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EXPONENTIABILITY VIA DOUBLE CATEGORIES

SUSAN NIEFIELD

Abstract. For a small category B and a double category $\mathbb{D}$, let $\mathrm{Lax}_N(B,\mathbb{D})$ denote the category whose objects are vertical normal lax functors $B \longrightarrow \mathbb{D}$ and morphisms are horizontal lax transformations. It is well known that $\mathrm{Lax}_N(B,\mathbb{C}\mathrm{at}) \simeq \mathrm{Cat}/B$, where $\mathbb{C}\mathrm{at}$ is the double category of small categories, functors, and profunctors. In [19], we generalized this equivalence to certain double categories, in the case where B is a finite poset. In [23], Street showed that $Y \longrightarrow B$ is exponentiable in Cat/B if and only if the corresponding normal lax functor $B \longrightarrow \mathbb{C}\mathrm{at}$ is a pseudo-functor. Using our generalized equivalence, we show that a morphism $Y \longrightarrow B$ is exponentiable in $\mathbb{D}_0/B$ if and only if the corresponding normal lax functor $B \longrightarrow \mathbb{D}$ is a pseudo-functor *plus* an additional condition that holds for all $X \longrightarrow !B$ in Cat. Thus, we obtain a single theorem which yields characterizations of certain exponentiable morphisms of small categories, topological spaces, locales, and posets.

1. Introduction

Suppose $\mathcal{A}$ is a category with finite limits. An object Y of $\mathcal{A}$ is called *exponentiable* if the functor $- \times Y : \mathcal{A} \longrightarrow \mathcal{A}$ has a right adjoint, denoted by $(\ )^Y$. A morphism is called *exponentiable* if it is exponentiable in $\mathcal{A}/Y$.

Exponentiable morphisms in the category Cat of small categories were characterized independently by Giraud [6] and Conduché [2] as those functors satisfying a factorization lifting property now known as the *Giraud-Conduché condition*. Exponentiable morphisms in the category Top of topological spaces were characterized by the author in [14] (see also [15, 16, 17]) as those satisfying a somewhat technical condition (see Lemma 4.1 below) which was used to show that the inclusion of a subspace of B is exponentiable if and only if it is locally closed, and also to establish the exponentiability of perfect maps as well as locally compact spaces over a locally Hausdorff base.

The obstruction to exponentiability in each of these two categories is quite different. In Cat, the Giraud-Conduché condition is used to define composition of morphisms in the category that serves as the exponential, and the unit and counit follow. Whereas in Top, one can always define a candidate for the exponential for which the unit is continuous, but the extra condition is needed for the continuity of the counit.

There is a more recent characterization of exponentiability in Cat. In a 1986 handwritten manuscript, referenced in his 2001 unpublished note [23], Street used the equiv-

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alence (attributed to Bénabou) between Cat/B and a category $\mathrm{Lax_N}(B, Prof)$ to show that a functor $Y \longrightarrow B$ is exponentiable if and only if the corresponding normal lax functor $B \longrightarrow Prof$ is a pseudo-functor. Objects of $\mathrm{Lax_N}(B, Prof)$ are normal lax functors from B to the bicategory $Prof$ of small categories and profunctors, and morphisms of $\mathrm{Lax_N}(B, Prof)$ are functor-valued lax transformations. Thus, Bénabou's equivalence can by viewed as taking place in the realm of double categories (in the sense of [4] or [7]). In particular, for a double category $\mathbb{D}$, we can consider the category $\mathrm{Lax_N}(B, \mathbb{D})$ whose objects are vertical normal lax functors and morphisms are horizontal lax transformations.

In [19], we established the equivalence between $\mathrm{Lax_N}(B, \mathbb{D})$ and $\mathbb{D}_0/\Gamma_B 1$, for certain double categories $\mathbb{D}$, in the case where B is a finite poset and the constant functor $\mathbb{D}_0 \longrightarrow \mathrm{Lax_N}(B, \mathbb{D})$ admits a left adjoint Γ_B. When $B = \mathbb{2}$, we know $\Gamma_B 1 = \mathbb{2}$ in Cat and Pos. It is the Sierpinski space $\mathbb{2}$ in Top and the Sierpinski locale S in Loc. The poset $B = \mathbb{2}$ was also used in [19] to define open and closed inclusions in $\mathbb{D}_0$ and obtain a general construction of exponentials for locally closed inclusions over an *arbitrary* base, which we then applied to Cat, Top, Loc, and Pos.

In this paper, we characterize the exponentiable objects $Y\colon B \longrightarrow \mathbb{D}$ of $\mathrm{Lax_N}(B, \mathbb{D})$ when B is the 3-element linearly-ordered poset, $\mathrm{Lax_N}(B, \mathbb{D})$ has finite limits, and $\mathbb{D}$ has, what we call, a *zero object*. Double categories $\mathbb{D}$ with these properties include $\mathbb{C}\mathrm{at}$, $\mathbb{P}\mathrm{os}$, $\mathbb{T}\mathrm{op}$, $\mathbb{L}\mathrm{oc}$, and $\mathbb{R}\mathrm{el}$. In particular, we show that Y is exponentiable if and only if $- \times Y$ preserves pseudo-functors, Y_b is exponentiable, for all $b \in B$, and $Y_b \longrightarrow\!\!\!\!\!\bullet\;\, Y_c$ is exponentiable as an object of $\mathbb{D}_1$, for all $b < c$ in B. Using the equivalence established in [19], we thus obtain a characterization of exponentiability in $\mathbb{D}_0/\Gamma_B 1$, for a general poset B, which applies to Cat, Pos, Top, and Loc, with the additional assumption that B is finite in the latter two cases. Note that every vertical morphism is exponentiable in $\mathbb{D}_1$, when $\mathbb{D}$ is $\mathbb{C}\mathrm{at}$ or $\mathbb{P}\mathrm{os}$.

We proceed as follows. In Section 2, we recall the definition of double category, and introduce zero objects as well as the double categories that will be considered throughout. The definition of $\mathrm{Lax_N}(B, \mathbb{D})$ and the characterization of its exponentiable objects are presented in Section 3. We conclude, in Section 4, by characterizing exponentiable objects of $\mathbb{D}_1$, for the two remaining cases, namely, $\mathbb{T}\mathrm{op}$ and $\mathbb{L}\mathrm{oc}$.

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2. Double Categories

A *double category* $\mathbb{D}$ is a weak internal category

$$\mathbb{D}_1 \times_{\mathbb{D}_0} \mathbb{D}_1 \xrightarrow{c} \mathbb{D}_1 \overset{d_0}{\underset{d_1}{\underset{\longrightarrow}{\overset{\longrightarrow}{\longleftarrow \Delta}}}} \mathbb{D}_0$$

in CAT. It consists of objects (those of $\mathbb{D}_0$), two types of morphisms: horizontal (morphisms of $\mathbb{D}_0$) and vertical (objects of $\mathbb{D}_1$ with domain and codomain given by d_0 and d_1), and cells (morphisms of $\mathbb{D}_1$) of the form

$$\begin{tikzcd} X_0 \arrow[r, "f_0"] \arrow[d, "m"', "\bullet" marking] \arrow[dr, phantom, "\rightarrow"] & Y_0 \arrow[d, "n", "\bullet"' marking] \\ X_1 \arrow[r, "f_1"'] & Y_1 \end{tikzcd} \tag{1}$$

Composition of morphisms and identities are defined horizontally in $\mathbb{D}_0$ and vertically using c and Δ, respectively. Cell composition is given horizontally in $\mathbb{D}_1$ and vertically via c. Cells in which the horizontal morphisms are identities are called *special cells*.

There are five double categories of interest in this paper.

(E1) $\mathbb{C}$at has small categories as objects, functors and profunctors as horizontal and vertical morphisms, respectively, and natural transformations $m \longrightarrow n(f_0-, f_1-)$ as cells of the form (1).

(E2) $\mathbb{T}$op has topological spaces as objects and continuous maps as horizontal morphisms. Vertical morphisms $m: X_0 \longrightarrow\!\!\!\!\!\bullet\;\; X_1$ are finite intersection-preserving maps $m: \mathcal{O}(X_0) \longrightarrow \mathcal{O}(X_1)$ on the open set lattices, and there is a cell of the form (1) if and only if $f_1^{-1}n \subseteq mf_0^{-1}$.

(E3) $\mathbb{L}$oc has locales as objects, locale morphisms (in the sense of [11]) as horizontal morphisms, and finite meet-preserving maps as vertical morphisms. There is a cell of the form (1) if and only if $f_1^* n \leq m f_0^*$.

(E4) $\mathbb{P}$os has partially-ordered sets as objects and order-preserving maps as horizontal morphisms. Vertical morphisms $m: X_0 \longrightarrow\!\!\!\!\!\bullet\;\; X_1$ are order ideals (i.e., up-sets) $m \subseteq X_0^{op} \times X_1$, and there is a cell of the form (1) if and only if $(x_0, x_1) \in m \Rightarrow (f_0(x_0), f_1(x_1)) \in n$.

(E5) $\mathbb{R}$el has sets as objects, functions and relations as horizontal and vertical morphisms, respectively, and there is a cell of the form (1) if and only if $(x_0, x_1) \in m \Rightarrow (f_0(x_0), f_1(x_1)) \in n$.

Our most general result, Theorem 3.3, will follow from properties shared by the five double categories. Although (E1)–(E5) are all framed bicategories (in the sense of [21]) and the first four have $\mathbb{2}$-glueing (in the sense of [19]), these conditions will not be used until we apply the main theorem to obtain the exponentiability results in (E1)–(E5).

An object 0 of $\mathbb{D}$ is called a *zero object* if it is horizontally initial, vertically both initial and terminal, and there exists a unique cell

$$\begin{tikzcd} X_0 \arrow[r, "f_0"] \arrow[d, "\bullet" marking] & Y_0 \arrow[dd, "n", "\bullet"' marking] \\ 0 \arrow[d, "\bullet" marking] \arrow[r, phantom, "\rightarrow"] & \\ X_1 \arrow[r, "f_1"'] & Y_1 \end{tikzcd}$$

for all f_0, f_1, n. Note that the latter condition implies that $X_0 \twoheadrightarrow 0 \twoheadrightarrow X_1$ is an initial object in the category of vertical morphisms $X_0 \twoheadrightarrow X_1$ and special cells.

The double categories in the five examples each have an initial object which is a zero object. In fact, if $\mathbb{D}$ is any framed bicategory, then any horizontal initial object which is vertically initial and terminal is easily seen to be a zero object.

3. Exponentiability of Normal Lax Functors

Suppose B is a poset and $\mathbb{D}$ is a double category. A *vertical normal lax functor* $X\colon B \longrightarrow \mathbb{D}$ consists of an object X_b of $\mathbb{D}$, for every $b \in B$, a vertical morphism $X_{bc}\colon X_b \twoheadrightarrow X_c$, for every $b < c$, and a special cell $X_{cd}X_{bc} \longrightarrow X_{bd}$, called a *comparison cell*, for every $b < c < d$, satisfying the usual coherence conditions. In particular, we are assuming our normal lax functors are *strict* normal in the sense that X_{bb} is the vertical identity morphism on X_b, for all b. A normal lax functor for which the comparison cells are all isomorphisms is called a *pseudo-functor*. A *horizontal lax transformation* $f\colon X \longrightarrow Y\colon B \longrightarrow \mathbb{D}$ consists of a horizontal morphism $f_b\colon X_b \longrightarrow Y_b$, for all $b \in B$, and a cell

$$\begin{tikzcd}
X_b \arrow[r, "f_b"] \arrow[d, "X_{bc}"'] \arrow[dr, phantom, "\rightarrow"] & Y_b \arrow[d, "Y_{bc}"] \\
X_c \arrow[r, "f_c"'] & Y_c
\end{tikzcd}$$

for every $b < c$, compatible with the comparison cells for X and Y. Vertical normal lax functors and horizontal transformations form a category which we denote by $\mathrm{Lax_N}(B, \mathbb{D})$. Note that $\mathrm{Lax_N}(\mathbb{1}, \mathbb{D}) \cong \mathbb{D}_0$ and $\mathrm{Lax_N}(\mathbb{2}, \mathbb{D}) \cong \mathbb{D}_1$.

If $\mathbb{D}$ has a zero object, then pseudo-functors $X\colon \mathbb{3} \longrightarrow \mathbb{D}$ can be described as follows. Given *any* normal lax functor $X\colon \mathbb{3} \longrightarrow \mathbb{D}$, there is a commutative diagram in $\mathrm{Lax_N}(\mathbb{3}, \mathbb{D})$ of the form

$$\begin{tikzcd}[column sep=small]
& \begin{smallmatrix} 0 \\ \downarrow \\ X_1 \\ \downarrow \\ 0 \end{smallmatrix} \arrow[dl] \arrow[dr] & \\
\begin{smallmatrix} X_0 \\ \downarrow \\ X_1 \\ \downarrow \\ 0 \end{smallmatrix} \arrow[dr] & & \begin{smallmatrix} 0 \\ \downarrow \\ X_1 \\ \downarrow \\ X_2 \end{smallmatrix} \arrow[dl] \\
& X &
\end{tikzcd} \tag{2}$$

where the morphisms and cells are identities or uniquely determined by the definition of zero object. If X is a functor, i.e., $X_{12}X_{01} = X_{02}$ and the comparison cell is the identity, then one can show that (2) is a pushout in $\mathrm{Lax_N}(\mathbb{3}, \mathbb{D})$. Thus, we get:

3.1. LEMMA. *The diagram (2) is a pushout in* $\mathrm{Lax_N}(\mathbb{3}, \mathbb{D})$ *if and only if* X *is a pseudo-functor.*

PROOF. Since (2) is a pushout when X is a functor, it follows that such a diagram is a pushout if and only if there is an isomorphism from X to the corresponding functor $X_0 \xrightarrow{\bullet} X_1 \xrightarrow{\bullet} X_2$ such that the cells $X_{01} \longrightarrow X_{01}$ and $X_{12} \longrightarrow X_{12}$ are identities if and only if X is a pseudo-functor. ∎

3.2. LEMMA. *If $\mathbb{D}$ has a zero object and A is a subposet of B, then the restriction functor $(\ )_A\colon \mathrm{Lax_N}(B,\mathbb{D}) \longrightarrow \mathrm{Lax_N}(A,\mathbb{D})$ has a left adjoint L_A such that $(\ )_A L_A = id$.*

PROOF. Given $X\colon A \longrightarrow \mathbb{D}$, define

$$(L_A X)_b = \begin{cases} X_b & \text{if } b \in A \\ 0 & \text{otherwise} \end{cases}$$

and given $b < c$, let $(L_A X)_{bc} = X_{bc}$, if $b, c \in A$, and let $(L_A X)_{bc}$ be the unique vertical morphism to or from 0, otherwise. That $L_A X$ is a normal lax functor follows directly from the definition of zero object, and the result easily follows. ∎

We will use the following notation for the functors L_A and $(\ )_A$ in some special cases. We write L_b and $(\ )_b$ when $A = \{b\}$ and L_{bc} when A is the subposet with two elements $b < c$. Similarly, we use the notation L_{bcd} in the case where $b < c < d$. We also write $L_b\colon \mathbb{D}_0 \longrightarrow \mathbb{D}_1$ and $(\ )_b\colon \mathbb{D}_1 \longrightarrow \mathbb{D}_0$ for the functors induced by the isomorphisms $\mathrm{Lax_N}(\mathbb{1}, \mathbb{D}) \cong \mathbb{D}_0$ and $\mathrm{Lax_N}(\mathbb{2}, \mathbb{D}) \cong \mathbb{D}_1$, where $b = 0, 1$.

Suppose $\mathrm{Lax_N}(B,\mathbb{D})$ has finite products. Then so does $\mathrm{Lax_N}(A,\mathbb{D})$, for all subposets $A \subseteq B$, since $(\ )_A$ has a left adjoint by Lemma 3.2. In particular, $\mathbb{D}_0$ has finite products and $(X \times Y)_b \cong X_b \times Y_b$ in $\mathbb{D}_0$, for all $b \in B$ and $X, Y \in \mathrm{Lax_N}(B,\mathbb{D})$.

Note that if $\mathbb{D}_0$ has chosen products, we do not necessarily know that we can take $(X \times Y)_b$ to be the chosen product. However, we will assume $\mathbb{D}$ is *horizontally invariant*, in the sense of [7], and then this problem disappears. The five double categories (E1)–(E5) of interest all have companions and conjoints, and hence are horizontally invariant.

Given $b < c < d$, we have cells

$$\begin{tikzcd}
X_b \times Y_b \arrow[r] \arrow[d, "X_{bc}\times Y_{bc}"'] & X_b \times Y_b \arrow[dd, "X_{bd}\times Y_{bd}"] \\
X_c \times Y_c \arrow[r, phantom, "\xrightarrow{\theta_{bcd}}"] \arrow[d, "X_{cd}\times Y_{cd}"'] & \\
X_d \times Y_d \arrow[r] & X_d \times Y_d
\end{tikzcd}$$

where $\pi_1\theta_{bcd}$ is $\pi_1 \cdot \pi_1\colon (X_{cd} \times Y_{cd}) \cdot (X_{bc} \times Y_{bc}) \longrightarrow X_{bd} \cdot X_{bc}$ followed by the comparison cell $X_{bd} \cdot X_{bc} \longrightarrow X_{bd}$, and $\pi_2\theta_{bcd}$ is defined similarly.

We will say $- \times Y_{bcd}$ *preserves pseudo-functors* if θ_{bcd} is invertible, whenever X is a pseudo-functor in $\mathrm{Lax_N}(\{b,c,d\},\mathbb{D})$. In the case where B is the 3-element totally-ordered set $\mathbb{3} = \{0,1,2\}$, we will say $- \times Y$ preserves pseudo-functors, when $- \times Y_{012}$ does.

Note that if Y_{bcd} preserves pseudo-functors, for all $b < c < d$, then Y itself is necessarily a pseudo-functor, provided that $\mathrm{Lax_N}(B,\mathbb{D})$ has a pseudo-functorial terminal object, e.g., $\mathbb{D}$ has a double terminal (in the sense of [7]). This is the case for the five double categories under consideration.

3.3. Theorem. *Suppose $\mathbb{D}$ is a horizontally invariant double category with a zero object such that $0 \times X \cong 0$, for all X in $\mathbb{D}$, and $\mathrm{Lax_N}(\mathfrak{3}, \mathbb{D})$ has finite limits. Then $Y \colon \mathfrak{3} \longrightarrow \mathbb{D}$ is exponentiable in $\mathrm{Lax_N}(\mathfrak{3}, \mathbb{D})$ if and only if*

(i) Y_b is exponentiable in $\mathbb{D}_0$, for all b;

(ii) $Y_b \xrightarrow{\bullet} Y_c$ is exponentiable in $\mathbb{D}_1$, for all $b < c$; and

(iii) $- \times Y$ preserves pseudo-functors.

Proof. Suppose (i), (ii), and (iii) hold. By the remark following Lemma 3.2, we know that $(X \times Y)_{bc} \colon X_b \times Y_b \xrightarrow{\bullet} X_c \times Y_c$ is the product of X_{bc} and Y_{bc} in $\mathbb{D}_1$, for all $b < c$. Given $Z \colon \mathfrak{3} \longrightarrow \mathbb{D}$, consider the exponential $Z_{bc}^{Y_{bc}}$ in $\mathbb{D}_1$. Then $(Z_{bc}^{Y_{bc}})_b \cong Z_b^{Y_b}$, since there are natural bijections

$$\begin{aligned}
\mathbb{D}_0(X \times Y_b, Z_b) &\cong \mathbb{D}_0(X \times Y_b, (Z_{bc})_0) \\
&\cong \mathbb{D}_1(L_0(X \times Y_b), Z_{bc}) \\
&\cong \mathbb{D}_1(L_0 X \times Y_{bc}, Z_{bc}) \\
&\cong \mathbb{D}_1(L_0 X, Z_{bc}^{Y_{bc}}) \\
&\cong \mathbb{D}_0(X, (Z_{bc}^{Y_{bc}})_0)
\end{aligned}$$

where the third bijection follows from the isomorphism $0 \times Y_c \cong 0$. Similarly, $(Z_{bc}^{Y_{bc}})_c \cong Z_c^{Y_c}$, and by horizontal invariance, we can assume these isomorphisms are equalities.

Consider the exponential $Z_{bc}^{Y_{bc}} \colon Z_b^{Y_b} \xrightarrow{\bullet} Z_c^{Y_c}$ in $\mathbb{D}_1$. Thus, $b \mapsto Z_b^{Y_b}$ becomes a lax functor $\mathfrak{3} \longrightarrow \mathbb{D}$ via the cell on the left which corresponds to the diagram on the right by exponentiability of $Y_{02} \colon Y_0 \xrightarrow{\bullet} Y_2$

$$\begin{tikzcd}
Z_0^{Y_0} \arrow[r, "id"] \arrow[d, "Z_{01}^{Y_{01}}"'] & Z_0^{Y_0} \arrow[dd, "Z_{02}^{Y_{02}}"] \\
Z_1^{Y_1} \arrow[r, phantom, "\rightarrow"] \arrow[d, "Z_{12}^{Y_{12}}"'] & {} \\
Z_2^{Y_2} \arrow[r, "id"'] & Z_2^{Y_2}
\end{tikzcd}
\qquad
\begin{tikzcd}
Z_0^{Y_0} \times Y_0 \arrow[r, "id"] \arrow[dd, "(Z_{12}^{Y_{12}} \cdot Z_{01}^{Y_{01}}) \times Y_{02}"'] & Z_0^{Y_0} \times Y_0 \arrow[r, "\varepsilon_0"] \arrow[d, "Z_{01}^{Y_{01}} \times Y_{01}"'] & Z_0 \arrow[r, "id"] \arrow[d, "Z_{01}"] & Z_0 \arrow[dd, "Z_{02}"] \\
{} \arrow[r, phantom, "\rightarrow"] & Z_1^{Y_1} \times Y_1 \arrow[r, "\varepsilon_1"] \arrow[d, "Z_{12}^{Y_{12}} \times Y_{12}"'] & Z_1 \arrow[r, phantom, "\rightarrow"] \arrow[d, "Z_{12}"] & {} \\
Z_2^{Y_2} \times Y_2 \arrow[r, "id"'] & Z_2^{Y_2} \times Y_2 \arrow[r, "\varepsilon_2"'] & Z_2 \arrow[r, "id"'] & Z_2
\end{tikzcd}$$

where there is a cell in the left rectangle of the right diagram, since $- \times Y$ preserves pseudo-functors. Note that we are applying the latter to the pseudo-functor $b \mapsto Z_b^{Y_b}$ with $Z_{12}^{Y_{12}} \cdot Z_{01}^{Y_{01}} \colon Z_0^{Y_0} \xrightarrow{\bullet} Z_2^{Y_2}$ and the identity comparison cell. With this definition, it is not difficult to show that the unit and counit for $(\ )^{Y_{bc}}$ extend to ones for $(\ )^{Y}$, and it follows that Y is exponentiable in $\mathrm{Lax_N}(\mathfrak{3}, \mathbb{D})$.

Conversely, suppose Y is exponentiable in $\mathrm{Lax_N}(\mathfrak{3}, \mathbb{D})$. Then arguments analogous to the one proving $(Z_{bc}^{Y_{bc}})_b \cong Z_b^{Y_b}$ in the first half of the proof, show (i) and (ii) hold. To see that $-\times Y$ preserves pseudo-functors, suppose $X\colon \mathbb{3}\longrightarrow \mathbb{D}$ is a pseudo-functor. Then

$$\begin{tikzcd}
 & & 0 \arrow[d] & & \\
 & & X_1 \arrow[d] \arrow[ddll] \arrow[ddrr] & & \\
X_0 \arrow[d] & & 0 & & 0 \arrow[d] \\
X_1 \arrow[d] \arrow[drr] & & & & X_1 \arrow[d] \arrow[dll] \\
0 & & X & & X_2
\end{tikzcd}$$

is a pushout, by Lemma 3.1. Since $-\times Y$ preserves pushouts, it follows that

$$\begin{tikzcd}
 & & 0 \arrow[d] & & \\
 & & X_1\times Y_1 \arrow[d] \arrow[ddll] \arrow[ddrr] & & \\
X_0\times Y_0 \arrow[d] & & 0 & & 0 \arrow[d] \\
X_1\times Y_1 \arrow[d] \arrow[drr] & & & & X_1\times Y_1 \arrow[d] \arrow[dll] \\
0 & & X\times Y & & X_2\times Y_2
\end{tikzcd}$$

is a pushout, as well, and the desired result follows. ∎

Note that the obstruction to extending the above proof to a general poset B is that the construction $b\mapsto Z_b^{Y_b}$ may not be a lax functor since the coherence condition relative to associativity need not hold. We will see that this is not a problem in $\mathbb{C}\mathrm{at}$, or for certain double categories including the other four of interest here.

Using Bénabou's equivalence $\mathrm{Lax}_{\mathrm{N}}(B,\mathbb{C}\mathrm{at})\simeq \mathrm{Cat}/B$, the fact that (i) and (ii) always hold, and $-\times Y_{bcd}$ preserves pseudo-functors, for all pseudo-functors $Y\colon B\longrightarrow \mathbb{C}\mathrm{at}$, we get Street's characterization [23], as a consequence of Theorem 3.3 as follows.

3.4. Corollary. *A functor $Y\longrightarrow B$ is exponentiable in* Cat *if and only if the corresponding vertical normal lax functor $B\longrightarrow\mathbb{C}\mathrm{at}$ is a pseudo-functor.*

Proof. Suppose $Y\longrightarrow B$ is exponentiable in Cat. Then so is $\mathbb{3}\times_B Y\longrightarrow \mathbb{3}$, for all functors $f\colon\mathbb{3}\longrightarrow B$, since pulling back along f preserves exponentiability. Thus, the corresponding normal lax functor, denoted via abuse of notation by $Y_f\colon\mathbb{3}\longrightarrow\mathbb{C}\mathrm{at}$, is exponentiable in $\mathrm{Lax}_{\mathrm{N}}(\mathbb{3},\mathbb{C}\mathrm{at})$, and hence, satisfies (iii) of Theorem 3.3. Since $\mathbb{C}\mathrm{at}$ has a double terminal object, it follows that that Y_f is a pseudo-functor, and so $B\longrightarrow\mathbb{C}\mathrm{at}$ is as well.

Conversely, suppose the corresponding vertical normal lax functor $Y\colon B\longrightarrow\mathbb{C}\mathrm{at}$ is a pseudo-functor. Then so is $Y_f\colon\mathbb{3}\longrightarrow\mathbb{C}\mathrm{at}$, for all $f\colon\mathbb{3}\longrightarrow B$. Since (i) and (ii) of Theorem 3.3 hold, in any case, and every pseudo-functor satisfies (iii), it follows that Y_f is exponentiable in $\mathrm{Lax}_{\mathrm{N}}(\mathbb{3},\mathbb{C}\mathrm{at})$. To see that Y is exponentiable in $\mathrm{Lax}_{\mathrm{N}}(B,\mathbb{C}\mathrm{at})$, we need only show that the construction $b\mapsto Z_b^{Y_b}$ given in Theorem 3.3 is coherent relative to associativity.

For $\beta\colon b \to c$ in B, the exponential $Z_\beta^{Y_\beta}\colon Z_b^{Y_b} \xrightarrow{\bullet} Z_c^{Y_c}$ can be described as follows. Identifying $Z_b^{Y_b}$ with the usual functor category, elements of the set $Z_\beta^{Y_\beta}(\sigma_b, \sigma_c)$ correspond to cells in $\mathbb{D}$ of the form

$$\begin{tikzcd} Y_b \arrow[r, "\sigma_b"] \arrow[d, "Y_\beta"', "\bullet" marking] \arrow[dr, phantom, "\rightarrow"] & Z_b \arrow[d, "Z_\beta", "\bullet" marking] \\ Y_c \arrow[r, "\sigma_c"'] & Z_c \end{tikzcd}$$

Unravelling the proof of Theorem 3.3, the comparison cells $Z_\gamma^{Y_\gamma} \cdot Z_\beta^{Y_\beta} \longrightarrow Z_{\gamma\beta}^{Y_{\gamma\beta}}$ are induced by the diagram

$$\begin{tikzcd} Y_b \arrow[r, "id"] \arrow[dd, "Y_{\gamma\beta}"', "\bullet" marking] & Y_b \arrow[r, "\sigma_b"] \arrow[d, "Y_\beta"', "\bullet" marking] \arrow[dr, phantom, "\rightarrow"] & Z_b \arrow[r, "id"] \arrow[d, "Z_\beta", "\bullet" marking] & Z_b \arrow[dd, "Z_{\gamma\beta}", "\bullet" marking] \\ {} \arrow[r, phantom, "\overset{\varphi^{-1}}{\rightarrow}"] & Y_c \arrow[r, "\sigma_c"] \arrow[d, "Y_\gamma"', "\bullet" marking] \arrow[dr, phantom, "\rightarrow"] & Z_c \arrow[d, "Z_\gamma", "\bullet" marking] \arrow[r, phantom, "\overset{\psi}{\rightarrow}"] & {} \\ Y_d \arrow[r, "id"'] & Y_d \arrow[r, "\sigma_d"'] & Z_d \arrow[r, "id"'] & Z_d \end{tikzcd}$$

where φ and ψ are the comparison cells for Y and Z, respectively, and so coherence easily follows. ■

Next, we prove a corollary that gives exponentiability results for $\mathbb{P}$os, $\mathbb{T}$op, $\mathbb{L}$oc, and $\mathbb{R}$el. First, we recall (from [7]) a property of double categories which is shared by these four examples and eliminates the coherence problem in the general version of Theorem 3.3.

A double category $\mathbb{D}$ is called *flat* if its cells are determined by their domains and codomains. In this case, there is at most one cell

$$\begin{tikzcd} X_0 \arrow[r, "f_0"] \arrow[d, "m"', "\bullet" marking] \arrow[dr, phantom, "\rightarrow"] & Y_0 \arrow[d, "n", "\bullet" marking] \\ X_1 \arrow[r, "f_1"'] & Y_1 \end{tikzcd}$$

for all f_0, m, n, f_1.

3.5. Corollary. *Suppose B is a poset, $\mathbb{D}$ is a flat horizontally invariant double category with a zero object such that $0 \times X \cong 0$, for all X in $\mathbb{D}$, and $\mathrm{Lax_N}(B, \mathbb{D})$ has finite limits. Then $Y\colon B \longrightarrow \mathbb{D}$ is exponentiable in $\mathrm{Lax_N}(B, \mathbb{D})$ if and only if*

(i) Y_b is exponentiable in $\mathbb{D}_0$, for all b;

(ii) $Y_b \xrightarrow{\bullet} Y_c$ is exponentiable in $\mathbb{D}_1$, for all $b < c$; and

(iii) $- \times Y_{bcd}$ preserves pseudo-functors, for all $b < c < d$.

Proof. Given (i)–(iii), consider $b \mapsto Z_b^{Y_b}$, as defined in Theorem 3.3. Since $\mathbb{D}$ is flat, this construction is coherent relative to associativity. Thus, we get a normal lax functor $B \longrightarrow \mathbb{D}$, and it follows that $Y\colon B \longrightarrow \mathbb{D}$ is exponentiable in $\mathrm{Lax_N}(B, \mathbb{D})$.

Suppose $Y\colon B \longrightarrow \mathbb{D}$ is exponentiable in $\mathrm{Lax_N}(B, \mathbb{D})$. Then, by an argument analogous to the one proving $(Z_{bc}^{Y_{bc}})_b \cong Z_b^{Y_b}$ in the proof Theorem 3.3, we see that Y_{bcd} is exponentiable in $\mathrm{Lax_N}(\mathfrak{Z}, \mathbb{D})$, for all $b < c < d$, and the desired result follows. ∎

When B is a poset, Bénabou's equivalence is easily seen to restrict to $\mathbb{P}\mathrm{os}$, yielding $\mathrm{Lax_N}(B, \mathbb{P}\mathrm{os}) \simeq \mathrm{Pos}/B$. As in $\mathbb{C}\mathrm{at}$, we know (i) and (ii) always hold (see [16]), and $- \times Y_{bcd}$ preserves pseudo-functors, for all pseudo-functors $Y\colon B \longrightarrow \mathbb{P}\mathrm{os}$. Applying Corollary 3.5, we get:

3.6. Corollary. *A morphism $Y \longrightarrow B$ is exponentiable in* Pos *if and only if the corresponding vertical normal lax functor $B \longrightarrow \mathbb{P}\mathrm{os}$ is a pseudo-functor.*

In [19], we showed that if $\mathbb{D}$ is a double category satisfying certain conditions, then Bénabou's equivalence generalizes to $\mathrm{Lax_N}(B, \mathbb{D}) \simeq \mathbb{D}_0/\Gamma_B 1$, for every finite poset B, where Γ_B is left adjoint to the constant functor $\mathbb{D}_0 \longrightarrow \mathrm{Lax_N}(B, \mathbb{D})$. Examples include $\mathbb{C}\mathrm{at}, \mathbb{P}\mathrm{os}, \mathbb{T}\mathrm{op}$, and $\mathbb{L}\mathrm{oc}$. In any case, we know pulling back preserves exponentiability in $\mathbb{D}_0$, and so using this equivalence, we can replace (i) and (ii) of Corollary 3.5 by the single condition that $Y_b \twoheadrightarrow Y_c$ is exponentiable in $\mathbb{D}_1$, for all $b \leq c$.

As noted above, the finiteness condition is not necessary in $\mathbb{C}\mathrm{at}$ and $\mathbb{P}\mathrm{os}$. Whether it is necessary in $\mathbb{T}\mathrm{op}$ and $\mathbb{L}\mathrm{oc}$ is an open question. However, we know that $\Gamma_B 1$ is the Alexandroff space on B (in the sense of [1], i.e., open sets are downward closed) in $\mathbb{T}\mathrm{op}$, and $\Gamma_B 1$ is the locale $\downarrow\mathrm{Cl}(B)$ of down-sets of B in $\mathbb{L}\mathrm{oc}$. Since every finite T_0 space is the Alexandroff space of its poset of points with the specialization order (see [11]), we get the following two corollaries:

3.7. Corollary. *The following are equivalent for a finite T_0 space B and a continuous map $q\colon Y \longrightarrow B$ with corresponding vertical normal lax functor $n\colon B \longrightarrow \mathbb{T}\mathrm{op}$.*

(a) $q\colon Y \longrightarrow B$ is exponentiable in Top*.*

(b) $n\colon B \longrightarrow \mathbb{T}\mathrm{op}$ is exponentiable in $\mathrm{Lax_N}(B, \mathbb{T}\mathrm{op})$.

(c) $Y_b \twoheadrightarrow Y_c$ is exponentiable in $\mathbb{T}\mathrm{op}_1$, for all $b \leq c$ and $- \times n_{bcd}$ preserves pseudo-functors, for all $b < c < d$.

3.8. Corollary. *The following are equivalent for a finite poset B and a locale morphism $q\colon Y \longrightarrow \downarrow\mathrm{Cl}(B)$ with corresponding vertical normal lax functor $n\colon B \longrightarrow \mathbb{L}\mathrm{oc}$.*

(a) $q\colon Y \longrightarrow \downarrow\mathrm{Cl}(B)$ is exponentiable in Loc*.*

(b) $n\colon B \longrightarrow \mathbb{L}\mathrm{oc}$ is exponentiable in $\mathrm{Lax_N}(B, \mathbb{L}\mathrm{oc})$.

(c) $Y_b \twoheadrightarrow Y_c$ *is exponentiable in* $\mathbb{L}\mathrm{oc}_1$*, for all* $b \leq c$ *and* $- \times n_{bcd}$ *preserves pseudo-functors, for all* $b < c < d$.

In the next section, we will characterize exponentiability in $\mathbb{T}\mathrm{op}_1$ and $\mathbb{L}\mathrm{oc}_1$, to get more complete versions of Corollaries 3.7 and 3.8. In fact, in both cases, we will also show that $- \times n_{bcd}$ preserves pseudo-functors, for all $b < c < d$, whenever n is a pseudo-functor such that $n_{bc}\colon Y_b \twoheadrightarrow Y_c$ is exponentiable, for all $b \leq c$.

Now, we will use Lemma 3.1 to characterize pseudo-functors, when Γ/B is an equivalence of categories. As noted earlier, $\mathbb{P}\mathrm{os}$ satisfies this condition for any poset B. Also, $\mathbb{T}\mathrm{op}$ and $\mathbb{L}\mathrm{oc}$ do when B is finite, and $\mathbb{C}\mathrm{at}$ does for any small category B. Our proof below, assumes B is a poset, but can be adapted to apply to $\mathbb{C}\mathrm{at}$ for a general B.

Suppose B is a poset and $n\colon B \longrightarrow \mathbb{D}$. We adopt the following abuse of notation. Given $b < c$, let Y_{bc} denote the image of $n_{bc}\colon Y_b \twoheadrightarrow Y_c$ under $\Gamma_2\colon \mathrm{Lax}_{\mathrm{N}}(\mathbb{2}, \mathbb{D}) \longrightarrow \mathbb{D}_0$, and similarly, Y_{bcd} for $b < c < d$ and Γ_3. If $Y \longrightarrow \Gamma_B 1$ is the image of n under Γ_B, then $Y_{bc} \cong \Gamma_B L_{bc} 1 \times_{\Gamma_B 1} Y$ and $Y_{bcd} \cong \Gamma_B L_{bcd} 1 \times_{\Gamma_B 1} Y$, when $\mathbb{D}_0$ is $\mathbb{C}\mathrm{at}$, $\mathbb{P}\mathrm{os}$, $\mathbb{T}\mathrm{op}$, and $\mathbb{L}\mathrm{oc}$. In any case, one can show that there is a commutative diagram

$$\begin{tikzcd} & Y_c \arrow[dl] \arrow[dr] & \\ Y_{bc} \arrow[dr] & & Y_{cd} \arrow[dl] \\ & Y_{bcd} & \end{tikzcd} \tag{3}$$

in $\mathbb{D}_0$, for all $b < c < d$.

3.9. Proposition. *Suppose* $\mathbb{D}$ *has* 0 *and* 1, $\Gamma/\mathbb{3}\colon \mathrm{Lax}_{\mathrm{N}}(\mathbb{3}, \mathbb{D}) \longrightarrow \mathbb{D}_0/\Gamma_3 1$ *is equivalence of categories, and* B *is a poset. Then a normal lax functor* $n\colon B \longrightarrow \mathbb{D}$ *is a pseudo-functor if and only if the diagram (3) is a pushout in* $\mathbb{D}_0$*, for all* $b < c < d$.

Proof. First, n is a pseudo-functor if and only if $n_{bcd}\colon \mathbb{3} \longrightarrow \mathbb{D}$ is, for all $b < c < d$ if and only if the diagram (2) from Lemma 3.1 is a pushout, for all $b < c < d$. Since $\Gamma/\mathbb{3}\colon \mathrm{Lax}_{\mathrm{N}}(\mathbb{3}, \mathbb{D}) \longrightarrow \mathbb{D}_0/\Gamma_3 1$ is an equivalence, the latter holds if and only if (3) is a pushout, for all $b < c < d$. ∎

We conclude this section by turning our attention to $\mathbb{R}\mathrm{el}$. In this case, the functor $\Gamma/B\colon \mathrm{Lax}_{\mathrm{N}}(B, \mathbb{R}\mathrm{el}) \longrightarrow \mathbb{R}\mathrm{el}_0/\Gamma_B 1$ is not an equivalence unless $B = \mathbb{1}$, since it is not difficult to show that $\Gamma_B 1$ is a one-point set so that $\mathbb{R}\mathrm{el}_0/\Gamma_B 1 \cong \mathrm{Set}$. Thus, the theorem from [19] does not apply. However, $\mathrm{Lax}_{\mathrm{N}}(B, \mathbb{R}\mathrm{el})$ is equivalent to the category Pos_d/B of posets with discrete fibers over B (see [18]), and $\mathbb{R}\mathrm{el}_1$ is easily seen to be cartesian closed. Applying Corollary 3.5, we get:

3.10. Corollary. *Suppose* B *is a poset. Then* $Y \longrightarrow B$ *is exponentiable in* Pos_d/B *if and only if the corresponding vertical normal lax functor* $B \longrightarrow \mathbb{R}\mathrm{el}$ *is a pseudo-functor.*

4. Exponentiability in $\mathbb{T}\mathrm{op}_1$ and $\mathbb{L}\mathrm{oc}_1$

A space Y is exponentiable in Top if and only if $\mathcal{O}(Y)$ is a continuous lattice (in the sense of Scott [20]) if and only if 2^Y exists in Top, where 2 denotes the Sierpinski space $\{0,1\}$, with $\{0\}$ open but not $\{1\}$. In this case, $2^Y \cong \mathcal{O}(Y)$ with the Scott topology, which is defined as follows.

Recall that a subset H of a complete lattice L is called *Scott open* if $\uparrow H = H$ and $\vee S \in H \Rightarrow \vee F \in H$, for some finite $F \subseteq S$. The set ΣL of Scott open subsets is called the *Scott topology* on L. Given $u, v \in L$, we say u is *way below* v, and write $u << v$, if $v \le \vee S \Rightarrow u \le \vee F$, for some finite $F \subseteq S$. Then L is a *continuous lattice* if it satisfies $v = \vee\{u \mid u << v\}$. A locale which is a continuous lattice is also called *locally compact*.

The characterization of exponentiable spaces has appeared in many forms, but was first achieved in 1970 when Day and Kelly [3] proved that $-\times Y$ preserves quotient maps precisely when $\mathcal{O}(Y)$ is a continuous lattice. By Freyd's Special Adjoint Functor Theorem, $-\times Y$ has a right adjoint if and only if it preserves quotient maps, for then it preserves all colimits (since coproducts are preserved in any case). The "technical condition" for exponentiability in Top$/B$, proved in [14] and referred to in the introduction, reduces to the Day/Kelly characterization when $B = 1$, and has the following form when B is a poset with the down-set topology.

Suppose $q\colon Y \longrightarrow B$ is a continuous map. Then $H \subseteq \coprod_{b\in B} \mathcal{O}(Y_b)$ is called *fiberwise Scott open* provided that H_b is Scott open, for all $b \in B$, and $V_c \in H_c \Rightarrow V_b \in H_b$, for all $b < c$ and $V \in \mathcal{O}(Y)$. With this topology, $\coprod_{b\in B} \mathcal{O}(Y_b)$ becomes a space over B via the projection. Consider the (not necessarily continuous) function $\varepsilon\colon \big(\coprod_{b\in B} \mathcal{O}(Y_b)\big) \times_B Y \longrightarrow 2$ defined by

$$\varepsilon(V_b, y) = \begin{cases} 0 & \text{if } y \in V_b \\ 1 & \text{if } y \notin V_b \end{cases}$$

Then, from [14], we get:

4.1. LEMMA. *The following are equivalent for $q\colon Y \longrightarrow B$ in* Top.

(a) $q\colon Y \longrightarrow B$ is exponentiable in Top.

(b) The map $\varepsilon\colon \big(\coprod_{b\in B} \mathcal{O}(Y_b)\big) \times_B Y \longrightarrow 2$ is continuous.

(c) For all $V_b \in \mathcal{O}(Y_b)$ and $y_b \in V_b$, there exists H fiberwise Scott open such that $V_b \in H_b$ and y_b is in the interior in Y of the set $\bigcup_{b\in B}(\cap H_b)$.

We will use Lemma 4.1, in the case where $B = 2$, to show that the continuous lattice characterization of exponentiable objects in Top and Loc generalizes to $\mathbb{T}\mathrm{op}_1$ and $\mathbb{L}\mathrm{oc}_1$. But, first we use this lemma to prove that $-\times n_{bcd}$ preserves pseudo-functors, for all $b < c < d$, whenever n is a pseudo-functor such that $n_{bc}\colon Y_b \longrightarrow\!\!\!\!\!\!\bullet\;\; Y_c$ is exponentiable, for all $b \le c$, thus removing the extra condition in Corollary 3.7 (and hence, in Corollary 3.8, as well). We begin by recalling the equivalence between $\mathrm{Lax}_{\mathrm{N}}(B, \mathbb{T}\mathrm{op})$ and Top$/B$, for a finite poset B.

Given $m: B \longrightarrow \mathbb{T}\mathrm{op}$, write $m_{bc}: X_b \twoheadrightarrow X_c$, for $b < c$. Let $X = \coprod_{b \in B} X_b$, with U *open* if U_b is open in X_b, for all b, and $U_c \subseteq m_{bc} U_b$, for all $b < c$. A horizontal transformation $f: m \longrightarrow n$ gives rise to a continuous map $f: X \longrightarrow Y$ in the obvious way. Note that B is the space associated with the terminal object of $\mathrm{Lax}_{\mathrm{N}}(B, \mathbb{T}\mathrm{op})$, and so the projection $X \longrightarrow B$ is continuous. Thus, we get $\Gamma_B: \mathrm{Lax}_{\mathrm{N}}(B, \mathbb{T}\mathrm{op}) \longrightarrow \mathrm{Top}$, which is left adjoint to the constant functor and induces an equivalence $\Gamma/B: \mathrm{Lax}_{\mathrm{N}}(B, \mathbb{T}\mathrm{op}) \longrightarrow \mathrm{Top}/B$, whose pseudo-inverse is defined as follows (see [19] for details).

For $X \longrightarrow B$, let $m_b = X_b$, the fiber over X at b. Then the inclusion $i_b: X_b \longrightarrow X$ induces a locale morphism $i_b: \mathcal{O}(X_b) \longrightarrow \mathcal{O}(X)$ defined by $i_b^* = i_b^{-1}$ and $i_{b*}(U_b) = [U_b \cup (X \setminus X_b)]^\circ$. Given $b < c$, the composition $\mathcal{O}(X_b) \xrightarrow{i_{b*}} \mathcal{O}(X) \xrightarrow{i_c^*} \mathcal{O}(X_c)$ is a vertical morphism in $\mathbb{T}\mathrm{op}$ which we denote by $m_{bc}: X_b \twoheadrightarrow X_c$. Thus, we get a normal lax functor $m: B \longrightarrow \mathbb{T}\mathrm{op}$, and hence a functor $\Phi_B: \mathrm{Top}/B \longrightarrow \mathrm{Lax}_{\mathrm{N}}(B, \mathbb{T}\mathrm{op})$ which is pseudo-inverse to Γ/B.

Suppose $q: Y \longrightarrow B$ corresponds with $n: B \longrightarrow \mathbb{T}\mathrm{op}$. We would like to describe the normal lax functor related to the space $\coprod_{b \in B} \mathcal{O}(Y_b)$ with the fiberwise Scott topology over B. Of course, the fiber over b is $\mathcal{O}(Y_b)$ with the Scott topology $\Sigma\mathcal{O}(Y_b)$. Using the description of open sets of Y arising from the equivalence given above, one can show that H is fiberwise Scott open if and only if H_b is Scott open, for all b, and $n_{bc} U_b \in H_c \Rightarrow U_b \in H_b$, for all $b < c$, if and only if H_b is Scott open, for all b, and $n_{bc}^{-1} H_c \subseteq H_b$, for all $b < c$.

Given $b < c$, consider $\hat{n}_{bc}: \mathcal{O}(Y_b) \twoheadrightarrow \mathcal{O}(Y_c)$, defined by

$$\hat{n}_{bc} H_b = \bigcup \{ H_c \in \Sigma\mathcal{O}(Y_c) \mid n_{bc}^{-1} H_c \subseteq H_b \}$$

It is not difficult to show that $\hat{n}_{bc}$ preserves finite intersections, but $\hat{n}$ does not necessarily define a lax functor $b \mapsto \mathcal{O}(Y_b)$, since $n_{bc}^{-1} H_c$ need not be Scott open, when H_c is. The latter holds precisely when $n_{bc}: \mathcal{O}(Y_b) \longrightarrow \mathcal{O}(Y_c)$ preserves directed unions. Such a function is called *Scott continuous*.

4.2. Lemma. *If n is a pseudo-functor and $n_{bc}: Y_b \twoheadrightarrow Y_c$ preserves directed unions, for all $b < c$, then $\hat{n}: B \longrightarrow \mathbb{T}\mathrm{op}$ is a pseudo-functor, and $\Gamma_B \hat{n} = \coprod_{b \in B} \mathcal{O}(Y_b)$, with the fiberwise Scott topology.*

Proof. First, we show that $\hat{n}$ is a pseudo-functor. Since $n_{bc}: \mathcal{O}(Y_b) \longrightarrow \mathcal{O}(Y_c)$ preserves directed unions, it is Scott continuous, and so $n_{bc}^{-1} H_c$ is Scott open, for all H_c Scott open in $\mathcal{O}(Y_c)$. Then $n_{bc}^{-1}: \Sigma\mathcal{O}(Y_b) \longrightarrow \Sigma\mathcal{O}(Y_c)$ is left adjoint to $\hat{n}_{bc}$, by definition of the latter. Since n is a pseudo-functor, we know that $n_{bd} = n_{cd} n_{bc}$, and so $n_{bd}^{-1} = n_{bc}^{-1} n_{cd}^{-1}$. Thus, it follows that $\hat{n}_{bd} = \hat{n}_{cd} \hat{n}_{bc}$, as desired.

It remains to show that $\Gamma_B \hat{n} = \coprod_{b \in B} \mathcal{O}(Y_b)$ with the fiberwise Scott topology. By definition, $H \subseteq \Gamma_B \hat{n}$ is open if and only if H_b is open in $\mathcal{O}(Y_b)$, for all b, and $H_c \subseteq \hat{n}_{bc} H_b$, for all $b < c$, if and only if H_b is open in $\mathcal{O}(Y_b)$, for all b, and $n_{bc}^{-1} H_c \subseteq H_b$, for all $b < c$, and the desired result follows. ∎

4.3. Lemma. *If $q: Y \longrightarrow \mathbb{2}$ is exponentiable in* Top*, then the corresponding $n: Y_0 \xrightarrow{\bullet} Y_1$ preserves directed unions.*

Proof. Suppose $\{U_\alpha\}$ is directed, and consider $\bigcup nU_\alpha \subseteq n\,(\bigcup U_\alpha)$. Now, $\mathcal{O}(Y_1)$ is a continuous lattice, since pulling back preserves exponentiability, and so given $y_1 \in n\,(\bigcup U_\alpha)$, there exists $V_1 \in \mathcal{O}(Y_1)$ such that $y_1 \in V_1 << n\,(\bigcup U_\alpha)$. Since q is exponentiable, applying Lemma 4.1 with $B = \mathbb{2}$, there exists $H \subseteq \mathcal{O}(Y_0) \bigsqcup \mathcal{O}(Y_1)$ fiberwise Scott open such that $V_1 \in H_1$ and $y_1 \in W \subseteq (\cap H_0) \cup (\cap H_1)$, for some W open in Y. Then H_1 is Scott open, $V_1 \in H_1$, and $V_1 \subseteq n\,(\bigcup U_\alpha)$, and so $n\,(\bigcup U_\alpha) \in H_1$. Since H is fiberwise Scott open, we know $\bigcup U_\alpha \in H_0$, and so $U_\alpha \in H_0$, for some α. Also, $W_0 \subseteq U_\alpha$, since $W_0 \subseteq \cap H_0$. Thus, $y_1 \in W_1 \subseteq nW_0 \subseteq nU_\alpha \subseteq \bigcup nU_\alpha$, and it follows that $n\,(\bigcup U_\alpha) \subseteq \bigcup nU_\alpha$, as desired. ∎

4.4. Theorem. *The following are equivalent for a finite T_0 space B and a continuous map $q: Y \longrightarrow B$ with corresponding vertical normal lax functor $n: B \longrightarrow \mathbb{T}\mathrm{op}$.*

(a) $q: Y \longrightarrow B$ is exponentiable in Top.

(b) $n: B \longrightarrow \mathbb{T}\mathrm{op}$ is exponentiable in $\mathrm{Lax}_{\mathrm{N}}(B, \mathbb{T}\mathrm{op})$.

(c) $n: B \longrightarrow \mathbb{T}\mathrm{op}$ is a pseudo-functor and $Y_b \xrightarrow{\bullet} Y_c$ is exponentiable in $\mathbb{T}\mathrm{op}_1$, for all $b \leq c$.

Proof. As before, we know (a)⇒(b)⇒(c). To show that (c)⇒(a), suppose n is a pseudo-functor and $Y_b \xrightarrow{\bullet} Y_c$ is exponentiable in $\mathbb{T}\mathrm{op}_1$, for all $b \leq c$. By Lemma 4.1, it suffices to show that $\varepsilon: \big(\bigsqcup_{b\in B} \mathcal{O}(Y_b)\big) \times_B Y \longrightarrow \mathbb{2}$ is continuous. By Lemmas 4.2 and 4.3, we know $\bigsqcup_{b\in B} \mathcal{O}(Y_b) \longrightarrow B$ corresponds to $\hat{n}: B \longrightarrow \mathbb{T}\mathrm{op}$ defined above, and so ε is continuous. Since $Y_b \xrightarrow{\bullet} Y_c$ is exponentiable in $\mathbb{T}\mathrm{op}_1$, for all $b \leq c$, we know Y_b is exponentiable in Top, and so $\varepsilon_b: \mathcal{O}(Y_b) \times Y_b \longrightarrow \mathbb{2}$ is continuous, for all b. Thus, we have a cell

$$\begin{tikzcd} \mathcal{O}(Y_b) \times Y_b \arrow[r, "\varepsilon_b"] \arrow[d, "\hat{n}_{bc} \times n_{bc}"', "\bullet" marking] \arrow[dr, phantom, "\supseteq"] & \mathbb{2} \arrow[d, "\mathrm{id}^\bullet", "\bullet" marking] \\ \mathcal{O}(Y_c) \times Y_c \arrow[r, "\varepsilon_c"'] & \mathbb{2} \end{tikzcd}$$

for each $b < c$, since n_{bc} is exponentiable in $\mathbb{T}\mathrm{op}_1$. Applying Γ/B to the associated morphism $\hat{n} \times n \longrightarrow \mathbb{2}$ in $\mathrm{Lax}_{\mathrm{N}}(B, \mathbb{T}\mathrm{op})$, the desired result follows. ∎

To generalize continuity to vertical morphisms in $\mathbb{T}\mathrm{op}$ and $\mathbb{L}\mathrm{oc}$, we first note that there is a connection between the way-below relation and the Scott topology ΣL on L, namely, $u << v$ if and only if there exists $H \in \Sigma L$ such that $v \in H$ and $u \leq \wedge H$ (since $u << v$, for all $u \leq \wedge H$ and $v \in H$). It is this condition that we generalize.

Suppose $n: L_0 \xrightarrow{\bullet} L_1$ is in $\mathbb{L}\mathrm{oc}$, and define $\hat{n}: \Sigma L_0 \xrightarrow{\bullet} \Sigma L_1$ by

$$\hat{n} H_0 = \bigcup \{H_1 \in \Sigma L_1 \mid n^{-1} H_1 \subseteq H_0\}$$

Although n is not necessarily continuous in the Scott topology, i.e., $n^{-1}H_1$ need not be Scott open when H_1 is, one can show that $H_1 \subseteq \hat{n} H_0 \iff n^{-1} H_1 \subseteq H_0$. Note that this definition of $\hat{n}$ agrees with the one defined above for $\mathbb{T}\mathrm{op}$.

Given $u_1, v_1 \in L_1$ and $H_0 \in \Sigma L_0$, we say u_1 is *way below* v_1 *relative to* H_0, written $u_1 <<_{H_0} v_1$, if $u_1 << v_1$ in L_1, $v_1 \in \hat{n} H_0$, and $u_1 \le n(\wedge H_0)$. Then $n\colon L_0 \twoheadrightarrow L_1$ is called *doubly continuous* if L_0 is continuous and L_1 satisfies

$$v_1 = \bigvee \{u_1 \mid u_1 <<_{H_0} v_1, \text{ for some } H_0 \in \Sigma L_0\}$$

4.5. Lemma. *$n\colon Y_0 \twoheadrightarrow Y_1$ is exponentiable in $\mathbb{T}\mathrm{op}_1$ if and only if $n\colon \mathcal{O}(Y_0) \twoheadrightarrow \mathcal{O}(Y_1)$ is doubly continuous in* $\mathbb{L}\mathrm{oc}$.

Proof. Suppose $n\colon Y_0 \twoheadrightarrow Y_1$ corresponds to $q\colon Y \longrightarrow 2$ via $\mathbb{T}\mathrm{op}_1 \simeq \mathrm{Top}/2$. It suffices to show that $q\colon Y \longrightarrow 2$ is exponentiable in Top if and only if $n\colon Y_0 \twoheadrightarrow Y_1$ is doubly continuous.

Suppose $q\colon Y \longrightarrow 2$ is exponentiable. Then Y_0 is exponentiable in Top, since the pullback of an exponentiable map is exponentiable, and so $\mathcal{O}(Y_0)$ is a continuous lattice. To see that $n\colon \mathcal{O}(Y_0) \twoheadrightarrow \mathcal{O}(Y_1)$ is doubly continuous, suppose $V_1 \in \mathcal{O}(Y_1)$ and $y_1 \in V_1$. Then, by Lemma 4.1, there exists H fiberwise Scott open such that $V_1 \in H_1$ and $y_1 \in U \subseteq (\cap H_0) \cup (\cap H_1)$, for some $U \in \mathcal{O}(Y)$. We claim that $U_1 <<_{H_0} V_1$. First, $U_1 << V_1$ in $\mathcal{O}(Y_1)$, since $U_1 \subseteq \cap H_1$ and $V_1 \in H_1$. Also, $U_1 \subseteq n(U_0) \subseteq n(\wedge H_0)$, since U is open in Y and $U_0 \subseteq \cap H_0$. Finally, since H is fiberwise Scott open, we know $n^{-1} H_1 \subseteq H_0$, and so $V_1 \in H_1 \subseteq \hat{n} H_0$. Thus, $U_1 <<_{H_0} V_1$, as desired.

Conversely, suppose $n\colon \mathcal{O}(Y_0) \twoheadrightarrow \mathcal{O}(Y_1)$ is doubly continuous. We will show that $q\colon Y \longrightarrow 2$ satisfies Lemma 4.1(c). Given $y_0 \in V_0 \in \mathcal{O}(Y_0)$, there exists $U_0 << V_0$ such that $y_0 \in U_0$. Take $H_0 = \{W_0 \mid U_0 << W_0\}$ and $H_1 = \emptyset$. Then H is fiberwise Scott open, $V_0 \in H_0$, and $y_0 \in U_0 \subseteq (\cap H_0) \cup (\cap H_1)$. Given $y_1 \in V_1 \in \mathcal{O}(Y_1)$, since n is doubly continuous, there exist H_0 Scott open and $U_1 \in \mathcal{O}(Y_1)$ such that $y_1 \in U_1$ and $U_1 <<_{H_0} V_1$. Take $H_1 = \{W_1 \in \hat{n} H_0 \mid U_1 << W_1\}$. Then $H = H_0 \sqcup H_1$ is fiberwise Scott open, since $H_1 \subseteq \hat{n} H_0 \Rightarrow n^{-1} H_1 \subseteq H_0$; $V_1 \in H_1$, since $V_1 \in \hat{n} H_0$ and $U_1 << V_1$; and $y_1 \in (\wedge H_0) \cup U_1 \subseteq (\cap H_0) \cup (\cap H_1)$ and $(\wedge H_0) \cup U_1$ is open, since $U_1 \subseteq n(\wedge H_0)$ by definition of $U_1 <<_{H_0} V_1$. Therefore, $q\colon Y \longrightarrow ! < 125 > 2$ is exponentiable in Top, as desired. ∎

In [9], Hyland showed that a locale L is exponentiable in Loc if and only if L is locally compact (i.e., a continuous lattice) if and only if the exponential S^Y exists in Loc, where S denotes the Sierpinski locale. This result is constructive so it applies to internal locales in any topos, in particular, in the topos $\mathrm{Sh}(B)$ of set-valued sheaves on the locale B. Moreover, Joyal and Tierney [12] showed that $q \mapsto q_* \Omega_L$ sets up an equivalence between Loc/B and the category $\mathrm{Loc}(\mathrm{Sh}(B))$ of internal locales in $\mathrm{Sh}(B)$, where Ω_L is the subobject classifier of $\mathrm{Sh}(L)$ and $q\colon \mathrm{Sh}(L) \longrightarrow \mathrm{Sh}(B)$ is the geometric morphism induced by $q\colon L \longrightarrow B$. Thus, $q\colon L \longrightarrow B$ is exponentiable in Loc if and only if $q_* \Omega_L$ is locally compact in $\mathrm{Loc}(\mathrm{Sh}(B))$

4.6. Theorem. *The following are equivalent for $n\colon L_0 \twoheadrightarrow L_1$ in $\mathbb{L}\mathrm{oc}$ with corresponding morphism $q\colon L \longrightarrow S$ in* Loc.

(a) $n\colon L_0 \twoheadrightarrow L_1$ is exponentiable in $\mathbb{L}\mathrm{oc}_1$.

(b) $q\colon L \longrightarrow S$ *is exponentiable in* Loc.

(c) $n\colon L_0 \longrightarrow\!\!\!\!\!\bullet\ L_1$ *is doubly continuous.*

(d) $q_*\Omega_L$ *is locally compact in* Loc(Sh(S)).

PROOF. $(a) \Leftrightarrow (b) \Leftrightarrow (d)$ follows from $\mathbb{L}\mathrm{oc}_1 \simeq \mathrm{Loc}/S \simeq \mathrm{Loc}(\mathrm{Sh}(S))$.
$(b) \Rightarrow (c)$ Suppose $q\colon L \longrightarrow S$ is exponentiable in Loc. Then L_0 and L_1 are exponentiable in Loc, and so $L_0 \cong \mathcal{O}(Y_0)$ and $L_1 \cong \mathcal{O}(Y_1)$, for some locally compact sober spaces Y_0 and Y_1 such that $L \cong \mathcal{O}(Y)$, where $Y \cong \Gamma_2 n$ and n also denotes the induced vertical morphism $n\colon Y_0 \longrightarrow\!\!\!\!\!\bullet\ Y_1$ in $\mathbb{T}\mathrm{op}$. To show that $n\colon L_0 \longrightarrow\!\!\!\!\!\bullet\ L_1$ is doubly continuous, by Lemma 4.5 it suffices to show that $n\colon Y_0 \longrightarrow\!\!\!\!\!\bullet\ Y_1$ is exponentiable in $\mathbb{T}\mathrm{op}_1$, or equivalently, $Y \longrightarrow 2$ is exponentiable in Top.

First, we show that $\mathcal{O}(X) \times_S \mathcal{O}(Y)$ has enough points so that $\mathcal{O}(X) \times_S \mathcal{O}(Y) \cong \mathcal{O}(X \times_2 Y)$, for all $X \longrightarrow 2$. It is easy to see that each point of $\mathcal{O}(X) \times_S \mathcal{O}(Y)$ factors through $\mathcal{O}(X_0) \times \mathcal{O}(Y_0)$ or $\mathcal{O}(X_1) \times \mathcal{O}(Y_1)$, and the latter locales are spatial since $\mathcal{O}(Y_0)$ and $\mathcal{O}(Y_1)$ are locally compact [10].

Then Y is exponentiable in Top/2, by Lemma 4.1, since

$$\begin{aligned}
\mathrm{Top}/2(X \times_2 Y, 2 \times 2) &\cong \mathrm{Loc}/S(\mathcal{O}(X \times_2 Y), S \times S)\\
&\cong \mathrm{Loc}/S(\mathcal{O}(X) \times_S \mathcal{O}(Y), S \times S)\\
&\cong \mathrm{Loc}/S(\mathcal{O}(X), (S \times S)^{\mathcal{O}(Y)})\\
&\cong \mathrm{Top}/2(X, \mathrm{pt}((S \times S)^{\mathcal{O}(Y)}))
\end{aligned}$$

where pt is right adjoint to $\mathcal{O}$ [11]. Thus, $n\colon Y_0 \longrightarrow\!\!\!\!\!\bullet\ Y_1$ is exponentiable in $\mathbb{T}\mathrm{op}_1$, and it follows that $n\colon L_0 \longrightarrow\!\!\!\!\!\bullet\ L_1$ is doubly continuous by Lemma 4.5.

$(c) \Rightarrow (d)$ Suppose $n\colon L_0 \longrightarrow\!\!\!\!\!\bullet\ L_1$ is doubly continuous. Then L_0 and L_1 are continuous lattices, and so $n\colon \mathcal{O}(Y_0) \longrightarrow\!\!\!\!\!\bullet\ \mathcal{O}(Y_1)$, for some sober spaces Y_0 and Y_1 such that $L \cong \mathcal{O}(Y)$, where $Y \cong \Gamma_2 n$. To see that $q_*\Omega_L$ is locally compact in Loc(Sh(S)) it suffices to show that $q_*\Omega_Y$ is locally compact in Loc(Sh(2)), or equivalently, for all V open in Y, $V = \vee I$, where I is the ideal $I = \{U \mid U << V\}$ in Sh(2). Note that $I(\{0\}) = \{U_0 \mid U_0 << V_0\}$ and $I(2) = \{U \mid U << V \text{ in } \mathcal{O}(Y) \text{ and } U_0 << V_0 \text{ in } \mathcal{O}(Y_0)\}$.

Suppose V is open in Y. If $V_1 = \emptyset$, then $V \in \mathcal{O}(Y_0)$, and the result follows by continuity of $\mathcal{O}(Y_0)$. Otherwise, since $n\colon \mathcal{O}(Y_0) \longrightarrow\!\!\!\!\!\bullet\ \mathcal{O}(Y_1)$ is doubly continuous, for all $y_1 \in V_1$, there exists H_0 Scott open in $\mathcal{O}(Y_0)$ and $U_1 \in \mathcal{O}(Y_1)$ such that $y_1 \in U_1$ and $U_1 <<_{H_0} V_1$, i.e., $V_1 \in \hat{n}H_0$ and $U_1 \subseteq n(\wedge H_0)$. Consider $U = (\wedge H_0) \cup U_1$. It suffices to show that $U << V$ in $\mathcal{O}(Y)$ and $U_0 << V_0$ in $\mathcal{O}(Y_0)$, for then $V = \vee I$, as desired.

To see that $U_0 << V_0$, suppose $V_0 \subseteq \cup_A W_\alpha$ in $\mathcal{O}(Y_0)$. Since $V_1 \subseteq n(V_0) \subseteq n(\cup_A W_\alpha)$ and $\hat{n}H_0$ is Scott open, we know $n(\cup_A W_\alpha) \in \hat{n}H_0$, and so $\cup_A W_\alpha \in H_0$ by definition of $\hat{n}$. Thus, $\cup_F W_\alpha \in H_0$, for some finite $F \subseteq A$, and it follows that $U_0 \subseteq \cup_F W_\alpha$, as desired.

To see that $U << V$, suppose $V \subseteq \cup_A W_\alpha$ in $\mathcal{O}(Y)$. Then $V_0 \subseteq \cup_A (W_\alpha)_0$, and so $U_0 \subseteq \cup_{F_0} (W_\alpha)_0$, for some finite $F_0 \subseteq A$, as above. Also, $U_1 \subseteq \cup_{F_1} (W_\alpha)_1$, for some finite $F_1 \subseteq A$, since $U_1 <<_{H_0} V_1$. Taking $F = F_0 \cup F_1$, it follows that $U \subseteq \cup_F W_\alpha$, as desired. ∎

Note that a single theorem for exponentiability in Top/$\mathbb{2}$ can be obtained by combining the conditions of Lemma 4.1 for $B = \mathbb{2}$ with Lemma 4.5, and adding "$q_*\Omega_Y$ is locally compact in Loc(Sh($\mathbb{2}$))." from Theorem 4.6. We can also add "$Y : B \longrightarrow \mathbb{T}\mathrm{op}$ is a pseudo-functor and $Y_b \longrightarrow Y_c$ is doubly continuous, for all $b \le c$." to the conditions in Theorem 4.4, and "$Y : B \longrightarrow \mathbb{L}\mathrm{oc}$ is a pseudo-functor and $Y_b \longrightarrow Y_c$ is doubly continuous, for all $b \le c$." to those in Corollary 3.8.

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Department of Mathematics
Union College
Schenectady, NY 12308

Email: niefiels@union.edu

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