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LAWVERE COMPLETENESS AS A TOPOLOGICAL PROPERTY

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Abstract. Lawvere's notion of completeness for quantale-enriched categories has been extended to the theory of lax algebras under the name of L-completeness. In this paper we introduce the corresponding morphism concept and examine its properties. We explore some important relativized topological concepts like separatedness, denseness, compactness and compactification with respect to L-complete morphisms. Moreover, we show that separated L-complete morphisms belong to a factorization system.

1. Introduction

The concept of a $(\mathbb{T}, V)$-category introduced in [11],[8],[10] is a simultaneous generalization of a V-enriched category [22] and a lax Eilenberg-Moore $\mathbb{T}$-algebra. The notion has its roots in Lawvere's interpretation of metric spaces as enriched categories [23] and the Manes-Barr representation of topological spaces as relational algebras [25],[2]. It provides a framework to study metric spaces, topological spaces and approach spaces [24] in an algebraic manner.

Lawvere's 1973 paper [23] describes Cauchy completeness of metric spaces by adjoint modules. A corresponding concept for $(\mathbb{T}, V)$-categories was introduced under the name of L-completeness in [9], which was followed by the development of the concepts of L-separation and L-closure [18]. In this context, to a large extend, L-completeness behaves similarly to compactness. To give a couple of examples, L-completeness is inherited by the L-closed subsets; secondly, for any subset of an L-separated $(\mathbb{T}, V)$-category L-completeness implies L-closedness. In topology the morphism notion for compactness leads to proper maps. Inspired by the interplay between compactness and L-completeness at the level of objects, we introduce a morphism notion for L-completeness which will be the counterpart of proper maps in this context. To establish the analogy between compactness and L-completeness further and rather rigourously we choose to explore topological concepts for $(\mathbb{T}, V)$-categories using this class of maps.

Early instances of the development of topological concepts in a category appear in [27],[26],[16]. More recently, as presented in [7], given a category equipped with a Dikranjan-Giuli closure operator [12] and a proper factorization system, one can pursue topological notions in that category by using a distinguished class of "closed morphisms".

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In fact many of these notions can be expressed by using "proper maps" which are stably "closed" [19]. Having developed an analogue of proper maps in the context of completeness, we put L-complete morphisms to work in a topological framework. Our investigation reveals that the topological concepts, like separation and denseness, can be recovered by L-complete maps, while compactness and compactification naturally translate into L-completeness and L-completion. For example, it is known that any continuous map of topological spaces with compact domain and Hausdorff codomain is proper. For $(!\mathbb{T}, V)$-categories any morphism with an L-complete domain and an L-separated codomain is L-complete. Likewise, the (Antiperfect, Perfect) factorization of continuous maps of Tychonoff spaces [15], [31], [6], [32] is obtained with the help of the left adjoint Stone-Čech compactification functor. Here "antiperfect maps" are the maps which are sent to isomorphisms by the compactification functor. Replacing the notion of compactification by L-completion, we obtain a similar factorization system for $(\mathbb{T}, V)$-categories, where perfect maps are replaced by L-complete and L-separated maps. In the place of the antiperfect maps we now have the morphisms which are sent to isomorphisms by the left adjoint L-completion functor.

Lastly, in [18] we see that L-completeness is equivalent to L-injectivity for objects. Recent work by Cagliari, Clementino and Mantovani [3] shows that for T_0 topological spaces injective maps [4] with respect to completely flat embeddings [13] are exactly fibrewise sober maps, which is also the characterization of L-complete maps in this setting. Encouraged by this result in one of the main examples of $(\mathbb{T}, V)$-categories, we also develop a morphism notion for L-injectivity and show that it is equivalent to L-completeness.

2. Preliminaries

In this section we provide the preliminary concepts for the $(\mathbb{T}, V)$-categories as well as the basic results concerning L-completeness, L-separation and L-closure, which originally appeared in [18], [17], [9]. We refer the reader to these sources for the proofs and more details.

2.1. The quantale V We let $V = (V, \otimes, k)$ to be a commutative unital quantale, in other words a complete lattice with a commutative binary operation $\otimes$ and a unit element k where tensoring preserves suprema in each variable. For $u \in V$, $u \otimes (\_)$ has a right adjoint $u \multimap (\_)$ defined by

$$v \leq u \multimap w \iff v \otimes u \leq w$$

for any $v, w \in V$. We assume that the quantale V is nontrivial, i.e. $V \neq 1$ or, equivalently, $k \neq \bot$. The main examples of nontrivial quantales in our context are $\mathbf{2} = (\{0,1\}, \wedge, 1)$, the extended nonnegative real numbers $\mathbb{P}_+ = ([0,\infty]^{\mathrm{op}}, +, 0)$ and $\mathbb{P}_{\max} = ([0,\infty]^{\mathrm{op}}, \max, 0)$.

A *V-relation* $r : X \to\!\!\!\!\!\!+\; Y$ is a map $r : X \times Y \to V$. Given another V-relation $s : Y \to\!\!\!\!\!\!+\; Z$ one defines the composite $s.r : X \to\!\!\!\!\!\!+\; Z$ by

$$s.r(x,z) = \bigvee_{y \in Y} r(x,y) \otimes s(y,z).$$

Any map $f : X \to Y$ induces a V-relation $f_\circ : X \nrightarrow Y$ where

$$f_\circ(x,y) = \begin{cases} k & \text{if } f(x) = y, \\ \bot & \text{else.} \end{cases}$$

Sets together with V-relations form the category V-**Rel** where $(1_X)_\circ : X \nrightarrow X$ acts as the identity V-relation for any set X. So one has the functor $(\_)_\circ : \mathbf{Set} \to V\text{-}\mathbf{Rel}$ which takes a map f to the V-relation $f_\circ$. Since $k \neq \bot$, this functor is faithful, so it is safe to write f instead of $f_\circ$.

The hom-sets of V-**Rel** carry the pointwise order of V, i.e. for $q, r : X \nrightarrow Y$ $q \le r$ if and only if $q(x,y) \le r(x,y)$ for all $x \in X, y \in Y$. Hence V-**Rel** is a 2-category. This allows one to consider adjunctions in V-**Rel**. One says that $r : X \nrightarrow Y$ is left adjoint to $s : Y \nrightarrow X$, denoted by $r \dashv s$, if $r.s \le 1_Y$ and $s.r \ge 1_X$.

There is an order-preserving involution $V\text{-}\mathbf{Rel} \to (V\text{-}\mathbf{Rel})^{\mathrm{op}}$ which maps the objects identically and sends a V-relation $r : X \nrightarrow Y$ to its opposite relation $r^\circ : Y \nrightarrow X$ given by $r^\circ(y,x) = r(x,y)$. For a map $f : X \to Y$, one has $f \dashv f^\circ$.

2.2. The topological theory $\mathcal{T}$ We assume that $\mathcal{T} = (\mathbb{T}, V, \xi)$ is a strict topological theory [17]. This means that $\mathbb{T} = (T, e, m)$ is a **Set** monad where T and m satisfy the Beck-Chevalley condition, i.e. T sends pullbacks to weak pullbacks and every naturality square of m is a weak pullback. Furthermore, $\xi : TV \to V$ is a map which is compatible with the monad $\mathbb{T}$ and the quantale V, which means

1. $1_V = \xi.e_V$,
2. $\xi.T\xi = \xi.m_V$,
3. $k.!_1 = \xi.Tk$,
4. $\otimes.\langle \xi.T\pi_1, \xi.T\pi_2 \rangle = \xi.T(\otimes)$,
5. $(\xi_X)_X : P_V \to P_V T$ is a natural transformation.

In the last condition $P_V : \mathbf{Set} \to \mathbf{Set}$ is the V-powerset functor defined by $P_V(X) = V^X$ on objects. Given $f : X \to Y$, $P_V(f)(\varphi)(y) = \bigvee_{x \in f^{-1}(y)} \varphi(x)$ for $\varphi \in V^X$ and $\xi_X : P_V(X) \to P_V T(X)$ is defined by $\xi_X(\varphi) = \xi.T\varphi$. We also assume that the functor T sends singletons to singletons.

2.3. Examples.

1. *$\mathcal{I}_V = (\mathbb{1}, V, 1_V)$ is a strict topological theory for any quantale V. Here $\mathbb{1}$ stands for the identity monad.*
2. *$\mathcal{U}_2 = (\mathbb{U}, 2, \xi_2)$ is a strict topological theory with the ultrafilter monad $\mathbb{U} = (U, e, m)$ and $\xi_2 : U(2) \to 2$ which is basically the identity map.*

3. *$\mathcal{U}_{\mathbb{P}_+} = (\mathbb{U}, \mathbb{P}_+, \xi_{\mathbb{P}_+})$ is a strict topological theory with $\xi_{\mathbb{P}_+} : U(\mathbb{P}_+) \to \mathbb{P}_+$ where $\xi_{\mathbb{P}_+}(\mathfrak{x}) = \inf\{v \in V \mid [0, v] \in \mathfrak{x}\}$.*

With these assumptions one extends the functor T : **Set** $\to$ **Set** to $T_\xi : V$-**Rel** $\to$ V-**Rel** [17]. For $r : X \nrightarrow Y$, $T_\xi r : TX \nrightarrow TY$ is given by

$$T_\xi r(\mathfrak{x}, \mathfrak{y}) = \bigvee\{\xi.Tr(\mathfrak{w}) \mid \mathfrak{w} \in T(X \times Y) : T\pi_1(\mathfrak{w}) = \mathfrak{x}, T\pi_2(\mathfrak{w}) = \mathfrak{y}\}.$$

2.4. Proposition. [17] *Given any map $f : X \to Y$, any V-relations $r, s : X \nrightarrow Y$ the following assertions hold.*

1. *$T_\xi f = Tf$.*

2. *$T_\xi(r)^\circ = T_\xi(r^\circ)$.*

3. *$r \leq s$ implies $T_\xi r \leq T_\xi s$.*

4. *$e_Y.r \leq T_\xi r.e_X$.*

5. *$m_Y.{T_\xi}^2 r = T_\xi r.m_X$.*

2.5. $\mathcal{T}$-categories For the sake of economy we will simply write $\mathcal{T}$-category instead of writing $(\mathbb{T}, V)$-category. We will apply this principle to related concepts as well.

A *$\mathcal{T}$-relation* r from X to Y, denoted by $r : X \rightharpoondown Y$, is a V-relation $r : TX \nrightarrow Y$. Composition of two $\mathcal{T}$-relations $r : X \rightharpoondown Y$ and $s : Y \rightharpoondown Z$ is given by the *Kleisli convolution*, $s \circ r := s.\widehat{T}r.m_X^\circ$. One orders $\mathcal{T}$-relations $q, r : X \rightharpoondown Y$ by considering them as V-relations $TX \nrightarrow Y$. Kleisli convolution is an associative operation that respects the order on $\mathcal{T}$-relations . Furthermore, for any $r : X \rightharpoondown Y$ one has $r \circ e_X^\circ = r$ and $e_Y^\circ \circ r \geq r$.

A *$\mathcal{T}$-category* (X, a) is a set X together with a $\mathcal{T}$-relation $a : X \rightharpoondown X$ which satisfies the conditions ${e_X}^\circ \leq a$ and $a \circ a \leq a$. Expressed elementwise these conditions mean

$$k \leq a(e_X(x), x) \qquad \& \qquad T_\xi(\mathfrak{X}, \mathfrak{x}) \otimes a(\mathfrak{x}, x) \leq a(m_X(\mathfrak{X}), x)$$

for all $\mathfrak{X} \in T^2X, \mathfrak{x} \in TX, x \in X$. A $\mathcal{T}$-functor $f : (X, a) \to (Y, b)$ is a map from X to Y that satisfies $f.a \leq b.Tf$ or, equivalently

$$a(\mathfrak{x}, x) \leq b(Tf(\mathfrak{x}), f(x))$$

for all $\mathfrak{x} \in TX, x \in X$. $\mathcal{T}$-categories together with $\mathcal{T}$-functors form the category $\mathcal{T}$-**Cat**.

2.6. Examples.

1. *$\mathcal{I}_V$-**Cat** is isomorphic to **Ord**, **Met**, **UMet** for $V = 2, \mathbb{P}_+, \mathbb{P}_{max}$ respectively. Here **Ord** is the category of (pre)ordered sets, **Met** is the category of (pre)metric spaces [23] and **UMet** is the category of (pre)ultrametric spaces.*

2. *$\mathcal{U}_2$-**Cat** is isomorphic to **Top** [2]. As shown in [8], $\mathcal{U}_{\mathbb{P}_+}$-**Cat** is isomophic to the category **App** of approach spaces [24].*

The quantale V becomes a $\mathcal{T}$-category with the structure map $\hom_\xi : TV \times V \to V$ defined by $\hom_\xi(\mathfrak{x}, v) = \xi(\mathfrak{x}) \multimap v$.

For each $\mathcal{T}$-category (X, a), one has the *free Eilenberg-Moore algebra* $|X| = (TX, m_X)$ and the *dual $\mathcal{T}$-category* X^{op}; one defines $X^{\mathrm{op}} = \mathsf{A}(\mathsf{M}(X)^{\mathrm{op}})$, where $\mathsf{M} : \mathcal{T}\text{-}\mathbf{Cat} \to V\text{-}\mathbf{Cat}$ is given by $\mathsf{M}(X) = (TX, T_\xi a.m_X{}^\circ)$ and $\mathsf{A} : V\text{-}\mathbf{Cat} \to \mathcal{T}\text{-}\mathbf{Cat}$ is given by $\mathsf{A}(X) = (X, e_X{}^\circ.T_\xi b)$ for a V-category (X, b).

Given $\mathcal{T}$-categories (X, a) and (Y, b) one forms their tensor product $(X, a) \otimes (Y, b) = (X \times Y, a \otimes b)$ where

$$a \otimes b(\mathfrak{w}, (x, y)) = a(T\pi_1(\mathfrak{w}), x) \otimes b(T\pi_2(\mathfrak{w}), y)$$

for $\mathfrak{w} \in T(X \times Y), (x, y) \in X \times Y$. The singleton set together with the constant relation k, denoted by (E, k), is the $\otimes$-neutral object. In general $\mathcal{T}\text{-}\mathbf{Cat}$ is not a closed category but for a $\mathcal{T}$-category (X, a), $X \otimes (\_)$ has a right adjoint $(\_)^X$ whenever $a.T_\xi a = a.m_X$ [17]. We call such a $\mathcal{T}$-category (X, a) *tensor exponentiable*. Given another $\mathcal{T}$-category (Y, b), the underlying set of the exponential object $(Y^X, [\![a, b]\!])$ is the set of all $\mathcal{T}$-functors from X to Y. The structure $[\![a, b]\!]$ is defined by

$$[\![a, b]\!](\mathfrak{p}, h) = \bigvee \left\{v \in V \mid \forall \mathfrak{q} \in T\pi_{Y^X}^{-1}(\mathfrak{p}), x \in X;\ a(T\pi_X(\mathfrak{q}), x) \otimes v \leq b(Tev(\mathfrak{q}), h(x))\right\}$$

where $\mathfrak{p} \in T(Y^X), h \in Y^X$ and $ev : Y^X \times X \to Y$ is the evaluation map.

As the forgetful functor from $\mathcal{T}\text{-}\mathbf{Cat}$ to $\mathbf{Set}$ is topological, the limits in $\mathcal{T}\text{-}\mathbf{Cat}$ are formed in $\mathbf{Set}$ with the appropriate structure on them. In particular we will denote the cartesian product of (X, a) and (Y, b) by $(X \times Y, a \times b)$ where $a \times b = (\pi_1{}^\circ.a.T\pi_1) \wedge (\pi_2{}^\circ.b.T\pi_2)$, i.e.

$$a \times b(\mathfrak{w}, (x, y)) = a(T\pi_1(\mathfrak{w}), x) \wedge b(T\pi_2(\mathfrak{w}), y)$$

for all $\mathfrak{w} \in T(X \times Y), (x, y) \in X \times Y$. The terminal object is the singleton set with the constant relation $\top$, which will be denoted by $1 = (1, \top)$.

2.7. $\mathcal{T}$-MODULES Let (X, a), (Y, b) be $\mathcal{T}$-categories and $\varphi : X \rightharpoondown Y$ be a $\mathcal{T}$-relation. φ is called a *$\mathcal{T}$-module* if $\varphi \circ a \leq \varphi$ and $b \circ \varphi \leq \varphi$. In such a case we write $\varphi : X \rightsquigarrow Y$. $\mathcal{T}$-categories and $\mathcal{T}$-modules with the Kleisli convolution form the category $\mathcal{T}\text{-}\mathbf{Mod}$. Since trivially $\varphi \circ a \geq \varphi$ and $b \circ \varphi \geq \varphi$, one actually has $\varphi \circ a = \varphi$ and $b \circ \varphi = \varphi$. As a result $a : (X, a) \rightsquigarrow (X, a)$ functions as the identity morphism of (X, a) in $\mathcal{T}\text{-}\mathbf{Mod}$.

There are two important functors: the lower star functor $(\_)_* : \mathcal{T}\text{-}\mathbf{Cat} \to \mathcal{T}\text{-}\mathbf{Mod}$ and the upper star functor $(\_)^* : (\mathcal{T}\text{-}\mathbf{Cat})^{\mathrm{op}} \to \mathcal{T}\text{-}\mathbf{Mod}$. These functors are identical on objects and they take a $\mathcal{T}$-functor $f : (X, a) \to (Y, b)$ to $f_* = b.Tf : (X, a) \rightsquigarrow (Y, b)$ and $f^* = f^\circ.b : (Y, b) \rightsquigarrow (X, a)$ respectively. Observe that with this notation $a = 1_X{}^* = 1_{X*}$.

$\mathcal{T}\text{-}\mathbf{Mod}$ is a 2-category as $\mathcal{T}$-modules inherit the order on $\mathcal{T}$-relations. This allows one to consider adjunctions in $\mathcal{T}\text{-}\mathbf{Mod}$. For any $\mathcal{T}$-functor $f : X \to Y$ one has $f_* \dashv f^*$. f is called *fully faithful* if $f^* \circ f_* = 1_X{}^*$ and *L-dense* if $f_* \circ f^* = 1_Y{}^*$. Composites of fully faithful (L-dense) $\mathcal{T}$-functors are fully faithful (L-dense). The following proposition will be useful in the sequel.

2.8. Proposition. [18] *Let $f:(X,a)\to(Y,b)$, $g:(Y,b)\to(Z,c)$ be $\mathcal{T}$-functors.*

1. *If $g.f$ is fully faithful then f is fully faithful.*
2. *If $g.f$ is L-dense then g is L-dense.*
3. *If $g.f$ is L-dense and g is fully faithful then f is L-dense.*

A $\mathcal{T}$-functor which is both fully faithful and L-dense is called an *L-equivalence*. L-equivalences are isomorphisms in $\mathcal{T}$**-Mod**. Given $\mathcal{T}$-functors $f,g:X\to Y$ one says that f and g are equivalent if $f_*=g_*$ or, equivalently, $f^*=g^*$, we write $f\simeq g$ in this case.

There is a close relationship between $\mathcal{T}$-modules and $\mathcal{T}$-functors. Suppose that (X,a), (Y,b) are $\mathcal{T}$-categories and $\psi:X \mathrel{-\!\!\!\rightharpoonup} Y$ is a $\mathcal{T}$-relation. Then $\psi:(X,a)\rightsquigarrow(Y,b)$ is a $\mathcal{T}$-module if and only if both $\psi:|X|\otimes Y\to V$ and $\psi:X^{\mathrm{op}}\otimes Y\to V$ are $\mathcal{T}$-functors. In particular for any $\mathcal{T}$-category (X,a), $a:X\rightsquigarrow X$ can be seen as a $\mathcal{T}$-functor $a:|X|\otimes X\to V$. Since $|X|$ is tensor exponentiable one can consider its mate $\boldsymbol{y}=\ulcorner a\urcorner:X\to V^{|X|}$, the *Yoneda functor* of X. It is given by $\boldsymbol{y}(x)=a(\_,x)$. Given $\psi\in V^{|X|}$, $\psi:X^{\mathrm{op}}\to V$ is a $\mathcal{T}$-functor if and only if $\psi(\mathfrak{x})=[\![m_X,\hom_\xi]\!](T\boldsymbol{y}(\mathfrak{x}),\psi)$ for all $\mathfrak{x}\in TX$. This corresponds to the Yoneda lemma for $\mathcal{T}$-categories. One defines the $\mathcal{T}$-category $(\widehat{X},\widehat{a})$ as

$$\widehat{X}=\{\psi\in V^{|X|}\mid \psi:X^{\mathrm{op}}\to V \text{ is a } \mathcal{T}\text{-functor}\}$$

where $\widehat{a}$ is the restriction of $[\![m_X,\hom_\xi]\!]$ to $\widehat{X}$. Letting $\psi=\boldsymbol{y}(x)$ in the above formulation implies that the Yoneda functor $\boldsymbol{y}:(X,a)\to(\widehat{X},\widehat{a})$ is fully faithful.

2.9. L-separation, L-completeness Let $X=(X,a)$ be a $\mathcal{T}$-category. One says that X is *L-separated* if given any $\mathcal{T}$-functors $f,g:Z\to X$, $f\simeq g$ implies $f=g$. In this context it is enough to consider the $\otimes$-neutral object E in the place of Z. Hence X is L-separated if and only if for any $x,y\in X$, $x\simeq y$ implies $x=y$ or, equivalently, the Yoneda functor $\boldsymbol{y}_X:(X,a)\to(\widehat{X},\widehat{a})$ is injective. Since the $\mathcal{T}$-category $(V,\hom_\xi)$ is L-separated, also $V^{|X|}$ and $\widehat{X}$ are L-separated for any $\mathcal{T}$-category X.

In **Top** an object is L-separated if and only if it is T_0. **Top** is a coreflective subcategory of **App**. An approach space is L-separated if and only if its coreflection is a T_0 topological space.

Given a $\mathcal{T}$-category $X=(X,a)$, one says that X is *L-complete* [9] if for any adjunction $\varphi\dashv\psi:X\rightsquigarrow Z$ there exists a $\mathcal{T}$-functor $f:Z\to X$ such that $\varphi=f_*$ or, equivalently, $\psi=f^*$. If one assumes the axiom of choice, Z can be replaced by E. In that case X is L-complete if and only if, given any adjunction $\varphi\dashv\psi:X\rightsquigarrow E$, there exists $x\in X$ such that $\varphi=x_*$, or $\psi=x^*$. As shown in [18], $V^{|X|}$ and $\widehat{X}$ are L-complete for any $\mathcal{T}$-category X.

A topological space is L-complete if and only if it is weakly sober, i.e. every irreducible closed set can be written as the closure of a point.

2.10. PROPOSITION. [18] *Let (X,a) be a $\mathcal{T}$-category. A $\mathcal{T}$-module $\psi : X \rightsquigarrow E$ is a right adjoint if and only if*

$$k \leq \bigvee_{\mathfrak{x}\in TX} \psi(\mathfrak{x}) \otimes T_\xi \widehat{a}(e_{T\widehat{X}}.e_{\widehat{X}}(\psi), T\boldsymbol{y}(\mathfrak{x})).$$

In this case ψ has a left adjoint $\varphi : E \rightsquigarrow X$ where $\varphi(x) = \widehat{a}(e_{\widehat{X}}(\psi), \boldsymbol{y}(x))$.

Let $\psi : |X| \to V$ be a $\mathcal{T}$-functor. One says that ψ is *tight* if $\psi : X^{\mathrm{op}} \to V$ is a $\mathcal{T}$-functor and as a $\mathcal{T}$-module $\psi : X \rightsquigarrow E$ is a right adjoint. We will denote the collection of tight $\mathcal{T}$-functors by $\widetilde{X}$, considered as a subobject of $\widehat{X}$.

A $\mathcal{T}$-category $X = (X,a)$ is called *L-injective* if given any $\mathcal{T}$-functor $h : Y \to X$ and any L-equivalence $f : Y \to Z$, there exists a $\mathcal{T}$-functor $g : Z \to X$ such that $g.f \simeq h$.

2.11. PROPOSITION. [18] *Let $X = (X,a)$ be a $\mathcal{T}$-category. X is L-complete if and only if X is L-injective.*

2.12. L-CLOSURE In $\mathcal{T}$-**Cat**, L-dense $\mathcal{T}$-functors are epimorphisms up to $\simeq$, i.e. a $\mathcal{T}$-functor $m : M \to X$ is L-dense if and only if for all $\mathcal{T}$-functors $f, g : X \to Y$ with $f.m = g.m$ one has $f \simeq g$.

Let (X,a) be a $\mathcal{T}$-category and $M \subseteq X$. One defines the *L-closure* of M in X by

$$\overline{M} = \{x \in X \mid \forall f, g : X \to Y,\ f_{|M} = g_{|M} \Rightarrow f(x) \simeq g(x)\}$$

So $\overline{M}$ is the largest subset of X for which the inclusion map $m : M \hookrightarrow X$ is L-dense. The definition also implies that a $\mathcal{T}$-functor $f : X \to Y$ is L-dense if and only if $\overline{f(X)} = Y$.

For topological spaces, L-closure is precisely the b-closure [1], [30].

2.13. PROPOSITION. [18] *Let (X,a) be a $\mathcal{T}$-category, $M \subseteq X$ and $x \in X$. Suppose that $m : M \hookrightarrow X$ is the inclusion map. Then the following are equivalent:*

1. *$x \in \overline{M}$;*
2. *$k \leq \bigvee_{\mathfrak{x}\in TM} a(\mathfrak{x}, x) \otimes T_\xi a(Te_X.e_X(x), \mathfrak{x})$;*
3. *$m^* \circ x_* \dashv x^* \circ m_*$;*
4. *$x_* : E \rightsquigarrow X$ factors through $m_* : M \rightsquigarrow X$ by a morphism $\varphi : E \rightsquigarrow M$ in $\mathcal{T}$-**Mod**.*

The L-closure is an extensive, monotone, idempotent and hereditary closure operator. A subset M of a $\mathcal{T}$-category X is called *L-closed* if $\overline{M} = M$.

One can formulate L-separatedness via L-closure. Let X be a $\mathcal{T}$-category and $\Delta \subseteq X \times X$ be its diagonal. Then $\overline{\Delta} = \{(x,y) \in X \times X \mid x \simeq y\}$. As a result X is L-separated if and only if the diagonal Δ is L-closed in $X \times X$.

Exploring the relationship between L-closure and L-completeness leads to interesting results.

2.14. Proposition. [18] *Let X be a $\mathcal{T}$-category and $M \subseteq X$.*

1. *If X is L-complete and M is L-closed, then M is L-complete.*
2. *If X is L-separated and M is L-complete, then M is L-closed.*

A closer inspection of $\widetilde{X}$ reveals that $\widetilde{X}$ is the L-closure of $\mathcal{y}(X)$ in $\widehat{X}$. So $\widetilde{X}$ is an L-closed subset of $\widehat{X}$ which is L-complete and L-separated. As a result $\widetilde{X}$ is L-complete and L-separated. Moreover, any $\mathcal{T}$-functor $f : X \to Y$ with Y L-complete and L-separated can be extended to a $\mathcal{T}$-functor $g : \widetilde{X} \to Y$ in a unique way. Hence $\mathcal{T}\text{-}\mathbf{Cat}_{\mathrm{cpl}\ \&\ \mathrm{sep}}$, the full subcategory of L-complete and L-separated $\mathcal{T}$-categories, is a reflective subcategory of $\mathcal{T}\text{-}\mathbf{Cat}$ with the reflection maps $\mathcal{y}_X : X \to \widetilde{X}$.

3. L-completeness, L-separation and L-injectivity for morphisms

3.1. Definition. *Let $f : (X,a) \to (Y,b)$ be a $\mathcal{T}$-functor. We say that f is L-complete if for any left adjoint $\mathcal{T}$-module $\varphi : Z \rightsquigarrow X$ and any $\mathcal{T}$-functor $h : Z \to Y$ such that $f_* \circ \varphi = h_*$, there exists a $\mathcal{T}$-functor $g : Z \to X$ for which $\varphi = g_*$ and $f.g = h$.*

Recall the lower star functor of the previous section. Since $f_* \dashv f^*$ for any $\mathcal{T}$-functor f, one has $(\_)_* : \mathcal{T}\text{-}\mathbf{Cat} \to \mathcal{T}\text{-}\mathbf{Mod}_l$ where $\mathcal{T}\text{-}\mathbf{Mod}_l$ is the subcategory of $\mathcal{T}\text{-}\mathbf{Mod}$ whose morphisms are the left adjoint $\mathcal{T}$-modules. Taking the functor $(\_)_*$ into account, we see that a $\mathcal{T}$-functor is L-complete if and only if it is a $(\_)_*$-quasi cartesian morphism where quasi refers to that fact the morphism g in the definition is only unique up to $\simeq$.

The above definition can be equivalently expressed with the upper star notation, i.e. f is L-complete if given any right adjoint $\mathcal{T}$-module $\psi : X \rightsquigarrow Z$ and any $\mathcal{T}$ functor $h : Z \to Y$ such that $\psi \circ f^* = h^*$, there exists a $\mathcal{T}$-functor $g : Z \to X$ for which $\psi = g^*$ and $f.g = h$. Now considering $(\_)^* : \mathcal{T}\text{-}\mathbf{Cat} \to \mathcal{T}\text{-}\mathbf{Mod}_r$ where $\mathcal{T}\text{-}\mathbf{Mod}_r$ is the subcategory of $\mathcal{T}\text{-}\mathbf{Mod}$ whose morphisms are the right adjoint $\mathcal{T}$-modules, we conclude that a $\mathcal{T}$-functor is L-complete if and only if it is a $(\_)^*$-quasi cocartesian morphism.

Assuming the axiom of choice one can replace Z by E.

3.2. Proposition. *For a $\mathcal{T}$-functor $f : (X,a) \to (Y,b)$ the following are equivalent:*

1. *$f : (X,a) \to (Y,b)$ is L-complete;*
2. *Given any left adjoint $\mathcal{T}$-module $\varphi : E \rightsquigarrow X$ and any $y \in Y$ such that $f_* \circ \varphi = y_*$, there exists $x \in X$ with $\varphi = x_*$ and $f(x) = y$;*
3. *Given any right adjoint $\mathcal{T}$-module $\psi : X \rightsquigarrow E$ and any $y \in Y$ such that $\psi \circ f^* = y^*$, there exists $x \in X$ with $\psi = x^*$ and $f(x) = y$.*

3.3. Examples.

1. *In* **Ord**, *a monotone map $f : X \to Y$ is L-complete if and only if given any $x \in X$ for which $f(x)$ is isomorphic to some $y \in Y$, i.e. $f(x) \leq y$ and $y \leq f(x)$, there exists $z \in f^{-1}(\{y\})$ that is isomorphic to x.*

2. *In* **Met**, *a nonexpansive map $f : X \to Y$ is L-complete if and only if for any Cauchy sequence (x_n) in X with $(f(x_n))$ converging to some $y \in Y$, there exists $x \in f^{-1}(\{y\})$ such that (x_n) converges to x.*

3. *In* **Top**, *a continuous map $f : X \to Y$ is L-complete if and only if for any irreducible closed set $A \subseteq X$ with $\overline{f(A)} = \overline{\{y\}}$ for some $y \in Y$, there exists $x \in f^{-1}(\{y\})$ such that $A = \overline{\{x\}}$. We call such maps weakly fibrewise sober. In case that the point x is unique, f is called a fibrewise sober map [28].*

3.4. Proposition.

1. *L-complete morphisms are closed under composition and contain all isomorphisms.*

2. *If $g.f$ is L-complete and f is an L-equivalence then g is L-complete.*

Most importantly, like proper maps in topology, L-complete maps are pullback stable.

3.5. Proposition. *L-complete $\mathcal{T}$-functors are stable under pullback.*

Proof. Let $g : (Y, b) \to (Z, c)$ be an L-complete $\mathcal{T}$-functor. Consider its pullback along a $\mathcal{T}$-functor $f : (X, a) \to (Z, c)$.

$$\begin{tikzcd}
(X \times_Z Y, a \times b) \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] \arrow[dr, phantom, very near start, "\lrcorner"] & (Y, b) \arrow[d, "g"] \\
(X, a) \arrow[r, "f"'] & (Z, c)
\end{tikzcd}$$

We need to show that π_1 is L-complete. So assume that $(\pi_1)_* \circ \varphi = (x_0)_*$ for some $x_0 \in X$ and $\varphi \dashv \psi : (X \times_Z Y, a \times b) \rightsquigarrow (E, k)$. Then we have the following commutative diagram in $\mathcal{T}$-**Mod**.

$$\begin{tikzcd}
(E, k) \arrow[r, squiggly, "\varphi"] \arrow[dr, squiggly, "(x_0)_*"'] & (X \times_Z Y, a \times b) \arrow[r, squiggly, "(\pi_2)_*"] \arrow[d, squiggly, "(\pi_1)_*"] & (Y, b) \arrow[d, squiggly, "g_*"] \\
 & (X, a) \arrow[r, squiggly, "f_*"'] & (Z, c)
\end{tikzcd}$$

So $g_* \circ (\pi_2)_* \circ \varphi = f_* \circ (x_0)_* = (f(x_0))_*$ where $(\pi_2)_* \circ \varphi \dashv \psi \circ (\pi_2)^*$. Since g is L-complete there exists $y_0 \in Y$ such that $(\pi_2)_* \circ \varphi = (y_0)_*$ and $g(y_0) = f(x_0)$. Hence $(x_0, y_0) \in X \times_Z Y$.

Claim: $\varphi = (x_0, y_0)_*$.

Since φ is a $\mathcal{T}$-module, one has $\varphi = (a \times b) \circ \varphi = (a \times b).T_\xi\varphi.m_E{}^\circ$. So for any $(x, y) \in X \times_Z Y$,

$$\begin{aligned}
\varphi(x,y) &= \bigvee_{\mathfrak{w} \in T(X \times_Z Y)} T_\xi\varphi(\mathfrak{w}) \otimes (a \times b)(\mathfrak{w}, (x,y)) \\
&= \bigvee_{\mathfrak{w} \in T(X \times_Z Y)} T_\xi\varphi(\mathfrak{w}) \otimes (a(T\pi_1(\mathfrak{w}), x) \wedge b(T\pi_2(\mathfrak{w}), y)) \qquad (\star) \\
&\leq \bigvee_{\mathfrak{w} \in T(X \times_Z Y)} T_\xi\varphi(\mathfrak{w}) \otimes a(T\pi_1(\mathfrak{w}), x) \\
&= (\pi_1)_* \circ \varphi(x) = (x_0)_*(x) = a(e_X(x_0), x).
\end{aligned}$$

Similarly $\varphi(x, y) \leq b(e_Y(y_0), y)$. Hence

$$\varphi(x,y) \leq a(e_X(x_0), x) \wedge b(e_Y(y_0), y) = (x_0, y_0)_*(x, y).$$

To obtain the other inequality, consider $(\star)$. Letting $\mathfrak{w} = e_{X \times_Z Y}(x_0, y_0)$ we get

$$\begin{aligned}
\varphi(x,y) &\geq T_\xi\varphi(e_{X \times_Z Y}(x_0,y_0)) \otimes (a(T\pi_1(e_{X \times_Z Y}(x_0,y_0)), x) \wedge b(T\pi_2(e_{X \times_Z Y}(x_0,y_0)), y)) \\
&= T_\xi\varphi(e_{X \times_Z Y}(x_0,y_0)) \otimes (a(e_X(x_0), x) \wedge b(e_Y(y_0), y)) \\
&\geq \varphi(x_0, y_0) \otimes (x_0, y_0)_*(x, y).
\end{aligned}$$

Observe that if $\varphi(x_0, y_0) \geq k$, then we are done. In the remaining part of the proof we will show this inequality.

We know that $(x_0)^* = \psi \circ (\pi_1)^*$. So for any $\mathfrak{x} \in TX$

$$a(\mathfrak{x}, x_0) = \bigvee_{\mathfrak{w} \in T(X \times_Z Y)} T_\xi a(m_X{}^\circ(\mathfrak{x}), T\pi_1(\mathfrak{w})) \otimes \psi(\mathfrak{w}).$$

Considering $\mathfrak{x} = T\pi_1(\mathfrak{w})$, we get

$$T_\xi a(m_X{}^\circ.T\pi_1(\mathfrak{w}), T\pi_1(\mathfrak{w})) \otimes \psi(\mathfrak{w}) \leq a(T\pi_1(\mathfrak{w}), x_0).$$

As T is order preserving, we have $1_{TX} = Te_X{}^\circ.m_X{}^\circ \leq T_\xi a.m_X{}^\circ$. So we have that $k \leq T_\xi a(m_X{}^\circ.T\pi_1(\mathfrak{w}), T\pi_1(\mathfrak{w}))$. Hence $\psi(\mathfrak{w}) \leq a(T\pi_1(\mathfrak{w}), x_0)$. Similarly $(y_0)^* = \psi \circ (\pi_2)^*$ gives $\psi(\mathfrak{w}) \leq b(T\pi_2(\mathfrak{w}), y_0)$. Then,

$$\psi(\mathfrak{w}) \leq a(T\pi_1(\mathfrak{w}), x_0) \wedge b(T\pi_2(\mathfrak{w}), y_0) = a \times b(\mathfrak{w}, (x_0, y_0)) \qquad \forall \mathfrak{w} \in T(X \times_Z Y). \quad (\dagger)$$

Consider the structure $\widehat{a \times b}$ on $\widehat{X \times_Z Y}$. Since $(\widehat{a \times b}) \circ (\widehat{a \times b}) \leq \widehat{a \times b}$ we have,

$$\begin{aligned}
&T_\xi(\widehat{a \times b})(e_{T(\widehat{X \times_Z Y})}.e_{\widehat{X \times_Z Y}}(\psi), T\boldsymbol{y}(\mathfrak{w})) \otimes \widehat{a \times b}(T\boldsymbol{y}(\mathfrak{w}), \boldsymbol{y}(x_0, y_0)) \\
&\qquad\qquad \leq \widehat{a \times b}(m_{\widehat{X \times_Z Y}}.e_{T(\widehat{X \times_Z Y})}.e_{\widehat{X \times_Z Y}}(\psi), \boldsymbol{y}(x_0, y_0)).
\end{aligned}$$

As $\boldsymbol{y}$ is fully faithful and $\varphi \dashv \psi$,

$$\begin{aligned} T_\xi(\widehat{a\times b})(e_{T(\widehat{X\times_Z Y})}.e_{\widehat{X\times_Z Y}}(\psi), T\boldsymbol{y}(\mathfrak{w}))\otimes a\times b(\mathfrak{w},(x_0,y_0)) &\leq \widehat{a\times b}(e_{\widehat{X\times_Z Y}}(\psi),\boldsymbol{y}(x_0,y_0)) \\ &= \varphi(x_0,y_0). \end{aligned}$$

Together with (†) we obtain,

$$T_\xi(\widehat{a\times b})(e_{T(\widehat{X\times_Z Y})}.e_{\widehat{X\times_Z Y}}(\psi), T\boldsymbol{y}(\mathfrak{w}))\otimes \psi(\mathfrak{w}) \leq \varphi(x_0,y_0)$$

$$T_\xi(\widehat{a\times b})(e_{T(\widehat{X\times_Z Y})}.e_{\widehat{X\times_Z Y}}(\psi), T\boldsymbol{y}(\mathfrak{w})) \leq \psi(\mathfrak{w}) \multimap \varphi(x_0,y_0).$$

Since ψ is a right adjoint $\mathcal{T}$-module we have,

$$\begin{aligned} k &\leq \bigvee_{\mathfrak{w}\in T(X\times_Z Y)} \psi(\mathfrak{w})\otimes T_\xi(\widehat{a\times b})(e_{T(\widehat{X\times_Z Y})}.e_{\widehat{X\times_Z Y}}(\psi), T\boldsymbol{y}(\mathfrak{w})) \\ &\leq \bigvee_{\mathfrak{w}\in T(X\times_Z Y)} \psi(\mathfrak{w})\otimes (\psi(\mathfrak{w})\multimap \varphi(x_0,y_0)) \\ &\leq \varphi(x_0,y_0). \end{aligned}$$

■

Every $\mathcal{T}$-category (X,a) has the L-completion $(\widetilde{X},\widehat{a})$ consisting of tight $\mathcal{T}$-functors. Let $\mathcal{Y} : \mathcal{T}\text{-}\mathbf{Cat} \to \mathcal{T}\text{-}\mathbf{Cat}$ be the L-completion functor. To see the action of $\mathcal{Y}$ on morphisms, recall that $\widetilde{X}$ can also be seen as the collection of right adjoint $\mathcal{T}$-modules from X to E. So given a $\mathcal{T}$-functor $f : (X,a) \to (Y,b)$, one has $\mathcal{Y}(f) = \widetilde{f}$ where $\widetilde{f}(\psi) = \psi \circ f^*$ for $\psi \in \widetilde{X}$. The family of the Yoneda functors $\boldsymbol{y}_X : X \to \widetilde{X}$ form a natural transformation $\boldsymbol{y} : 1_{\mathcal{T}\text{-}\mathbf{Cat}} \to \mathcal{Y}$. This gives us another way to characterize L-complete morphisms.

3.6. Proposition. *Let $f : X \to Y$ be a $\mathcal{T}$-functor. Then f is L-complete if and only if the naturality square*

$$\begin{tikzcd} X \arrow[r, "\boldsymbol{y}_X"] \arrow[d, "f"'] & \widetilde{X} \arrow[d, "\widetilde{f}"] \\ Y \arrow[r, "\boldsymbol{y}_Y"'] & \widetilde{Y} \end{tikzcd} \tag{1}$$

is a weak pullback.

Proof. The naturality square is a weak pullback if and only if for any $\psi \in \widetilde{X}$ and $y \in Y$ such that $\widetilde{f}(\psi) = \boldsymbol{y}_Y(y)$, there exists $x \in X$ satisfying $\psi = \boldsymbol{y}_X(x)$ and $f(x) = y$. This is equivalent to saying that for any right adjoint $\mathcal{T}$-module $\psi : X \rightsquigarrow E$ and $y \in Y$ with $\psi \circ f^* = y^*$ there exists $x \in X$ such that $\psi = x^*$ and $f(x) = y$. That is equivalent to f being L-complete. ■

The following result will be useful in the next section.

3.7. LEMMA. *Let $f : X \to Y$ be a fully faithful $\mathcal{T}$-functor. Then f is L-complete if and only if $f(X)$ is L-closed in Y.*

PROOF. Assume that $f(X)$ is L-closed in Y. Let $\varphi \dashv \psi : X \rightsquigarrow E$ such that $f_* \circ \varphi = y_*$. Considering the canonical factorization $X \overset{f'}{\twoheadrightarrow} f(X) \overset{i}{\hookrightarrow} Y$ of f, we can write $i_* \circ f'_* \circ \varphi = y_*$. Then by Prop. 2.13, we have $y \in \overline{f(X)}$. Since $f(X)$ is L-closed, $y \in f(X)$. So there exists $x \in X$ such that $f(x) = y$. Then $f_* \circ \varphi = y_* = f_* \circ x_*$. Since f is fully faithful $\varphi = x_*$. Hence f is L-complete.

Now assume that f is L-complete. Write $f = i.f'$ as above. As f' is surjective, it is L-dense. It is also fully faithful since f is fully faithful. Hence f' is an L-equivalence. Then by Prop. 3.4, i is L-complete. To show that $f(X)$ is L-closed, take $y \in \overline{f(X)}$. By Prop. 2.13, $i^* \circ y_* \dashv y^* \circ i_*$. Then $1_E^* \le y^* \circ i_* \circ i^* \circ y_*$. Composing both sides with y_* and taking advantage of the adjunctions we obtain $y_* = i_* \circ i^* \circ y_*$. Since $i^* \circ y_* : E \rightsquigarrow f(X)$ is a left adjoint $\mathcal{T}$-module and i is L-complete, there exists $f(x) \in f(X)$ such that $i(f(x)) = f(x) = y$. Hence $y \in f(X)$ and $f(X)$ is L-closed. ∎

We now look at L-separation and L-injectivity for morphisms. Let Y be a $\mathcal{T}$-category. Consider the comma category $\mathcal{T}\text{-}\mathbf{Cat}/Y$ whose objects of are the $\mathcal{T}$-functors with the codomain Y. A morphism from $k : Z \to Y$ to $f : X \to Y$ in this category is a $\mathcal{T}$-functor $g : Z \to X$ such that $f.g = k$.

3.8. DEFINITION. *We call a $\mathcal{T}$-functor $f : X \to Y$ L-separated if f is an L-separated object in $\mathcal{T}\text{-}\mathbf{Cat}/Y$. This means, given any morphisms $g, h : k \to f$ such that $g \simeq h$, one has $g = h$.*

So a $\mathcal{T}$-functor $f : X \to Y$ is L-separated if and only if given any $\mathcal{T}$-functors $g, h : Z \to X$ such that $g \simeq h$ and $f.g = f.h$, one has $f = g$. It is sufficient to consider the $\otimes$-neutral object E instead of Z. Hence $f : X \to Y$ is L-separated if and only if $x \simeq w$ and $f(x) = f(w)$ implies $x = w$ for any $x, w \in X$.

In the previous section we have seen the concept of L-injectivity which coincides with L-completeness at the level of objects. We now develop its morphism notion which in fact will be equivalent to L-completeness for morphisms. We will delay the proof of this fact to the end of the next section.

3.9. DEFINITION. *We call a $\mathcal{T}$-functor $f : X \to Y$ L-injective if f is an L-injective object in $\mathcal{T}\text{-}\mathbf{Cat}/Y$. This means, given any morphism $j : k \to f$ and any L-equivalence $i : k \to h$, there exists a morphism $g : h \to f$ such that $g.i \simeq j$.*

Observe that $f : X \to Y$ L-injective if and only if given any commutative square

$$\begin{tikzcd} A \arrow[r, "j"] \arrow[d, "i"'] & X \arrow[d, "f"] \\ B \arrow[r, "h"'] \arrow[ur, dashed, "g"] & Y \end{tikzcd}$$

in $\mathcal{T}$**-Cat** where i is an L-equivalence, there exists $g : B \to X$ such that $f.g = h$ and $g.i \simeq j$. The commutative square above corresponds to the morphisms $j : f.j \to f$ and $i : f.j \to h$. There exists $g : B \to X$ with the desired properties if and only if there exists $g : h \to f$ such that $g.i \simeq j$.

So L-injective $\mathcal{T}$-functors are the morphisms in $\mathcal{T}$**-Cat** which have the weak right lifting property with respect to L-equivalences.

4. Topology with respect to L-complete morphisms

One can develop topological notions in a category by using a distinguished class $\mathcal{F}$ of "closed morphisms" [7]. In fact, as shown in [19] many of these notions can be expressed by using "proper maps" which are stably "closed". Having an analogue of proper maps in the context of completeness, we explore the option of doing topology in $\mathcal{T}$**-Cat** using L-complete morphisms.

Letting $\mathcal{F}$ to be $\mathcal{L} = \{$L-complete $\mathcal{T}$-functors$\}$, we see that $\mathcal{L}$-separation and $\mathcal{L}$-denseness coincide with L-separation and L-denseness, while $\mathcal{L}$-compactness and $\mathcal{L}$-compactification translate into L-completeness and L-completion.

$\mathcal{T}$**-Cat** has the proper factorization system $(\mathcal{E}, \mathcal{M})$ where $\mathcal{E} = \{$Surjective $\mathcal{T}$-functors$\}$ and $\mathcal{M} = \{$Injective and fully faithful $\mathcal{T}$-functors$\}$.

The first topological notion we will explore is compactness. A topological space X is compact if and only if the unique map $!_X : X \to 1$ is proper. Taking this fact as a reference, a $\mathcal{T}$-category X is called *$\mathcal{L}$-compact* if and only if $!_X : X \to 1$ is in $\mathcal{L}$.

4.1. Proposition. *X is $\mathcal{L}$-compact if and only if X is L-complete.*

Proof. By Prop 3.6, $!_X : X \to 1$ is L-complete if and only if

$$\begin{tikzcd} X \arrow[r, "y_X"] \arrow[d, "!_X"'] & \widetilde{X} \arrow[d, "\widetilde{!}_X"] \\ 1 \arrow[r, "y_1"'] & \widetilde{1} \end{tikzcd}$$

is a weak pullback. Since $\widetilde{1}$ is the terminal object, this is equivalent to saying that y_X is surjective. y_X is surjective if and only if X is L-complete. ∎

L-completeness is carried forward by L-equivalences and backward by L-complete morphisms.

4.2. Proposition.

1. *For any L-complete $f : X \to Y$ with Y L-complete, X is L-complete.*

2. *For any L-equivalence $f : X \to Y$ with X L-complete, Y is L-complete.*

By taking advantage of the framework in [7], [19] one obtains an analogue of the Kuratowski-Mrowka theorem [7] for L-completeness.

4.3. Proposition. *For a $\mathcal{T}$-category X, the following are equivalent:*

1. *X is L-complete;*

2. *For any $\mathcal{T}$-category Y, the projection $X \times Y \to Y$ is L-complete;*

3. *For any L-complete $\mathcal{T}$-category Y, $X \times Y$ is L-complete.*

Proof. (1) $\Rightarrow$ (2) $X \times Y \to Y$ is a pullback of $!_X : X \to 1$. (2) $\Rightarrow$ (3) By Prop. 4.2. (3) $\Rightarrow$ (1) Take $Y = 1$. ∎

Consider $f : X \to Y$ as an object of the comma category $\mathcal{T}$-**Cat**$/Y$. In that case one sees that the unique map $!_f : f \to 1_Y$ going to the terminal object is f itself. So $f : X \to Y$ is L-complete if and only if it is an $\mathcal{L}$-compact object in $\mathcal{T}$-**Cat**$/Y$.

Now we investigate separation. Let $f : X \to Y$ be a a $\mathcal{T}$-functor. In accordance with [7], [19] one says that f is *$\mathcal{L}$-separated* if $\delta_f = \langle 1_X, 1_X \rangle : X \to X \times_Y X$ is in $\mathcal{L}$.

4.4. Proposition. [7]

1. *$\mathcal{L}$-separated maps are closed under composition and contain all monomorphisms.*

2. *$\mathcal{L}$-separated maps are stable under pullback.*

3. *If $g.f$ is $\mathcal{L}$-separated then f is $\mathcal{L}$-separated.*

Similar to the case for $\mathcal{L}$-compactness, one calls a $\mathcal{T}$-category X *$\mathcal{L}$-separated* if $!_X : X \to 1$ is $\mathcal{L}$-separated.

4.5. Proposition. *Let $f : X \to Y$ be a $\mathcal{T}$-functor. f is $\mathcal{L}$-separated if and only if it is L-separated.*

Proof. f is $\mathcal{L}$-separated if and only if δ_f is L-complete. Since δ_f is fully faithful, it enough to consider whether $\delta_f(X)$ is L-closed by Lemma 3.7. Observe that

$$\overline{\delta_f(X)}^{X \times_Y X} = \overline{\delta_f(X)}^{X \times X} \cap (X \times_Y X) = \{(x, z) \mid x \simeq z, f(x) = f(z)\}.$$

So $\delta_f(X)$ is L-closed if and only if $x \simeq z$ and $f(x) = f(z)$ implies $x = z$ for any $x, z \in X$. This is equivalent to f being L-separated. ∎

4.6. Corollary. *Let X be a $\mathcal{T}$-category. X is $\mathcal{L}$-separated if and only if X is L-separated.*

As in the case of $\mathcal{L}$-compactness, L-separated $\mathcal{T}$-functors $f : X \to Y$ are the $\mathcal{L}$-separated objects of $\mathcal{T}$-$\mathbf{Cat}/Y$.

The general setting of [7], [19] allows one to obtain the following.

4.7. Proposition. *For a $\mathcal{T}$-category X, the following are equivalent:*

1. *X is L-separated;*
2. *Any morphism $f : X \to Y$ is L-separated;*
3. *There exists an L-separated morphism $f : X \to Y$ with Y L-separated;*
4. *For any $\mathcal{T}$-category Y, the projection $X \times Y \to Y$ is L-separated;*
5. *For any L-separated $\mathcal{T}$-category Y, $X \times Y$ is L-separated.*

It is well known that a continuous map between a compact domain and a Hausdorff codomain is proper. The analogous statement about the maps with $\mathcal{L}$-compact domain and $\mathcal{L}$-separated codomain [7] gives us the following result.

4.8. Proposition. *Any $\mathcal{T}$-functor $f : X \to Y$ with X L-complete and Y L-separated is L-complete.*

The proposition also works in the comma category for morphisms with an $\mathcal{L}$-compact domain and an $\mathcal{L}$-separated codomain.

4.9. Corollary. *Suppose that $g.f$ is L-complete and g is L-separated, then f is L-complete.*

Compactification of a topological space X is provided by a dense embedding of X into a compact Hausdorff space Y. Before investigating what compactifications are relative to the L-complete morphisms, one needs a notion for $\mathcal{L}$-dense maps. Let $f : X \to Y$ be a $\mathcal{T}$-functor. Following [7], one says that f is *$\mathcal{L}$-dense* if and only if in any factorization $f = i.g$ where $i \in \mathcal{L} \cap \mathcal{M}$, i is an isomorphism.

4.10. Proposition. *Let $f : X \to Y$ be a $\mathcal{T}$-functor. f is $\mathcal{L}$-dense if and only if f is L-dense.*

Proof. Let f be $\mathcal{L}$-dense. Consider the $X \xrightarrow{f'} \overline{f(X)} \xhookrightarrow{i} Y$ factorization for f. Since i is fully faithful and $\overline{f(X)}$ is L-closed in Y, i is L-complete by Lemma 3.7. Then $i \in \mathcal{L} \cap \mathcal{M}$. As f is $\mathcal{L}$-dense, i becomes an isomorphism. Hence $\overline{f(X)} = Y$ and f is L-dense.

Now suppose that f is L-dense. Consider any factorization $X \xrightarrow{g} Z \xrightarrow{i} Y$ of f where $i \in \mathcal{L} \cap \mathcal{M}$. By Prop 2.8, i is L-dense. So $\overline{i(Z)} = Y$. Since i is fully faithful and L-complete $i(Z)$ is L-closed by Lemma 3.7. Hence $i(Z) = Y$, $i \in \mathcal{E}$. So i is an isomorphism. ∎

An $\mathcal{L}$-compactification of a $\mathcal{T}$-category X will be an L-dense embedding $i : X \hookrightarrow K$ where K is L-complete and L-separated. But such an embedding exists only for L-separated objects by Prop 4.7. To extend the notion to a larger collection of objects we drop the condition that i is an embedding. We ask i to be fully faithful instead.

4.11. Definition. *Let X be a $\mathcal{T}$-category. An $\mathcal{L}$-compactification of X is given by an L-equivalence $i : X \to K$ where K is $\mathcal{L}$-compact and $\mathcal{L}$-separated.*

We are particularly interested in a functorial $\mathcal{L}$-compactification of $\mathcal{T}$-categories. By that we mean a functor $\Gamma : \mathcal{T}\text{-}\mathbf{Cat} \to \mathcal{T}\text{-}\mathbf{Cat}_{\mathrm{cpl}\ \&\ \mathrm{sep}}$ which comes with a natural transformation $\{\gamma_X : X \to \Gamma X\}_{X \in \mathcal{T}\text{-}\mathbf{Cat}}$ where each γ_X is an L-equivalence and each ΓX is L-complete and L-separated.

4.12. Proposition. *The L-completion functor $\mathcal{Y} : \mathcal{T}\text{-}\mathbf{Cat} \to \mathcal{T}\text{-}\mathbf{Cat}_{cpl\,\&\,sep}$ together with the natural transformation $\{y_X : X \to \mathcal{Y}(X)\}_{X \in \mathcal{T}\text{-}\mathbf{Cat}}$ is a functorial $\mathcal{L}$-compactification.*

4.13. Examples. *In* **Met**, *$\mathcal{L}$-compactification of a generalized metric space is its Cauchy completion. In* **Top**, *$\mathcal{L}$-compactification takes the form of soberification where $\mathcal{Y}$ is the soberification functor [21].*

Working in the comma category one can extend the $\mathcal{L}$-compactification notion to morphisms.

4.14. Definition. *Let $f : X \to Y$ be a $\mathcal{T}$-functor. $\mathcal{L}$-compactification of f is given by an L-equivalence $i : f \to g$ where g is $\mathcal{L}$-compact and $\mathcal{L}$-separated.*

An $\mathcal{L}$-compactification of a morphism is its L-completion. The functorial L-completion $\mathcal{Y}$ for objects provides such an L-completion for morphisms. To see this let $f : X \to Y$ be any $\mathcal{T}$-functor. Consider the following diagram:

$$\begin{tikzcd}
X \arrow[dr, "i"] \arrow[drr, bend left, "y_X"] \arrow[ddr, bend right, "f"'] & & \\
& Y \times_{\widetilde{Y}} \widetilde{X} \arrow[r, "\pi_2"'] \arrow[d, "\pi_1"] \arrow[dr, phantom, "\lrcorner", very near start] & \widetilde{X} \arrow[d, "\widetilde{f}"] \\
& Y \arrow[r, "y_Y"'] & \widetilde{Y}
\end{tikzcd} \tag{2}$$

By Prop 4.7 and 4.8, $\widetilde{f}$ is L-complete and L-separated. π_1 is L-complete and L-separated as it is a pullback of $\widetilde{f}$. On the other hand, π_2 is fully faithful as a pullback of y_Y. Since y_X is fully faithful and L-dense, so is i by Prop 2.8. Hence $i : f \to \pi_1$ is an

L-equivalence where π_1 is L-complete and L-separated. This means that $i : f \to \pi_1$ is an L-completion of f.

The morphism counterpart of a compact Hausdorff space is a proper, separated map which is also called a *perfect map*. In accordance with [7] one calls a $\mathcal{T}$-functor *$\mathcal{L}$-perfect* if and only if it is $\mathcal{L}$-compact and $\mathcal{L}$-separated as an object of the comma category. In other words a $\mathcal{T}$-functor is $\mathcal{L}$-perfect if and only if it is L-complete and L-separated.

The Isbell-Henriksen theorem [14] which describes the perfect maps in topology translates into a characterization of $\mathcal{L}$-perfect maps.

4.15. PROPOSITION. *For a $\mathcal{T}$-functor $f : X \to Y$, the following are equivalent:*

1. *f is L-complete and L-separated;*
2. *In any factorization $f = g.i$ with i L-equivalence and g L-separated, i is an isomorphism;*
3. *The naturality square (1) is a pullback.*

Considering the functors $(\_)_* : \mathcal{T}\textbf{-Cat} \to \mathcal{T}\textbf{-Mod}_l$ and $(\_)^* : \mathcal{T}\textbf{-Cat} \to \mathcal{T}\textbf{-Mod}_r$, one also sees that an L-complete and L-separated $\mathcal{T}$-functor is a $(\_)_*$-cartesian morphism or, equivalently, a $(\_)^*$-cocartesian morphism.

The (Antiperfect, Perfect) factorization of the continuous maps of Tychonoff spaces [15], [31], [6], [32] is obtained with the help of the left adjoint Stone-Cech compactification functor. Here an "antiperfect map" stands for a map which is sent to an isomorphism by the compactification functor. Analogously in our context the reflector $\mathcal{Y}$ is simple in the sense of [5] and it induces the factorization system $(\mathscr{E}, \mathscr{M})$ for $\mathcal{T}\textbf{-Cat}$. Here $\mathscr{E}$ is the collection of morphisms that are mapped to isomorphisms by $\mathcal{Y}$ and $\mathscr{M}$ is the collection of L-complete and L-separated morphisms. These types of factorization systems are also studied in [20], [29].

4.16. LEMMA. *Let $f : X \to Y$ be a $\mathcal{T}$-functor. $\mathcal{Y}(f)$ is an isomorphism if and only if f is an L-equivalence.*

PROOF. Suppose that $\mathcal{Y}(f) = \widetilde{f}$ is an isomorphism. Then $\widetilde{f}$ is an L-equivalence. The naturality square (1) gives $\widetilde{f}.y_X = y_Y.f$ where $y_X, y_Y, \widetilde{f}$ are L-equivalences. Then by Prop 2.8, f is an L-equivalence.

Conversely suppose that f is an L-equivalence. Define $\breve{f} : \widetilde{Y} \to \widetilde{X}$ by $\breve{f}(\psi) = \psi \circ f_*$ for any right adjoint $\mathcal{T}$-module $\psi : Y \rightsquigarrow E$. Then $\breve{f} = (\widetilde{f})^{-1}$. ∎

4.17. THEOREM. *Let $\mathscr{E}$ be the collection of L-equivalences and $\mathscr{M}$ be the collection of L-complete and L-separated morphisms. Then $(\mathscr{E}, \mathscr{M})$ is a factorization system for $\mathcal{T}\textbf{-Cat}$.*

PROOF. Given Prop 4.15, Lemma 4.16 and the fact that $\mathcal{T}\textbf{-Cat}_{\text{cpl \& sep}}$ is a reflective subcategory of $\mathcal{T}\textbf{-Cat}$ with the reflector $\mathcal{Y}$, it follows from Theorem 4.1 of [5] . ∎

4.18. Examples.

1. *Let $f : X \to Y$ be a nonexpansive map in* **Met**. *f is an L-equivalence if and only if it is a dense isometry. f is L-complete and L-separated if and only if for any Cauchy sequence (x_n) in X with $(f(x_n))$ converging to some $y \in Y$, there exists a unique $x \in f^{-1}(\{y\})$ such that (x_n) converges to x. Since L-complete and L-separated maps are the analogues of perfect maps, in* **Met** *we will refer to these morphisms as L-perfect maps. So (Dense isometry, L-perfect) is a factorization system in* **Met**.

2. *Let $f : X \to Y$ be a continuous map in* **Top**. *f is an L-equivalence if and only if it is an isomorphism as a continuous map in the category of locales. f is L-complete and L-separated if and only if it is fibrewise sober [28]. Let's denote the canonical functor from the category of topological spaces to the category of locales by $\mathsf{L} : \mathbf{Top} \to \mathbf{Loc}$. Then $(\mathsf{L}^{-1}(Iso), Fibrewise\ sober)$ is a factorization system in* **Top**.

As a consequence of the Theorem 4.17, L-equivalences are orthogonal to L-complete and L-separated morphisms. The existence of the lifting is actually due to the equivalence of L-injectivity and L-completeness at the level of morphisms, which we prove now.

4.19. Theorem. *Let $f : X \to Y$ be a $\mathcal{T}$-functor. f is L-complete if and only if f is L-injective.*

Proof. First assume that f is L-injective. We know that f is L-complete if the diagram (2) is a weak pullback or, equivalently, the induced map i is surjective. So it will be enough to show that i is a retraction.

Consider the following commutative square.

$$\begin{tikzcd} X \arrow[r, "1_X"] \arrow[d, "i"'] & X \arrow[d, "f"] \\ Y \times_{\widetilde{Y}} \widetilde{X} \arrow[r, "\pi_1"'] \arrow[ur, dashed, "m"] & Y \end{tikzcd}$$

It is shown that i is an L-equivalence. Since f is L-injective, there exists $m : Y \times_{\widetilde{Y}} \widetilde{X} \to X$ such that $m.i \simeq 1_X$ and $\pi_1.i.m = \pi_1$. Then we have $\pi_1.i.m.i = \pi_1.i$ with $i.m.i \simeq i$. Since π_1 is L-separated, $i.m.i = i$. As i is L-dense, we get $i.m \simeq 1_{Y \times_{\widetilde{Y}} \widetilde{X}}$. Now we have $\pi_1.i.m = \pi_1.1_{Y \times_{\widetilde{Y}} \widetilde{X}}$ with $i.m \simeq 1_{Y \times_{\widetilde{Y}} \widetilde{X}}$. Using again the fact that π_1 is L-separated we obtain $i.m = 1_{Y \times_{\widetilde{Y}} \widetilde{X}}$.

Conversely assume that f is L-complete. Suppose that we have the following commutative square,

$$\begin{tikzcd} A \arrow[r, "j"] \arrow[d, "i"'] & X \arrow[d, "f"] \\ B \arrow[r, "h"'] \arrow[ur, dashed, "g"] & Y \end{tikzcd}$$

where i is an L-equivalence. Then $f_* \circ j_* = h_* \circ i_*$. As i is an L-equivalence, we get $f_* \circ j_* \circ i^* = h_*$ and $j_* \circ i^* \dashv i_* \circ j^*$. Since f is L-complete, there exists $g : B \to X$ such that $j_* \circ i^* = g_*$ and $f.g = h$. Hence $j \simeq g.i$ and f is L-injective. ■

4.20. Remark. In the full subcategory of L-separated $\mathcal{T}$-categories, L-complete maps are exactly the injective maps with respect to L-equivalences in the sense that the lifting g makes the above diagram strictly commutative.

In particular, in the category of T_0 topological spaces L-complete maps are the injective maps with respect to L-equivalences. In this context L-equivalences are precisely completely flat embeddings [13] and L-complete maps are precisely fiberwise sober maps. Hence we see that fiberwise sober maps are the injective maps with respect to completely flat embeddings in the category of T_0 topological spaces as stated in [3].

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