_{\mathbb{1}})$. We have:

$$\begin{aligned}
\gamma(i_{\mathbb{1}}) &= \big[\mathbb{1} \xrightarrow{j_{\mathbb{1}}} \underline{\mathcal{C}}(\mathbb{1}, \mathbb{1}) \xrightarrow{\underline{\mathcal{C}}(1, i_{\mathbb{1}})} \underline{\mathcal{C}}(\mathbb{1}, \underline{\mathcal{C}}(\mathbb{1}, \mathbb{1}))\big] \\
&= \big[\mathbb{1} \xrightarrow{j_{\mathbb{1}}} \underline{\mathcal{C}}(\mathbb{1}, \mathbb{1}) \xrightarrow{i_{\underline{\mathcal{C}}(\mathbb{1},\mathbb{1})}} \underline{\mathcal{C}}(\mathbb{1}, \underline{\mathcal{C}}(\mathbb{1}, \mathbb{1}))\big] && \text{(Proposition 2.4)} \\
&= \big[\mathbb{1} \xrightarrow{j_{\mathbb{1}}} \underline{\mathcal{C}}(\mathbb{1}, \mathbb{1}) \xrightarrow{L^{\mathbb{1}}_{\mathbb{1}\mathbb{1}}} \underline{\mathcal{C}}(\underline{\mathcal{C}}(\mathbb{1}, \mathbb{1}), \underline{\mathcal{C}}(\mathbb{1}, \mathbb{1})) \xrightarrow{\underline{\mathcal{C}}(j_{\mathbb{1}}, 1)} \underline{\mathcal{C}}(\mathbb{1}, \underline{\mathcal{C}}(\mathbb{1}, \mathbb{1}))\big] && \text{(axiom CC2)} \\
&= \big[\mathbb{1} \xrightarrow{j_{\underline{\mathcal{C}}(\mathbb{1},\mathbb{1})}} \underline{\mathcal{C}}(\underline{\mathcal{C}}(\mathbb{1}, \mathbb{1}), \underline{\mathcal{C}}(\mathbb{1}, \mathbb{1})) \xrightarrow{\underline{\mathcal{C}}(j_{\mathbb{1}}, 1)} \underline{\mathcal{C}}(\mathbb{1}, \underline{\mathcal{C}}(\mathbb{1}, \mathbb{1}))\big] && \text{(axiom CC1)} \\
&= \big[\mathbb{1} \xrightarrow{j_{\mathbb{1}}} \underline{\mathcal{C}}(\mathbb{1}, \mathbb{1}) \xrightarrow{\underline{\mathcal{C}}(1, j_{\mathbb{1}})} \underline{\mathcal{C}}(\mathbb{1}, \underline{\mathcal{C}}(\mathbb{1}, \mathbb{1}))\big] && \text{(dinaturality of } j) \\
&= \gamma(j_{\mathbb{1}}).
\end{aligned}$$

The proposition is proven. ■

2.6. Corollary. $\big[\mathcal{C}(\mathbb{1}, X) \xrightarrow{\gamma} \mathcal{C}(\mathbb{1}, \underline{\mathcal{C}}(\mathbb{1}, X)) \xrightarrow{\mathcal{C}(\mathbb{1}, i_X^{-1})} \mathcal{C}(\mathbb{1}, X)\big] = 1_{\mathcal{C}(\mathbb{1}, X)}$.

Proof. An element $f \in \mathcal{C}(\mathbb{1}, X)$ is mapped by the left hand side to the composite

$$\mathbb{1} \xrightarrow{j_{\mathbb{1}}} \underline{\mathcal{C}}(\mathbb{1}, \mathbb{1}) \xrightarrow{\underline{\mathcal{C}}(1, f)} \underline{\mathcal{C}}(\mathbb{1}, X) \xrightarrow{i_X^{-1}} X,$$

which is equal to

$$\big[\mathbb{1} \xrightarrow{j_{\mathbb{1}}} \underline{\mathcal{C}}(\mathbb{1}, \mathbb{1}) \xrightarrow{i_{\mathbb{1}}^{-1}} \mathbb{1} \xrightarrow{f} X\big] = f$$

by the naturality of i_X^{-1}, and because $j_{\mathbb{1}} = i_{\mathbb{1}} : \mathbb{1} \to \underline{\mathcal{C}}(\mathbb{1}, \mathbb{1})$ by Proposition 2.5. The corollary is proven. ■

2.7. Proposition. *The following diagram commutes:*

$$\begin{tikzcd}
\mathcal{C}(Y,Z) \arrow[r, "\underline{\mathcal{C}}(X,-)"] \arrow[d, "\gamma"'] & \mathcal{C}(\underline{\mathcal{C}}(X,Y),\underline{\mathcal{C}}(X,Z)) \arrow[d, "\gamma"] \\
\mathcal{C}(\mathbb{1},\underline{\mathcal{C}}(Y,Z)) \arrow[r, "\mathcal{C}(\mathbb{1},L^X_{YZ})"] & \mathcal{C}(\mathbb{1},\underline{\mathcal{C}}(\underline{\mathcal{C}}(X,Y),\underline{\mathcal{C}}(X,Z)))
\end{tikzcd}$$

Proof. For each $f \in \mathcal{C}(Y,Z)$, we have:

$$\begin{aligned}
\gamma(\underline{\mathcal{C}}(1,f)) &= \big[\mathbb{1} \xrightarrow{j_{\underline{\mathcal{C}}(X,Y)}} \underline{\mathcal{C}}(\underline{\mathcal{C}}(X,Y),\underline{\mathcal{C}}(X,Y)) \xrightarrow{\underline{\mathcal{C}}(1,\underline{\mathcal{C}}(1,f))} \underline{\mathcal{C}}(\underline{\mathcal{C}}(X,Y),\underline{\mathcal{C}}(X,Z))\big] \\
&= \big[\mathbb{1} \xrightarrow{j_Y} \underline{\mathcal{C}}(Y,Y) \xrightarrow{L^X_{YY}} \underline{\mathcal{C}}(\underline{\mathcal{C}}(X,Y),\underline{\mathcal{C}}(X,Y)) \xrightarrow{\underline{\mathcal{C}}(1,\underline{\mathcal{C}}(1,f))} \underline{\mathcal{C}}(\underline{\mathcal{C}}(X,Y),\underline{\mathcal{C}}(X,Z))\big] \\
&= \big[\mathbb{1} \xrightarrow{j_Y} \underline{\mathcal{C}}(Y,Y) \xrightarrow{\underline{\mathcal{C}}(1,f)} \underline{\mathcal{C}}(Y,Z) \xrightarrow{L^X_{YZ}} \underline{\mathcal{C}}(\underline{\mathcal{C}}(X,Y),\underline{\mathcal{C}}(X,Z))\big] \\
&= \mathcal{C}(\mathbb{1},L^X_{YZ})(\gamma(f))
\end{aligned}$$

where the second equality is by the axiom CC1, and the third equality is by the naturality of L^X_{YZ} in Z. ∎

2.8. Proposition. *For each $f \in \mathcal{C}(X,Y)$, $g \in \mathcal{C}(Y,Z)$, we have*

$$\gamma(f \cdot g) = \gamma(f) \cdot \underline{\mathcal{C}}(1,g) = \gamma(g) \cdot \underline{\mathcal{C}}(f,1).$$

Proof. Indeed, $\gamma(f \cdot g) = j_X \cdot \underline{\mathcal{C}}(1, f \cdot g) = j_X \cdot \underline{\mathcal{C}}(1,f) \cdot \underline{\mathcal{C}}(1,g) = \gamma(f) \cdot \underline{\mathcal{C}}(1,g)$, proving the first equality. Let us prove the second equality. We have:

$$\begin{aligned}
\gamma(f) \cdot \underline{\mathcal{C}}(1,g) &= \big[\mathbb{1} \xrightarrow{j_X} \underline{\mathcal{C}}(X,X) \xrightarrow{\underline{\mathcal{C}}(1,f)} \underline{\mathcal{C}}(X,Y) \xrightarrow{\underline{\mathcal{C}}(1,g)} \underline{\mathcal{C}}(X,Z)\big] \\
&= \big[\mathbb{1} \xrightarrow{j_Y} \underline{\mathcal{C}}(Y,Y) \xrightarrow{\underline{\mathcal{C}}(f,1)} \underline{\mathcal{C}}(X,Y) \xrightarrow{\underline{\mathcal{C}}(1,g)} \underline{\mathcal{C}}(X,Z)\big] \qquad \text{(dinaturality of } j) \\
&= \big[\mathbb{1} \xrightarrow{j_Y} \underline{\mathcal{C}}(Y,Y) \xrightarrow{\underline{\mathcal{C}}(1,g)} \underline{\mathcal{C}}(Y,Z) \xrightarrow{\underline{\mathcal{C}}(f,1)} \underline{\mathcal{C}}(X,Z)\big] \qquad \text{(functoriality of } \underline{\mathcal{C}}(-,-)) \\
&= \gamma(g) \cdot \underline{\mathcal{C}}(f,1).
\end{aligned}$$

The proposition is proven. ∎

We now recall the definitions of closed functor and closed natural transformation following [2, Section 2].

2.9. Definition. *Let $\mathcal{C}$ and $\mathcal{D}$ be closed categories. A* closed functor *$\Phi = (\phi, \hat{\phi}, \phi^0) : \mathcal{C} \to \mathcal{D}$ consists of the following data:*

- *a functor $\phi : \mathcal{C} \to \mathcal{D}$;*
- *a natural transformation $\hat{\phi} = \hat{\phi}_{X,Y} : \phi\underline{\mathcal{C}}(X,Y) \to \underline{\mathcal{D}}(\phi X, \phi Y)$;*
- *a morphism $\phi^0 : \mathbb{1} \to \phi\mathbb{1}$.*

These data are subject to the following axioms.

CF1. The following equation holds true:

$$\bigl[\mathbb{1} \xrightarrow{\phi^0} \phi\mathbb{1} \xrightarrow{\phi j_X} \phi\underline{\mathcal{C}}(X,X) \xrightarrow{\hat{\phi}} \underline{\mathcal{D}}(\phi X,\phi X)\bigr] = j_{\phi X}.$$

CF2. The following equation holds true:

$$\bigl[\phi X \xrightarrow{\phi i_X} \phi\underline{\mathcal{C}}(\mathbb{1},X) \xrightarrow{\hat{\phi}} \underline{\mathcal{D}}(\phi\mathbb{1},\phi X) \xrightarrow{\underline{\mathcal{D}}(\phi^0,1)} \underline{\mathcal{D}}(\mathbb{1},\phi X)\bigr] = i_{\phi X}.$$

CF3. The following diagram commutes:

$$\begin{tikzcd}
\phi\underline{\mathcal{C}}(Y,Z) \arrow[r, "\phi L^X_{YZ}"] \arrow[d, "\hat{\phi}"'] & \phi\underline{\mathcal{C}}(\underline{\mathcal{C}}(X,Y),\underline{\mathcal{C}}(X,Z)) \arrow[r, "\hat{\phi}"] & \underline{\mathcal{D}}(\phi\underline{\mathcal{C}}(X,Y),\phi\underline{\mathcal{C}}(X,Z)) \arrow[d, "\underline{\mathcal{D}}(1,\hat{\phi})"] \\
\underline{\mathcal{D}}(\phi Y,\phi Z) \arrow[r, "L^{\phi X}_{\phi Y,\phi Z}"] & \underline{\mathcal{D}}(\underline{\mathcal{D}}(\phi X,\phi Y),\underline{\mathcal{D}}(\phi X,\phi Z)) \arrow[r, "\underline{\mathcal{D}}(\hat{\phi},1)"] & \underline{\mathcal{D}}(\phi\underline{\mathcal{C}}(X,Y),\underline{\mathcal{D}}(\phi X,\phi Z))
\end{tikzcd}$$

2.10. Proposition. *Let $\mathcal{V}$ be a closed category. There is a closed functor $E = (e,\hat{e},e^0) : \mathcal{V} \to \mathcal{S}$, where:*

- $e = \mathcal{V}(\mathbb{1},-) : \mathcal{V} \to \mathcal{S}$;
- $\hat{e} = \bigl[\mathcal{V}(\mathbb{1},\underline{\mathcal{V}}(X,Y)) \xrightarrow{\gamma^{-1}} \mathcal{V}(X,Y) \xrightarrow{\mathcal{V}(\mathbb{1},-)} \mathcal{S}(\mathcal{V}(\mathbb{1},X),\mathcal{V}(\mathbb{1},X))\bigr]$;
- $e^0 : \{*\} \to \mathcal{V}(\mathbb{1},\mathbb{1})$, $* \mapsto 1_{\mathbb{1}}$.

Proof. Let us check the axioms CF1–CF3. The reader is referred to Example 2.2 for a description of the structure of a closed category on $\mathcal{S}$.

CF1 We must prove that the composite

$$\{*\} \xrightarrow{e^0} \mathcal{V}(\mathbb{1},\mathbb{1}) \xrightarrow{\mathcal{V}(\mathbb{1},j_X)} \mathcal{V}(\mathbb{1},\underline{\mathcal{V}}(X,X)) \xrightarrow{\gamma^{-1}} \mathcal{V}(X,X) \xrightarrow{\mathcal{V}(\mathbb{1},-)} \mathcal{S}(\mathcal{V}(\mathbb{1},X),\mathcal{V}(\mathbb{1},X))$$

equals $j_{\mathcal{V}(\mathbb{1},X)}$, which is obvious, as the image of $*$ is $\mathcal{V}(\mathbb{1},\gamma^{-1}(j_X)) = \mathcal{V}(\mathbb{1},1_X) = 1_{\mathcal{V}(\mathbb{1},X)}$, which is precisely $j_{\mathcal{V}(\mathbb{1},X)}(*)$.

CF2 We must prove the following equation:

$$\begin{aligned}\bigl[\mathcal{V}(\mathbb{1},X) &\xrightarrow{\mathcal{V}(\mathbb{1},i_X)} \mathcal{V}(\mathbb{1},\underline{\mathcal{V}}(\mathbb{1},X)) \\ &\xrightarrow{\gamma^{-1}} \mathcal{V}(\mathbb{1},X) \\ &\xrightarrow{\mathcal{V}(\mathbb{1},-)} \mathcal{S}(\mathcal{V}(\mathbb{1},\mathbb{1}),\mathcal{V}(\mathbb{1},X)) \\ &\xrightarrow{\mathcal{S}(e^0,1)} \mathcal{S}(\{*\},\mathcal{V}(\mathbb{1},X))\bigr] = i_{\mathcal{V}(\mathbb{1},X)}.\end{aligned}$$

By Corollary 2.6 the left hand side is equal to

$$\mathcal{V}(\mathbb{1}, X) \xrightarrow{\mathcal{V}(\mathbb{1},-)} \mathcal{S}(\mathcal{V}(\mathbb{1},\mathbb{1}),\mathcal{V}(\mathbb{1},X)) \xrightarrow{\mathcal{S}(e^0,1)} \mathcal{S}(\{*\},\mathcal{V}(\mathbb{1},X)),$$

and so it maps an element $f \in \mathcal{V}(\mathbb{1}, X)$ to the function $\{*\} \to \mathcal{V}(\mathbb{1}, X)$, $* \mapsto f$, which is precisely $i_{\mathcal{V}(\mathbb{1},X)}(f)$.

CF3 We must prove that the exterior of the following diagram commutes:

$$\begin{tikzcd}
\mathcal{V}(\mathbb{1},\underline{\mathcal{V}}(Y,Z)) \arrow[r, "\mathcal{V}(\mathbb{1},L^X_{YZ})"] \arrow[d, "\gamma^{-1}"'] & \mathcal{V}(\mathbb{1},\underline{\mathcal{V}}(\underline{\mathcal{V}}(X,Y),\underline{\mathcal{V}}(X,Z))) \arrow[d, "\gamma^{-1}"] \\
\mathcal{V}(Y,Z) \arrow[r, "\underline{\mathcal{V}}(X,-)"] \arrow[d, "\mathcal{V}(\mathbb{1},-)"'] & \mathcal{V}(\underline{\mathcal{V}}(X,Y),\underline{\mathcal{V}}(X,Z)) \arrow[d, "\mathcal{V}(\mathbb{1},-)"] \\
\mathcal{S}(\mathcal{V}(\mathbb{1},Y),\mathcal{V}(\mathbb{1},Z)) \arrow[d, "L^{\mathcal{V}(\mathbb{1},X)}_{\mathcal{V}(\mathbb{1},Y),\mathcal{V}(\mathbb{1},Z)}"'] & \mathcal{S}(\mathcal{V}(\mathbb{1},\underline{\mathcal{V}}(X,Y)),\mathcal{V}(\mathbb{1},\underline{\mathcal{V}}(X,Z))) \arrow[d, "\mathcal{S}(1,\gamma^{-1})"] \\
\mathcal{S}(\mathcal{S}(\mathcal{V}(\mathbb{1},X),\mathcal{V}(\mathbb{1},Y)),\mathcal{S}(\mathcal{V}(\mathbb{1},X),\mathcal{V}(\mathbb{1},Z))) \arrow[d, "\mathcal{S}(\mathcal{V}(\mathbb{1},-),1)"'] & \mathcal{S}(\mathcal{V}(\mathbb{1},\underline{\mathcal{V}}(X,Y)),\mathcal{V}(X,Z)) \arrow[d, "\mathcal{S}(1,\mathcal{V}(\mathbb{1},-))"] \\
\mathcal{S}(\mathcal{V}(X,Y),\mathcal{S}(\mathcal{V}(\mathbb{1},X),\mathcal{V}(\mathbb{1},Z))) \arrow[r, "\mathcal{S}(\gamma^{-1},1)"] & \mathcal{S}(\mathcal{V}(\mathbb{1},\underline{\mathcal{V}}(X,Y)),\mathcal{S}(\mathcal{V}(\mathbb{1},X),\mathcal{V}(\mathbb{1},Z)))
\end{tikzcd}$$

The upper square commutes by Proposition 2.7. Let us prove that so does the remaining region. Taking an element $f \in \mathcal{V}(Y, Z)$ and tracing it along the top-right path we obtain the function

$$\begin{aligned} &\mathcal{V}(\mathbb{1},\underline{\mathcal{V}}(X,Y)) \to \mathcal{S}(\mathcal{V}(\mathbb{1},X),\mathcal{V}(\mathbb{1},Z)), \\ &\qquad g \mapsto \big(h \mapsto h \cdot \gamma^{-1}(g \cdot \underline{\mathcal{V}}(1,f))\big), \end{aligned}$$

whereas pushing f along the left-bottom path yields the function

$$\begin{aligned} &\mathcal{V}(\mathbb{1},\underline{\mathcal{V}}(X,Y)) \to \mathcal{S}(\mathcal{V}(\mathbb{1},X),\mathcal{V}(\mathbb{1},Z)), \\ &\qquad g \mapsto \big(h \mapsto h \cdot \gamma^{-1}(g) \cdot f\big). \end{aligned}$$

These two functions are equal by Proposition 2.8. The proposition is proven. ∎

2.11. Definition. *Let* $\Phi = (\phi, \hat{\phi}, \phi^0), \Psi = (\psi, \hat{\psi}, \psi^0) : \mathcal{C} \to \mathcal{D}$ *be closed functors. A* closed natural transformation $\eta : \Phi \to \Psi : \mathcal{C} \to \mathcal{D}$ *is a natural transformation* $\eta : \phi \to \psi : \mathcal{C} \to \mathcal{D}$ *satisfying the following axioms.*

CN1. The following equation holds true:

$$\big[\mathbb{1} \xrightarrow{\phi^0} \phi\mathbb{1} \xrightarrow{\eta_{\mathbb{1}}} \psi\mathbb{1}\big] = \psi^0.$$

CN2. The following diagram commutes:

$$\begin{tikzcd}
\phi\underline{\mathcal{C}}(X,Y) \arrow[rr, "\hat{\phi}"] \arrow[d, "\eta_{\underline{\mathcal{C}}(X,Y)}"'] & & \underline{\mathcal{D}}(\phi X,\phi Y) \arrow[d, "\underline{\mathcal{D}}(1,\eta_Y)"] \\
\psi\underline{\mathcal{C}}(X,Y) \arrow[r, "\hat{\psi}"] & \underline{\mathcal{D}}(\psi X,\psi Y) \arrow[r, "\underline{\mathcal{D}}(\eta_X,1)"] & \underline{\mathcal{D}}(\phi X,\psi Y)
\end{tikzcd}$$

Closed categories, closed functors, and closed natural transformations form a 2-category [2, Theorem 4.2], which we shall denote by **ClCat**. The composite of closed functors $\Phi = (\phi,\hat{\phi},\phi^0) : \mathcal{C} \to \mathcal{D}$ and $\Psi = (\psi,\hat{\psi},\psi^0) : \mathcal{D} \to \mathcal{E}$ is defined to be $\mathrm{X} = (\chi,\hat{\chi},\chi^0) : \mathcal{C} \to \mathcal{E}$, where:

- χ is the composite $\mathcal{C} \xrightarrow{\phi} \mathcal{D} \xrightarrow{\psi} \mathcal{E}$;
- $\hat{\chi}$ is the composite $\psi\phi\underline{\mathcal{C}}(X,Y) \xrightarrow{\psi\hat{\phi}} \psi\underline{\mathcal{D}}(\phi X,\phi Y) \xrightarrow{\hat{\psi}} \underline{\mathcal{E}}(\psi\phi X,\psi\phi Y)$;
- χ^0 is the composite $\mathbb{1} \xrightarrow{\psi^0} \psi\mathbb{1} \xrightarrow{\psi\phi^0} \psi\phi\mathbb{1}$.

Compositions of closed natural transformations are defined in the usual way.

We can enrich in closed categories. Below we recall some enriched category theory for closed categories mainly following [2, Section 5].

2.12. Definition. *Let $\mathcal{V}$ be a closed category. A* $\mathcal{V}$-category $\mathcal{A}$ *consists of the following data:*

- *a set* $\operatorname{Ob}\mathcal{A}$ *of objects;*
- *for each $X,Y \in \operatorname{Ob}\mathcal{A}$, an object $\mathcal{A}(X,Y)$ of $\mathcal{V}$;*
- *for each $X \in \operatorname{Ob}\mathcal{A}$, a morphism $j_X : \mathbb{1} \to \mathcal{A}(X,X)$ in $\mathcal{V}$;*
- *for each $X,Y,Z \in \operatorname{Ob}\mathcal{A}$, a morphism $L^X_{YZ} : \mathcal{A}(Y,Z) \to \underline{\mathcal{V}}(\mathcal{A}(X,Y),\mathcal{A}(X,Z))$ in $\mathcal{V}$.*

These data are to satisfy axioms [2, VC1–VC3]. If $\mathcal{A}$ and $\mathcal{B}$ are $\mathcal{V}$-categories, a $\mathcal{V}$-functor $F : \mathcal{A} \to \mathcal{B}$ *consists of the following data:*

- *a function $\operatorname{Ob}F : \operatorname{Ob}\mathcal{A} \to \operatorname{Ob}\mathcal{B}$, $X \mapsto FX$;*
- *for each $X,Y \in \operatorname{Ob}\mathcal{A}$, a morphism $F = F_{XY} : \mathcal{A}(X,Y) \to \mathcal{B}(FX,FY)$ in $\mathcal{V}$.*

These data are subject to axioms [2, VF1–VF2].

2.13. Example. By [2, Theorem 5.2] a closed category $\mathcal{V}$ gives rise to a category $\underline{\mathcal{V}}$ if we take the objects of $\underline{\mathcal{V}}$ to be those of $\mathcal{V}$, take $\underline{\mathcal{V}}(X, Y)$ to be the internal Hom-object, and take for j and L those of the closed category $\mathcal{V}$. Furthermore, if $\mathcal{A}$ is a $\mathcal{V}$-category and X is an object of $\mathcal{A}$, then we get a $\mathcal{V}$-functor $L^X : \mathcal{A} \to \underline{\mathcal{V}}$ if we take $L^X Y = \mathcal{A}(X, Y)$ and $(L^X)_{YZ} = L^X_{YZ}$. In particular, for each $X \in \operatorname{Ob}\mathcal{V}$, there is a $\mathcal{V}$-functor $L^X : \underline{\mathcal{V}} \to \underline{\mathcal{V}}$ such that $L^X Y = \underline{\mathcal{V}}(X, Y)$ and $(L^X)_{YZ} = L^X_{YZ}$.

There is also a notion of $\mathcal{V}$-natural transformation. We recall it in a particular case, namely for $\mathcal{V}$-functors $\mathcal{A} \to \underline{\mathcal{V}}$.

2.14. Definition. *Let $F, G : \mathcal{A} \to \underline{\mathcal{V}}$ be $\mathcal{V}$-functors. A* $\mathcal{V}$-natural transformation *$\alpha : F \to G : \mathcal{A} \to \underline{\mathcal{V}}$ is a collection of morphisms $\alpha_X : FX \to GX$ in $\mathcal{V}$, for each $X \in \operatorname{Ob}\mathcal{A}$, such that the diagram*

$$\begin{tikzcd} \mathcal{A}(X,Y) \arrow[r, "F_{XY}"] \arrow[d, "G_{XY}"'] & \underline{\mathcal{V}}(FX,FY) \arrow[d, "\underline{\mathcal{V}}(1,\alpha_Y)"] \\ \underline{\mathcal{V}}(GX,GY) \arrow[r, "\underline{\mathcal{V}}(\alpha_X,1)"] & \underline{\mathcal{V}}(FX,GY) \end{tikzcd}$$

commutes, for each $X, Y \in \operatorname{Ob}\mathcal{A}$.

2.15. Example. By [2, Proposition 8.4] if $f \in \mathcal{V}(X, Y)$, the morphisms

$$\underline{\mathcal{V}}(f, 1) : \underline{\mathcal{V}}(Y, Z) \to \underline{\mathcal{V}}(X, Z), \quad Z \in \operatorname{Ob}\mathcal{V},$$

are components of a $\mathcal{V}$-natural transformation $L^f : L^Y \to L^X : \underline{\mathcal{V}} \to \underline{\mathcal{V}}$.

By [2, Theorem 10.2] $\mathcal{V}$-categories, $\mathcal{V}$-functors, and $\mathcal{V}$-natural transformations form a 2-category, which we shall denote by $\mathcal{V}$-**Cat**.

2.16. Proposition. [2, Proposition 6.1] *If $\Phi = (\phi, \hat{\phi}, \phi^0) : \mathcal{V} \to \mathcal{W}$ is a closed functor and $\mathcal{A}$ is a $\mathcal{V}$-category, the following data define a $\mathcal{W}$-category $\Phi_*\mathcal{A}$:*

- $\operatorname{Ob}\Phi_*\mathcal{A} = \operatorname{Ob}\mathcal{A}$;
- $(\Phi_*\mathcal{A})(X, Y) = \phi\mathcal{A}(X, Y)$;
- $j_X = \big[\mathbb{1} \xrightarrow{\phi^0} \phi\mathbb{1} \xrightarrow{\phi j_X} \phi\mathcal{A}(X, X)\big]$;
- $L^X_{YZ} = \big[\phi\mathcal{A}(Y, Z) \xrightarrow{\phi L^X_{YZ}} \phi\underline{\mathcal{V}}(\mathcal{A}(X, Y), \mathcal{A}(X, Z)) \xrightarrow{\hat{\phi}} \underline{\mathcal{W}}(\phi\mathcal{A}(X, Y), \phi\mathcal{A}(X, Z))\big]$.

2.17. Example. Let us study the effect of the closed functor E from Proposition 2.10 on $\mathcal{V}$-categories. Let $\mathcal{A}$ be a $\mathcal{V}$-category. Then the ordinary category $E_*\mathcal{A}$ has the same set of objects as $\mathcal{A}$ and its Hom-sets are $(E_*\mathcal{A})(X, Y) = \mathcal{V}(\mathbb{1}, \mathcal{A}(X, Y))$. The morphism j_X for the category $E_*\mathcal{A}$ is given by the composite

$$\{*\} \xrightarrow{e^0} \mathcal{V}(\mathbb{1}, \mathbb{1}) \xrightarrow{\mathcal{V}(\mathbb{1}, j_X)} \mathcal{V}(\mathbb{1}, \mathcal{A}(X, X)),$$

i.e., $1_X \in (E_*\mathcal{A})(X,X)$ identifies with j_X. The morphism L^X_{YZ} for the category $E_*\mathcal{A}$ is given by the composite

$$\begin{aligned} \mathcal{V}(\mathbb{1},\mathcal{A}(Y,Z)) &\xrightarrow{\mathcal{V}(\mathbb{1},L^X)} \mathcal{V}(\mathbb{1},\underline{\mathcal{V}}(\mathcal{A}(X,Y),\mathcal{A}(X,Z))) \\ &\xrightarrow{\gamma^{-1}} \mathcal{V}(\mathcal{A}(X,Y),\mathcal{A}(X,Z)) \\ &\xrightarrow{\mathcal{V}(\mathbb{1},-)} \mathcal{S}(\mathcal{A}(\mathbb{1},\mathcal{A}(X,Y)),\mathcal{V}(\mathbb{1},\mathcal{A}(X,Z))). \end{aligned}$$

It follows that composition in $E_*\mathcal{A}$ is given by

$$\mathcal{V}(\mathbb{1},\mathcal{A}(X,Y)) \times \mathcal{V}(\mathbb{1},\mathcal{A}(Y,Z)) \to \mathcal{V}(\mathbb{1},\mathcal{A}(X,Z)), \quad (f,g) \mapsto f \cdot \gamma^{-1}(g \cdot L^X_{YZ}).$$

2.18. Proposition. *The bijections $\gamma : \mathcal{V}(X,Y) \to \mathcal{V}(\mathbb{1},\underline{\mathcal{V}}(X,Y))$ define an isomorphism of categories $\gamma : \mathcal{V} \to E_*\underline{\mathcal{V}}$ identical on objects.*

Proof. For each $X \in \operatorname{Ob}\mathcal{V}$, we have $\gamma(1_X) = j_X$, so γ preserves identities. Let us check that it also preserves composition. For each $f \in \mathcal{V}(X,Y)$, $g \in \mathcal{V}(Y,Z)$, we have $\gamma(f)\cdot\gamma(g) = \gamma(f)\cdot\gamma^{-1}(\gamma(g)\cdot L^X_{YZ})$. By Proposition 2.7, $\gamma(g)\cdot L^X_{YZ} = \gamma(\underline{\mathcal{V}}(1,g))$, therefore $\gamma(f)\cdot\gamma(g) = \gamma(f)\cdot\underline{\mathcal{V}}(1,g) = \gamma(f\cdot g)$ by Proposition 2.8. The proposition is proven. ∎

2.19. Theorem. *Every closed category is isomorphic to a closed category in the sense of Eilenberg and Kelly.*

More precisely, for every closed category $\mathcal{V}$ in the sense of Definition 2.1 there is a closed category $\mathcal{W}$ in the sense of Eilenberg and Kelly such that $\mathcal{W}$, when viewed as a closed category in the sense of Definition 2.1, is isomorphic as a closed category to $\mathcal{V}$.

Proof. Let $\mathcal{V}$ be a closed category. Take $\mathcal{W} = E_*\underline{\mathcal{V}}$. The isomorphism γ from Proposition 2.18 allows us to translate the structure of a closed category from $\mathcal{V}$ to $\mathcal{W}$. Thus the unit object of $\mathcal{W}$ is that of $\mathcal{V}$, the internal Hom-functor is given by the composite

$$\underline{\mathcal{W}}(-,-) = \bigl[\mathcal{W}^{\mathrm{op}} \times \mathcal{W} \xrightarrow{(\gamma^{\mathrm{op}}\times\gamma)^{-1}} \mathcal{V}^{\mathrm{op}} \times \mathcal{V} \xrightarrow{\underline{\mathcal{V}}(-,-)} \mathcal{V} \xrightarrow{\gamma} \mathcal{W}\bigr].$$

In particular, $\underline{\mathcal{W}}(X,Y) = \underline{\mathcal{V}}(X,Y)$ for each pair of objects X and Y. The transformations i_X, j_X, L^X_{YZ} for $\mathcal{W}$ are just $\gamma(i_X)$, $\gamma(j_X)$, $\gamma(L^X_{YZ})$ respectively. The category $\mathcal{W}$ admits a functor $W : \mathcal{W} \to \mathcal{S}$ such that the diagram

$$\begin{tikzcd} \mathcal{W}^{\mathrm{op}} \times \mathcal{W} \arrow[r, "\underline{\mathcal{W}}(-,-)"] \arrow[dr, "\mathcal{W}(-,-)"'] & \mathcal{W} \arrow[d, "W"] \\ & \mathcal{S} \end{tikzcd}$$

commutes, namely $W = \bigl[\mathcal{W} \xrightarrow{\gamma^{-1}} \mathcal{V} \xrightarrow{E} \mathcal{S}\bigr]$. The commutativity on objects is obvious. Let us check that it also holds on morphisms. Let $f \in \mathcal{W}(X,Y)$, $h \in \mathcal{W}(U,V)$; i.e., suppose that $f : \mathbb{1} \to \underline{\mathcal{V}}(X,Y)$ and $h : \mathbb{1} \to \underline{\mathcal{V}}(U,V)$ are morphisms in $\mathcal{V}$. Then the map $\mathcal{W}(f,g) : \mathcal{W}(Y,U) \to \mathcal{W}(X,V)$ is given by $g \mapsto f \cdot g \cdot h$, where the composition is taken in $\mathcal{W}$. We must show that it is equal to the map

$$\mathcal{V}(\mathbb{1}, \underline{\mathcal{V}}(\gamma^{-1}(f), \gamma^{-1}(h))) : \mathcal{V}(\mathbb{1}, \underline{\mathcal{V}}(Y,U)) \to \mathcal{V}(\mathbb{1}, \underline{\mathcal{V}}(X,V)), \quad g \mapsto g \cdot \underline{\mathcal{V}}(\gamma^{-1}(f), \gamma^{-1}(h)).$$

We have:

$$\begin{aligned} g \cdot \underline{\mathcal{V}}(\gamma^{-1}(f), \gamma^{-1}(h)) &= \gamma(\gamma^{-1}(g)) \cdot \underline{\mathcal{V}}(\gamma^{-1}(f), 1) \cdot \underline{\mathcal{V}}(1, \gamma^{-1}(h)) && \text{(functoriality of } \underline{\mathcal{V}}(-,-)) \\ &= \gamma(\gamma^{-1}(f) \cdot \gamma^{-1}(g)) \cdot \underline{\mathcal{V}}(1, \gamma^{-1}(h)) && \text{(Proposition 2.8)} \\ &= \gamma(\gamma^{-1}(f) \cdot \gamma^{-1}(g) \cdot \gamma^{-1}(h)) && \text{(Proposition 2.8)} \\ &= f \cdot g \cdot h, && \text{(Proposition 2.18)} \end{aligned}$$

hence the assertion. The functor W also satisfies the axiom CC5'. Indeed, we need to show that

$$W i_{\underline{\mathcal{W}}(X,X)} = \mathcal{V}(\mathbb{1}, i_{\underline{\mathcal{V}}(X,X)}) : \mathcal{V}(\mathbb{1}, \underline{\mathcal{V}}(X,X)) \to \mathcal{V}(\mathbb{1}, \underline{\mathcal{V}}(\mathbb{1}, \underline{\mathcal{V}}(X,X)))$$

maps $j_X \in \mathcal{V}(\mathbb{1}, \underline{\mathcal{V}}(X,X))$ to $\gamma(j_X) \in \mathcal{V}(\mathbb{1}, \underline{\mathcal{V}}(\mathbb{1}, \underline{\mathcal{V}}(X,X)))$. In other words, we need to show that the diagram

$$\begin{tikzcd} \mathbb{1} \arrow[r, "j_X"] \arrow[d, "j_{\mathbb{1}}"'] & \underline{\mathcal{V}}(X,X) \arrow[d, "i_{\underline{\mathcal{V}}(X,X)}"] \\ \underline{\mathcal{V}}(\mathbb{1},\mathbb{1}) \arrow[r, "\underline{\mathcal{V}}(1,j_X)"] & \underline{\mathcal{V}}(\mathbb{1}, \underline{\mathcal{V}}(X,X)) \end{tikzcd}$$

commutes. However $j_{\mathbb{1}} = i_{\mathbb{1}} : \mathbb{1} \to \underline{\mathcal{V}}(\mathbb{1},\mathbb{1})$ by Proposition 2.5, so the above diagram is commutative by the naturality of i. The theorem is proven. ■

Finally, let us recall from [2] the representation theorem for $\mathcal{V}$-functors $\mathcal{A} \to \underline{\mathcal{V}}$.

2.20. Proposition. [2, Corollary 8.7] *Suppose that $\mathcal{V}$ is a closed category in the sense of Eilenberg and Kelly; i.e., it is equipped with a functor $V : \mathcal{V} \to \mathcal{S}$ satisfying the axioms CC0 and CC5'. Let $T : \mathcal{A} \to \underline{\mathcal{V}}$ be a $\mathcal{V}$-functor, and let W be an object of $\mathcal{A}$. Then the map*[1]

$$\Gamma : \mathcal{V}\text{-}\mathbf{Cat}(\mathcal{A}, \underline{\mathcal{V}})(L^W, T) \to VTW, \quad p \mapsto (Vp_W)1_W,$$

is a bijection.

2.21. Example. For each $f \in VL^XY = V\underline{\mathcal{V}}(X,Y) = \mathcal{V}(X,Y)$, the $\mathcal{V}$-natural transformation $L^f : L^Y \to L^X : \underline{\mathcal{V}} \to \underline{\mathcal{V}}$ from Example 2.15 is uniquely determined by the condition $(V(L^f)_Y)1_Y = f$.

[1] It is denoted by Γ' in [2, Corollary 8.7].

3. Closed multicategories

We begin by briefly recalling the notions of multicategory, multifunctor, and multinatural transformation. The reader is referred to the excellent book by Leinster [11] or to [1, Chapter 3] for a more elaborate introduction to multicategories.

3.1. Definition. *A* multigraph C *is a set* $\operatorname{Ob}\mathsf{C}$*, whose elements are called* objects *of* C*, together with a set* $\mathsf{C}(X_1,\dots,X_n;Y)$ *for each* $n\in\mathbb{N}$ *and* $X_1,\dots,X_n,Y\in\operatorname{Ob}\mathsf{C}$*. Elements of* $\mathsf{C}(X_1,\dots,X_n;Y)$ *are called* morphisms *and written as* $X_1,\dots,X_n\to Y$*. If* $n=0$*, elements of* $\mathsf{C}(;Y)$ *are written as* $()\to Y$*. A morphism of multigraphs* $F:\mathsf{C}\to\mathsf{D}$ *consists of a function* $\operatorname{Ob}F:\operatorname{Ob}\mathsf{C}\to\operatorname{Ob}\mathsf{D}$*,* $X\mapsto FX$*, and functions*

$$F=F_{X_1,\dots,X_n;Y}:\mathsf{C}(X_1,\dots,X_n;Y)\to\mathsf{D}(FX_1,\dots,FX_n;FY),\quad f\mapsto Ff,$$

for each $n\in\mathbb{N}$ *and* $X_1,\dots,X_n,Y\in\operatorname{Ob}\mathsf{C}$*.*

3.2. Definition. *A* multicategory C *consists of the following data:*

- *a multigraph* C*;*
- *for each* $n,k_1,\dots,k_n\in\mathbb{N}$ *and* $X_{ij},Y_i,Z\in\operatorname{Ob}\mathsf{C}$*,* $1\le i\le n$*,* $1\le j\le k_i$*, a function*

  $$\prod_{i=1}^n\mathsf{C}(X_{i1},\dots,X_{ik_i};Y_i)\times\mathsf{C}(Y_1,\dots,Y_n;Z)\to\mathsf{C}(X_{11},\dots,X_{1k_1},\dots,X_{n1},\dots,X_{nk_n};Z),$$

  called composition *and written* $(f_1,\dots,f_n,g)\mapsto(f_1,\dots,f_n)\cdot g$*;*
- *for each* $X\in\operatorname{Ob}\mathsf{C}$*, an element* $1^{\mathsf{C}}_X\in\mathsf{C}(X;X)$*, called the* identity *of* X*.*

These data are subject to the obvious associativity and identity axioms.

3.3. Example. A strict monoidal category $\mathcal{C}$ gives rise to a multicategory $\widehat{\mathcal{C}}$ as follows:

- $\operatorname{Ob}\widehat{\mathcal{C}}=\operatorname{Ob}\mathcal{C}$;
- for each $n\in\mathbb{N}$ and $X_1,\dots,X_n,Y\in\operatorname{Ob}\mathcal{C}$, $\widehat{\mathcal{C}}(X_1,\dots,X_n;Y)=\mathcal{C}(X_1\otimes\cdots\otimes X_n,Y)$; in particular $\widehat{\mathcal{C}}(;Y)=\mathcal{C}(\mathbb{1},Y)$, where $\mathbb{1}$ is the unit object of $\mathcal{C}$;
- for each n, k_1, …, $k_n\in\mathbb{N}$ and X_{ij}, Y_i, $Z\in\operatorname{Ob}\mathcal{C}$, $1\le i\le n$, $1\le j\le k_i$, the composition map

  $$\prod_{i=1}^n\mathcal{C}(X_{i1}\otimes\cdots\otimes X_{ik_i},Y_i)\times\mathcal{C}(Y_1\otimes\cdots\otimes Y_n,Z)\\ \to\mathcal{C}(X_{11}\otimes\cdots\otimes X_{1k_1}\otimes\cdots\otimes X_{n1}\otimes\cdots\otimes X_{nk_n},Z)$$

  is given by $(f_1,\dots,f_n,g)\mapsto(f_1\otimes\cdots\otimes f_n)\cdot g$;
- for each $X\in\operatorname{Ob}\mathcal{C}$, $1^{\widehat{\mathcal{C}}}_X=1^{\mathcal{C}}_X\in\widehat{\mathcal{C}}(X;X)=\mathcal{C}(X,X)$.

3.4. Definition. *Let* C *and* D *be multicategories. A* multifunctor $F : \mathsf{C} \to \mathsf{D}$ *is a morphism of the underlying multigraphs that preserves composition and identities.*

3.5. Definition. *Suppose that* $F, G : \mathsf{C} \to \mathsf{D}$ *are multifunctors. A* multinatural transformation $r : F \to G : \mathsf{C} \to \mathsf{D}$ *is a family of morphisms* $r_X \in \mathsf{D}(FX; GX)$, $X \in \operatorname{Ob}\mathsf{C}$, *such that* $Ff \cdot r_Y = (r_{X_1}, \dots, r_{X_n}) \cdot Gf$, *for each* $f \in \mathsf{C}(X_1, \dots, X_n; Y)$.

Multicategories, multifunctors, and multinatural transformations form a 2-category, which we shall denote by **Multicat**.

3.6. Definition. [1, Definition 4.7] *A multicategory* C *is called* closed *if for each* $m \in \mathbb{N}$ *and* $X_1, \dots, X_m, Z \in \operatorname{Ob}\mathsf{C}$ *there exist an object* $\underline{\mathsf{C}}(X_1, \dots, X_m; Z)$, *called* internal Hom-object, *and an* evaluation *morphism*

$$\mathrm{ev}^{\mathsf{C}} = \mathrm{ev}^{\mathsf{C}}_{X_1,\dots,X_m;Z} : X_1, \dots, X_m, \underline{\mathsf{C}}(X_1, \dots, X_m; Z) \to Z$$

such that, for each $Y_1, \dots, Y_n \in \operatorname{Ob}\mathsf{C}$, *the function*

$$\varphi^{\mathsf{C}} = \varphi^{\mathsf{C}}_{X_1,\dots,X_m;Y_1,\dots,Y_n;Z} : \mathsf{C}(Y_1, \dots, Y_n; \underline{\mathsf{C}}(X_1, \dots, X_m; Z)) \to \mathsf{C}(X_1, \dots, X_m, Y_1, \dots, Y_n; Z)$$

that sends a morphism $f : Y_1, \dots, Y_n \to \underline{\mathsf{C}}(X_1, \dots, X_m; Z)$ *to the composite*

$$X_1, \dots, X_m, Y_1, \dots, Y_n \xrightarrow{1^{\mathsf{C}}_{X_1},\dots,1^{\mathsf{C}}_{X_m},f} X_1, \dots, X_m, \underline{\mathsf{C}}(X_1, \dots, X_m; Z) \xrightarrow{\mathrm{ev}^{\mathsf{C}}_{X_1,\dots,X_m;Z}} Z$$

is bijective. Let **ClMulticat** *denote the full 2-subcategory of* **Multicat** *whose objects are closed multicategories.*

3.7. Remark. Notice that for $m = 0$ an object $\underline{\mathsf{C}}(;Z)$ and a morphism $\mathrm{ev}^{\mathsf{C}}_{;Z}$ with the required property always exist. Namely, we may (and we shall) always take $\underline{\mathsf{C}}(;Z) = Z$ and $\mathrm{ev}^{\mathsf{C}}_{;Z} = 1^{\mathsf{C}}_Z : Z \to Z$. With these choices $\varphi^{\mathsf{C}}_{;Y_1,\dots,Y_n;Z} : \mathsf{C}(Y_1, \dots, Y_n; Z) \to \mathsf{C}(Y_1, \dots, Y_n; Z)$ is the identity map.

3.8. Example. Let $\mathcal{C}$ be a strict monoidal category, and let $\widehat{\mathcal{C}}$ be the associated multicategory, see Example 3.3. It is easy to see that the multicategory $\widehat{\mathcal{C}}$ is closed if and only if $\mathcal{C}$ is closed as a monoidal category.

3.9. Proposition. *Suppose that for each pair of objects* $X, Z \in \operatorname{Ob}\mathsf{C}$ *there exist an object* $\underline{\mathsf{C}}(X; Z)$ *and a morphism* $\mathrm{ev}^{\mathsf{C}}_{X;Z} : X, \underline{\mathsf{C}}(X; Z) \to Z$ *of* C *such that the function* $\varphi^{\mathsf{C}}_{X;Y_1,\dots,Y_n;Z}$ *is a bijection, for each finite sequence* $Y_1, \dots, Y_n$ *of objects of* C. *Then* C *is a closed multicategory.*

Proof. Define internal Hom-objects $\underline{\mathsf{C}}(X_1, \dots, X_m; Z)$ and evaluations

$$\mathrm{ev}^{\mathsf{C}}_{X_1,\dots,X_m;Z} : X_1, \dots, X_m, \underline{\mathsf{C}}(X_1, \dots, X_m; Z) \to Z$$

by induction on m. For $m = 0$ choose $\underline{\mathsf{C}}(;Z) = Z$ and $\mathrm{ev}^{\mathsf{C}}_{;Z} = 1^{\mathsf{C}}_Z : Z \to Z$ as explained above. For $m = 1$ we are already given $\underline{\mathsf{C}}(X; Z)$ and $\mathrm{ev}^{\mathsf{C}}_{X;Z}$. Assume that we have defined $\underline{\mathsf{C}}(X_1, \dots, X_k; Z)$ and $\mathrm{ev}^{\mathsf{C}}_{X_1,\dots,X_k;Z}$ for each $k < m$, and that the function

$$\varphi^{\mathsf{C}}_{X_1,\dots,X_k;Y_1,\dots,Y_n;Z} : \mathsf{C}(Y_1, \dots, Y_n; \underline{\mathsf{C}}(X_1, \dots, X_k; Z)) \to \mathsf{C}(X_1, \dots, X_k, Y_1, \dots, Y_n; Z)$$

is a bijection, for each $k < m$ and for each finite sequence $Y_1, \dots, Y_n$ of objects of C. For $X_1, \dots, X_m$, $Z \in \operatorname{Ob} \mathsf{C}$ define

$$\underline{\mathsf{C}}(X_1, \dots, X_m; Z) \overset{\text{def}}{=} \underline{\mathsf{C}}(X_m; \underline{\mathsf{C}}(X_1, \dots, X_{m-1}; Z)).$$

The evaluation morphism $\mathrm{ev}^{\mathsf{C}}_{X_1,\dots,X_m;Z}$ is given by the composite

$$\begin{array}{c}
X_1, \dots, X_m, \underline{\mathsf{C}}(X_m; \underline{\mathsf{C}}(X_1, \dots, X_{m-1}; Z)) \\
\Big\downarrow {\scriptstyle 1^{\mathsf{C}}_{X_1}, \dots, 1^{\mathsf{C}}_{X_{m-1}}, \mathrm{ev}^{\mathsf{C}}_{X_m; \underline{\mathsf{C}}(X_1, \dots, X_{m-1}; Z)}} \\
X_1, \dots, X_{m-1}, \underline{\mathsf{C}}(X_1, \dots, X_{m-1}; Z) \\
\Big\downarrow {\scriptstyle \mathrm{ev}^{\mathsf{C}}_{X_1, \dots, X_{m-1}; Z}} \\
Z.
\end{array}$$

It is easy to see that with these choices the function $\varphi^{\mathsf{C}}_{X_1,\dots,X_m;Y_1,\dots,Y_n;Z}$ decomposes as

$$\begin{array}{c}
\mathsf{C}(Y_1, \dots, Y_n; \underline{\mathsf{C}}(X_1, \dots, X_m; Z)) \\
\wr \Big\downarrow {\scriptstyle \varphi^{\mathsf{C}}_{X_m; Y_1, \dots, Y_n; \underline{\mathsf{C}}(X_1, \dots, X_{m-1}; Z)}} \\
\mathsf{C}(X_m, Y_1, \dots, Y_n; \underline{\mathsf{C}}(X_1, \dots, X_{m-1}; Z)) \\
\wr \Big\downarrow {\scriptstyle \varphi^{\mathsf{C}}_{X_1, \dots, X_{m-1}; X_m, Y_1, \dots, Y_n; Z}} \\
\mathsf{C}(X_1, \dots, X_m, Y_1, \dots, Y_n; Z),
\end{array}$$

hence it is a bijection, and the induction goes through. ■

3.10. Remark. Lambek defined [7, p. 106] a (left) closed multicategory as one having, for each pair of objects X and Z, an internal Hom-object $X \backslash Z$ together with a morphism $\ell : X, X \backslash Z \to Z$ such that the induced mappings

$$[Y_1, \dots, Y_n; X \backslash Z] \to [X, Y_1, \dots, Y_n; Z]$$

are bijective; here $[-;-]$ denotes the Hom-set in the multicategory. Up to the obvious notational changes, this is precisely the condition of Proposition 3.9. Therefore, Lambek's definition of closedness is equivalent to ours.

3.11. Notation. For each morphism $f : X_1, \dots, X_n \to Y$ with $n \geq 1$, denote by $\langle f \rangle$ the morphism $(\varphi_{X_1; X_2, \dots, X_n; Z})^{-1}(f) : X_2, \dots, X_n \to \underline{\mathsf{C}}(X_1; Y)$. In other words, $\langle f \rangle$ is uniquely determined by the equation

$$\big[X_1, X_2, \dots, X_n \xrightarrow{1^{\mathsf{C}}_{X_1}, \langle f \rangle} X_1, \underline{\mathsf{C}}(X_1; Y) \xrightarrow{\mathrm{ev}^{\mathsf{C}}_{X_1; Y}} Y\big] = f.$$

Clearly we can enrich in multicategories. We leave it as an easy exercise for the reader to spell out the definitions of categories and functors enriched in a multicategory V.

3.12. Proposition. *A closed multicategory C gives rise to a C-category $\underline{\mathsf{C}}$ as follows. The objects of $\underline{\mathsf{C}}$ are those of C. For each pair $X, Y \in \operatorname{Ob}\mathsf{C}$, the* Hom*-object $\underline{\mathsf{C}}(X;Y)$ is the internal* Hom*-object of C. For each $X, Y, Z \in \operatorname{Ob}\mathsf{C}$, the composition morphism $\mu_{\underline{\mathsf{C}}} : \underline{\mathsf{C}}(X;Y), \underline{\mathsf{C}}(Y;Z) \to \underline{\mathsf{C}}(X;Z)$ is uniquely determined by requiring the commutativity in the diagram*

$$\begin{tikzcd}
X, \underline{\mathsf{C}}(X;Y), \underline{\mathsf{C}}(Y;Z) \arrow[r, "1^{\mathsf{C}}_X, \mu_{\underline{\mathsf{C}}}"] \arrow[d, "\mathrm{ev}^{\mathsf{C}}_{X;Y}, 1^{\mathsf{C}}_{\underline{\mathsf{C}}(Y;Z)}"'] & X, \underline{\mathsf{C}}(X;Z) \arrow[d, "\mathrm{ev}^{\mathsf{C}}_{X;Z}"] \\
Y, \underline{\mathsf{C}}(Y;Z) \arrow[r, "\mathrm{ev}^{\mathsf{C}}_{Y;Z}"] & Z
\end{tikzcd}$$

The identity of an object $X \in \operatorname{Ob}\mathsf{C}$ is $1^{\underline{\mathsf{C}}}_X = \langle 1^{\mathsf{C}}_X \rangle : () \to \underline{\mathsf{C}}(X;X)$.

Proof. The proof is similar to that for a closed monoidal category. ■

3.13. Notation. For each morphism $f : X_1, \dots, X_n \to Y$ and object Z of a closed multicategory C, there exists a unique morphism $\underline{\mathsf{C}}(f;Z) : \underline{\mathsf{C}}(Y;Z) \to \underline{\mathsf{C}}(X_1, \dots, X_n; Z)$ such that the diagram

$$\begin{tikzcd}
X_1, \dots, X_n, \underline{\mathsf{C}}(Y;Z) \arrow[r, "1^{\mathsf{C}}_{X_1}, \dots, 1^{\mathsf{C}}_{X_n}, \underline{\mathsf{C}}(f;Z)"] \arrow[d, "f, 1^{\mathsf{C}}_{\underline{\mathsf{C}}(Y;Z)}"'] & X_1, \dots, X_n, \underline{\mathsf{C}}(X_1, \dots, X_n; Z) \arrow[d, "\mathrm{ev}^{\mathsf{C}}_{X_1, \dots, X_n; Z}"] \\
Y, \underline{\mathsf{C}}(Y;Z) \arrow[r, "\mathrm{ev}^{\mathsf{C}}_{Y;Z}"] & Z
\end{tikzcd}$$

in C is commutative. In particular, if $n = 0$, then $\underline{\mathsf{C}}(f;Z) = (f, 1^{\mathsf{C}}_{\underline{\mathsf{C}}(Y;Z)}) \cdot \mathrm{ev}^{\mathsf{C}}_{Y;Z}$. If $n = 1$, then $\underline{\mathsf{C}}(f;Z) = \langle (f, 1^{\mathsf{C}}_{\underline{\mathsf{C}}(Y;Z)}) \cdot \mathrm{ev}^{\mathsf{C}}_{Y;Z} \rangle$. For each sequence of morphisms $f_1 : X_1 \to Y_1$, $\dots$, $f_n : X_n \to Y_n$ in C there is a unique morphism $\underline{\mathsf{C}}(f_1, \dots, f_n; Z) : \underline{\mathsf{C}}(Y_1, \dots, Y_n; Z) \to \underline{\mathsf{C}}(X_1, \dots, X_n; Z)$ such that the diagram

$$\begin{tikzcd}
X_1, \dots, X_n, \underline{\mathsf{C}}(Y_1, \dots, Y_n; Z) \arrow[r, "1^{\mathsf{C}}_{X_1}, \dots, 1^{\mathsf{C}}_{X_n}, \underline{\mathsf{C}}(f_1, \dots, f_n; Z)"] \arrow[d, "f_1, \dots, f_n, 1^{\mathsf{C}}_{\underline{\mathsf{C}}(Y_1, \dots, Y_n; Z)}"'] & X_1, \dots, X_n, \underline{\mathsf{C}}(X_1, \dots, X_n; Z) \arrow[d, "\mathrm{ev}^{\mathsf{C}}_{X_1, \dots, X_n; Z}"] \\
Y_1, \dots, Y_n, \underline{\mathsf{C}}(Y_1, \dots, Y_n; Z) \arrow[r, "\mathrm{ev}^{\mathsf{C}}_{Y_1, \dots, Y_n; Z}"] & Z
\end{tikzcd}$$

in C is commutative. Similarly, for each morphism $g : Y \to Z$ in C, there exists a unique morphism $\underline{\mathsf{C}}(X_1, \dots, X_n; g) : \underline{\mathsf{C}}(X_1, \dots, X_n; Y) \to \underline{\mathsf{C}}(X_1, \dots, X_n; Z)$ such that the diagram

$$\begin{tikzcd}
X_1, \dots, X_n, \underline{\mathsf{C}}(X_1, \dots, X_n; Y) \arrow[r, "1^{\mathsf{C}}_{X_1}, \dots, 1^{\mathsf{C}}_{X_n}, \underline{\mathsf{C}}(X_1, \dots, X_n; g)"] \arrow[d, "\mathrm{ev}^{\mathsf{C}}_{X_1, \dots, X_n; Y}"'] & X_1, \dots, X_n, \underline{\mathsf{C}}(X_1, \dots, X_n; Z) \arrow[d, "\mathrm{ev}^{\mathsf{C}}_{X_1, \dots, X_n; Z}"] \\
Y \arrow[r, "g"] & Z
\end{tikzcd}$$

in C is commutative. In particular, if $n = 0$, then our conventions force $\underline{\mathsf{C}}(;g) = g$. If $n = 1$, then $\underline{\mathsf{C}}(X;g) = \langle \mathrm{ev}^{\mathsf{C}}_{X;Y} \cdot g\rangle$.

3.14. Lemma. *Suppose that* $f_1 : X_1^1, \dots, X_1^{k_1} \to Y_1$, $\dots$, $f_n : X_n^1, \dots, X_n^{k_n} \to Y_n$, *and* $g : Y_1, \dots, Y_n \to Z$ *are morphisms in a closed multicategory* C.

(a) If $k_1 = 0$, *i.e.,* f_1 *is a morphism* $() \to Y_1$, *then* $(f_1, \dots, f_n)\cdot g$ *is equal to the composite*

$$X_2^1, \dots, X_2^{k_2}, \dots, X_n^1, \dots, X_n^{k_n} \xrightarrow{f_2,\dots,f_n} Y_2, \dots, Y_n \xrightarrow{\langle g\rangle} \underline{\mathsf{C}}(Y_1;Z) \xrightarrow{\underline{\mathsf{C}}(f_1;Z)} \underline{\mathsf{C}}(;Z) = Z.$$

(b) If $k_1 = 1$, *i.e.,* f_1 *is a morphism* $X_1^1 \to Y_1$, *then* $\langle (f_1, \dots, f_n) \cdot g\rangle$ *is equal to the composite*

$$X_2^1, \dots, X_2^{k_2}, \dots, X_n^1, \dots, X_n^{k_n} \xrightarrow{f_2,\dots,f_n} Y_2, \dots, Y_n \xrightarrow{\langle g\rangle} \underline{\mathsf{C}}(Y_1;Z) \xrightarrow{\underline{\mathsf{C}}(f_1;Z)} \underline{\mathsf{C}}(X_1^1;Z).$$

(c) If $k_1 \geq 1$, *then* $\langle (f_1, \dots, f_n) \cdot g\rangle$ *is equal to the composite*

$$\begin{aligned} X_1^2, \dots, X_1^{k_1}, X_2^1, \dots, X_2^{k_2}, \dots, X_n^1, \dots, X_n^{k_n} &\xrightarrow{\langle f_1\rangle, f_2,\dots,f_n} \underline{\mathsf{C}}(X_1^1;Y_1), Y_2, \dots, Y_n \\ &\xrightarrow{1,\langle g\rangle} \underline{\mathsf{C}}(X_1^1;Y_1), \underline{\mathsf{C}}(Y_1;Z) \\ &\xrightarrow{\mu_{\underline{\mathsf{C}}}} \underline{\mathsf{C}}(X_1^1;Z). \end{aligned}$$

(d) if $n = 1$, *then* $\langle f_1 \cdot g\rangle = \big[X_1^2, \dots, X_1^{k_1} \xrightarrow{\langle f_1\rangle} \underline{\mathsf{C}}(X_1^1;Y_1) \xrightarrow{\underline{\mathsf{C}}(X_1^1;g)} \underline{\mathsf{C}}(X_1^1;Z)\big]$.

Proof. The proofs are easy and consist of checking the definitions. For example, in order to prove (a) note that

$$\underline{\mathsf{C}}(f_1;Z) = \big[\underline{\mathsf{C}}(Y_1;Z) \xrightarrow{f_1, 1^{\mathsf{C}}_{\underline{\mathsf{C}}(Y_1;Z)}} Y_1, \underline{\mathsf{C}}(Y_1;Z) \xrightarrow{\mathrm{ev}^{\mathsf{C}}_{Y_1;Z}} Z\big],$$

therefore the composite in (a) is equal to

$$\begin{aligned} &\big[X_2^1, \dots, X_2^{k_2}, \dots, X_n^1, \dots, X_n^{k_n} \xrightarrow{f_2,\dots,f_n} Y_2, \dots, Y_n \\ &\qquad \xrightarrow{\langle g\rangle} \underline{\mathsf{C}}(Y_1;Z) \xrightarrow{f_1, 1^{\mathsf{C}}_{\underline{\mathsf{C}}(Y_1;Z)}} Y_1, \underline{\mathsf{C}}(Y_1;Z) \xrightarrow{\mathrm{ev}^{\mathsf{C}}_{Y_1;Z}} Z\big] \\ &= \big[X_2^1, \dots, X_2^{k_2}, \dots, X_n^1, \dots, X_n^{k_n} \xrightarrow{f_1, f_2,\dots,f_n} Y_1, Y_2, \dots, Y_n \xrightarrow{1^{\mathsf{C}}_{Y_1}, \langle g\rangle} Y_1, \underline{\mathsf{C}}(Y_1;Z) \xrightarrow{\mathrm{ev}^{\mathsf{C}}_{Y_1;Z}} Z\big]. \end{aligned}$$

The last two arrows compose to $\varphi^{\mathsf{C}}_{Y_1;Y_2,\dots,Y_n;Z}(\langle g\rangle) = g : Y_1, \dots, Y_n \to Z$, hence the whole composite is equal to $(f_1, \dots, f_n) \cdot g$. ■

3.15. Lemma. *Let $f : X \to Y$ and $g : Y \to Z$ be morphisms in a closed multicategory C. Then for each $W \in \operatorname{Ob}\mathsf{C}$ holds $\underline{\mathsf{C}}(W; f \cdot g) = \underline{\mathsf{C}}(W; f) \cdot \underline{\mathsf{C}}(W; g)$.*

Proof. The composite $\underline{\mathsf{C}}(W; f) \cdot \underline{\mathsf{C}}(W; g)$ can be written as

$$\underline{\mathsf{C}}(W;X) \xrightarrow{\langle \mathrm{ev}^{\mathsf{C}}_{W;X} \cdot f\rangle} \underline{\mathsf{C}}(W;Y) \xrightarrow{\underline{\mathsf{C}}(W;g)} \underline{\mathsf{C}}(W;Z),$$

which is equal to $\langle \mathrm{ev}^{\mathsf{C}}_{W;X} \cdot f \cdot g\rangle = \underline{\mathsf{C}}(W; f \cdot g)$ by Proposition 3.14, (d). ∎

3.16. Lemma. *Let $f : W \to X$ and $g : X \to Y$ be morphisms in a closed multicategory C. Then for each $Z \in \operatorname{Ob}\mathsf{C}$ holds $\underline{\mathsf{C}}(f \cdot g; Z) = \underline{\mathsf{C}}(g; Z) \cdot \underline{\mathsf{C}}(f; Z)$.*

Proof. The composite $\underline{\mathsf{C}}(g; Z) \cdot \underline{\mathsf{C}}(f; Z)$ can be written as

$$\underline{\mathsf{C}}(Y;Z) \xrightarrow{\langle (g, 1^{\mathsf{C}}_{\underline{\mathsf{C}}(Y;Z)}) \cdot \mathrm{ev}^{\mathsf{C}}_{Y;Z}\rangle} \underline{\mathsf{C}}(X;Z) \xrightarrow{\underline{\mathsf{C}}(f;Z)} \underline{\mathsf{C}}(W;Z),$$

which is equal to $\langle (f, 1^{\mathsf{C}}_{\underline{\mathsf{C}}(Y;Z)}) \cdot ((g, 1^{\mathsf{C}}_{\underline{\mathsf{C}}(Y;Z)}) \cdot \mathrm{ev}^{\mathsf{C}}_{Y;Z})\rangle = \langle (f \cdot g, 1^{\mathsf{C}}_{\underline{\mathsf{C}}(Y;Z)}) \cdot \mathrm{ev}^{\mathsf{C}}_{Y;Z}\rangle = \underline{\mathsf{C}}(f \cdot g; Z)$ by Proposition 3.14, (b). ∎

3.17. Lemma. *Let $f : W \to X$ and $g : Y \to Z$ be morphisms in a closed multicategory C. Then $\underline{\mathsf{C}}(f; Y) \cdot \underline{\mathsf{C}}(W; g) = \underline{\mathsf{C}}(X; g) \cdot \underline{\mathsf{C}}(f; Z)$.*

Proof. Both sides of the equation are equal to $\langle (f, 1^{\mathsf{C}}_{\underline{\mathsf{C}}(X;Y)}) \cdot \mathrm{ev}^{\mathsf{C}}_{X;Y} \cdot g\rangle$ by Proposition 3.14, (b),(d). ∎

It follows from Lemmas 3.15–3.17 that there exists a functor $\underline{\mathsf{C}}(-,-) : \mathsf{C}^{\mathrm{op}} \times \mathsf{C} \to \mathsf{C}$, $(X, Y) \mapsto \underline{\mathsf{C}}(X;Y)$, defined by the formula $\underline{\mathsf{C}}(f; g) = \underline{\mathsf{C}}(f; Y) \cdot \mathsf{C}(W; g) = \underline{\mathsf{C}}(X; g) \cdot \underline{\mathsf{C}}(f; Z)$ for each pair of morphisms $f : W \to X$ and $g : Y \to Z$ in C.

For each $X, Y, Z \in \operatorname{Ob}\mathsf{C}$ there is a morphism $L^X_{YZ} : \underline{\mathsf{C}}(Y; Z) \to \underline{\mathsf{C}}(\underline{\mathsf{C}}(X; Y); \underline{\mathsf{C}}(X; Z))$ uniquely determined by the equation

$$\big[\underline{\mathsf{C}}(X;Y), \underline{\mathsf{C}}(Y;Z) \xrightarrow{1, L^X_{YZ}} \underline{\mathsf{C}}(X;Y), \underline{\mathsf{C}}(\underline{\mathsf{C}}(X;Y); \underline{\mathsf{C}}(X;Z)) \xrightarrow{\mathrm{ev}^{\mathsf{C}}} \underline{\mathsf{C}}(X;Z)\big] = \mu_{\underline{\mathsf{C}}}. \tag{3.1}$$

3.18. Proposition. *There is a C-functor $L^X : \underline{\mathsf{C}} \to \underline{\mathsf{C}}$, $Y \mapsto \underline{\mathsf{C}}(X; Y)$, with the action on* Hom*-objects given by $L^X_{YZ} : \underline{\mathsf{C}}(Y; Z) \to \underline{\mathsf{C}}(\underline{\mathsf{C}}(X; Y); \underline{\mathsf{C}}(X; Z))$.*

Proof. That so defined L^X preserves identities is a consequence of the identity axiom. The compatibility with composition is established as follows. Consider the diagram

$$\begin{tikzcd}
\begin{array}{l}\underline{\mathsf{C}}(X;Y),\\ \underline{\mathsf{C}}(Y;Z),\\ \underline{\mathsf{C}}(Z;W)\end{array} \arrow[r, "{1,L^X_{YZ},L^X_{ZW}}"] \arrow[d, "{1,\mu_{\underline{\mathsf{C}}}}"'] & \begin{array}{l}\underline{\mathsf{C}}(X;Y),\\ \underline{\mathsf{C}}(\underline{\mathsf{C}}(X;Y);\underline{\mathsf{C}}(X;Z)),\\ \underline{\mathsf{C}}(\underline{\mathsf{C}}(X;Z);\underline{\mathsf{C}}(X;W))\end{array} \arrow[r, "{\mathrm{ev}^{\mathsf{C}},1}"] \arrow[d, "{1,\mu_{\underline{\mathsf{C}}}}"] & \begin{array}{l}\underline{\mathsf{C}}(X;Z),\\ \underline{\mathsf{C}}(\underline{\mathsf{C}}(X;Z);\underline{\mathsf{C}}(X;W))\end{array} \arrow[d, "\mathrm{ev}^{\mathsf{C}}"] \\
\begin{array}{l}\underline{\mathsf{C}}(X;Y),\\ \underline{\mathsf{C}}(Y;W)\end{array} \arrow[r, "{1,L^X_{YW}}"] & \begin{array}{l}\underline{\mathsf{C}}(X;Y),\\ \underline{\mathsf{C}}(\underline{\mathsf{C}}(X;Y);\underline{\mathsf{C}}(X;W))\end{array} \arrow[r, "\mathrm{ev}^{\mathsf{C}}"] & \underline{\mathsf{C}}(X;W)
\end{tikzcd}$$

By the definition of L^X the exterior expresses the associativity of $\mu_{\underline{\mathsf{C}}}$. The right square is the definition of $\mu_{\underline{\mathsf{C}}}$. By the closedness of C the square

$$\begin{tikzcd}
\underline{\mathsf{C}}(Y;Z),\underline{\mathsf{C}}(Z;W) \arrow[r, "L^X_{YZ},L^X_{ZW}"] \arrow[d, "\mu_{\underline{\mathsf{C}}}"'] & \underline{\mathsf{C}}(\underline{\mathsf{C}}(X;Y);\underline{\mathsf{C}}(X;Z)),\underline{\mathsf{C}}(\underline{\mathsf{C}}(X;Z);\underline{\mathsf{C}}(X;W)) \arrow[d, "\mu_{\underline{\mathsf{C}}}"] \\
\underline{\mathsf{C}}(Y;W) \arrow[r, "L^X_{YW}"] & \underline{\mathsf{C}}(\underline{\mathsf{C}}(X;Y);\underline{\mathsf{C}}(X;W))
\end{tikzcd}$$

is commutative, hence the assertion. ■

3.19. Definition. [1, Section 4.18] *Let* C, D *be multicategories. Let* $F : \mathsf{C} \to \mathsf{D}$ *be a multifunctor. For each* $X_1, \dots, X_m, Z \in \operatorname{Ob}\mathsf{C}$, *define a morphism in* D

$$\underline{F}_{X_1,\dots,X_m;Z} : F\underline{\mathsf{C}}(X_1,\dots,X_m;Z) \to \underline{\mathsf{D}}(FX_1,\dots,FX_m;FZ)$$

as the only morphism that makes the diagram

$$\begin{tikzcd}
 & FX_1\dots,FX_m,\underline{\mathsf{D}}(FX_1,\dots,FX_m;FZ) \arrow[d, "\mathrm{ev}^{\mathsf{D}}_{FX_1,\dots,FX_m;FZ}"] \\
FX_1,\dots,FX_m,F\underline{\mathsf{C}}(X_1,\dots,X_m;Z) \arrow[ur, "1^{\mathsf{D}}_{FX_1},\dots,1^{\mathsf{D}}_{FX_m},\underline{F}_{X_1,\dots,X_m;Z}"] \arrow[r, "F\,\mathrm{ev}^{\mathsf{C}}_{X_1,\dots,X_m;Z}"'] & FZ
\end{tikzcd}$$

commute. It is called the closing transformation *of the multifunctor* F.

The following properties of closing transformations can be found in [1, Section 4.18]. To keep the exposition self-contained we include their proofs here.

3.20. Proposition. [1, Lemma 4.19] *The diagram*

$$\begin{tikzcd}
\mathsf{C}\big(Y_1,\dots,Y_n;\underline{\mathsf{C}}(X_1,\dots,X_m;Z)\big) \arrow[r, "F"] \arrow[dd, "\varphi^{\mathsf{C}}_{X_1,\dots,X_m;Y_1,\dots,Y_n;Z}"'] & \mathsf{D}\big(FY_1,\dots,FY_n;F\underline{\mathsf{C}}(X_1,\dots,X_m;Z)\big) \arrow[d, "\mathsf{D}(1;\underline{F}_{X_1,\dots,X_m;Z})"] \\
 & \mathsf{D}\big(FY_1,\dots,FY_n;\underline{\mathsf{D}}(FX_1,\dots,FX_m;FZ)\big) \arrow[d, "\varphi^{\mathsf{D}}_{FX_1,\dots,FX_m;FY_1,\dots,FY_n;FZ}"] \\
\mathsf{C}\big(X_1,\dots,X_m,Y_1,\dots,Y_n;Z\big) \arrow[r, "F"] & \mathsf{D}\big(FX_1,\dots,FX_m,FY_1,\dots,FY_n;FZ\big)
\end{tikzcd} \tag{3.2}$$

commutes, for each $m, n \in \mathbb{N}$ *and objects* $X_i, Y_j, Z \in \operatorname{Ob}\mathsf{C}$, $1 \le i \le m$, $1 \le j \le n$.

Proof. Pushing an arbitrary morphism $g : Y_1, \dots, Y_n \to \underline{\mathsf{C}}(X_1, \dots, X_m; Z)$ along the top-right path produces the composite

$$\begin{aligned} FX_1, \dots, FX_m, FY_1, \dots, FY_n &\xrightarrow{1^{\mathsf{D}}_{FX_1}, \dots, 1^{\mathsf{D}}_{FX_m}, Fg} FX_1, \dots, FX_m, F\underline{\mathsf{C}}(X_1, \dots, X_m; Z) \\ &\xrightarrow{1^{\mathsf{D}}_{FX_1}, \dots, 1^{\mathsf{D}}_{FX_m}, \underline{F}_{(X_i);Z}} FX_1, \dots, FX_m, \mathsf{D}(FX_1, \dots, FX_m; FZ) \\ &\xrightarrow{\mathrm{ev}^{\mathsf{D}}_{FX_1, \dots, FX_m; FZ}} FZ. \end{aligned}$$

The composition of the last two arrows is equal to $F \,\mathrm{ev}^{\mathsf{C}}_{X_1, \dots, X_m; Z}$ by the definition of $\underline{F}_{X_1, \dots, X_m; Z}$. Since F preserves composition and identities, the above composite equals

$$F\big((1^{\mathsf{C}}_{X_1}, \dots, 1^{\mathsf{C}}_{X_m}, g) \cdot \mathrm{ev}^{\mathsf{C}}_{X_1, \dots, X_m; Z}\big) = F\big(\varphi_{X_1, \dots, X_m; Y_1, \dots, Y_n; Z}(g)\big),$$

hence the assertion. ∎

Let $F : \mathsf{V} \to \mathsf{W}$ be a multifunctor, and let $\mathcal{C}$ be a V-category. We obtain a W-category $F_*\mathcal{C}$ with the same set of objects if we define its Hom-objects by $(F_*\mathcal{C})(X, Y) = F\mathcal{C}(X, Y)$, and identities and composition by respectively $1^{F_*\mathcal{C}}_X = F(1^{\mathcal{C}}_X) : () \to F\mathcal{C}(X, X)$ and $\mu_{F_*\mathcal{C}} = F(\mu_{\mathcal{C}}) : F\mathcal{C}(X, Y), F\mathcal{C}(Y, Z) \to F\mathcal{C}(X, Z)$.

3.21. Proposition. [cf. [1, Proposition 4.21]] *Let $F : \mathsf{C} \to \mathsf{D}$ be a multifunctor between closed multicategories. There is D-functor $\underline{F} : F_*\underline{\mathsf{C}} \to \underline{\mathsf{D}}$, $X \mapsto FX$, such that*

$$\underline{F}_{X;Y} : (F_*\underline{\mathsf{C}})(X; Y) = F\underline{\mathsf{C}}(X; Y) \to \underline{\mathsf{D}}(FX; FY)$$

is the closing transformation, for each $X, Y \in \mathrm{Ob}\,\mathsf{C}$.

Proof. First, let us check that $\underline{F}$ preserves identities. In other words, we must prove the equation

$$\big[() \xrightarrow{F1^{\mathsf{C}}_X} F\underline{\mathsf{C}}(X; X) \xrightarrow{\underline{F}_{X,X}} \underline{\mathsf{D}}(FX; FX)\big] = 1^{\mathsf{D}}_{FX}.$$

Let us check that the left hand side solves the equation that determines the right hand side. We have:

$$\begin{aligned} &\big[FX \xrightarrow{1^{\mathsf{C}}_{FX}, F1^{\mathsf{C}}_X} FX, F\underline{\mathsf{C}}(X; X) \xrightarrow{1^{\mathsf{C}}_{FX}, \underline{F}_{X,X}} FX, \underline{\mathsf{D}}(FX; FX) \xrightarrow{\mathrm{ev}^{\mathsf{D}}} FX\big] \\ &\quad = \big[FX \xrightarrow{1^{\mathsf{C}}_{FX}, F1^{\mathsf{C}}_X} FX, F\underline{\mathsf{C}}(X; X) \xrightarrow{F\,\mathrm{ev}^{\mathsf{C}}} FX\big] = F[(1^{\mathsf{C}}_{FX}, 1^{\mathsf{C}}_X) \cdot \mathrm{ev}^{\mathsf{C}}] = F1^{\mathsf{C}}_X = 1^{\mathsf{D}}_{FX}. \end{aligned}$$

To show that $\underline{F}$ preserves composition, we must show that the diagram

$$\begin{tikzcd} F\underline{\mathsf{C}}(X; X), F\underline{\mathsf{C}}(Y; Z) \arrow[r, "F\mu_{\underline{\mathsf{C}}}"] \arrow[d, "\underline{F}_{X,Y}, \underline{F}_{Y,Z}"'] & F\underline{\mathsf{C}}(X; Z) \arrow[d, "\underline{F}_{X,Z}"] \\ \underline{\mathsf{D}}(FX; FY), \underline{\mathsf{D}}(FY; FZ) \arrow[r, "\mu_{\underline{\mathsf{D}}}"] & \underline{\mathsf{D}}(FX; FZ) \end{tikzcd} \tag{3.3}$$

$$
\begin{tikzcd}
 & FX, F\underline{\mathsf{C}}(X;Y), F\underline{\mathsf{C}}(Y;Z) \arrow[dl, "F\,\mathrm{ev}^{\mathsf{C}}_{X;Y},1"'] \arrow[dr, "F\mu_{\underline{\mathsf{C}}}"] \arrow[dd, "1{,}\underline{F}_{X;Y}{,}\underline{F}_{Y;Z}"] & \\
FY, F\underline{\mathsf{C}}(Y;Z) \arrow[dd, "1{,}\underline{F}_{Y;Z}"] \arrow[ddddr, bend right, "F\,\mathrm{ev}^{\mathsf{C}}_{Y;Z}"'] & & FX, F\underline{\mathsf{C}}(X;Z) \arrow[dd, "1{,}\underline{F}_{X,Z}"'] \arrow[ddddl, bend left, "F\,\mathrm{ev}^{\mathsf{C}}_{X;Z}"] \\
 & FX, \underline{\mathsf{D}}(FX;FY), \underline{\mathsf{D}}(FY;FZ) \arrow[dl, "\mathrm{ev}^{\mathsf{D}}_{FX;FY},1"] \arrow[dr, "1{,}\mu_{\underline{\mathsf{D}}}"'] & \\
FY, \underline{\mathsf{D}}(FY;FZ) \arrow[dr, "\mathrm{ev}^{\mathsf{D}}_{FY;FZ}"] & & FX, \underline{\mathsf{D}}(FX;FZ) \arrow[dl, "\mathrm{ev}^{\mathsf{D}}_{FX;FZ}"'] \\
 & FZ &
\end{tikzcd}
$$

Figure 3.1

commutes. This follows from the diagram displayed on Figure 3.1. The lower diamond is the definition of $\mu_{\underline{\mathsf{D}}}$. The exterior commutes by the definition of $\mu_{\underline{\mathsf{C}}}$ and because F preserves composition. The left upper diamond and both triangles commute by the definition of the closing transformation. ∎

3.22. Lemma. [1, Lemma 4.25] *Let* C, D, E *be closed multicategories and* $\mathsf{C} \xrightarrow{F} \mathsf{D} \xrightarrow{G} \mathsf{E}$ *multifunctors. Then*

$$
\begin{aligned}
\underline{G \circ F}_{X_1,\dots,X_m;Y} = \big[GF\underline{\mathsf{C}}(X_1,\dots,X_m;Y) &\xrightarrow{G\underline{F}_{X_1,\dots,X_m;Y}} G\underline{\mathsf{D}}(FX_1,\dots,FX_m;FY) \\
&\xrightarrow{\underline{G}_{FX_1,\dots,FX_m;FY}} \underline{\mathsf{E}}(GFX_1,\dots,GFX_m;GFY) \big].
\end{aligned}
$$

Proof. This follows from the commutative diagram

$$\begin{tikzcd}
GFX_1,\dots,GFX_m,\underline{\mathsf{E}}(GFX_1,\dots,GFX_m;GFY) \arrow[d, "1^{\mathsf{E}}_{GFX_1},\dots,1^{\mathsf{E}}_{GFX_m},\underline{G}_{FX_1,\dots,FX_m;FY}"'] \arrow[dr, "\mathrm{ev}^{\mathsf{E}}_{GFX_1,\dots,GFX_m;GFY}"] & \\
GFX_1,\dots,GFX_m,G\underline{\mathsf{D}}(FX_1,\dots,FX_m;FY) \arrow[r, "G\,\mathrm{ev}^{\mathsf{D}}_{FX_1,\dots,FX_m;FY}"] \arrow[d, "1^{\mathsf{E}}_{GFX_1},\dots,1^{\mathsf{E}}_{GFX_m},G\underline{F}_{X_1,\dots,X_m;Y}"'] & GFY \\
GFX_1,\dots,GFX_m,GF\underline{\mathsf{C}}(X_1,\dots,X_m;Y) \arrow[ur, "GF\,\mathrm{ev}^{\mathsf{C}}_{X_1,\dots,X_m;Y}"'] &
\end{tikzcd}$$

The upper triangle is the definition of $\underline{G}_{FX_1,\dots,FX_m;FY}$, the lower triangle commutes by the definition of $\underline{F}_{X_1,\dots,X_m;Y}$ and because G preserves composition. ∎

3.23. Proposition. [1, Lemma 4.24] *Let $\nu : F \to G : \mathsf{C} \to \mathsf{D}$ be a multinatural transformation of multifunctors between closed multicategories. Then the diagram*

$$\begin{tikzcd}
F\underline{\mathsf{C}}(X_1,\dots,X_m;Y) \arrow[r, "\underline{F}_{X_1,\dots,X_m;Y}"] \arrow[d, "\nu_{\underline{\mathsf{C}}(X_1,\dots,X_m;Y)}"'] & \underline{\mathsf{D}}(FX_1,\dots,FX_m;FY) \arrow[dd, "\underline{\mathsf{D}}(FX_1,\dots,FX_m;\nu_Y)"] \\
G\underline{\mathsf{C}}(X_1,\dots,X_m;Y) \arrow[d, "\underline{G}_{X_1,\dots,X_m;Y}"'] & \\
\underline{\mathsf{D}}(GX_1,\dots,GX_m;GY) \arrow[r, "\underline{\mathsf{D}}(\nu_{X_1},\dots,\nu_{X_m};GY)"] & \underline{\mathsf{D}}(FX_1,\dots,FX_m;GY)
\end{tikzcd} \tag{3.4}$$

is commutative.

Proof. The claim follows from the diagram displayed on Figure 3.2. Its exterior commutes by the multinaturality of ν. The quadrilateral in the middle is the definition of $\underline{\mathsf{D}}(\nu_{X_1},\dots,\nu_{X_m};GY)$. The trapezoid on the right is the definition of $\underline{\mathsf{D}}(FX_1,\dots,FX_m;\nu_Y)$. The triangles commute by the definition of closing transformation. ∎

4. From closed multicategories to closed categories

A closed category comes equipped with a distinguished object $\mathbb{1}$. We want to produce a closed category out of a closed multicategory, so we need a notion of a closed multicategory

$$
\begin{tikzcd}[column sep=large, row sep=large]
\begin{array}{c} FX_1,\ldots,FX_m, \\ F\underline{\mathsf{C}}(X_1,\ldots,X_m;Y) \end{array} \arrow[rrr, "F\,\mathrm{ev}^{\mathsf{C}}"] \arrow[rr, "1^{\mathsf{D}}_{FX_1},\ldots,1^{\mathsf{D}}_{FX_m},\underline{F}"] \arrow[d, "1^{\mathsf{D}}_{FX_1},\ldots,1^{\mathsf{D}}_{FX_m},\nu_{\underline{\mathsf{C}}(X_1,\ldots,X_m;Y)}"] & & & FY \arrow[dddd, "\nu_Y"'] \\
\begin{array}{c} FX_1,\ldots,FX_m, \\ G\underline{\mathsf{C}}(X_1,\ldots,X_m;Y) \end{array} \arrow[dr, "1^{\mathsf{D}}_{FX_1},\ldots,1^{\mathsf{D}}_{FX_m},\underline{G}"] \arrow[ddd, "\nu_{X_1},\ldots,\nu_{X_m},1^{\mathsf{D}}_{G\underline{\mathsf{C}}(X_1,\ldots,X_m;Y)}"] & & \begin{array}{c} FX_1,\ldots,FX_m, \\ \underline{\mathsf{D}}(FX_1,\ldots,FX_m;FY) \end{array} \arrow[ur, "\mathrm{ev}^{\mathsf{D}}"] \arrow[d, "1^{\mathsf{D}}_{FX_1},\ldots,1^{\mathsf{D}}_{FX_m},\underline{\mathsf{D}}(FX_1,\ldots,FX_m;\nu_Y)"'] & \\
& \begin{array}{c} FX_1,\ldots,FX_m, \\ \underline{\mathsf{D}}(GX_1,\ldots,GX_m;GY) \end{array} \arrow[r, "1^{\mathsf{D}}_{FX_1},\ldots,1^{\mathsf{D}}_{FX_m},\underline{\mathsf{D}}(\nu_{X_1},\ldots,\nu_{X_m};GY)"] \arrow[d, "\nu_{X_1},\ldots,\nu_{X_m},1^{\mathsf{D}}_{\underline{\mathsf{D}}(GX_1,\ldots,GX_m;GY)}"] & \begin{array}{c} FX_1,\ldots,FX_m, \\ \underline{\mathsf{D}}(FX_1,\ldots,FX_m;GY) \end{array} \arrow[ddr, "\mathrm{ev}^{\mathsf{D}}"'] & \\
& \begin{array}{c} GX_1,\ldots,GX_m, \\ \underline{\mathsf{D}}(GX_1,\ldots,GX_m;GY) \end{array} \arrow[drr, "\mathrm{ev}^{\mathsf{D}}"] & & \\
\begin{array}{c} GX_1,\ldots,GX_m, \\ G\underline{\mathsf{C}}(X_1,\ldots,X_m;Y) \end{array} \arrow[ur, "1^{\mathsf{D}}_{GX_1},\ldots,1^{\mathsf{D}}_{GX_m},\underline{G}"'] \arrow[rrr, "G\,\mathrm{ev}^{\mathsf{C}}"'] & & & GY
\end{tikzcd}
$$

Figure 3.2

with a unit object. We introduce it in somewhat ad hoc fashion, which is sufficient for our purposes though. Similarly to closedness, possession of a unit object is a property of a closed multicategory rather than additional data.

4.1. DEFINITION. *Let* C *be a closed multicategory. A* unit object *of* C *is an object* $\mathbb{1} \in \operatorname{Ob}\mathsf{C}$ *together with a morphism* $u : () \to \mathbb{1}$ *such that, for each* $X \in \operatorname{Ob}\mathsf{C}$*, the morphism*

$$\underline{\mathsf{C}}(u; 1) : \underline{\mathsf{C}}(\mathbb{1}; X) \to \underline{\mathsf{C}}(; X) = X$$

is an isomorphism.

4.2. REMARK. Suppose that $\mathbb{1}$ is a unit object of a closed multicategory C. Then $\mathsf{C}(u; X) : \mathsf{C}(\mathbb{1}; X) \to \mathsf{C}(; X)$ is a bijection, as follows from the equation

$$\big[\mathsf{C}(; \underline{\mathsf{C}}(\mathbb{1}; X)) \xrightarrow[\sim]{\varphi^{\mathsf{C}}} \mathsf{C}(\mathbb{1}; X) \xrightarrow{\mathsf{C}(u;X)} \mathsf{C}(; X)\big] = \mathsf{C}(; \underline{\mathsf{C}}(u; X)),$$

which is an immediate consequence of the definitions. The bijectivity of $\mathsf{C}(u; X)$ can be stated as the following universal property: for each morphism $f : () \to X$, there exists a unique morphism $\overline{f} : \mathbb{1} \to X$ such that $u \cdot \overline{f} = f$. In particular, a unit object, if it exists, is unique up to isomorphism.

4.3. PROPOSITION. *A closed multicategory* C *with a unit object gives rise to a closed category* $(\mathcal{C}, \underline{\mathcal{C}}(-,-), \mathbb{1}, i, j, L)$*, where:*

- $\mathcal{C}$ *is the underlying category of the multicategory* C*;*
- $\underline{\mathcal{C}}(X, Y) = \underline{\mathsf{C}}(X; Y)$*, for each* $X, Y \in \operatorname{Ob}\mathsf{C}$*;*
- $\mathbb{1}$ *is the unit object of* C*;*
- $i_X = \big(\underline{\mathsf{C}}(u; X)\big)^{-1} : X = \underline{\mathsf{C}}(; X) \to \underline{\mathsf{C}}(\mathbb{1}; X)$*;*
- $j_X = \overline{1^{\mathsf{C}}_X} : \mathbb{1} \to \underline{\mathsf{C}}(X; X)$ *is a unique morphism such that* $\big[() \xrightarrow{u} \mathbb{1} \xrightarrow{j_X} \underline{\mathsf{C}}(X; X)\big] = 1^{\mathsf{C}}_X$*;*
- $L^X_{YZ} : \underline{\mathsf{C}}(Y; Z) \to \underline{\mathsf{C}}(\underline{\mathsf{C}}(X; Y); \underline{\mathsf{C}}(X; Z))$ *is determined uniquely by equation* (3.1).

We shall call $\mathcal{C}$ the *underlying closed category* of C. Usually we do not distinguish notationally between a closed multicategory and its underlying closed category; this should lead to minimal confusion.

Proof. We leave it as an easy exercise for the reader to show the naturality of i_X, j_X, and L^X_{YZ}, and proceed directly to checking the axioms.

CC1. By Remark 4.2 the equation

$$\big[\mathbb{1} \xrightarrow{j_Y} \underline{\mathsf{C}}(Y;Y) \xrightarrow{L^X_{YY}} \underline{\mathsf{C}}(\underline{\mathsf{C}}(X;Y);\underline{\mathsf{C}}(X;Y))\big] = j_{\underline{\mathsf{C}}(X;Y)}$$

is equivalent to the equation

$$\big[() \xrightarrow[1^{\mathsf{C}}_Y]{u\cdot j_Y} \underline{\mathsf{C}}(Y,Y) \xrightarrow{L^X_{YY}} \underline{\mathsf{C}}(\underline{\mathsf{C}}(X;Y);\underline{\mathsf{C}}(X;Y))\big] = u \cdot j_{\underline{\mathsf{C}}(X;Y)} = 1^{\mathsf{C}}_{\underline{\mathsf{C}}(X;Y)},$$

which expresses the fact that the C-functor L^X preserves identities.

CC2. The equation in question

$$\big[\underline{\mathsf{C}}(X;Y) \xrightarrow{L^X_{XY}} \underline{\mathsf{C}}(\underline{\mathsf{C}}(X;X);\underline{\mathsf{C}}(X;Y)) \xrightarrow{\underline{\mathsf{C}}(j_X;1)} \underline{\mathsf{C}}(\mathbb{1};\underline{\mathsf{C}}(X;Y))\big] = i_{\underline{\mathsf{C}}(X;Y)} = (\underline{\mathsf{C}}(u;1))^{-1}$$

is equivalent to

$$\big[\underline{\mathsf{C}}(X;Y) \xrightarrow{L^X_{XY}} \underline{\mathsf{C}}(\underline{\mathsf{C}}(X;X);\underline{\mathsf{C}}(X;Y)) \xrightarrow[\underline{\mathsf{C}}(1^{\mathsf{C}}_X;1)]{\underline{\mathsf{C}}(u\cdot j_X;1)} \underline{\mathsf{C}}(;\underline{\mathsf{C}}(X;Y)) = \underline{\mathsf{C}}(X;Y)\big] = 1^{\mathsf{C}}_{\underline{\mathsf{C}}(X;Y)}.$$

The left hand side is equal to

$$\big[\underline{\mathsf{C}}(X;Y) \xrightarrow{L^X_{YZ}} \underline{\mathsf{C}}(\underline{\mathsf{C}}(X;X);\underline{\mathsf{C}}(X;Y)) \xrightarrow{1^{\mathsf{C}}_X,1} \underline{\mathsf{C}}(X;X),\underline{\mathsf{C}}(\underline{\mathsf{C}}(X;X);\underline{\mathsf{C}}(X;Y)) \xrightarrow{\mathrm{ev}^{\mathsf{C}}} \underline{\mathsf{C}}(X;Y)\big]$$
$$= \big[\underline{\mathsf{C}}(X;Y) \xrightarrow{1^{\mathsf{C}}_X,1} \underline{\mathsf{C}}(X;X),\underline{\mathsf{C}}(X;Y) \xrightarrow{1,L^X_{YZ}} \underline{\mathsf{C}}(X;X),\underline{\mathsf{C}}(\underline{\mathsf{C}}(X;X);\underline{\mathsf{C}}(X;Y)) \xrightarrow{\mathrm{ev}^{\mathsf{C}}} \underline{\mathsf{C}}(X;Y)\big]$$
$$= \big[\underline{\mathsf{C}}(X;Y) \xrightarrow{1^{\mathsf{C}}_X,1} \underline{\mathsf{C}}(X;X),\underline{\mathsf{C}}(X;Y) \xrightarrow{\mu_{\underline{\mathsf{C}}}} \underline{\mathsf{C}}(X;Y)\big] = 1^{\mathsf{C}}_{\underline{\mathsf{C}}(X;Y)}$$

by the identity axiom in the C-category $\underline{\mathsf{C}}$.

CC3. The commutativity of the diagram

$$\begin{tikzcd}
\underline{\mathsf{C}}(U;V) \arrow[rr, "L^Y_{UV}"] \arrow[d, "L^X_{UV}"'] & & \underline{\mathsf{C}}(\underline{\mathsf{C}}(Y;U);\underline{\mathsf{C}}(Y;V)) \arrow[dd, "\underline{\mathsf{C}}(1;L^X_{YV})"] \\
\underline{\mathsf{C}}(\underline{\mathsf{C}}(X;U);\underline{\mathsf{C}}(X;V)) \arrow[d, "L^{\underline{\mathsf{C}}(X;Y)}_{\underline{\mathsf{C}}(X;U),\underline{\mathsf{C}}(X;V)}"'] & & \\
\underline{\mathsf{C}}(\underline{\mathsf{C}}(\underline{\mathsf{C}}(X;Y);\underline{\mathsf{C}}(X;U));\underline{\mathsf{C}}(\underline{\mathsf{C}}(X;Y);\underline{\mathsf{C}}(X;V))) \arrow[rr, "\underline{\mathsf{C}}(L^X_{YU};1)"] & & \underline{\mathsf{C}}(\underline{\mathsf{C}}(Y;U);\underline{\mathsf{C}}(\underline{\mathsf{C}}(X;Y);\underline{\mathsf{C}}(X;V)))
\end{tikzcd}$$

is equivalent by closedness to the commutativity of the exterior of the diagram displayed on Figure 4.1, which just expresses the fact that the C-functor $L^X : \underline{\mathsf{C}} \to \underline{\mathsf{C}}$ preserves composition and which is part of the assertion of Proposition 3.18.

$$
\begin{tikzcd}
 & \underline{\mathsf{C}}(Y;U),\underline{\mathsf{C}}(U;V) \arrow[rrr, "\mu_{\underline{\mathsf{C}}}"] \arrow[drr, "1{,}L^Y_{UV}"] \arrow[dd, "1{,}L^X_{UV}"] \arrow[dddddl, "L^X_{YU}{,}L^X_{UV}"'] & & & \underline{\mathsf{C}}(Y;V) \arrow[ddddd, "L^X_{YV}"] \\
 & & & \underline{\mathsf{C}}(Y;U),\underline{\mathsf{C}}(\underline{\mathsf{C}}(Y;U);\underline{\mathsf{C}}(Y;V)) \arrow[ur, "\mathrm{ev}^{\mathsf{C}}"] \arrow[d, "1{,}\underline{\mathsf{C}}(1;L^X_{YV})"] & \\
 & \underline{\mathsf{C}}(Y;U),\underline{\mathsf{C}}(\underline{\mathsf{C}}(X;U);\underline{\mathsf{C}}(X;V)) \arrow[dr, "1{,}L^{\underline{\mathsf{C}}(X;Y)}_{\underline{\mathsf{C}}(X;U),\underline{\mathsf{C}}(X;V)}"'] \arrow[dddl, "L^X_{YU}{,}1"'] & & \underline{\mathsf{C}}(Y;U),\underline{\mathsf{C}}(\underline{\mathsf{C}}(Y;U);\underline{\mathsf{C}}(\underline{\mathsf{C}}(X;Y);\underline{\mathsf{C}}(X;V))) \arrow[dddr, "\mathrm{ev}^{\mathsf{C}}"] & \\
 & & \underline{\mathsf{C}}(Y;U),\underline{\mathsf{C}}(\underline{\mathsf{C}}(\underline{\mathsf{C}}(X;Y);\underline{\mathsf{C}}(X;U));\underline{\mathsf{C}}(\underline{\mathsf{C}}(X;Y);\underline{\mathsf{C}}(X;V))) \arrow[ur, "1{,}\underline{\mathsf{C}}(L^X_{YU};1)"'] \arrow[d, "L^X_{YU}{,}1"] & & \\
 & & \underline{\mathsf{C}}(\underline{\mathsf{C}}(X;Y);\underline{\mathsf{C}}(X;U)),\underline{\mathsf{C}}(\underline{\mathsf{C}}(\underline{\mathsf{C}}(X;Y);\underline{\mathsf{C}}(X;U));\underline{\mathsf{C}}(\underline{\mathsf{C}}(X;Y);\underline{\mathsf{C}}(X;V))) \arrow[drr, "\mathrm{ev}^{\mathsf{C}}"] & & \\
\underline{\mathsf{C}}(\underline{\mathsf{C}}(X;Y);\underline{\mathsf{C}}(X;U)),\underline{\mathsf{C}}(\underline{\mathsf{C}}(X;U);\underline{\mathsf{C}}(X;V)) \arrow[urr, "1{,}L^{\underline{\mathsf{C}}(X;Y)}_{\underline{\mathsf{C}}(X;U),\underline{\mathsf{C}}(X;V)}"] \arrow[rrrr, "\mu_{\underline{\mathsf{C}}}"] & & & & \underline{\mathsf{C}}(\underline{\mathsf{C}}(X;Y);\underline{\mathsf{C}}(X;V))
\end{tikzcd}
$$

Figure 4.1

CC4. The equation in question

$$\big[\underline{\mathsf{C}}(Y;Z) \xrightarrow{L^{\mathbb{1}}} \underline{\mathsf{C}}(\underline{\mathsf{C}}(\mathbb{1};Y);\underline{\mathsf{C}}(\mathbb{1};Z)) \xrightarrow{\underline{\mathsf{C}}(i_Y;1)} \underline{\mathsf{C}}(Y;\underline{\mathsf{C}}(\mathbb{1};Z))\big] = \underline{\mathsf{C}}(1;i_Z)$$

is equivalent to the equation

$$\big[\underline{\mathsf{C}}(Y;Z) \xrightarrow{L^{\mathbb{1}}} \underline{\mathsf{C}}(\underline{\mathsf{C}}(\mathbb{1};Y);\underline{\mathsf{C}}(\mathbb{1};Z)) \xrightarrow{\underline{\mathsf{C}}(1;\underline{\mathsf{C}}(u;1))} \underline{\mathsf{C}}(\underline{\mathsf{C}}(\mathbb{1};Y);Z)\big] = \underline{\mathsf{C}}(\underline{\mathsf{C}}(u;1);1).$$

The latter follows by closedness from the commutative diagram

$$\begin{tikzcd}
\underline{\mathsf{C}}(\mathbb{1};Y),\underline{\mathsf{C}}(Y;Z) \arrow[rrr, "{1,L^{\mathbb{1}}}"] \arrow[drr, "\mu_{\underline{\mathsf{C}}}"] \arrow[ddr, "{u,1,1}"] \arrow[ddd, "{\underline{\mathsf{C}}(u;1),1}"'] & & & \underline{\mathsf{C}}(\mathbb{1};Y),\underline{\mathsf{C}}(\underline{\mathsf{C}}(\mathbb{1};Y),\underline{\mathsf{C}}(\mathbb{1};Z)) \arrow[dl, "\mathrm{ev}^{\mathsf{C}}"'] \arrow[d, "{1,\underline{\mathsf{C}}(1;\underline{\mathsf{C}}(u;1))}"] \\
& & \underline{\mathsf{C}}(\mathbb{1};Z) \arrow[d, "{u,1}"] \arrow[ddr, "\underline{\mathsf{C}}(u;1)"] & \underline{\mathsf{C}}(\mathbb{1};Y),\underline{\mathsf{C}}(\underline{\mathsf{C}}(\mathbb{1};Y);Z) \arrow[dd, "\mathrm{ev}^{\mathsf{C}}"] \\
& \mathbb{1},\underline{\mathsf{C}}(\mathbb{1};Y),\underline{\mathsf{C}}(Y;Z) \arrow[r, "{1,\mu_{\underline{\mathsf{C}}}}"] \arrow[dl, "{\mathrm{ev}^{\mathsf{C}},1}"] & \mathbb{1},\underline{\mathsf{C}}(\mathbb{1};Z) \arrow[dr, "\mathrm{ev}^{\mathsf{C}}"] & \\
Y,\underline{\mathsf{C}}(Y;Z) \arrow[rrr, "\mathrm{ev}^{\mathsf{C}}"] & & & Z
\end{tikzcd}$$

in which the bottom quadrilateral is the definition of $\mu_{\underline{\mathsf{C}}}$, the right hand side quadrilateral is the definition of the morphism $\underline{\mathsf{C}}(1;\underline{\mathsf{C}}(u;1))$, the top triangle is the definition of $L^{\mathbb{1}}$, and the remaining triangles commute by the definition of $\underline{\mathsf{C}}(u;1)$.

CC5. A straightforward computation shows that the composite

$$\mathsf{C}(X;Y) \xrightarrow{\gamma} \mathsf{C}(\mathbb{1};\underline{\mathsf{C}}(X;Y)) \xrightarrow[\sim]{\mathsf{C}(u;1)} \mathsf{C}(;\underline{\mathsf{C}}(X;Y)) \xrightarrow[\sim]{\varphi^{\mathsf{C}}} \mathsf{C}(X;Y)$$

is the identity map, which readily implies that γ is a bijection.

The proposition is proven. ∎

4.4. Proposition. *Let* C *and* D *be closed multicategories with unit objects. Let* $\mathcal{C}$ *and* $\mathcal{D}$ *denote the corresponding underlying closed categories. A multifunctor* $F : \mathsf{C} \to \mathsf{D}$ *gives rise to a closed functor* $\Phi = (\phi, \hat{\phi}, \phi^0) : \mathcal{C} \to \mathcal{D}$*, where:*

- $\phi : \mathcal{C} \to \mathcal{D}$ *is the underlying functor of the multifunctor* F*;*
- $\hat{\phi} = \hat{\phi}_{X,Y} = \underline{F}_{X,Y} : F\underline{\mathsf{C}}(X;Y) \to \underline{\mathsf{D}}(FX;FY)$ *is the closing transformation;*
- $\phi^0 = \overline{Fu} : \mathbb{1} \to F\mathbb{1}$ *is a unique morphism such that* $\big[() \xrightarrow{u} \mathbb{1} \xrightarrow{\phi^0} F\mathbb{1}\big] = Fu$*.*

PROOF. Let us check the axioms.

CF1. By Remark 4.2 the equation

$$\bigl[\mathbb{1} \xrightarrow{\phi^0} F\mathbb{1} \xrightarrow{Fj_X} F\underline{\mathsf{C}}(X;X) \xrightarrow{\underline{F}} \underline{\mathsf{D}}(FX;FX)\bigr] = j_{FX}$$

is equivalent to the equation

$$\bigl[() \xrightarrow{u} \mathbb{1} \xrightarrow{\phi^0} F\mathbb{1} \xrightarrow{Fj_X} F\underline{\mathsf{C}}(X;X) \xrightarrow{\underline{F}} \underline{\mathsf{D}}(FX;FX)\bigr] = u \cdot j_{FX} = 1^{\mathsf{D}}_{FX}.$$

Since $u \cdot \phi^0 \cdot Fj_X = Fu \cdot Fj_X = F(u \cdot j_X) = F1^{\mathsf{C}}_X$, the above equation simply expresses the fact that the D-functor $\underline{F} : F_*\underline{\mathsf{C}} \to \underline{\mathsf{D}}$ preserves identities, which is part of Proposition 3.21.

CF2. The equation in question

$$\bigl[FX \xrightarrow[F\underline{\mathsf{C}}(u;1)^{-1}]{Fi_X} F\underline{\mathsf{C}}(\mathbb{1};X) \xrightarrow{\underline{F}} \underline{\mathsf{D}}(F\mathbb{1};FX) \xrightarrow{\underline{\mathsf{D}}(\phi^0;1)} \underline{\mathsf{D}}(\mathbb{1};FX)\bigr] = i_{FX} = \underline{\mathsf{D}}(u;1)^{-1}$$

is equivalent to

$$\bigl[F\underline{\mathsf{C}}(\mathbb{1};X) \xrightarrow{\underline{F}} \underline{\mathsf{D}}(F\mathbb{1};FX) \xrightarrow{\underline{\mathsf{D}}(\phi^0;1)} \underline{\mathsf{D}}(\mathbb{1};FX) \xrightarrow{\underline{\mathsf{D}}(u;1)} \underline{\mathsf{D}}(;FX) = FX\bigr] = F\underline{\mathsf{C}}(u;1). \quad (4.1)$$

The composition of the last two arrows is equal to $\underline{\mathsf{D}}(Fu;1)$. Hence the left hand side of the above equation is equal to

$$\begin{aligned}
&\bigl[F\underline{\mathsf{C}}(\mathbb{1};X) \xrightarrow{\underline{F}} \underline{\mathsf{D}}(F\mathbb{1};FX) \xrightarrow{\underline{\mathsf{D}}(Fu;1)} \underline{\mathsf{D}}(;FX) = FX\bigr] \\
&\qquad = \bigl[F\underline{\mathsf{C}}(\mathbb{1};X) \xrightarrow{\underline{F}} \underline{\mathsf{D}}(F\mathbb{1};FX) \xrightarrow{Fu,1} F\mathbb{1},\underline{\mathsf{D}}(F\mathbb{1};FX) \xrightarrow{\mathrm{ev}^{\mathsf{D}}} FX\bigr] \\
&\qquad = \bigl[F\underline{\mathsf{C}}(\mathbb{1};X) \xrightarrow{Fu,1} F\mathbb{1},F\underline{\mathsf{C}}(\mathbb{1};X) \xrightarrow{1,\underline{F}} F\mathbb{1},\underline{\mathsf{D}}(F\mathbb{1};FX) \xrightarrow{\mathrm{ev}^{\mathsf{D}}} FX\bigr] \\
&\qquad = \bigl[F\underline{\mathsf{C}}(\mathbb{1};X) \xrightarrow{Fu,1} F\mathbb{1},F\underline{\mathsf{C}}(\mathbb{1};X) \xrightarrow{F\,\mathrm{ev}^{\mathsf{C}}} FX\bigr] = F((u,1)\cdot \mathrm{ev}^{\mathsf{C}}) = F\underline{\mathsf{C}}(u;1).
\end{aligned}$$

CF3. We must prove that the diagram

$$\begin{tikzcd}
F\underline{\mathsf{C}}(Y;Z) \arrow[r, "FL^X"] \arrow[d, "\underline{F}"'] & F\underline{\mathsf{C}}(\underline{\mathsf{C}}(X;Y);\underline{\mathsf{C}}(X;Z)) \arrow[r, "\underline{F}"] & \underline{\mathsf{D}}(F\underline{\mathsf{C}}(X;Y);F\underline{\mathsf{C}}(X;Z)) \arrow[d, "\underline{\mathsf{D}}(1;\underline{F})"] \\
\underline{\mathsf{D}}(FY;FZ) \arrow[r, "L^{FX}"] & \underline{\mathsf{D}}(\underline{\mathsf{D}}(FX;FY);\underline{\mathsf{D}}(FX;FZ)) \arrow[r, "\underline{\mathsf{D}}(\underline{F};1)"] & \underline{\mathsf{D}}(F\underline{\mathsf{C}}(X;Y);\underline{\mathsf{D}}(FX;FZ))
\end{tikzcd}$$

commutes. By closedness, this is equivalent to the commutativity of the exterior of the diagram displayed on Figure 4.2, which expresses the fact that the D-functor $\underline{F} : F_*\underline{\mathsf{C}} \to \underline{\mathsf{D}}$ preserves composition and which is part of Proposition 3.21.

The proposition is proven. ∎

$$
\begin{tikzcd}
F\underline{\mathsf{C}}(X;Y),F\underline{\mathsf{C}}(Y;Z) \arrow[rrrrrr, "\underline{F},\underline{F}"] \arrow[drr, "1{,}\underline{F}"] \arrow[ddr, "1{,}FL^X"'] \arrow[dddddd, "F\mu_{\underline{\mathsf{C}}}"'] & & & & & & \underline{\mathsf{D}}(FX;FY),\underline{\mathsf{D}}(FY;FZ) \arrow[ddll, "1{,}L^{FX}"] \arrow[dddddd, "\mu_{\underline{\mathsf{D}}}"] \\
& & F\underline{\mathsf{C}}(X;Y),\underline{\mathsf{D}}(FY;FZ) \arrow[urrrr, "\underline{F}{,}1"] \arrow[ddr, "1{,}L^{FX}"'] & & & & \\
& F\underline{\mathsf{C}}(X;Y),F\underline{\mathsf{C}}(\underline{\mathsf{C}}(X;Y);\underline{\mathsf{C}}(X;Z)) \arrow[ddddl, "F\,\mathrm{ev}^{\mathsf{C}}"'] \arrow[ddr, "1{,}\underline{F}"'] & & & \underline{\mathsf{D}}(FX;FY),\underline{\mathsf{D}}(\underline{\mathsf{D}}(FX;FY);\underline{\mathsf{D}}(FX;FZ)) \arrow[ddddrr, "\mathrm{ev}^{\mathsf{D}}"] & & \\
& & & F\underline{\mathsf{C}}(X;Y),\underline{\mathsf{D}}(\underline{\mathsf{D}}(FX;FY);\underline{\mathsf{D}}(FX;FZ)) \arrow[ur, "\underline{F}{,}1"'] \arrow[ddr, "1{,}\underline{\mathsf{D}}(\underline{F};1)"] & & & \\
& & F\underline{\mathsf{C}}(X;Y),\underline{\mathsf{D}}(F\underline{\mathsf{C}}(X;Y);F\underline{\mathsf{C}}(X;Z)) \arrow[ddll, "\mathrm{ev}^{\mathsf{D}}"'] \arrow[drr, "1{,}\underline{\mathsf{D}}(1;\underline{F})"'] & & & & \\
& & & & F\underline{\mathsf{C}}(X;Y),\underline{\mathsf{D}}(F\underline{\mathsf{C}}(X;Y);\underline{\mathsf{D}}(FX;FZ)) \arrow[drr, "\mathrm{ev}^{\mathsf{D}}"] & & \\
F\underline{\mathsf{C}}(X;Z) \arrow[rrrrrr, "\underline{F}"] & & & & & & \underline{\mathsf{D}}(FX;FZ)
\end{tikzcd}
$$

4.5. Proposition. *A multinatural transformation $t : F \to G : \mathsf{C} \to \mathsf{D}$ of multifunctors between closed multicategories with unit objects gives rise to a closed natural transformation given by the same components.*

Proof. Let $\Phi = (\phi, \hat{\phi}, \phi^0), \Psi = (\psi, \hat{\psi}, \psi^0) : \mathcal{C} \to \mathcal{D}$ be closed functors induced by the multifunctors F and G respectively. The axiom CN1 reads

$$\big[\mathbb{1} \xrightarrow{\phi^0} F\mathbb{1} \xrightarrow{t_{\mathbb{1}}} G\mathbb{1}\big] = \psi^0.$$

It is equivalent to the equation

$$\big[() \xrightarrow{u} \mathbb{1} \xrightarrow{\phi^0} F\mathbb{1} \xrightarrow{t_{\mathbb{1}}} G\mathbb{1}\big] = u \cdot \psi^0,$$

i.e., to the equation $Fu \cdot t_{\mathbb{1}} = Gu$, which is a consequence of the multinaturality of t. The axiom CN2 is a particular case of Proposition 3.23. ∎

Let $\mathbf{ClMulticat}^u$ denote the full 2-subcategory of **ClMulticat** whose objects are closed multicategories with a unit object. Note that a 2-category is the same thing as a **Cat**-category. Thus we can speak about **Cat**-functors between 2-categories. These are sometimes called strict 2-functors; they preserve composition of 1-morphisms and identity 1-morphisms on the nose.

4.6. Proposition. *Propositions 4.3, 4.4, and 4.5 define a* **Cat**-*functor*

$$U : \mathbf{ClMulticat}^u \to \mathbf{ClCat}.$$

Proof. It is obvious that composition of 2-morphisms and identity 2-morphisms are preserved. It is also clear that the identity multifunctor induces the closed identity functor. Finally, composition of 1-morphisms is preserved by Lemma 3.22. ∎

5. From closed categories to closed multicategories

In this section we prove our main result.

5.1. Theorem. *The* **Cat**-*functor* $U : \mathbf{ClMulticat}^u \to \mathbf{ClCat}$ *is a* **Cat**-*equivalence.*

We have to prove that U is bijective on 1-morphisms and 2-morphisms, and that it is essentially surjective; the latter means that for each closed category $\mathcal{V}$ there is a closed multicategory with a unit object such that its underlying closed category is isomorphic (as a closed category) to $\mathcal{V}$.

5.2. The surjectivity of U on 1-morphisms Let C and D be closed multicategories with unit objects. Denote their underlying closed categories by the same symbols. Let $\Phi = (\phi, \hat{\phi}, \phi^0) : \mathsf{C} \to \mathsf{D}$ be a closed functor. We are going to define a multifunctor

$F : \mathsf{C} \to \mathsf{D}$ whose underlying closed functor is Φ. Define $FX = \phi X$, for each $X \in \operatorname{Ob}\mathsf{C}$. For each $Y \in \operatorname{Ob}\mathsf{C}$, the map $F_{;Y} : \mathsf{C}(;Y) \to \mathsf{D}(;\phi Y)$ is defined via the diagram

$$\begin{tikzcd}
\mathsf{C}(;Y) \arrow[rr, "F_{;Y}"] & & \mathsf{D}(;\phi Y) \\
\mathsf{C}(\mathbb{1};Y) \arrow[u, "\mathsf{C}(u;1)", "\wr"'] \arrow[r, "\phi"] & \mathsf{D}(\phi\mathbb{1};\phi Y) \arrow[r, "\mathsf{D}(\phi^0;1)"] & \mathsf{D}(\mathbb{1};\phi Y) \arrow[u, "\wr", "\mathsf{D}(u;1)"']
\end{tikzcd}$$

Recall that for a morphism $f : () \to Y$ we denote by $\overline{f} : \mathbb{1} \to Y$ a unique morphism such that $u \cdot \overline{f} = f$. Then the commutativity in the above diagram means that

$$Ff = \bigl[() \xrightarrow{u} \mathbb{1} \xrightarrow{\phi^0} \phi\mathbb{1} \xrightarrow{\phi(\overline{f})} \phi Y\bigr], \tag{5.1}$$

for each $f : () \to Y$. For $n \geq 1$ and $X_1, \dots, X_n, Y \in \operatorname{Ob}\mathsf{C}$, the map

$$F_{X_1,\dots,X_n;Y} : \mathsf{C}(X_1, \dots, X_n; Y) \to \mathsf{D}(\phi X_1, \dots, \phi X_n; \phi Y)$$

is defined inductively by requesting the commutativity in the diagram

$$\begin{tikzcd}
\mathsf{C}(X_2, \dots, X_n; \underline{\mathsf{C}}(X_1;Y)) \arrow[r, "F_{X_2,\dots,X_n;\underline{\mathsf{C}}(X_1;Y)}"] \arrow[dd, "\varphi^{\mathsf{C}}"', "\wr"] & \mathsf{D}(\phi X_2, \dots, \phi X_n; \phi\underline{\mathsf{C}}(X_1;Y)) \arrow[d, "\mathsf{D}(1;\hat{\phi})"] \\
& \mathsf{D}(\phi X_2, \dots, \phi X_n; \underline{\mathsf{D}}(\phi X_1;\phi Y)) \arrow[d, "\wr"', "\varphi^{\mathsf{D}}"] \\
\mathsf{C}(X_1, \dots, X_n; Y) \arrow[r, "F_{X_1,\dots,X_n;Y}"] & \mathsf{D}(\phi X_1, \dots, \phi X_n; \phi Y)
\end{tikzcd} \tag{5.2}$$

5.3. Lemma. *The following diagram commutes*

$$\begin{tikzcd}
\mathsf{C}(;\underline{\mathsf{C}}(X;Y)) \arrow[r, "F_{;\underline{\mathsf{C}}(X;Y)}"] \arrow[d, "\varphi^{\mathsf{C}}"', "\wr"] & \mathsf{D}(;\phi\underline{\mathsf{C}}(X;Y)) \arrow[r, "\mathsf{D}(;\hat{\phi})"] & \mathsf{D}(;\underline{\mathsf{D}}(\phi X;\phi Y)) \arrow[d, "\wr"', "\varphi^{\mathsf{D}}"] \\
\mathsf{C}(X;Y) \arrow[rr, "\phi"] & & \mathsf{D}(\phi X;\phi Y)
\end{tikzcd}$$

In particular, $F_{X;Y} = \phi_{X,Y} : \mathsf{C}(X;Y) \to \mathsf{D}(\phi X;\phi Y)$.

Proof. Equivalently, the exterior of the diagram

$$\begin{tikzcd}
\mathsf{C}(;\underline{\mathsf{C}}(X;Y)) \arrow[rr, "F_{;\underline{\mathsf{C}}(X;Y)}"] & & \mathsf{D}(;\phi\underline{\mathsf{C}}(X;Y)) \arrow[r, "\mathsf{D}(;\hat{\phi})"] & \mathsf{D}(;\underline{\mathsf{D}}(\phi X;\phi Y)) \\
\mathsf{C}(\mathbb{1};\underline{\mathsf{C}}(X;Y)) \arrow[u, "\mathsf{C}(u;1)", "\wr"'] \arrow[r, "\phi"] & \mathsf{D}(\phi\mathbb{1};\phi\underline{\mathsf{C}}(X;Y)) \arrow[r, "\mathsf{D}(\phi^0;1)"] & \mathsf{D}(\mathbb{1};\phi\underline{\mathsf{C}}(X;Y)) \arrow[u, "\mathsf{D}(u;1)"', "\wr"] \arrow[r, "\mathsf{D}(1;\hat{\phi})"] & \mathsf{D}(\mathbb{1};\underline{\mathsf{D}}(\phi X;\phi Y)) \arrow[u, "\mathsf{D}(u;1)"', "\wr"] \\
\mathsf{C}(X;Y) \arrow[u, "\gamma"] \arrow[uu, bend left=60, "(\varphi^{\mathsf{C}})^{-1}"] \arrow[rrr, "\phi"] & & & \mathsf{D}(\phi X;\phi Y) \arrow[u, "\gamma"'] \arrow[uu, bend right=60, "(\varphi^{\mathsf{D}})^{-1}"']
\end{tikzcd}$$

commutes. The upper pentagon is the definition of $F_{;\underline{\mathsf{C}}(X;Y)}$. The bottom hexagon commutes. Indeed, taking $f \in \mathsf{C}(X;Y)$ and tracing it along the left-top path yields

$$\begin{aligned}
\phi^0 \cdot \phi(j_X) \cdot \phi\underline{\mathsf{C}}(1;f) \cdot \hat{\phi} &= \phi^0 \cdot \phi(j_X) \cdot \hat{\phi} \cdot \mathsf{D}(1;\phi(f)) && \text{(naturality of } \hat{\phi}\text{)} \\
&= j_{\phi X} \cdot \underline{\mathsf{D}}(1;\phi(f)), && \text{(axiom CF1)}
\end{aligned}$$

which is precisely the image of f along the bottom-right path. ■

5.4. Lemma. *For each $f : () \to Y$ and $Z \in \operatorname{Ob}\mathsf{C}$, the diagram*

$$\begin{tikzcd}
\phi\underline{\mathsf{C}}(Y;Z) \arrow[r, "\phi\underline{\mathsf{C}}(f;1)"] \arrow[d, "\hat{\phi}"'] & \phi\underline{\mathsf{C}}(;Z) = \phi Z \arrow[d, equal] \\
\underline{\mathsf{D}}(\phi Y;\phi Z) \arrow[r, "\underline{\mathsf{D}}(Ff;1)"] & \underline{\mathsf{D}}(;\phi Z) = \phi Z
\end{tikzcd}$$

commutes.

Proof. By definition,

$$Ff = \big[() \xrightarrow{u} \mathbb{1} \xrightarrow{\phi^0} \phi\mathbb{1} \xrightarrow{\phi(\overline{f})} \phi Y\big].$$

The diagram

$$\begin{tikzcd}
\phi\underline{\mathsf{C}}(Y;Z) \arrow[r, "\phi\underline{\mathsf{C}}(\overline{f};1)"] \arrow[d, "\hat{\phi}"'] \arrow[rrr, bend left, "\phi\underline{\mathsf{C}}(f;1)"] & \phi\underline{\mathsf{C}}(\mathbb{1};Z) \arrow[r, "\phi\underline{\mathsf{C}}(u;1)"] \arrow[d, "\hat{\phi}"] & \phi\underline{\mathsf{C}}(;Z) \arrow[r, equal] & \phi Z \arrow[d, equal] \\
\underline{\mathsf{D}}(\phi Y;\phi Z) \arrow[r, "\underline{\mathsf{D}}(\phi(\overline{f});1)"] \arrow[rrr, bend right, "\underline{\mathsf{D}}(Ff;1)"'] & \underline{\mathsf{D}}(\phi\mathbb{1};\phi Z) \arrow[r, "\underline{\mathsf{D}}(u\cdot\phi^0;1)"] & \underline{\mathsf{D}}(;\phi Z) \arrow[r, equal] & \phi Z
\end{tikzcd}$$

commutes. Indeed, the left square commutes by the naturality of $\hat{\phi}$, while the commutativity of the right square is a consequence of the axiom CF2, see (4.1). ■

With the notation of Lemma 3.14, we can rewrite the commutativity condition in diagram (5.2) as a recursive formula for the multigraph morphism F:

$$Ff = \varphi^{\mathsf{D}}(F((\varphi^{\mathsf{C}})^{-1}(f)) \cdot \hat{\phi}) = \varphi^{\mathsf{D}}(F\langle f\rangle \cdot \hat{\phi}),$$

for each $f : X_1, \dots, X_n \to Y$ with $n \geq 1$, or equivalently

$$\langle Ff\rangle = \big[\phi X_2, \dots, \phi X_n \xrightarrow{F\langle f\rangle} \phi\underline{\mathsf{C}}(X_1;Y) \xrightarrow{\hat{\phi}} \underline{\mathsf{D}}(\phi X_1; \phi Y)\big]. \tag{5.3}$$

5.5. Lemma. *For each $X, Y, Z \in \operatorname{Ob}\mathsf{C}$, the diagram*

$$\begin{tikzcd}
\phi\underline{\mathsf{C}}(X;Y), \phi\underline{\mathsf{C}}(Y;Z) \arrow[r, "F\mu_{\underline{\mathsf{C}}}"] \arrow[d, "\hat{\phi},\hat{\phi}"'] & \phi\underline{\mathsf{C}}(X;Z) \arrow[d, "\hat{\phi}"] \\
\underline{\mathsf{D}}(\phi X;\phi Y), \underline{\mathsf{D}}(\phi Y;\phi Z) \arrow[r, "\mu_{\underline{\mathsf{D}}}"] & \underline{\mathsf{D}}(\phi X;\phi Z)
\end{tikzcd}$$

commutes.

Proof. It suffices to prove the equation

$$\langle F\mu_{\underline{\mathsf{C}}} \cdot \hat{\phi}\rangle = \langle(\hat{\phi}, \hat{\phi}) \cdot \mu_{\underline{\mathsf{D}}}\rangle.$$

By Lemma 3.14,(c), the left hand side is equal to

$$\phi\underline{\mathsf{C}}(Y;Z) \xrightarrow{\langle F\mu_{\underline{\mathsf{C}}}\rangle} \underline{\mathsf{D}}(\phi\underline{\mathsf{C}}(X;Y); \phi\underline{\mathsf{C}}(X;Z)) \xrightarrow{\underline{\mathsf{D}}(1;\hat{\phi})} \underline{\mathsf{D}}(\phi\underline{\mathsf{C}}(X;Y); \underline{\mathsf{D}}(\phi X;\phi Z)),$$

while the right hand side is equal to

$$\phi\underline{\mathsf{C}}(Y;Z) \xrightarrow{\hat{\phi}} \underline{\mathsf{D}}(\phi Y;\phi Z) \xrightarrow{\langle\mu_{\underline{\mathsf{D}}}\rangle} \underline{\mathsf{D}}(\underline{\mathsf{D}}(\phi X;\phi Y); \underline{\mathsf{D}}(\phi Y;\phi Z)) \xrightarrow{\underline{\mathsf{D}}(\hat{\phi};1)} \underline{\mathsf{D}}(\phi\underline{\mathsf{C}}(X;Y); \underline{\mathsf{D}}(\phi X;\phi Z))$$

by Lemma 3.14,(b). Note that $\langle\mu_{\underline{\mathsf{D}}}\rangle = (\varphi^{\mathsf{D}})^{-1}(\mu_{\underline{\mathsf{D}}}) = L^{\phi X}$. Furthermore, by (5.3),

$$\begin{aligned}
\langle F\mu_{\underline{\mathsf{C}}}\rangle &= \big[\phi\underline{\mathsf{C}}(Y;Z) \xrightarrow{\phi\langle\mu_{\underline{\mathsf{C}}}\rangle} \phi\underline{\mathsf{C}}(\underline{\mathsf{C}}(X;Y); \underline{\mathsf{C}}(X;Z)) \xrightarrow{\hat{\phi}} \underline{\mathsf{D}}(\phi\underline{\mathsf{C}}(X;Y); \phi\underline{\mathsf{C}}(X;Z))\big] \\
&= \big[\phi\underline{\mathsf{C}}(Y;Z) \xrightarrow{\phi L^X} \phi\underline{\mathsf{C}}(\underline{\mathsf{C}}(X;Y); \underline{\mathsf{C}}(X;Z)) \xrightarrow{\hat{\phi}} \underline{\mathsf{D}}(\phi\underline{\mathsf{C}}(X;Y); \phi\underline{\mathsf{C}}(X;Z))\big],
\end{aligned}$$

therefore the equation in question is simply the axiom CF3. ■

5.6. Proposition. *The multigraph morphism $F : \mathsf{C} \to \mathsf{D}$ is a multifunctor, and its underlying closed functor is Φ.*

Proof. Trivially, F preserves identities since so does ϕ. Let us prove that F preserves composition. The proof is in three steps.

5.7. Lemma. *F preserves composition of the form $X_1, \dots, X_k \xrightarrow{f} Y \xrightarrow{g} Z$.*

Proof. The proof is by induction on k. There is nothing to prove in the case $k = 1$. Suppose that $k = 0$ and we are given composable morphisms

$$() \xrightarrow{f} X \xrightarrow{g} Y.$$

Then since $u \cdot \overline{fg} = f \cdot g = (u \cdot \overline{f}) \cdot g = u \cdot (\overline{f} \cdot g)$, it follows that $\overline{f \cdot g} = \overline{f} \cdot g$. By formula (5.1),

$$F(f \cdot g) = u \cdot \phi^0 \cdot \phi(\overline{f \cdot g}) = u \cdot \phi^0 \cdot \phi(\overline{f} \cdot g) = u \cdot \phi^0 \cdot \phi(\overline{f}) \cdot \phi(g) = Ff \cdot Fg.$$

Suppose that $k > 1$. Then

$$\begin{aligned}
\langle F(f \cdot g)\rangle &= F\langle f \cdot g\rangle \cdot \hat{\phi} && \text{(formula (5.3))} \\
&= F(\langle f\rangle \cdot \underline{\mathsf{C}}(1; g)) \cdot \hat{\phi} && \text{(Lemma 3.14,(c))} \\
&= F\langle f\rangle \cdot \phi\underline{\mathsf{C}}(1; g) \cdot \hat{\phi} && \text{(induction hypothesis)} \\
&= F\langle f\rangle \cdot \hat{\phi} \cdot \underline{\mathsf{D}}(1; \phi(g)) && \text{(naturality of } \hat{\phi}) \\
&= \langle Ff\rangle \cdot \underline{\mathsf{D}}(1; Fg) && \text{(formula (5.3))} \\
&= \langle Ff \cdot Fg\rangle, && \text{(Lemma 3.14,(c))}
\end{aligned}$$

and induction goes through. ∎

5.8. Lemma. *F preserves composition of the form*

$$X_1^1, \dots, X_1^{k_1}, X_2^1, \dots, X_2^{k_2} \xrightarrow{f_1, f_2} Y_1, Y_2 \xrightarrow{g} Z$$

.

Proof. The proof is by induction on k_1. If $k_1 = 0$, then by Lemma 3.14,(a),

$$(f_1, f_2) \cdot g = \big[X_2^1, \dots, X_2^{k_2} \xrightarrow{f_2} Y_2 \xrightarrow{\langle g\rangle} \underline{\mathsf{C}}(Y_1; Z) \xrightarrow{\underline{\mathsf{C}}(f_1;1)} \underline{\mathsf{C}}(; Z) = Z\big],$$

therefore

$$\begin{aligned}
F((f_1, f_2) \cdot g) &= Ff_2 \cdot \phi\langle g\rangle \cdot \phi\underline{\mathsf{C}}(f_1; 1) && \text{(Lemma 5.7)} \\
&= Ff_2 \cdot \phi\langle g\rangle \cdot \hat{\phi} \cdot \underline{\mathsf{D}}(\phi(f_1); 1) && \text{(Lemma 5.4)} \\
&= Ff_2 \cdot \langle Fg\rangle \cdot \underline{\mathsf{D}}(Ff_1; 1) && \text{(formula (5.3))} \\
&= (Ff_1, Ff_2) \cdot Fg. && \text{(Lemma 3.14,(a))}
\end{aligned}$$

If $k_1 = 1$, then by Lemma 3.14,(b),

$$\langle (f_1, f_2) \cdot g\rangle = \big[X_2^1, \dots, X_2^{k_2} \xrightarrow{f_2} Y_2 \xrightarrow{\langle g\rangle} \underline{\mathsf{C}}(Y_1; Z) \xrightarrow{\underline{\mathsf{C}}(f_1;1)} \underline{\mathsf{C}}(X_1^1; Z)\big],$$

therefore

$$
\begin{aligned}
\langle F((f_1,f_2)\cdot g)\rangle &= F\langle (f_1,f_2)\cdot g\rangle \cdot \hat\phi && \text{(formula (5.3))}\\
&= Ff_2\cdot \phi\langle g\rangle \cdot \phi\underline{\mathsf{C}}(f_1;1)\cdot \hat\phi && \text{(Lemma 5.7)}\\
&= Ff_2\cdot \phi\langle g\rangle \cdot \hat\phi\cdot \underline{\mathsf{D}}(\phi(f_1);1) && \text{(naturality of } \hat\phi)\\
&= Ff_2\cdot \langle Fg\rangle \cdot \underline{\mathsf{D}}(Ff_1;1) && \text{(formula (5.3))}\\
&= \langle (Ff_1,Ff_2)\cdot Fg\rangle, && \text{(Lemma 3.14,(b))}
\end{aligned}
$$

and hence $F((f_1,f_2)\cdot g) = (Ff_1,Ff_2)\cdot Fg$. Suppose that $k_1>1$. Then by Lemma 3.14,(c) $\langle (f_1,f_2)\cdot g\rangle$ is equal to the composite

$$
X_1^2,\dots,X_1^{k_1},X_2^1,\dots,X_2^{k_2} \xrightarrow{\langle f_1\rangle,f_2} \underline{\mathsf{C}}(X_1^1;Y_1),Y_2 \xrightarrow{1,\langle g\rangle} \underline{\mathsf{C}}(X_1^1;Y_1),\underline{\mathsf{C}}(Y_1;Z) \xrightarrow{\mu_{\underline{\mathsf{C}}}} \underline{\mathsf{C}}(X_1^1;Z),
$$

therefore

$$
\begin{aligned}
\langle F((f_1,f_2)\cdot g)\rangle &= F\langle (f_1,f_2)\cdot g\rangle \cdot \hat\phi && \text{(formula (5.3))}\\
&= (F\langle f_1\rangle, Ff_2)\cdot F((1,\langle g\rangle)\mu_{\underline{\mathsf{C}}})\cdot\hat\phi && \text{(induction hypothesis)}\\
&= (F\langle f_1\rangle, Ff_2)\cdot (1,F\langle g\rangle)\cdot F\mu_{\underline{\mathsf{C}}}\cdot\hat\phi && \text{(case } k_1=1)\\
&= (F\langle f_1\rangle, Ff_2)\cdot (1,F\langle g\rangle)\cdot (\hat\phi,\hat\phi)\cdot\mu_{\underline{\mathsf{D}}} && \text{(Lemma 5.5)}\\
&= (F\langle f_1\rangle\cdot\hat\phi, Ff_2)\cdot (1,F\langle g\rangle\cdot\hat\phi)\cdot\mu_{\underline{\mathsf{D}}} && \\
&= (\langle Ff_1\rangle, Ff_2)\cdot (1,\langle Fg\rangle)\cdot\mu_{\underline{\mathsf{D}}} && \text{(formula (5.3))}\\
&= \langle (Ff_1,Ff_2)\cdot Fg\rangle, && \text{(Lemma 3.14,(c))}
\end{aligned}
$$

hence $F((f_1,f_2)\cdot g) = (Ff_1,Ff_2)\cdot Fg$, and the lemma is proven. ∎

5.9. Lemma. *F preserves composition of the form*

$$
X_1^1,\dots,X_1^{k_1},\dots,X_n^1,\dots,X_n^{k_n} \xrightarrow{f_1,\dots,f_n} Y_1,\dots,Y_n \xrightarrow{g} Z. \tag{5.4}
$$

Proof. The proof is by induction on n, and for a fixed n by induction on k_1. We have worked out the cases $n=1$ and $n=2$ explicitly in Lemmas 5.7 and 5.8. Assume that F preserves an arbitrary composition of the form

$$
U_1^1,\dots,U_1^{l_1},\dots,U_{n-1}^1,\dots,U_{n-1}^{l_{n-1}} \xrightarrow{p_1,\dots,p_{n-1}} V_1,\dots,V_{n-1} \xrightarrow{q} W,
$$

and suppose we are given composite (5.4). We do induction on k_1. If $k_1=0$, then by Lemma 3.14,(a) $(f_1,\dots,f_n)\cdot g$ is equal to the composite

$$
X_2^1,\dots,X_2^{k_2},\dots,X_n^1,\dots,X_n^{k_n} \xrightarrow{f_2,\dots,f_n} Y_2,\dots,Y_n \xrightarrow{\langle g\rangle} \underline{\mathsf{C}}(Y_1;Z) \xrightarrow{\underline{\mathsf{C}}(f_1;1)} \underline{\mathsf{C}}(;Z)=Z,
$$

therefore

$$
\begin{aligned}
F((f_1,\dots,f_n)\cdot g) &= (Ff_2,\dots,Ff_n)\cdot F(\langle g\rangle\cdot\underline{\mathsf{C}}(f_1;1)) && \text{(induction hypothesis)}\\
&= (Ff_2,\dots,Ff_n)\cdot(F\langle g\rangle\cdot\phi\underline{\mathsf{C}}(f_1;1)) && \text{(Lemma 5.7)}\\
&= (Ff_2,\dots,Ff_n)\cdot(F\langle g\rangle\cdot\hat{\phi}\cdot\underline{\mathsf{D}}(\phi(f_1);1)) && \text{(Lemma 5.4)}\\
&= (Ff_2,\dots,Ff_n)\cdot(\langle Fg\rangle\cdot\underline{\mathsf{D}}(Ff_1;1)) && \text{(formula (5.3))}\\
&= (Ff_1,\dots,Ff_n)\cdot Fg. && \text{(Lemma 3.14,(a))}
\end{aligned}
$$

Suppose that $k_1=1$. Then by Lemma 3.14,(b) $\langle(f_1,\dots,f_n)\cdot g\rangle$ is equal to the composite

$$X_2^1,\dots,X_2^{k_2},\dots,X_n^1,\dots,X_n^{k_n}\xrightarrow{f_2,\dots,f_n}Y_2,\dots,Y_n\xrightarrow{\langle g\rangle}\underline{\mathsf{C}}(Y_1;Z)\xrightarrow{\underline{\mathsf{C}}(f_1;1)}\underline{\mathsf{C}}(X_1^1;Z),$$

therefore

$$
\begin{aligned}
\langle F((f_1,\dots,f_n)\cdot g)\rangle &= F\langle(f_1,\dots,f_n)\cdot g\rangle\cdot\hat{\phi} && \text{(formula (5.3))}\\
&= (Ff_2,\dots,Ff_n)\cdot F(\langle g\rangle\cdot\underline{\mathsf{C}}(f_1;1))\cdot\hat{\phi} && \text{(induction hypothesis)}\\
&= (Ff_2,\dots,Ff_n)\cdot F\langle g\rangle\cdot\phi\underline{\mathsf{C}}(f_1;1)\cdot\hat{\phi} && \text{(Lemma 5.7)}\\
&= (Ff_2,\dots,Ff_n)\cdot F\langle g\rangle\cdot\hat{\phi}\cdot\underline{\mathsf{D}}(\phi(f_1);1) && \text{(naturality of }\hat{\phi})\\
&= (Ff_2,\dots,Ff_n)\cdot\langle Fg\rangle\cdot\underline{\mathsf{D}}(Ff_1;1) && \text{(formula (5.3))}\\
&= \langle(Ff_1,\dots,Ff_n)\cdot Fg\rangle, && \text{(Lemma 3.14,(b))}
\end{aligned}
$$

and hence $F((f_1,\dots,f_n)\cdot g)=(Ff_1,\dots,Ff_n)\cdot Fg$. Suppose that $k_1>1$, then by Lemma 3.14,(c) $\langle(f_1,\dots,f_n)\cdot g\rangle$ is equal to the composite

$$
\begin{aligned}
X_1^2,\dots,X_1^{k_1},X_2^1,\dots,X_2^{k_2},\dots,X_n^1,\dots,X_n^{k_n}&\xrightarrow{\langle f_1\rangle,f_2,\dots,f_n}\underline{\mathsf{C}}(X_1^1;Y_1),Y_2,\dots,Y_n\\
&\xrightarrow{1,\langle g\rangle}\underline{\mathsf{C}}(X_1^1;Y_1),\underline{\mathsf{C}}(Y_1;Z)\\
&\xrightarrow{\mu_{\underline{\mathsf{C}}}}\underline{\mathsf{C}}(X_1^1;Z),
\end{aligned}
$$

therefore

$$
\begin{aligned}
\langle F((f_1,\dots,f_n)\cdot g)\rangle &= F\langle(f_1,\dots,f_n)\cdot g\rangle\cdot\hat{\phi} && \text{(formula (5.3))}\\
&= (F\langle f_1\rangle,Ff_2,\dots,Ff_n)\cdot F((1,\langle g\rangle)\mu_{\underline{\mathsf{C}}})\cdot\hat{\phi} && \text{(induction hypothesis)}\\
&= (F\langle f_1\rangle,Ff_2,\dots,Ff_n)\cdot(1,F[g])\cdot F\mu_{\underline{\mathsf{C}}}\cdot\hat{\phi} && \text{(Lemma 5.8)}\\
&= (F\langle f_1\rangle,Ff_2,\dots,Ff_n)\cdot(1,F\langle g\rangle)\cdot(\hat{\phi},\hat{\phi})\cdot\mu_{\underline{\mathsf{D}}} && \text{(Lemma 5.5)}\\
&= (F\langle f_1\rangle\cdot\hat{\phi},Ff_2,\dots,Ff_n)\cdot(1,F\langle g\rangle\cdot\hat{\phi})\cdot\mu_{\underline{\mathsf{D}}}\\
&= (\langle Ff_1\rangle,Ff_2,\dots,Ff_n)\cdot(1,\langle Fg\rangle)\cdot\mu_{\underline{\mathsf{D}}} && \text{(formula (5.3))}\\
&= \langle(Ff_1,\dots,Ff_n)\cdot Fg\rangle, && \text{(Lemma 3.14,(c))}
\end{aligned}
$$

hence $F((f_1,\dots,f_n)\cdot g)=(Ff_1,\dots,Ff_n)\cdot Fg$, and induction goes through. ∎

Thus we have proven that $F : \mathsf{C} \to \mathsf{D}$ is a multifunctor. By construction, its underlying functor is ϕ. Furthermore, the closing transformation $\underline{F}_{X;Y}$ coincides with $\hat{\phi}_{X,Y} : \phi\underline{\mathsf{C}}(X;Y) \to \underline{\mathsf{D}}(\phi X;\phi Y)$. Indeed, we first observe that $\underline{F}_{X,Y} = \langle F\,\mathrm{ev}^{\mathsf{C}}\rangle$, where $\mathrm{ev}^{\mathsf{C}} : X, \underline{\mathsf{C}}(X;Y) \to Y$ is the evaluation morphism. Further, by formula (5.3),

$$\underline{F}_{X,Y} = \langle F\,\mathrm{ev}^{\mathsf{C}}\rangle = \phi\langle \mathrm{ev}^{\mathsf{C}}\rangle \cdot \hat{\phi}_{X,Y} = \hat{\phi}_{X,Y},$$

since $\langle \mathrm{ev}^{\mathsf{C}}\rangle = 1 : \underline{\mathsf{C}}(X;Y) \to \underline{\mathsf{C}}(X;Y)$. Finally,

$$Fu = \bigl[() \xrightarrow{u} \mathbb{1} \xrightarrow{\phi^0} \phi\mathbb{1}\bigr].$$

Indeed, by formula (5.1),

$$Fu = \bigl[() \xrightarrow{u} \mathbb{1} \xrightarrow{\phi^0} \phi\mathbb{1} \xrightarrow{\phi(\overline{u})} \phi\mathbb{1}\bigr] = \bigl[() \xrightarrow{u} \mathbb{1} \xrightarrow{\phi^0} \phi\mathbb{1}\bigr],$$

since $\overline{u} = 1 : \mathbb{1} \to \mathbb{1}$. Thus we conclude that $F : \mathsf{C} \to \mathsf{D}$ is a multifunctor whose underlying closed functor is Φ. The proposition is proven. ■

5.10. The injectivity of U on 1-morphisms The following proposition shows that the **Cat**-functor U is injective on 1-morphisms.

5.11. Proposition. *Let $F, G : \mathsf{C} \to \mathsf{D}$ be multifunctors between closed multicategories with unit objects. Suppose that F and G induce the same closed functor $\Phi = (\phi, \hat{\phi}, \phi^0)$ between the underlying closed categories. Then $F = G$.*

Proof. By assumption, the underlying functors of the multifunctors F and G are the same and are equal to the functor ϕ. Let us prove that $Ff = Gf$, for each $f : X_1, \dots, X_n \to Y$. The proof is by induction on n. There is nothing to prove if $n = 1$. Suppose that $n = 0$, i.e., f is a morphism $() \to Y$. Then since F and G are multifunctors,

$$Ff = F(u \cdot \overline{f}) = Fu \cdot F\overline{f}, \qquad Gf = G(u \cdot \overline{f}) = Gu \cdot G\overline{f}.$$

Since F and G coincide on morphisms with one source object, it follows that $F\overline{f} = G\overline{f}$. Furthermore,

$$Fu = \bigl[() \xrightarrow{u} \mathbb{1} \xrightarrow{\phi^0} F\mathbb{1} = G\mathbb{1}\bigr] = Gu,$$

hence $Ff = Gf$. The induction step follows from the commutative diagram

$$\begin{tikzcd}
\mathsf{C}(X_2, \dots, X_n; \underline{\mathsf{C}}(X_1;Y)) \arrow[r, "F"] \arrow[dd, "\varphi^{\mathsf{C}}"', "\wr"] & \mathsf{D}(\phi X_2, \dots, \phi X_n; \phi\underline{\mathsf{C}}(X_1;Y)) \arrow[d, "\mathsf{D}(1;\hat{\phi})"] \\
& \mathsf{D}(\phi X_2, \dots, \phi X_n; \underline{\mathsf{D}}(\phi X_1;\phi Y)) \arrow[d, "\varphi^{\mathsf{D}}", "\wr"'] \\
\mathsf{C}(X_1, \dots, X_n; Y) \arrow[r, "F"] & \mathsf{D}(\phi X_1, \dots, \phi X_n; \phi Y)
\end{tikzcd}$$

and a similar diagram for G, which are particular cases of Proposition 3.20. ■

5.12. The bijectivity of U on 2-morphisms The following proposition implies that U is bijective on 2-morphisms.

5.13. Proposition. *Let $F,G:\mathsf{C}\to\mathsf{D}$ be multifunctors between closed multicategories with unit objects. Denote by $\Phi=(\phi,\hat{\phi},\phi^0)$ and $\Psi=(\psi,\hat{\psi},\psi^0)$ the corresponding closed functors. Let $r:\Phi\to\Psi$ be a closed natural transformation. Then r is also a multinatural transformation $F\to G:\mathsf{C}\to\mathsf{D}$.*

Proof. We must prove that, for each $f:X_1,\dots,X_n\to Y$, the equation

$$Ff\cdot r_Y=(r_{X_1},\dots,r_{X_n})\cdot Gf$$

holds true. The proof is by induction on n. Suppose that $n=0$, and that f is a morphism $()\to Y$. The axiom CN1

$$\big[\mathbb{1}\xrightarrow{\phi^0}F\mathbb{1}\xrightarrow{r_{\mathbb{1}}}G\mathbb{1}\big]=\psi^0$$

implies

$$\big[()\xrightarrow{Fu}F\mathbb{1}\xrightarrow{r_{\mathbb{1}}}G\mathbb{1}\big]=Gu.$$

It follows that

$$Ff\cdot r_Y=Fu\cdot F\overline{f}\cdot r_Y=Fu\cdot r_{\mathbb{1}}\cdot G\overline{f}=Gu\cdot G\overline{f}=Gf,$$

where the second equality is due to the naturality of r. There is nothing to prove in the case $n=1$. Suppose that $n>1$. It suffices to prove that

$$\langle Ff\cdot r_Y\rangle=\langle(r_{X_1},\dots,r_{X_n})\cdot Gf\rangle:FX_2,\dots,FX_n\to\underline{\mathsf{D}}(FX_1;GY).$$

By Lemma 3.14,(c), the left hand side expands out as $\langle Ff\rangle\cdot\underline{\mathsf{D}}(1;r_Y)$, which by formula (5.3) is equal to $F\langle f\rangle\cdot\hat{\phi}\cdot\underline{\mathsf{D}}(1;r_Y)$. By Lemma 3.14,(b), the right hand side of the equation in question is equal to $(r_{X_2},\dots,r_{X_n})\cdot\langle Gf\rangle\cdot\underline{\mathsf{D}}(r_{X_1};1)$, which by formula (5.3) is equal to $(r_{X_2},\dots,r_{X_n})\cdot G\langle f\rangle\cdot\hat{\psi}\cdot\underline{\mathsf{D}}(r_{X_1};1)$. By the induction hypothesis, the latter is equal to $F\langle f\rangle\cdot r_{\underline{\mathsf{C}}(X_1;Y)}\cdot\hat{\psi}\cdot\underline{\mathsf{D}}(r_{X_1};1)$. The required equation follows then from the axiom CN2. ■

5.14. The essential surjectivity of U Let us prove that for each closed category $\mathcal{V}$ there is a closed multicategory V with a unit object whose underlying closed category is isomorphic to $\mathcal{V}$. First of all, notice that by Theorem 2.19 we may (and we shall) assume in what follows that $\mathcal{V}$ is a closed category in the sense of Eilenberg and Kelly; i.e., that $\mathcal{V}$ is equipped with a functor $V:\mathcal{V}\to\mathcal{S}$ such that $V\underline{\mathcal{V}}(-,-)=\mathcal{V}(-,-):\mathcal{V}^{\mathrm{op}}\times\mathcal{V}\to\mathcal{S}$ and the axiom CC5' is satisfied. In particular, we can use the whole theory of closed categories developed in [2] without any modifications. We are now going to construct a closed multicategory V with a unit object whose underlying closed category is isomorphic to $\mathcal{V}$. The construction is based on ideas of Laplaza's paper [9].

We begin by recalling that for each object X of the category $\mathcal{V}$ one can assign a $\mathcal{V}$-functor $L^X:\underline{\mathcal{V}}\to\underline{\mathcal{V}}$, and for each $f\in V\underline{\mathcal{V}}(X,Y)=\mathcal{V}(X,Y)$ there is a unique $\mathcal{V}$-natural transformation $L^f : L^Y \to L^X : \underline{\mathcal{V}} \to \underline{\mathcal{V}}$ such that $(V(L^f)_Y)1_Y = f$, see Examples 2.13, 2.15, 2.21, or [2, Section 9]. Moreover, by [2, Proposition 9.2] the assignments $X \mapsto L^X$ and $f \mapsto L^f$ determine a fully faithful functor from the category $\mathcal{V}^{\mathrm{op}}$ to the category $\mathcal{V}\text{-}\mathbf{Cat}(\underline{\mathcal{V}},\underline{\mathcal{V}})$ of $\mathcal{V}$-functors $\underline{\mathcal{V}} \to \underline{\mathcal{V}}$ and their $\mathcal{V}$-natural transformations. For us it is more convenient to write it as functor from $\mathcal{V}$ to $\mathcal{V}\text{-}\mathbf{Cat}(\underline{\mathcal{V}},\underline{\mathcal{V}})^{\mathrm{op}}$. Note that the latter category is strict monoidal with the tensor product given by composition of $\mathcal{V}$-functors. More precisely, the tensor product of F and G in the given order is $FG = F \cdot G = G \circ F$. Consider the multicategory associated with $\mathcal{V}\text{-}\mathbf{Cat}(\underline{\mathcal{V}},\underline{\mathcal{V}})^{\mathrm{op}}$ (see Example 3.3) and consider its full submulticategory whose objects are $\mathcal{V}$-functors L^X, $X \in \operatorname{Ob}\mathcal{V}$. That is, in essence, our V. More precisely, $\operatorname{Ob}\mathsf{V} = \operatorname{Ob}\mathcal{V}$ and

$$\begin{aligned}\mathsf{V}(X_1,\dots,X_n;Y) &= \mathcal{V}\text{-}\mathbf{Cat}(\underline{\mathcal{V}},\underline{\mathcal{V}})^{\mathrm{op}}(L^{X_1}\cdot\dots\cdot L^{X_n}, L^Y)\\ &= \mathcal{V}\text{-}\mathbf{Cat}(\underline{\mathcal{V}},\underline{\mathcal{V}})(L^Y, L^{X_n}\circ\dots\circ L^{X_1}).\end{aligned}$$

Identities and composition coincide with those of the multicategory associated with the strict monoidal category $\mathcal{V}\text{-}\mathbf{Cat}(\underline{\mathcal{V}},\underline{\mathcal{V}})^{\mathrm{op}}$. Note that by Proposition 2.20 there is a bijection

$$\Gamma : \mathsf{V}(X_1,\dots,X_n;Y) \to (V\circ L^{X_n}\circ\dots\circ L^{X_1})Y, \quad f \mapsto (Vf_Y)1_Y.$$

5.15. Theorem. *The multicategory* V *is closed and has a unit object. The underlying closed category of* V *is isomorphic to* $\mathcal{V}$.

Proof. First, let us check that the multicategory V is closed. By Proposition 3.9, it suffices to prove that for each pair of objects X and Z there exist an internal Hom-object $\underline{\mathsf{V}}(X;Z)$ and an evaluation morphism $\mathrm{ev}^{\mathsf{V}}_{X;Z} : X, \underline{\mathsf{V}}(X;Z) \to Z$ such that the map

$$\varphi : \mathsf{V}(Y_1,\dots,Y_n;\underline{\mathsf{V}}(X;Z)) \to \mathsf{V}(X,Y_1,\dots,Y_n;Z), \quad f \mapsto (1_X, f)\cdot \mathrm{ev}^{\mathsf{V}}_{X;Z},$$

is bijective, for each sequence of objects $Y_1, \dots, Y_n$. We set $\underline{\mathsf{V}}(X;Z) = \underline{\mathcal{V}}(X,Z)$. The evaluation map $\mathrm{ev}^{\mathsf{V}}_{X;Z} : X, \underline{\mathsf{V}}(X;Z) \to Z$ is by definition a $\mathcal{V}$-natural transformation $L^Z \to L^{\underline{\mathcal{V}}(X,Z)}\circ L^X$. We define it by requesting $(V(\mathrm{ev}^{\mathsf{V}}_{X;Z})_Z)1_Z = 1_{\underline{\mathcal{V}}(X,Z)}$ (we extensively use the representation theorem for $\mathcal{V}$-functors in the form of Proposition 2.20). Let us check that the map φ is bijective. Note that the codomain of φ identifies via the map Γ with the set $(V\circ L^{Y_n}\circ\dots\circ L^{Y_1}\circ L^X)Z$, and that the domain of φ identifies via Γ with the set

$$(V\circ L^{Y_n}\circ\dots\circ L^{Y_1})\underline{\mathcal{V}}(X,Z) = (V\circ L^{Y_n}\circ\dots\circ L^{Y_1}\circ L^X)Z.$$

The bijectivity of φ follows readily from the diagram

$$\begin{tikzcd}[column sep=small] \mathsf{V}(Y_1,\dots,Y_n;\underline{\mathsf{V}}(X;Z)) \arrow[rr, "\varphi"] \arrow[dr, "\Gamma"'] & & \mathsf{V}(X,Y_1,\dots,Y_n;Z) \arrow[dl, "\Gamma"] \\ & (V\circ L^{Y_n}\circ\dots\circ L^{Y_1}\circ L^X)Z & \end{tikzcd}$$

whose commutativity we are going to establish. Take an $f \in \mathsf{V}(Y_1, \dots, Y_n; \underline{\mathsf{V}}(X;Z))$, i.e., a $\mathcal{V}$-natural transformation $f : L^{\underline{\mathcal{V}}(X,Z)} \to L^{Y_n} \circ \dots \circ L^{Y_1}$. Then $\varphi(f)$ is given by the composite

$$L^Z \xrightarrow{\mathrm{ev}^{\mathsf{V}}_{X;Z}} L^{\underline{\mathcal{V}}(X,Z)} \circ L^X \xrightarrow{fL^X} L^{Y_n} \circ \dots \circ L^{Y_1} \circ L^X.$$

Therefore, $\Gamma\varphi(f)$ is equal to

$$\big(V((fL^X) \circ \mathrm{ev}^{\mathsf{V}}_{X;Z})_Z\big)1_Z = \big(V(fL^X)_Z\big)\big(V\,\mathrm{ev}^{\mathsf{V}}_{X;Z}\big)1_Z = (Vf_{\underline{\mathcal{V}}(X,Z)})1_{\underline{\mathcal{V}}(X,Z)} = \Gamma(f).$$

Thus we conclude that V is a closed multicategory.

Let us check that $\mathbb{1} \in \mathrm{Ob}\,\mathcal{V}$ is a unit object of V. By definition, a morphism $u : () \to \mathbb{1}$ is a $\mathcal{V}$-natural transformation $L^{\mathbb{1}} \to \mathrm{Id}$. We let it be equal to i^{-1}, which is a $\mathcal{V}$-natural transformation by [2, Proposition 8.5]. Then for each object X of $\mathcal{V}$ holds

$$\underline{\mathsf{V}}(u;1) = (u,1) \cdot \mathrm{ev}^{\mathsf{V}}_{\mathbb{1};X} : \underline{\mathsf{V}}(\mathbb{1};X) \to X,$$

i.e., $\underline{\mathsf{V}}(u;1)$ is the V-natural transformation

$$L^X \xrightarrow{\mathrm{ev}^{\mathsf{V}}_{\mathbb{1};X}} L^{\underline{\mathcal{V}}(\mathbb{1};X)} \circ L^{\mathbb{1}} \xrightarrow{L^{\underline{\mathcal{V}}(\mathbb{1},X)}u} L^{\underline{\mathcal{V}}(\mathbb{1},X)}.$$

We claim that it coincides with $L^{i_X^{-1}}$ and hence is invertible. Indeed, applying Γ to the above composite we obtain

$$\begin{aligned}\big(V((L^{\underline{\mathcal{V}}(\mathbb{1},X)}u) \circ \mathrm{ev}^{\mathsf{V}}_{\mathbb{1};X})_X\big)1_X &= \big(V(L^{\underline{\mathcal{V}}(\mathbb{1},X)}u)_X\big)\big(V(\mathrm{ev}^{\mathsf{V}}_{\mathbb{1};X})_X\big)1_X \\ &= \mathcal{V}(\underline{\mathcal{V}}(\mathbb{1},X), u_X)1_{\underline{\mathcal{V}}(\mathbb{1},X)} \\ &= u_X = i_X^{-1}.\end{aligned}$$

Let us now describe the underlying closed category of the closed multicategory V. Its objects are those of $\mathcal{V}$, and for each pair of objects X and Y the set of morphisms from X to Y is $\mathsf{V}(X;Y) = \mathcal{V}\text{-}\mathbf{Cat}(\underline{\mathcal{V}}, \underline{\mathcal{V}})(L^Y, L^X)$. The unit object is $\mathbb{1}$ and the internal Hom-object $\underline{\mathsf{V}}(X;Y)$ coincides with $\underline{\mathcal{V}}(X,Y)$. For each object X, the identity morphism $1^{\mathsf{V}}_X : () \to \underline{\mathsf{V}}(X;X)$, i.e., a $\mathcal{V}$-natural transformation $L^{\underline{\mathcal{V}}(X,X)} \to \mathrm{Id}$, is found from the equation

$$\big[X \xrightarrow{1_X, 1^{\mathsf{V}}_X} X, \underline{\mathsf{V}}(X;X) \xrightarrow{\mathrm{ev}^{\mathsf{V}}_{X;X}} X\big] = 1_X,$$

or equivalently from the equation

$$\big[L^X \xrightarrow{\mathrm{ev}^{\mathsf{V}}_{X;X}} L^{\underline{\mathcal{V}}(X,X)} \circ L^X \xrightarrow{1^{\mathsf{V}}_X L^X} L^X\big] = \mathrm{id}.$$

Applying Γ to both sides we find that

$$\begin{aligned}\big(V((1^{\mathsf{V}}_X L^X) \circ \mathrm{ev}^{\mathsf{V}}_{X;X})_X\big)1_X &= \big(V(1^{\mathsf{V}}_X)_{\underline{\mathcal{V}}(X,X)}\big)\big(V(\mathrm{ev}^{\mathsf{V}}_{X;X})_X\big)1_X \\ &= V(1^{\mathsf{V}}_X)_{\underline{\mathcal{V}}(X,X)}1_{\underline{\mathcal{V}}(X,X)} \\ &= 1_X.\end{aligned}$$

Here $V(1^{\mathsf{V}}_X)_{\underline{\mathcal{V}}(X,X)} : \mathcal{V}(\underline{\mathcal{V}}(X,X), \underline{\mathcal{V}}(X,X)) \to \mathcal{V}(X,X)$. The morphism j_X of the underlying closed category of V is a $\mathcal{V}$-natural transformation $L^{\underline{\mathcal{V}}(X,X)} \to L^{\mathbb{1}}$; it is found from the equation

$$\big[L^{\underline{\mathcal{V}}(X,X)} \xrightarrow{j_X} L^{\mathbb{1}} \xrightarrow{u} \mathrm{Id}\big] = 1^{\mathsf{V}}_X.$$

Applying Γ to both sides we obtain

$$\big(V(u \circ j_X)_{\underline{\mathcal{V}}(X,X)}\big)1_{\underline{\mathcal{V}}(X,X)} = V(1^{\mathsf{V}}_X)_{\underline{\mathcal{V}}(X,X)}1_{\underline{\mathcal{V}}(X,X)},$$

i.e.,

$$\big(Vi^{-1}_{\underline{\mathcal{V}}(X,X)}\big)\big(V(j_X)_{\underline{\mathcal{V}}(X,X)}1_{\underline{\mathcal{V}}(X,X)}\big) = 1_X,$$

or equivalently

$$(V(j_X)_{\underline{\mathcal{V}}(X,X)})1_{\underline{\mathcal{V}}(X,X)} = (Vi_{\underline{\mathcal{V}}(X,X)})1_X = j_X,$$

where the last equality is the axiom CC5'. Therefore, $j_X = L^{j_X} : L^{\underline{\mathcal{V}}(X,X)} \to L^{\mathbb{1}}$. It also follows by construction that i_X for the underlying closed category of the closed multicategory V is $(\underline{\mathsf{V}}(u;1))^{-1} = (L^{i_X^{-1}})^{-1} = L^{i_X}$.

Let us compute the morphism $L^X_{YZ} : \underline{\mathsf{V}}(Y;Z) \to \underline{\mathsf{V}}(\underline{\mathsf{V}}(X;Y);\underline{\mathsf{V}}(X;Z))$. First note that $\mathrm{ev}^{\mathsf{V}}_{X;Y} : X, \underline{\mathsf{V}}(X;Y) \to Y$ is the $\mathcal{V}$-natural transformation $L^Y \to L^{\underline{\mathcal{V}}(X,Y)} \circ L^X$ with components

$$(\mathrm{ev}^{\mathsf{V}}_{X;Y})_Z = L^X_{YZ} : \underline{\mathcal{V}}(Y,Z) \to \underline{\mathcal{V}}(\underline{\mathcal{V}}(X,Y), \underline{\mathcal{V}}(X,Z)).$$

In other words, $\mathrm{ev}^{\mathsf{V}}_{X;Y} = L^X_{Y,-}$. Indeed, applying Γ to both side of the equation in question we obtain an equivalent equation

$$(V(\mathrm{ev}^{\mathsf{V}}_{X;Y})_Y)1_Y = (VL^X_{YY})1_Y.$$

Since $VL^X = \mathcal{V}(X,-)$, it follows that $(VL^X_{YY})1_Y = 1_{\underline{\mathcal{V}}(X,Y)}$, so that the above equation is just the definition of $\mathrm{ev}^{\mathsf{V}}_{X;Y}$.

The morphism $L^X_{YZ} : \underline{\mathsf{V}}(Y;Z) \to \underline{\mathsf{V}}(\underline{\mathsf{V}}(X;Y);\underline{\mathsf{V}}(X;Z))$ is uniquely determined by requesting that the diagram

$$\begin{tikzcd}
X, \underline{\mathsf{V}}(X;Y), \underline{\mathsf{V}}(Y;Z) \arrow[r, "{1,1,L^X_{YZ}}"] \arrow[dd, "{\mathrm{ev}^{\mathsf{V}}_{X;Y},1}"'] & X, \underline{\mathsf{V}}(X;Y), \underline{\mathsf{V}}(\underline{\mathsf{V}}(X;Y);\underline{\mathsf{V}}(X;Z)) \arrow[d, "{1,\mathrm{ev}^{\mathsf{V}}_{\underline{\mathsf{V}}(X;Y);\underline{\mathsf{V}}(X;Z)}}"] \\
 & X, \underline{\mathsf{V}}(X;Z) \arrow[d, "\mathrm{ev}^{\mathsf{V}}_{X;Z}"] \\
Y, \underline{\mathsf{V}}(Y;Z) \arrow[r, "\mathrm{ev}^{\mathsf{V}}_{Y;Z}"] & Z
\end{tikzcd}$$

in the multicategory V, or equivalently the diagram

$$\begin{tikzcd}
L^Z \arrow[rr, "\mathrm{ev}^{\mathsf{V}}_{X;Z}"] \arrow[dd, "\mathrm{ev}^{\mathsf{V}}_{Y;Z}"'] & & L^{\underline{\mathcal{V}}(X,Z)} \circ L^X \arrow[d, "\mathrm{ev}^{\mathsf{V}}_{\underline{\mathcal{V}}(X,Y);\underline{\mathcal{V}}(X,Z)}\, L^X"] \\
& & L^{\underline{\mathcal{V}}(\underline{\mathcal{V}}(X,Y),\underline{\mathcal{V}}(X,Z))} \circ L^{\underline{\mathcal{V}}(X,Y)} \circ L^X \arrow[d, "L^X_{YZ} L^{\underline{\mathcal{V}}(X,Y)} L^X"] \\
L^{\underline{\mathcal{V}}(Y,Z)} \circ L^Y \arrow[rr, "L^{\underline{\mathcal{V}}(Y,Z)}\, \mathrm{ev}^{\mathsf{V}}_{X;Y}"] & & L^{\underline{\mathcal{V}}(Y,Z)} \circ L^{\underline{\mathcal{V}}(X,Y)} \circ L^X
\end{tikzcd}$$

in the category $\mathcal{V}$-$\mathbf{Cat}(\underline{\mathcal{V}},\underline{\mathcal{V}})$ commute. Applying Γ to both paths in the latter diagram we obtain

$$\big(V(L^X_{YZ})_{\underline{\mathcal{V}}(\underline{\mathcal{V}}(X,Y),\underline{\mathcal{V}}(X,Z))}\big)\big(V(\mathrm{ev}^{\mathsf{V}}_{\underline{\mathcal{V}}(X,Y);\underline{\mathcal{V}}(X,Z)})_{\underline{\mathcal{V}}(X,Z)}\big)\big(V(\mathrm{ev}^{\mathsf{V}}_{X;Z})_Z\big)1_Z \\ = \mathcal{V}(\underline{\mathcal{V}}(Y,Z),(\mathrm{ev}^{\mathsf{V}}_{X;Y})_Z)(V(\mathrm{ev}^{\mathsf{V}}_{Y;Z})_Z)1_Z,$$

or equivalently

$$\big(V(L^X_{YZ})_{\underline{\mathcal{V}}(\underline{\mathcal{V}}(X,Y),\underline{\mathcal{V}}(X,Z))}\big)1_{\underline{\mathcal{V}}(\underline{\mathcal{V}}(X,Y),\underline{\mathcal{V}}(X,Z))} = (\mathrm{ev}^{\mathsf{V}}_{X;Y})_Z = L^X_{YZ}.$$

In other words, L^X_{YZ} for the underlying closed category of V is $L^{L^X_{YZ}}$.

Let us denote the underlying closed category of the multicategory V by the same symbol. There is a closed functor $(L,1,1) : \mathcal{V} \to \mathsf{V}$, where $L : \mathcal{V} \to \mathsf{V}$ is given by $X \mapsto X$, $f \mapsto L^f$, and the morphisms $\underline{\mathcal{V}}(X,Y) \to \underline{\mathsf{V}}(X;Y)$ and $\mathbb{1} \to \mathbb{1}$ are the identities. The axioms CF1–CF3 follow readily from the above description of the closed category V. Clearly, the functor L is an isomorphism. The theorem is proven. ■

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