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EXACT COMPLETIONS AND SMALL SHEAVES

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Abstract. We prove a general theorem which includes most notions of "exact completion" as special cases. The theorem is that "κ-ary exact categories" are a reflective sub-2-category of "κ-ary sites", for any regular cardinal κ. A κ-ary exact category is an exact category with disjoint and universal κ-small coproducts, and a κ-ary site is a site whose covering sieves are generated by κ-small families and which satisfies a solution-set condition for finite limits relative to κ.

In the unary case, this includes the exact completions of a regular category, of a category with (weak) finite limits, and of a category with a factorization system. When $\kappa = \omega$, it includes the pretopos completion of a coherent category. And when $\kappa = \mathfrak{K}$ is the size of the universe, it includes the category of sheaves on a small site, and the category of small presheaves on a locally small and finitely complete category. The $\mathfrak{K}$-ary exact completion of a large nontrivial site gives a well-behaved "category of small sheaves".

Along the way, we define a slightly generalized notion of "morphism of sites" and show that κ-ary sites are equivalent to a type of "enhanced allegory". This enables us to construct the exact completion in two ways, which can be regarded as decategorifications of "representable profunctors" (i.e. entire functional relations) and "anafunctors", respectively.

Contents

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1. Introduction

In this paper we show that the following "completion" operations are all instances of a single general construction.

(i) The free exact category on a category with (weak) finite limits, as in [CM82, Car95, CV98, HT96].
(ii) The exact completion of a regular category, as in [SC75, FS90, Car95, CV98, Lac99].
(iii) The pretopos completion of a coherent category, as in [FS90, Joh02], and its infinitary analogue.
(iv) The category of sheaves on a small site.
(v) The category of small presheaves on a locally small category satisfying the solution-set condition for finite diagrams, as in [DL07] (the solution-set condition makes the category of small presheaves finitely complete).

The existence of a relationship between the above constructions is not surprising. On the one hand, Giraud's theorem characterizes categories of sheaves as the infinitary pretoposes with a small generating set. It is also folklore that adding disjoint universal coproducts is the natural "higher-ary" generalization of exactness; this is perhaps most explicit in [Str84]. Furthermore, the category of sheaves on a small infinitary-coherent category agrees with its infinitary-pretopos completion, as remarked in [Joh02]. On the other hand, [HT96] showed that the free exact category on a category with weak finite limits can be identified with a full subcategory of its presheaf category, and [Lac99] showed that the exact completion of a regular category can similarly be identified with a full subcategory of the sheaves for its regular topology.

However, in other ways the above-listed constructions appear different. For instance, each has a universal property, but the universal properties are not all the same.

(i) The free exact category on a category with finite limits is a left adjoint to the forgetful functor. However, the free exact category on a category with *weak* finite limits represents "left covering" functors.
(ii) The exact completion of a regular category is a reflection.
(iii) The pretopos completion of a coherent category is also a reflection.
(iv) The category of sheaves on a small site is the classifying topos for flat cover-preserving functors.
(v) The category of small presheaves on a locally small category is its free cocompletion under small colimits.

In searching for a common generalization of these constructions, which also unifies their universal properties, we are led to introduce the following new definitions, relative to a regular cardinal κ.

1.1. DEFINITION. *A κ-**ary site** is a site whose covering sieves are generated by κ-small families, and which satisfies a certain weak solution-set condition for finite cones relative to κ (see §3).*

This includes the inputs to all the above constructions, as follows:

(i) The trivial topology on a category is unary precisely when the category has weak finite limits.
(ii) The regular topology on a regular category is also unary.
(iii) The coherent topology on a coherent category is ω-ary ("finitary"), and its infinitary analogue is $\mathfrak{K}$-ary, where $\mathfrak{K}$ is the size of the universe.
(iv) The topology of any small site is $\mathfrak{K}$-ary.
(v) The trivial topology on a large category is $\mathfrak{K}$-ary just when that category satisfies the solution-set condition for finite diagrams.

1.2. Definition. *A κ**-ary exact category** is a category with universally effective equivalence relations and disjoint universal κ-small coproducts.*

This includes the *outputs* of all the above constructions, as follows.

(i)-(ii) A unary exact category is an exact category in the usual sense.
(iii) An ω-ary exact category is a pretopos.
(iv) A $\mathfrak{K}$-ary exact category is an infinitary pretopos (a category satisfying the exactness conditions of Giraud's theorem).

1.3. Definition. *A **morphism of sites** is a functor $\mathbf{C} \to \mathbf{D}$ which preserves covering families and is "flat relative to the topology of $\mathbf{D}$" in the sense of [Koc89, Kar04].*

This is a slight generalization of the usual notion of "morphism of sites". The latter requires the functor to be "representably flat", which is equivalent to flatness relative to the trivial topology of the codomain. The two are equivalent if the sites have actual finite limits and subcanonical topologies. Our more general notion also includes all "dense inclusions" of sub-sites, and has other pleasing properties which the usual one does not (see §4 and §11).

Generalized morphisms of sites include all the relevant morphisms between all the above inputs, as follows:

(i) A morphism between sites with trivial topology is a flat functor. A morphism from a unary trivial site to an exact category is a left covering functor.
(ii) A morphism of sites between regular categories is a regular functor.
(iii) A morphism of sites between coherent categories is a coherent functor.
(iv) A morphism of sites from a small site to a Grothendieck topos (with its canonical topology) is a flat cover-preserving functor. A morphism of sites *between* Grothendieck toposes is the inverse image functor of a geometric morphism.

We can now state the general theorem which unifies all the above constructions.

1.4. Theorem. *κ-ary exact categories form a reflective sub-2-category of κ-ary sites. The reflector is called **κ-ary exact completion**.*

Besides its intrinsic interest, this has several useful consequences. For instance, if $\mathbf{C}$ satisfies the solution-set condition for finite limits, then its category of small presheaves is an infinitary pretopos. More generally, if $\mathbf{C}$ is a *large* site which is $\mathfrak{K}$-ary (this includes most large sites arising in practice), then its $\mathfrak{K}$-ary exact completion is a category of "small sheaves". For instance, any scheme can be regarded as a small sheaf on the large site of rings. The category of small sheaves shares many properties of the sheaf topos of a small site: it is an infinitary pretopos, it has a similar universal property, and it satisfies a "size-free" version of Giraud's theorem.

Additionally, by completing with successively larger κ, we can obtain information about ordinary sheaf toposes with "cardinality limits". For instance, the sheaves on any small ω-ary site form a coherent topos.

We can also find "κ-ary regular completions" sitting inside the κ-ary exact completion, in the usual way. This includes the classical regular completion of a category with (weak) finite limits as in [CM82, CV98, HT96], as well as variants such as the regular completion of a category with a factorization system from [Kel91] and the relative regular completion from [Hof04]. More generally, we can obtain the exact completions of [GL12] relative to a class of lex-colimits.

Finally, our approach to proving Theorem. 1.4 also unifies many existing proofs. There are three general methods used to construct the known exact completions.

(a) Construct a bicategory of binary relations from the input, complete it under certain colimits, then consider the category of "maps" (left adjoints) in the result.
(b) As objects take "κ-ary congruences" (many-object equivalence relations), and as morphisms take "congruence functors", perhaps with "weak equivalences" inverted.
(c) Find the exact completion as a subcategory of the category of (pre)sheaves.

The bicategories used in (a) are called *allegories* [FS90]. In order to generalize this construction to κ-ary sites, we are led to the following "enhanced" notion of allegory.

1.5. Definition. *A **framed allegory** is an allegory equipped with a category of "tight maps", each of which has an underlying map in the underlying allegory.*

Framed allegories are a "decategorification" of proarrow equipments [Woo82], framed bicategories [Shu08], and $\mathscr{F}$-categories [LS12]. We can then prove:

1.6. Theorem. *The 2-category of κ-ary sites is equivalent to a suitable 2-category of framed allegories.*

Besides further justifying the notion of "κ-ary site", this theorem allows us to construct the exact completion of κ-ary sites using analogues of all three of the above methods.

(a) We can build the corresponding framed allegory, forget the framing to obtain an ordinary allegory, then perform the usual completion under appropriate colimits and consider the category of maps. This construction is most convenient for obtaining the universal property of the exact completion.

(b) Alternatively, we can *first* complete a framed allegory under a corresponding type of "framed colimit", then forget the framing and consider the category of maps. The objects of this framed cocompletion are κ-ary congruences and its tight maps are "congruence functors". In this case, the last step is equivalent to constructing a category of fractions of the tight maps.

(c) The universal property of the exact completion, obtained from (a), induces a functor into the category of sheaves. Using description (b) of the exact completion, we show that this functor is fully faithful and identify its image.

We find it convenient to describe the completion operations in (a) and (b) in terms of enriched category theory, using ideas from [LS02, BLS12, LS12]. This also makes clear that (a) is a decategorification of the "enriched categories and modules" construction from [Str81a, CKW87], while the "framed colimits" in (b) are a decategorification of those appearing in [Woo85, LS12], and that all of these are an aspect of Cauchy or absolute cocompletion [Law74, Str83a]. The idea of "categorified sheaves", and the connection to Cauchy completion, is also explicit in [Str81a, CKW87], building on [Wal81, Wal82].

We hope that making these connections explicit will facilitate the study of exact completions of higher categories. Of particular interest is the fact that the "framed colimits" in (b) naturally produce decategorified versions of *functors*, in addition to the *profunctors* resulting from the colimits in (a).

1.7. Remark. There are a few other viewpoints on exact completion in the literature, such as that of [CW02, CW05], which seem not to be closely related to this paper.

1.8. Organization We begin in §2 with some preliminary definitions. Then we define the basic notions mentioned above: in §3 we define κ-ary sites, in §4 we define morphisms of sites, and in §5 we define κ-ary regularity and exactness. In §6 we recall the notion of allegory, define framed allegories, and prove Theorem. 1.6. Then in §7 we deduce Theorem. 1.4 using construction (a). In §8 and §9 we show the equivalence with constructions (b) and (c), respectively.

We will explain the relationship of our theory to each existing sort of exact completion as we develop the requisite technology in §§7–9. In §10, we discuss separately a couple of related notions which require a somewhat more in-depth treatment: the *postulated colimits* of [Koc89] and the *lex-colimits* of [GL12]. In particular, we show that the relative exact completions of [GL12] can also be generalized to (possibly large) κ-ary sites, and we derive the κ-ary regular completion as a special case.

In §11 we study *dense* morphisms of sites. There we further justify our generalized notion of "morphism of sites" by showing that every dense inclusion is a morphism of sites, and that every geometric morphism which lifts to a pair of sites of definition is determined by a morphism between those sites. Neither of these statements is true for the classical notion of "morphism of sites". Moreover, the latter is merely a special case of a fact about κ-ary exact completions for any κ; in particular it applies just as well to categories of small sheaves.

1.9. Foundational remarks We assume two "universes" $\mathbb{U}_1 \in \mathbb{U}_2$, and denote by $\mathfrak{K}$ the least cardinal number not in $\mathbb{U}_1$, or equivalently the cardinality of $\mathbb{U}_1$. These universes might be Grothendieck universes (i.e. $\mathfrak{K}$ might be inaccessible), but they might also be the class of all sets and the hyperclass of all classes (in a set theory such as NBG), or they might be small reflective models of ZFC (as in [Fef69]). In fact, $\mathfrak{K}$ might be any regular cardinal at all; all of our constructions will apply equally well to all regular cardinals κ, with $\mathfrak{K}$ as merely a special case. However, for comparison with standard notions, it is helpful to have one regular cardinal singled out to call "the size of the universe".

Regardless of foundational choices, we refer to objects (such as categories) in $\mathbb{U}_1$ as *small* and others as *large*, and to objects in $\mathbb{U}_2$ as *moderate* (following [Str81b]) and others as *very large*. In particular, small objects are also moderate. We write **Set** for the moderate category of small sets. *All categories, functors, and transformations in this paper will be moderate*, with a few exceptions such as the very large category **SET** of moderate sets. (We do not assume categories to be locally small, however.) But most of our 2-categories will be very large, such as the 2-category CAT of moderate categories.

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2. Preliminary notions

2.1. Arity classes In our notions of κ-ary site, κ-ary exactness, etc., κ does not denote exactly a regular cardinal, but rather something of the following sort.

2.2. Definition. *An* ***arity class*** *is a class κ of small cardinal numbers such that:*

(i) $1 \in \kappa$.

(ii) κ is closed under indexed sums: if $\lambda \in \kappa$ and $\alpha\colon \lambda \to \kappa$, then $\sum_{i\in\lambda} \alpha(i)$ is also in κ.

(iii) κ is closed under indexed decompositions: if $\lambda \in \kappa$ and $\sum_{i\in\lambda} \alpha(i) \in \kappa$, then each $\alpha(i)$ is also in κ.

We say that a set is ***κ-small*** *if its cardinality is in κ.*

2.3. Remark. Conditions (ii) and (iii) can be combined to say that if $\phi\colon I \to J$ is any function where J is κ-small, then I is κ-small if and only if all fibers of ϕ are κ-small. Also, if we assume (iii), then condition (i) is equivalent to nonemptiness of κ, since for any $\lambda \in \kappa$ we can write $\lambda = \sum_{i\in\lambda} 1$.

By induction, (ii) implies closure under iterated indexed sums: for any $n \geq 2$,

$$\sum_{i_1 \in \lambda_1} \sum_{i_2 \in \lambda_2(i_1)} \cdots \sum_{i_{n-1} \in \lambda_{n-1}(i_1,\ldots,i_{n-2})} \lambda_n(i_1,\ldots,i_{n-1})$$

is in κ if all the λ's are. Condition (i) can be regarded as the case $n = 0$ of this (the case $n = 1$ being just "$\lambda \in \kappa$ if $\lambda \in \kappa$"). I am indebted to Toby Bartels and Sridhar Ramesh for a helpful discussion of this point.

The most obvious examples are the following.

- The set $\{1\}$ is an arity class.
- The set $\{0, 1\}$ is an arity class.
- For any regular cardinal $\kappa \leq \mathfrak{K}$, the set of all cardinals strictly less than κ is an arity class, which we abusively denote also by κ. (We can include $\{0, 1\}$ in this clause if we allow 2 as a regular cardinal.) The cases of most interest are $\kappa = \omega$ and $\kappa = \mathfrak{K}$, which consist respectively of the *finite* or *small* cardinal numbers.

In fact, these are the only examples. For if κ contains any $\lambda > 1$, then it must be down-closed, since if $\mu \leq \nu$ and $\lambda > 1$ we can write ν as a λ-indexed sum containing μ. And clearly any down-closed arity class must arise from a regular cardinal (including possibly 2). So we could equally well have defined an arity class to be "either the set of all cardinals less than some regular cardinal, or the set $\{1\}$"; but the definition we have given seems less arbitrary.

2.4. Remark. For any κ, the full subcategory $\mathbf{Set}_\kappa \subseteq \mathbf{Set}$ consisting of the κ-small sets is closed under finite limits. A reader familiar with "indexed categories" will see that all our constructions can be phrased using "naively" $\mathbf{Set}_\kappa$-indexed categories, and suspect a generalization to $\mathbf{K}$-indexed categories for any finitely complete $\mathbf{K}$. We leave such a generalization to a later paper, along with potential examples such as [RR90, Fre12].

From now on, all definitions and constructions will be relative to an arity class κ, whether or not this is explicitly indicated in the notation. We sometimes say **unary**, **finitary**, and **infinitary** instead of $\{1\}$-ary, ω-ary, and $\mathfrak{K}$-ary respectively.

2.5. Remark. Let x and y be elements of some set I. The *subsingleton* $[\![x = y]\!]$ is a set that contains one element if $x = y$ and is empty otherwise. Then if I is κ-small, then so is $[\![x = y]\!]$. This is trivial unless $\kappa = \{1\}$, so we can prove this by cases. Alternatively, we can observe that $[\![x = y]\!]$ is a fiber of the diagonal map $I \to I \times I$, and both I and $I \times I$ are κ-small.

2.6. Matrices and families We now introduce some terminology and notation for families of morphisms. This level of abstraction is not strictly necessary, but otherwise the notation later on would become quite cumbersome.

By a *family* (of objects or morphisms) we always mean a *small-set-indexed* family. We will always use uppercase letters for families and lowercase letters for their elements, such as $X = \{x_i\}_{i\in I}$. We use braces to denote families, although they are not sets and in particular can contain duplicates. For example, the family $\{x, x\}$ has two elements. We further abuse notation by writing $x \in X$ to mean that there is a specified $i \in I$ such that $x = x_i$. We say $X = \{x_i\}_{i\in I}$ is **κ-ary** if I is κ-small.

If $\{X_i\}_{i\in I}$ is a family of families, we have a "disjoint union" family $\coprod X_i$, which is κ-ary if I is κ-small and each X_i is κ-ary.

2.7. Definition. *Let* $\mathbf{C}$ *be a category and* X *and* Y *families of objects of* $\mathbf{C}$*. A* ***matrix*** *from* X *to* Y*, written* $F\colon X \Rightarrow Y$*, is a family* $F = \{f_{xy}\}_{x\in X, y\in Y}$*, where each* f_{xy} *is a set of morphisms from* x *to* y *in* $\mathbf{C}$*. If* $G\colon Y \Rightarrow Z$ *is another matrix, then their composite* $GF\colon X \Rightarrow Z$ *is*

$$GF = \Big\{ \{\, gf \mid y \in Y, g \in g_{yz}, f \in f_{xy} \,\} \Big\}_{x\in X, z\in Z}.$$

Composition of matrices is associative and unital. Also, for any family of matrices $\{F_i\colon X_i \Rightarrow Y_i\}_{i\in I}$, we have a disjoint union matrix $\coprod F_i\colon \coprod X_i \Rightarrow \coprod Y_i$, defined by

$$\Big(\coprod F_i\Big)_{xy} = \begin{cases} (f_i)_{xy} & x \in X_i, y \in Y_i \\ \emptyset & x \in X_i, y \in Y_j, i \neq j. \end{cases}$$

2.8. Definition. *A matrix* $F\colon X \Rightarrow Y$ *is* ***κ-sourced*** *if* X *is* κ*-ary, and* ***κ-targeted*** *if* Y *is* κ*-ary. It is* ***κ-to-finite*** *if it is* κ*-sourced and* ω*-targeted.*

If $F\colon X \Rightarrow Y$ is a matrix and X' is a subfamily of X, we have an induced matrix $F|^{X'}\colon X' \Rightarrow Y$. Similarly, for a subfamily Y' of Y, we have $F|_{Y'}\colon X \Rightarrow Y'$.

2.9. Definition. *An* ***array*** *in* $\mathbf{C}$ *is a matrix each of whose entries is a singleton. A* ***sparse array*** *in* $\mathbf{C}$ *is a matrix each of whose entries is a subsingleton (i.e. contains at most one element).*

The composite of two (sparse) arrays $F\colon X \Rightarrow Y$ and $G\colon Y \Rightarrow Z$ is always defined as a matrix. It is a (sparse) array just when for all $x \in X$ and $z \in Z$, the composite $g_{yz}f_{xy}$ is independent of y.

We can identify objects of $\mathbf{C}$ with singleton families, and arrays between such families with single morphisms.

2.10. Definition. *A* ***cone*** *is an array with singleton domain, and a* ***cocone*** *is one with singleton codomain.*

Cones and cocones are sometimes called *sources* and *sinks* respectively, but this use of "source" has potential for confusion with the source (= domain) of a morphism. Another important sort of sparse array is the following.

2.11. Definition. *A* ***functional array*** *is a sparse array* $F\colon X \Rightarrow Y$ *such that for each* $x \in X$*, there is exactly one* $y \in Y$ *such that* f_{xy} *is nonempty.*

Thus, if $X = \{x_i\}_{i\in I}$ and $Y = \{y_j\}_{j\in J}$, a functional array $F\colon X \Rightarrow Y$ consists of a function $f\colon I \to J$ and morphisms $f_i\colon x_i \to y_{f(j)}$. We abuse notation further by writing

$f(x_i)$ for $y_{f(i)}$ and f_{x_i} for f_i, so that F consists of morphisms $f_x\colon x \to f(x)$. For instance, the *identity* functional array $X \Rightarrow X$ has $f(x) = x$ and $f_x = 1_x$ for each x.

Any cocone is functional, as is any disjoint union $\bigsqcup F_i$ of functional arrays. Conversely, any functional array $F\colon X \Rightarrow Y$ can be decomposed as

$$F = \bigsqcup F|_y\colon \bigsqcup X|_y \Longrightarrow \bigsqcup\{y\} = Y,$$

where $X|_y = \{x \in X\}_{f(x)=y}$ and each $F|_y\colon X|_y \Rightarrow y$ is the induced cocone. (This is a slight abuse of notation, as $F|_y$ might also refer to the sparse array $X \Rightarrow y$ which is empty at those $x \in X$ with $f(x) \neq y$, but the context will always disambiguate.)

Also, if $F\colon X \Rightarrow Y$ is functional and $G\colon Y \Rightarrow Z$ is any sparse array, then the composite matrix GF is also a sparse array. If G is also functional, then so is GF.

2.12. Remark. The category of κ-ary families of objects in $\mathbf{C}$ and functional arrays is the free completion of $\mathbf{C}$ under κ-ary coproducts.

Any functor $\mathbf{g}\colon \mathbf{D} \to \mathbf{C}$ gives rise to a family $\mathbf{g}(\mathbf{D}) \coloneqq \{\mathbf{g}(d)\}_{d\in\mathbf{D}}$ in $\mathbf{C}$.

2.13. Definition. *An array $F\colon X \Rightarrow \mathbf{g}(\mathbf{D})$ is* ***over*** $\mathbf{g}$ *if $\mathbf{g}(\delta)\circ f_{dx} = f_{d'x}$ for all $\delta\colon d \to d'$ in $\mathbf{D}$.*

This generalizes the standard notion of "cone over a functor."

2.14. Definition. *Let $F\colon X \Rightarrow Z$ and $G\colon Y \Rightarrow Z$ be arrays with the same target.*

(i) We say that F ***factors through*** *G or* ***refines*** *G if for every $x \in X$ there exists a $y \in Y$ and a morphism $h\colon x \to y$ such that $f_{zx} = g_{zy}h$ for all $z \in Z$. In this case we write $F \leq G$.*

(ii) If $F \leq G$ and $G \leq F$, we say F and G are ***equivalent****.*

Note that $F \leq G$ just when there exists a functional array $H\colon X \Rightarrow Y$ such that $F = GH$. We have a (possibly large) preorder of κ-sourced arrays with a fixed target, under the relation $\leq$.

2.15. κ-prelimits We mentioned in the introduction that a κ-ary site must satisfy a solution-set condition. We will define the actual condition in §3; here we define a preliminary, closely related notion.

2.16. Definition. *A* ***κ-prelimit*** *of a functor $\mathbf{g}\colon \mathbf{D} \to \mathbf{C}$ is a κ-sourced array T over $\mathbf{g}$ such that every cone over $\mathbf{g}$ factors through T.*

Note that if $P\colon X \Rightarrow Z$ and $Q\colon Y \Rightarrow Z$ are arrays over Z, then P factors through Q if and only if each cone $P|^x\colon x \Rightarrow Z$ does. Thus, Definition. 2.16 could equally well ask that every *array* over $\mathbf{g}$ factor through T. That is, a κ-prelimit of $\mathbf{g}$ is a κ-sourced array over $\mathbf{g}$ which is $\leq$-greatest among arrays over $\mathbf{g}$. Similar remarks apply to subsequent related definitions, such as Definition. 3.7.

In [FS90], the term *prelimit* refers to our $\mathfrak{K}$-prelimits.

2.17. Examples.

- Since $1 \in \kappa$, any **limit** of $\mathbf{g}$ is *a fortiori* a κ-prelimit.
- Recall that a **multilimit** of $\mathbf{g}$ is a set T of cones such that for any cone x, there exists a unique $t \in T$ such that x factors through t, and for this t the factorization is unique. Any κ-small multilimit is also a κ-prelimit.
- Recall that a **weak limit** of $\mathbf{g}$ is a cone such that any other cone factors through it, not necessarily uniquely. Since $1 \in \kappa$, any weak limit is a κ-prelimit. Conversely, $\{1\}$-prelimits are precisely weak limits.

Note that even if a limit, multilimit, or weak limit exists, it will not in general be the *only* κ-prelimit. In particular, if $\mathbf{g}$ has a limit T, then a κ-small family of cones over $\mathbf{g}$ is a κ-prelimit if and only if it contains some cone whose comparison map to T is split epic (in the category of cones).

2.18. Example. If there is a κ-small family which includes *all* cones over $\mathbf{g}$ (in particular, if $\kappa = \mathfrak{K}$ and $\mathbf{C}$ and $\mathbf{D}$ are small), then this family is a κ-prelimit.

2.19. Remark. Given $\mathbf{D}$, a category $\mathbf{C}$ has $\mathfrak{K}$-prelimits of all $\mathbf{D}$-shaped diagrams precisely when the diagonal functor $\mathbf{C} \to \mathbf{C}^{\mathbf{D}}$ has a *(co-)solution set*, as in (the dual form of) Freyd's General Adjoint Functor Theorem. In fact, the crucial lemma for the GAFT can be phrased as "if $\mathbf{C}$ is cocomplete and locally small, and $\mathbf{g}\colon \mathbf{D} \to \mathbf{C}$ has a $\mathfrak{K}$-prelimit, then it also has a limit." See also [FS90, §1.8] and [AR94, Ch. 4].

2.20. Remark. $\mathfrak{K}$-prelimits also appear in [DL07], though not by that name. There it is proven that $\mathbf{C}$ has $\mathfrak{K}$-prelimits if and only if its category of small presheaves is complete. We will deduce this by an alternative method in §9 and §11.

A **finite κ-prelimit** is a κ-prelimit of a finite diagram. Another important notion is the following. Given a cocone $P\colon V \Rightarrow u$ and a morphism $f\colon x \to u$, for each $v \in V$ let $Q^v\colon Y^v \Rightarrow \{x, v\}$ be a κ-prelimit of the cospan $x \xrightarrow{f} u \xleftarrow{p_v} v$, and let $Y = \bigsqcup Y^v$. Putting together the cocones $Q^v|_x\colon Y^v \Rightarrow x$, we obtain a cocone $Y \Rightarrow x$, which we denote f^*P and call a **κ-pre-pullback** of P along f. This is unique up to equivalence of cocones over x (in the sense of Definition. 2.14).

2.21. Classes of epimorphisms We now define several types of epimorphic cocones.

2.22. Definition. *Let $R\colon V \Rightarrow u$ be a cocone.*

*(i) R is **epic** if $fR = gR$ implies $f = g$. (Of course, a cone is **monic** if it is an epic cocone in the opposite category.)*

*(ii) R is **extremal-epic** if it is epic, and whenever $R = qP$ with $q\colon z \to u$ monic, it follows that q is an isomorphism.*

*(iii) R is **strong-epic** if it is epic, and whenever $FR = QP$, for $F\colon u \Rightarrow W$ a finite cone, $Q\colon z \Rightarrow W$ a finite monic cone, and $P\colon V \Rightarrow z$ any cocone, there exists $h\colon u \to z$ (necessarily unique) such that $hR = P$ and $Qh = F$.*

*(iv) R is **effective-epic** if whenever $Q\colon V \Rightarrow x$ is a cocone such that $r_{v_1}a = r_{v_2}b$ implies $q_{v_1}a = q_{v_2}b$, then Q factors as hR for a unique $h\colon u \to x$.*

It is standard to show that

$$\text{effective-epic} \Rightarrow \text{strong-epic} \Rightarrow \text{extremal-epic} \Rightarrow \text{epic}.$$

Note that if $\mathbf{C}$ lacks finite products, our notion of strong-epic is stronger than the usual one which only involves orthogonality to *single* monomorphisms. If $\mathbf{C}$ has all finite limits, then strong-epic and extremal-epic coincide.

Of particular importance are cocones with these properties that are "stable under pullback". Since we are not assuming the existence of actual pullbacks, defining this appropriately requires a little care.

2.23. DEFINITION. *Let $\mathscr{A}$ be a collection of κ-ary cocones in $\mathbf{C}$. We define $\mathscr{A}^\star$ to be the largest possible collection of κ-ary cocones $P\colon V \Rightarrow u$ such that*
(i) if $P \in \mathscr{A}^\star$, then $P \in \mathscr{A}$, and
(ii) if $P \in \mathscr{A}^\star$, then for any $f\colon x \to u$, there exists a $Q \in \mathscr{A}^\star$ such that $fQ \le P$.
*If $\mathscr{A}$ is the collection of cocones with some property X, we speak of the cocones in $\mathscr{A}^\star$ as being κ-**universally X**.*

This is a coinductive definition. The resulting *coinduction principle* says that to prove $\mathscr{B} \subseteq \mathscr{A}^\star$, for some collection of κ-ary cocones $\mathscr{B}$, it suffices to show that
(a) if $P \in \mathscr{B}$, then $P \in \mathscr{A}$, and
(b) if $P \in \mathscr{B}$, then for any $f\colon x \to u$, there exists a $Q \in \mathscr{B}$ such that $fQ \le P$.

2.24. DEFINITION. *A collection $\mathscr{A}$ of κ-ary cocones is **saturated** if whenever $P \in \mathscr{A}$ and $P \le Q$ for a κ-ary cocone Q, then also $Q \in \mathscr{A}$.*

2.25. LEMMA. *If $\mathscr{A}$ is saturated, so is $\mathscr{A}^\star$.*

PROOF. Let $\mathscr{B}$ be the collection of cocones Q such that $P \le Q$ for some $P \in \mathscr{A}^\star$. We want to show $\mathscr{B} \subseteq \mathscr{A}^\star$, and by coinduction it suffices to verify (a) and (b) above. Thus, suppose $Q \in \mathscr{B}$, i.e. $P \le Q$ for some $P \in \mathscr{A}^\star$. Since $\mathscr{A}$ is saturated, and $P \in \mathscr{A}$, we have $Q \in \mathscr{A}$, so (a) holds. And given f, since $P \in \mathscr{A}^\star$ we have an $R \in \mathscr{A}^\star$ (hence $R \in \mathscr{B}$) with $fR \le P$, whence $fR \le Q$. Thus (b) also holds. ∎

2.26. LEMMA. *Suppose that $\mathscr{A}^\star$ is saturated (for instance, if $\mathscr{A}$ is saturated) and that $\mathbf{C}$ has finite κ-prelimits. Then a κ-ary cocone $P\colon V \Rightarrow u$ lies in $\mathscr{A}^\star$ if and only if for any $f\colon x \to u$, some (hence any) κ-pre-pullback f^*P lies in $\mathscr{A}$.*

PROOF. Suppose $P \in \mathscr{A}^\star$. Then given f, we have some $Q \in \mathscr{A}^\star$ with $fQ \le P$. Thus $Q \le f^*P$, so (by saturation) $f^*P \in \mathscr{A}^\star$. Hence, in particular, $f^*P \in \mathscr{A}$.

For the converse, let $\mathscr{B}$ be the collection of κ-ary cocones P such that $f^*P \in \mathscr{A}$ for any f. By coinduction, to show that $\mathscr{B} \subseteq \mathscr{A}^\star$, it suffices to show (a) and (b) above. Since P is a κ-pre-pullback of itself along 1_u, we have (a) easily. For (b), we can take $Q = f^*P$. Then for any further $g\colon z \to x$, the κ-pre-pullback $g^*(f^*P)$ is also a κ-pre-pullback $(fg)^*P$, hence lies in $\mathscr{A}$; thus $f^*P \in \mathscr{B}$ as desired. ∎

It is easy to see that epic, extremal-epic, and strong-epic cocones are saturated. It seems that effective-epic cocones are not saturated in general, but κ-universally effective-epic cocones are, so that Lemma. 2.26 still applies; cf. for instance [Joh02, C2.1.6].

3. κ-ary sites

As suggested in the introduction, a *κ-ary site* is one whose covers are determined by κ-small families and which satisfies a solution-set condition. We begin with weakly κ-ary sites, which omit the solution-set condition. Recall that all categories we consider will be moderate.

3.1. Definition. *A* ***weakly κ-ary topology*** *on a category $\mathbf{C}$ consists of a class of κ-ary cocones $P\colon V \Rightarrow u$, called* ***covering families****, such that*

- *(i) For each object $u \in \mathbf{C}$, the singleton family $\{1_u\colon u \to u\}$ is covering.*
- *(ii) For any covering family $P\colon V \Rightarrow u$ and any morphism $f\colon x \to u$, there exists a covering family $Q\colon Y \Rightarrow x$ such that $fQ \le P$.*
- *(iii) If $P\colon V \Rightarrow u$ is a covering family and for each $v \in V$ we have a covering family $Q_v\colon W_v \Rightarrow v$, then $P(\bigsqcup Q_v)\colon W \Rightarrow u$ is a covering family.*
- *(iv) If $P\colon V \Rightarrow u$ is a covering family and $Q\colon W \Rightarrow u$ is a κ-ary cocone with $P \le Q$, then Q is also a covering family.*

If $\mathbf{C}$ is equipped with a weakly κ-ary topology, we call it a ***weakly κ-ary site****.*

3.2. Remark. Conditions (ii) and (iii) imply that for any covering families $P\colon V \Rightarrow u$ and $Q\colon W \Rightarrow u$, there exists a covering family $R\colon Z \Rightarrow u$ with $R \le P$ and $R \le Q$.

3.3. Remark. If we strengthen 3.1(ii) to require covering families to have *actual* pullbacks, as is common in the definition of "Grothendieck pretopology", then our *weakly unary topologies* become the *saturated singleton pretopologies* of [Rob] and the *quasi-topologies* of [Hof04].

We should first of all relate this definition to the usual notion of Grothendieck topology, which consists of a collection of *covering sieves* such that

- (a) For any u, the maximal sieve on u, which consists of all morphisms with target u, is covering.
- (b) If P is a covering sieve on u and $f\colon v \to u$, then the sieve $f^{-1}P = \{\, g\colon w \to v \mid fg \in P \,\}$ is also covering.
- (c) If P is a sieve on u such that the sieve $\{\, f\colon v \to u \mid f^{-1}P \text{ is covering } \}$ is covering, then P is also covering.

The general relationship between covering sieves and covering families is well-known (see, for instance, [Joh02, C2.1]), but it is worth making explicit here to show how the arity class κ enters. Recall that any cocone $P\colon V \Rightarrow u$ *generates* a sieve $\overline{P} = \{\, f\colon w \to u \mid f \le P \,\}$. We have $P \subseteq \overline{Q}$ if and only if $\overline{P} \subseteq \overline{Q}$, if and only if $P \le Q$.

3.4. Proposition. *For any category* $\mathbf{C}$*, there is a bijection between*

- *Weakly* κ*-ary topologies on* $\mathbf{C}$*, and*
- *Grothendieck topologies on* $\mathbf{C}$*, in the usual sense, such that every covering sieve contains a* κ*-small family which generates a covering sieve.*

Proof. First let $\mathbf{C}$ have a weakly κ-ary topology, and define a sieve to be covering if it contains a covering family. We show this is a Grothendieck topology in the usual sense.

For (a), the maximal sieve on u contains 1_u, hence is covering.

For (b), if P is a sieve on u containing a covering family P' and $f\colon v \to u$, then by 3.1(ii) there exists a covering family Q of v such that $fQ \le P'$, and hence $Q \le f^{-1}P$; thus the sieve $f^{-1}P$ is covering.

For (c), if $\{\, f\colon v \to u \mid f^{-1}P \text{ is covering} \,\}$ is covering, then it contains a covering family $F\colon V \Rightarrow u$. Moreover, since for each $v \in V$, the sieve $f_v^{-1}P$ is covering, it contains a covering family $G_v\colon W_v \Rightarrow v$. But then P contains $F\big(\bigsqcup G_v\big)$, hence is also covering.

Finally, it is clear that in this Grothendieck topology, any covering sieve contains a κ-small family which generates a covering sieve.

Now let $\mathbf{C}$ be given a Grothendieck topology satisfying the condition above, and define a κ-small cocone to be covering if it generates a covering sieve. We show that this defines a weakly κ-ary topology.

For (i), we note that the identity morphism generates the maximal sieve.

For (ii), suppose that the κ-small family $P\colon V \Rightarrow u$ generates a covering sieve $\overline{P}$, and let $f\colon x \to u$. Then the sieve $f^{-1}\overline{P}$ on x is covering, hence contains a κ-small family $Q\colon Y \Rightarrow x$ such that $\overline{Q}$ is covering. Since $Q \subseteq f^{-1}\overline{P}$, we have $fQ \le P$.

For (iii), let $R = \overline{P(\bigsqcup Q_v)}$, where $P\colon V \Rightarrow u$ and each $Q_v\colon W_v \Rightarrow v$ are covering families. Then each sieve $p_v^{-1}R$ contains the sieve $\overline{Q_v}$, which is covering, so it is also covering. Therefore, the sieve $\{\, f\colon v \to u \mid f^{-1}R \text{ is covering} \,\}$ contains the sieve $\overline{P}$, which is covering, so it is also covering. Thus, by (c), R is covering.

For (iv), if $P \le Q$, then $\overline{P} \subseteq \overline{Q}$, so if $\overline{P}$ is covering then so is $\overline{Q}$.

Finally, we prove the two constructions are inverse.

If we start with a weakly κ-ary topology, then any covering family P generates a covering sieve $\overline{P}$ since $P \subseteq \overline{P}$. Conversely, if Q is a κ-small family such that $\overline{Q}$ is a covering sieve, then by definition there exists a covering family P with $P \subseteq \overline{Q}$. That means that $P \le Q$, so by 3.1(iv), Q is covering.

In the other direction, if we start with a Grothendieck topology in terms of sieves, then any sieve R which contains a covering family P contains the sieve $\overline{P}$, which is covering; hence R is itself covering in the original topology. Conversely, if R is covering, then by assumption it contains a κ-small family P that generates a covering sieve, so that P is a covering family contained by R. ∎

Thus, we may unambiguously ask about a topology whether it "is weakly κ-ary." When interpreted in this sense, a weakly κ-ary topology is also weakly κ'-ary whenever $\kappa \subseteq \kappa'$. (When expressed with covering families, to pass from κ to κ' we need to "saturate".)

We have chosen to define κ-ary topologies in terms of covering families rather than sieves for several reasons. Firstly, in constructing the exact completion, there seems no way around working with κ-ary covering families to some extent, and constantly rephrasing things in terms of sieves would become tiresome. Secondly, covering families tend to make the constructions somewhat more explicit, especially for small values of κ. And thirdly, there may be foundational issues: a sieve on a large category is a large object, so that a collection of such sieves is an illegitimate object in ZFC.

We will consider some examples momentarily, but first we explain the solution-set condition that eliminates the adjective "weakly" from the notion of κ-ary site. For this we need a few more definitions. First of all, it is convenient to generalize the notion of covering family as follows.

3.5. Definition. *If U is a family of objects in a weakly κ-ary site, a* ***covering family*** *of U is a functional array $P\colon V \Rightarrow U$ such that each $P|_u\colon V|_u \Rightarrow u$ is a covering family.*

For instance, Definition. 3.1(iii) can then be rephrased as "the composite of two covering families is covering." We can also generalize 3.1(ii) as follows.

3.6. Lemma. *If $P\colon V \Rightarrow U$ is a covering family and $F\colon X \Rightarrow U$ is a functional array, then there exists a covering family $Q\colon Y \Rightarrow X$ such that $FQ \le P$.*

Proof. For each $x \in X$, there exists a covering family $Q_x\colon Y_x \Rightarrow x$ such that $f_x Q_x \le P_{f(x)}$; take $Q = \bigsqcup Q_x$. ∎

Thus, the category of κ-ary families and functional arrays in a (weakly) κ-ary site $\mathbf{C}$ inherits a weakly unary topology whose covers are those of Definition. 3.5. We will see in Example. 11.13 that this topology can be used to "factor" the κ-ary exact completion into a coproduct completion (recall Remark. 2.12) followed by unary exact completion.

3.7. Definition. *Let $\mathbf{C}$ be a weakly κ-ary site.*

- *(i) If $F\colon X \Rightarrow Z$ and $G\colon Y \Rightarrow Z$ are arrays in $\mathbf{C}$ with the same target, we say that F* ***factors locally through*** *G or* ***locally refines*** *G, and write $F \preceq G$, if there exists a covering family $P\colon V \Rightarrow X$ such that $FP \le G$.*
- *(ii) If $F \preceq G$ and $G \preceq F$, we say F and G are* ***locally equivalent****.*
- *(iii) A* ***local*** ***κ-prelimit*** *of $\mathbf{g}\colon \mathbf{D} \to \mathbf{C}$ is a κ-sourced array T over $\mathbf{g}$ such that every cone over $\mathbf{g}$ factors locally through T.*

3.8. Remark. Since identities cover, any κ-prelimit is also a local κ-prelimit. The converse holds if every covering family contains a split epic (see Example. 3.21).

3.9. Remark. If covering families in $\mathbf{C}$ are strong-epic and G is a monic cone, then $F \preceq G$ implies $F \le G$. In particular, in such a $\mathbf{C}$, locally equivalent monic cones are actually isomorphic, and any monic cone that is a local κ-prelimit is in fact a limit.

Local κ-prelimits are "closed under passage to covers."

3.10. Proposition. *In a weakly κ-ary site, if $T\colon L \Rightarrow X$ is a local κ-prelimit of a functor $\mathbf{g}$, and $P\colon M \Rightarrow L$ is a covering family, then TP is also a local κ-prelimit of $\mathbf{g}$.*

Proof. If $F\colon U \Rightarrow X$ is any array over $\mathbf{g}$, then by assumption we have a covering family $Q\colon V \Rightarrow U$ with $FQ \le T$. Thus, there is a functional array $H\colon V \Rightarrow L$ with $FQ = TH$. But by Lemma. 3.6, there is a covering family $R\colon W \Rightarrow V$ with $HR \le P$, whence $FQR = THR \le TP$, and QR is also a covering family. ∎

Conversely, assuming actual limits, any local κ-prelimit can be obtained in this way.

3.11. Proposition. *Suppose that $T\colon y \Rightarrow \mathbf{g}(\mathbf{D})$ is a limiting cone over $\mathbf{g}$ in a weakly κ-ary site, and $S\colon Z \Rightarrow \mathbf{g}(\mathbf{D})$ is a κ-sourced array over $\mathbf{g}$. Let $H\colon Z \Rightarrow y$ be the unique cocone such that $S = TH$. Then S is a local κ-prelimit of D if and only if H is covering.*

Proof. "If" is a special case of Proposition. 3.10. Conversely, if S is a local κ-prelimit, then $TP = SK$ for some covering $P\colon V \Rightarrow y$ and functional $K\colon V \Rightarrow Z$. Thus $THK = TP$, whence $HK = P$ since T is a limiting cone. This means $P \le H$, so H is covering. ∎

3.12. Corollary. *A local κ-prelimit of a single object is the same as a covering family of that object.*

Finally, we note two ways to construct local κ-prelimits from more basic ones.

3.13. Proposition. *If a weakly κ-ary site has local binary κ-pre-products and local κ-pre-equalizers, then it has all finite nonempty local κ-prelimits.*

Proof. This is basically like the same property for weak limits, [CV98, Prop. 1]. By induction, we can construct nonempty finite local κ-pre-products. Now, given a finite nonempty diagram $\mathbf{g}\colon \mathbf{D} \to \mathbf{C}$, let $P\colon X_0 \Rightarrow \mathbf{g}(\mathbf{D})$ be an array exhibiting X_0 as a local κ-pre-product of the finite family $\mathbf{g}(\mathbf{D})$. Enumerate the arrows of $\mathbf{D}$ as $h_1, \dots, h_m$; we will define a sequence of κ-ary families X_i in $\mathbf{C}$ and functional arrays

$$X_m \Rightarrow \cdots \Rightarrow X_1 \Rightarrow X_0. \tag{3.14}$$

Suppose we have constructed the sequence $X_i \Rightarrow \cdots \Rightarrow X_0$, and write u and v for the source and target of h_{i+1} respectively. Then we have an induced array $X_i \Rightarrow \mathbf{g}(\mathbf{D})$, and therefore in particular we have cocones $X_i \Rightarrow \mathbf{g}(u)$ and $X_i \Rightarrow \mathbf{g}(v)$. For each $x \in X_i$, let $E^x \Rightarrow x$ be a local κ-pre-equalizer of $x \to \mathbf{g}(u) \xrightarrow{\mathbf{g}(h_{i+1})} \mathbf{g}(v)$ and $x \to \mathbf{g}(v)$. Finally, define $X_{i+1} = \bigsqcup_x E^x$. This completes the inductive definition of (3.14). It is straightforward to verify that X_m is then a local κ-prelimit of $\mathbf{g}$. ∎

Unlike the case of ordinary limits, but like that of weak limits, it seems that local κ-pre-pullbacks and a local κ-pre-terminal-object do not suffice to construct all finite local κ-prelimits. We do, however, have the following.

3.15. Proposition. *If a weakly κ-ary site has local κ-pre-pullbacks and local κ-pre-equalizers, then it has all finite connected local κ-prelimits.*

Proof. Suppose $\mathbf{g}\colon \mathbf{D} \to \mathbf{C}$ is a finite connected diagram, and pick some object $u_0 \in \mathbf{D}$. For each $v \in \mathbf{D}$, let $\ell(v)$ denote the length of the shortest zigzag from u_0 to v. Now order the objects of $\mathbf{D}$ as

$$u_0, u_1, \ldots, u_n$$

in such a way that $\ell(u_i) \le \ell(u_{i+1})$ for all i. We will inductively define, for each i, a κ-ary family Y^i and an array

$$P^i\colon Y^i \Rightarrow \{\, \mathbf{g}(u_j) \mid j \le i \,\}\,. \tag{3.16}$$

Let $Y^0 = \{\mathbf{g}(u_0)\}$ and $P^0 = \{1_{\mathbf{g}(u_0)}\}$. For the inductive step, suppose given Y^i and P^i, choose a zigzag from u_0 to u_{i+1} of minimal length, and consider the final morphism in this zigzag, which connects some object v to u_{i+1}. By our choice of ordering, we have $v = u_j$ for some $j \le i$. If this morphism is directed $k\colon v \to u_{i+1}$, we let $Y^{i+1} = Y^i$ and define P^{i+1} by

$$p^{i+1}_{y,u_j} = \begin{cases} p^i_{y,u_j} & j \le i \\ \mathbf{g}(k) \circ p^i_{y,v} & j = i+1. \end{cases}$$

If, on the other hand, this morphism is directed $k\colon u_{i+1} \to v$, then for each $y \in Y^i$ consider a local κ-pre-pullback

$$\begin{tikzcd} Z^y \arrow[r, Rightarrow, "G"] \arrow[d, Rightarrow, "H"'] & y \arrow[d, "p^i_{y,v}"] \\ \mathbf{g}(u_{i+1}) \arrow[r, "\mathbf{g}(k)"'] & \mathbf{g}(v). \end{tikzcd}$$

Let $Y^{i+1} = \coprod_{y \in Y^i} Z^y$, with

$$p^{i+1}_{z,u_j} = \begin{cases} p^i_{g(z),u_j} \circ g_z & j \le i \\ h_z & j = i+1. \end{cases}$$

This completes the inductive definition of (3.16). We can use local κ-pre-equalizers as in the proof of Proposition. 3.15, starting from $P^n\colon Y^n \Rightarrow \mathbf{g}(\mathbf{D})$ instead of a local κ-pre-product X_0, to construct a local κ-prelimit of $\mathbf{g}$. ∎

Finally, we can define (strongly) κ-ary sites.

3.17. Definition. *A* ***κ-ary topology*** *on a category* $\mathbf{C}$ *is a weakly κ-ary topology for which* $\mathbf{C}$ *admits all finite local κ-prelimits. When* $\mathbf{C}$ *is equipped with a κ-ary topology, we call it a* ***κ-ary site****.*

The existence of finite local κ-prelimits may seem like a somewhat technical assumption. Its importance will become clearer with use, but we can say at this point that it is at least a generalization of the existence of weak limits, which is known to be necessary for the construction of the ordinary exact completion.

3.18. Remark. By a **locally κ-ary site** we will mean a weakly κ-ary site which admits finite *connected* local κ-prelimits. By Proposition. 3.13, any slice category of a locally κ-ary site is a κ-ary site. We are interested in these not because we have many examples of them, but because they will clarify the constructions in §6.

If $\kappa \subseteq \kappa'$, then any local κ-prelimit is also a local κ'-prelimit, and hence any κ-ary site is also κ'-ary. In particular, any κ-ary site is also $\mathfrak{K}$-ary. We now consider some examples.

3.19. Example. If $\mathbf{C}$ is a small category, then it automatically has $\mathfrak{K}$-prelimits, and every covering sieve is generated by a $\mathfrak{K}$-small family (itself). Therefore, every Grothendieck topology on a small category is $\mathfrak{K}$-ary. More generally, every topology on a κ-small category is κ-ary.

3.20. Example. If $\mathbf{C}$ has finite limits, or even finite κ-prelimits, then any weakly κ-ary topology is κ-ary. Thus, for finitely complete sites, the only condition to be κ-ary is that the topology be determined by κ-small covering families.

In particular, for a large category with finite limits, a topology is $\mathfrak{K}$-ary if and only if it is determined by small covering families. This is the case for many large sites arising in practice, such as topological spaces with the open cover topology, or $\mathbf{Ring}^{\mathrm{op}}$ with its Zariski or étale topologies.

3.21. Example. Consider the **trivial topology** on a category $\mathbf{C}$, in which a sieve is covering just when it contains a split epic. In this topology every sieve contains a single covering morphism (the split epic), so it is κ-ary just when $\mathbf{C}$ has local κ-prelimits. But as noted in Remark. 3.8, in this case local κ-prelimits reduce to plain κ-prelimits.

In particular, $\mathbf{C}$ admits a trivial unary topology if and only if it has weak finite limits. On the other hand, every small category admits a trivial $\mathfrak{K}$-ary topology, and any category with finite limits admits a trivial κ-ary topology for any κ.

3.22. Example. The intersection of any collection of weakly κ-ary topologies is again weakly κ-ary, so any collection of κ-ary cocones generates a smallest weakly κ-ary topology for which they are covering. If the category has finite κ-prelimits, then such an intersection is of course κ-ary.

3.23. Example. A topology is called **subcanonical** if every covering family is effective-epic, in the sense of Definition. 2.22. By Definition. 3.1(ii), every covering family in a subcanonical and weakly κ-ary topology must in fact be *κ-universally effective-epic* in the sense of Definition. 2.23. The collection of all κ-universally effective-epic cocones forms a weakly κ-ary topology on $\mathbf{C}$, which we call the **κ-canonical topology**. It may not be κ-ary, but it will be if (for instance) $\mathbf{C}$ has finite κ-prelimits, as is usually the case in practice. Note that unlike the situation for trivial topologies, the κ-canonical and κ'-canonical topologies rarely coincide for $\kappa \neq \kappa'$.

More generally, if $\mathscr{A}$ is a class of κ-ary cocones satisfying 3.1(i) and (iii), and $\mathscr{A}^\star$ is saturated, then $\mathscr{A}^\star$ is a weakly κ-ary topology.

If $\mathbf{C}$ has pullbacks, its $\{1\}$-canonical topology coincides with the "canonical singleton pretopology" of [Rob], and consists of the pullback-stable regular epimorphisms. If $\mathbf{C}$

is small, its $\mathfrak{K}$-canonical topology agrees with its canonical topology as usually defined (consisting of all universally effective-epic sieves). This is not necessarily the case if $\mathbf{C}$ is large, but it is if the canonical topology is small-generated, as in the next example.

3.24. Example. Suppose a $\mathbf{C}$ is a *small-generated site*[1], meaning that it is locally small, is equipped with a Grothendieck topology in the usual sense, and has a small (full) subcategory $\mathbf{D}$ such that every object of $\mathbf{C}$ admits a covering sieve generated by morphisms out of objects in $\mathbf{D}$. For instance, $\mathbf{C}$ might be a Grothendieck topos with its canonical topology. Then we claim that the topology of $\mathbf{C}$ is $\mathfrak{K}$-ary.

Firstly, given a finite diagram $\mathbf{g}\colon \mathbf{E} \to \mathbf{C}$ (or in fact any small diagram), consider the family P of all cones over $\mathbf{g}$ with vertex in $\mathbf{D}$. Since $\mathbf{D}$ is small and $\mathbf{C}$ is locally small, there are only a small number of such cones. Thus, we may put them together into a single array over $\mathbf{g}$, whose domain is a small family of objects of $\mathbf{D}$. This array is a local κ-prelimit of $\mathbf{g}$, since for any cone T over $\mathbf{g}$, we can cover its vertex by objects of $\mathbf{D}$, and each resulting cone will automatically factor through P. Thus $\mathbf{C}$ has local κ-prelimits.

Now suppose R is a covering sieve of an object $u \in \mathbf{C}$; we must show it contains a small family generating a covering sieve. Let P be the family of all morphisms $v \to u$ in R with $v \in \mathbf{D}$. Since $\mathbf{D}$ is small and $\mathbf{C}$ is locally small, P is small, and it is clearly contained in R. Thus, it remains to show $\overline{P}$ is a covering sieve.

Consider any morphism $r\colon w \to u$ in R. Then the sieve $r^{-1}\overline{P}$ contains all maps from objects of $\mathbf{D}$ to w, hence is covering. Thus the sieve $\left\{ r\colon w \to u \;\middle|\; r^{-1}\overline{P} \text{ is covering} \right\}$ contains R and hence is covering; thus $\overline{P}$ itself is covering.

3.25. Example. Suppose $\mathbf{C}$ has finite limits and a stable factorization system $(\mathcal{E}, \mathcal{M})$, where $\mathcal{M}$ consists of monos; I claim $\mathcal{E}$ is then a unary topology on $\mathbf{C}$. It clearly satisfies 3.1(i)–(iii) for $\kappa = \{1\}$. For 3.1(iv), suppose $fg \in \mathcal{E}$ and factor $f = me$ with $m \in \mathcal{M}$ and $e \in \mathcal{E}$. Then unique lifting gives an h with $hfg = eg$ and $mh = 1$. So m is split epic (by h) and monic (since it is in $\mathcal{M}$) and thus an isomorphism. Hence f, like e, is in $\mathcal{E}$.

4. Morphisms of sites

While the search for a general construction of exact completion has led us to *restrict* the notion of *site* by requiring κ-arity, it simultaneously leads us to *generalize* the notion of *morphism of sites*. Classically, a morphism of sites is defined to be a functor $\mathbf{f}\colon \mathbf{C} \to \mathbf{D}$ which preserves covering families and is *representably flat*. By the latter condition we mean that each functor $\mathbf{D}(d, \mathbf{f}-)$ is flat, which is to say that for any finite diagram $\mathbf{g}$ in $\mathbf{C}$, any cone over $\mathbf{fg}$ in $\mathbf{D}$ factors through the $\mathbf{f}$-image of some cone over $\mathbf{g}$.

A representably flat functor preserves all finite limits, and indeed all finite prelimits, existing in its domain. Conversely, if $\mathbf{C}$ has finite limits, or even finite prelimits, and $\mathbf{f}$ preserves them, it is representably flat. For our purposes, it is clearly natural to seek a notion analogously related to *local* finite prelimits. This leads us to the following definition

[1]Traditionally called an *essentially small site*, but this can be confusing since $\mathbf{C}$ itself need not be essentially small as a category (i.e. equivalent to a small category).

which was studied in [Koc89] (using the internal logic) and [Kar04] (who called it being *flat relative to the topology of* $\mathbf{D}$).

4.1. Definition. *Let* $\mathbf{C}$ *be any category and* $\mathbf{D}$ *any site. A functor* $\mathbf{f}\colon \mathbf{C} \to \mathbf{D}$ *is* ***covering-flat*** *if for any finite diagram* $\mathbf{g}$ *in* $\mathbf{C}$*, every cone over* $\mathbf{fg}$ *in* $\mathbf{D}$ *factors* locally *through the* $\mathbf{f}$*-image of some array over* $\mathbf{g}$*.*

4.2. Lemma. $\mathbf{f}\colon \mathbf{C} \to \mathbf{D}$ *is covering-flat if and only if for any finite diagram* $\mathbf{g}\colon \mathbf{E} \to \mathbf{C}$ *and any cone* T *over* $\mathbf{fg}$ *with vertex* u*, the sieve*

$$\{\, h\colon v \to u \mid \textit{there exists a cone } S \textit{ over } \mathbf{g} \textit{ such that } Th \le \mathbf{f}(S) \,\} \tag{4.3}$$

is a covering sieve of u *in* $\mathbf{D}$*.* ∎

4.4. Example. Any representably flat functor is covering-flat. The converse holds if $\mathbf{D}$ has a trivial topology.

4.5. Lemma. *If* $\mathbf{D}$ *has finite limits, then* $\mathbf{f}\colon \mathbf{C} \to \mathbf{D}$ *is covering-flat if and only if for any finite diagram* $\mathbf{g}\colon \mathbf{E} \to \mathbf{C}$*, the family of factorizations through* $\lim \mathbf{fg}$ *of the* $\mathbf{f}$*-images of all cones over* $\mathbf{g}$ *generates a covering sieve.*

Proof. When $u = \lim \mathbf{fg}$, the family in question generates the sieve (4.3), which is covering if $\mathbf{f}$ is covering-flat. Conversely, for any u, the sieve (4.3) is the pullback to u of the corresponding one for $\lim \mathbf{fg}$, so if the latter is covering, so is the former. ∎

4.6. Example. If $\mathbf{D}$ is a Grothendieck topos with its canonical topology, and $\mathbf{C}$ is small, then $\mathbf{f}\colon \mathbf{C} \to \mathbf{D}$ is covering-flat if and only if "$\mathbf{f}$ is representably flat" is true in the internal logic of $\mathbf{D}$. This is the sort of "flat functor" which Diaconescu's theorem says is classified by geometric morphisms $\mathbf{D} \to [\mathbf{C}^{\mathrm{op}}, \mathbf{Set}]$ (it is not the same as being representably flat).

If $\mathbf{C}$ has finite κ-prelimits for some κ, then in Lemma. 4.5 it suffices to consider the family of factorizations through $\lim \mathbf{fg}$ of the cones in some κ-prelimit of $\mathbf{g}$.

4.7. Example. If $\mathbf{C}$ has weak finite limits and $\mathbf{D}$ is a regular category with its regular topology, then $\mathbf{f}\colon \mathbf{C} \to \mathbf{D}$ is covering-flat if and only if for any weak limit t of a finite diagram $\mathbf{g}$ in $\mathbf{C}$, the induced map $\mathbf{f}(t) \to \lim \mathbf{fg}$ in $\mathbf{D}$ is regular epic. As observed in [Kar04], this is precisely the definition of *left covering* functors used in [CV98] (called $\aleph_0$-*flat* in [HT96]) to describe the universal property of regular and exact completions.

4.8. Proposition. *Let* $\mathbf{C}$ *be a* κ*-ary site,* $\mathbf{D}$ *any site, and* $\mathbf{f}\colon \mathbf{C} \to \mathbf{D}$ *a functor; the following are equivalent.*

(i) $\mathbf{f}$ *is covering-flat and preserves covering families.*

(ii) For any finite diagram $\mathbf{g}\colon \mathbf{E} \to \mathbf{C}$ *and any local finite* κ*-prelimit* T *of* $\mathbf{g}$*, the image* $\mathbf{f}(T)$ *is a local* κ*-prelimit of* $\mathbf{fg}$*.*

(iii) $\mathbf{f}$ *preserves covering families, and for any finite diagram* $\mathbf{g}\colon \mathbf{E} \to \mathbf{C}$ *there exists a local* κ*-prelimit* T *of* $\mathbf{g}$ *such that* $\mathbf{f}(T)$ *is a local* κ*-prelimit of* $\mathbf{fg}$*.*

Proof. Suppose (i) and let T be a local κ-prelimit of a finite diagram $\mathbf{g}\colon \mathbf{E} \to \mathbf{C}$. Then any cone S over $\mathbf{fg}$ factors locally through $\mathbf{f}(R)$ for some array R over $\mathbf{g}$. But since T is a local κ-prelimit, $R \preceq T$, and since $\mathbf{f}$ preserves covering families, it preserves $\preceq$. Thus $S \preceq \mathbf{f}(R) \preceq \mathbf{f}(T)$, so $\mathbf{f}(T)$ is a local prelimit of $\mathbf{fg}$; hence (i)$\Rightarrow$(ii).

Now suppose (ii). Since a local κ-prelimit of a single object is just a covering family of that object, $\mathbf{f}$ preserves covering families. Since $\mathbf{C}$ has finite local κ-prelimits, (iii) follows.

Finally, suppose (iii), and let $\mathbf{g}\colon \mathbf{E} \to \mathbf{C}$ be a finite diagram and S a cone over $\mathbf{fg}$. Let T be a local κ-prelimit of $\mathbf{g}$ such that $\mathbf{f}(T)$ is a local κ-prelimit of $\mathbf{fg}$. Then $S \preceq \mathbf{f}(T)$; hence $\mathbf{f}$ is covering-flat. ∎

4.9. Remark. If $\mathbf{C}$ and $\mathbf{D}$ have finite κ-prelimits and trivial κ-ary topologies, then Proposition. 4.8 reduces to the fact that a functor is representably flat if and only if it preserves these finite κ-prelimits.

4.10. Definition. *For sites $\mathbf{C}$ and $\mathbf{D}$, we say $\mathbf{f}\colon \mathbf{C} \to \mathbf{D}$ is a* ***morphism of sites*** *if it is covering-flat and preserves covering families.*

4.11. Corollary. *A functor $\mathbf{f}\colon \mathbf{C} \to \mathbf{D}$ between κ-ary sites is a morphism of sites if and only if it preserves covering families, local binary κ-pre-products, local κ-pre-equalizers, and local κ-pre-terminal-objects.*

Proof. "Only if" is clear, so suppose $\mathbf{f}$ preserves the aforementioned things. But then it preserves the construction of nonempty finite local κ-prelimits in Proposition. 3.13, and hence satisfies Proposition. 4.8(iii). ∎

We define the very large 2-category SITE_κ to consist of κ-ary sites, morphisms of sites, and arbitrary natural transformations. Note that for $\kappa \subseteq \kappa'$, we have $\mathrm{SITE}_\kappa \subseteq \mathrm{SITE}_{\kappa'}$ as a full sub-2-category. Since representably flat implies covering-flat, any morphism of sites in the classical sense is also one in our sense. We now show, following [FS90, 1.829] and [CV98, Prop. 20], that the converse is often true.

4.12. Lemma. *If $\mathbf{D}$ is a site in which all covering families are epic, then any covering-flat functor $\mathbf{f}\colon \mathbf{C} \to \mathbf{D}$ preserves finite monic cones.*

Proof. Suppose $T\colon x \Rightarrow U$ is a finite monic cone in $\mathbf{C}$, where U has cardinality n. Let $\mathbf{E}$ be the finite category such that a diagram of shape $\mathbf{E}$ consists of a family of n objects and two cones over it. Let $\mathbf{g}\colon \mathbf{E} \to \mathbf{C}$ be the diagram both of whose cones are T. Monicity of T says exactly that T together with two copies of 1_x is a limit of $\mathbf{g}$; call this cone T'.

Now suppose $h, k\colon z \rightrightarrows \mathbf{f}(x)$ satisfy $\mathbf{f}(T) \circ h = \mathbf{f}(T) \circ k$. Then h and k induce a cone S over $\mathbf{fg}$. Since $\mathbf{f}$ is covering-flat, this cone factors locally through the $\mathbf{f}$-image of some array over $\mathbf{g}$, and hence through $\mathbf{f}(T')$. This just means that z admits a covering family P such that $hP = kP$; but P is epic, so $h = k$. ∎

4.13. Proposition. *Suppose* $\mathbf{C}$ *has finite limits and all covering families in* $\mathbf{D}$ *are strong-epic. Then any covering-flat functor* $\mathbf{f}\colon \mathbf{C} \to \mathbf{D}$ *preserves finite limits.*

Proof. Let $\mathbf{g}\colon \mathbf{E} \to \mathbf{C}$ be a finite diagram and $T\colon x \Rightarrow \mathbf{g}(\mathbf{E})$ a limit cone. Then T is monic, so by Lemma. 4.12, $\mathbf{f}(T)$ is also monic. But by Proposition. 4.8, $\mathbf{f}(T)$ is a local κ-prelimit of $\mathbf{fg}$; thus by Remark. 3.9 it is a limit. ∎

Recall that if finite limits exist, every extremal-epic family is strong-epic.

4.14. Corollary. *If* $\mathbf{C}$ *is a finitely complete site and* $\mathbf{D}$ *a site in which covering families are strong-epic, then* $\mathbf{f}\colon \mathbf{C} \to \mathbf{D}$ *is a morphism of sites if and only if it preserves finite limits and covering families.*

However, for morphisms between arbitrary sites, our notion is more general than the usual one. It is easy to give boring examples of this.

4.15. Example. Let $\mathbf{C}$ be the terminal category, let $\mathbf{D}$ be the category $(0 \to 1)$ in which the morphism $0 \to 1$ is a cover, and let $\mathbf{f}\colon \mathbf{C} \to \mathbf{D}$ pick out the object 0. Then $\mathbf{f}$ is covering-flat, but not representably flat, since $\mathbf{C}$ has a terminal object but $\mathbf{f}$ does not preserve it. Note that $\mathbf{C}$ and $\mathbf{D}$ have finite limits and all covers in $\mathbf{D}$ are epic.

Our main reason for introducing the more general notion of morphism of sites is to state the universal property of exact completion. It is further justified, however, by the following observation.

4.16. Proposition. *For a small category* $\mathbf{C}$ *and a small site* $\mathbf{D}$*, a functor* $\mathbf{f}\colon \mathbf{C} \to \mathbf{D}$ *is covering-flat if and only if the composite*

$$[\mathbf{C}^{\mathrm{op}}, \mathbf{Set}] \xrightarrow{\mathrm{Lan}_{\mathbf{f}}} [\mathbf{D}^{\mathrm{op}}, \mathbf{Set}] \xrightarrow{\mathbf{a}} \mathrm{Sh}(\mathbf{D}) \tag{4.17}$$

preserves finite limits, where $\mathbf{a}$ *denotes sheafification. If* $\mathbf{C}$ *is moreover a site and* $\mathbf{f}$ *a morphism of sites, then*

$$\mathbf{f}^*\colon [\mathbf{D}^{\mathrm{op}}, \mathbf{Set}] \to [\mathbf{C}^{\mathrm{op}}, \mathbf{Set}]$$

takes $\mathrm{Sh}(\mathbf{D})$ *into* $\mathrm{Sh}(\mathbf{C})$*, so* $\mathbf{f}$ *induces a geometric morphism* $\mathrm{Sh}(\mathbf{D}) \to \mathrm{Sh}(\mathbf{C})$.

Note that representable-flatness of $\mathbf{f}$ is equivalent to $\mathrm{Lan}_{\mathbf{f}}$ preserving finite limits. (This certainly *implies* that (4.17) does so, since $\mathbf{a}$ always preserves finite limits.) This proposition can be proved explicitly, but we will deduce it from general facts about exact completion in §11.

4.18. Remark. When $\mathbf{C}$ and $\mathbf{D}$ are *locally* κ-ary sites (Remark. 3.18), we will consider the notion of a **pre-morphism of sites**, which we define to be a functor $\mathbf{f}\colon \mathbf{C} \to \mathbf{D}$ preserving finite *connected* local κ-prelimits. (The name is chosen by analogy with "pre-geometric morphisms", which are adjunctions between toposes whose left adjoint preserves finite connected limits. The prefix "pre-" here has unfortunately nothing to do with the "pre-" in "prelimit".) We write LSITE_κ for the 2-category of locally κ-ary sites, pre-morphisms of sites, and arbitrary natural transformations.

Arguing as in Corollary. 4.11 but using Proposition. 3.15, we see that $\mathbf{f}$ is a pre-morphism of sites just when it preserves covering families, local κ-pre-pullbacks, and local κ-pre-equalizers. Similarly, we have versions of Lemma. 4.12 for *nonempty* finite monic cones, and of Proposition. 4.13 and Corollary. 4.14 for *connected* finite limits.

5. Regularity and exactness

In this section we define the notions of κ*-ary regular* and κ*-ary exact* categories, which are the outputs of our completion operations. These are essentially relativizations to κ of the notions of "familially regular" and "familially exact" from [Str84].

5.1. Some operations on arrays We begin by defining some special arrays, and operations on arrays. Firstly, for any object x, we write $\Delta_x = \{1_x, 1_x\} \colon x \Rightarrow \{x, x\}$. Secondly, for any u, v we have a functional array $\sigma \colon \{u, v\} \Rightarrow \{v, u\}$ consisting of identities.

Thirdly, suppose $P \colon X \Rightarrow U$ is a κ-to-finite array in a κ-ary site, V is another finite family of objects, and $F \colon V \Rightarrow U$ is a functional array. For each $x \in X$, let $(F^*X)_x$ be a local κ-prelimit of the (finite) diagram consisting of V, U, all the morphisms in F, and the cone $P|^x$. Write $(F^*P)_x \colon (F^*X)_x \Rightarrow V$ for the cocone built from the projections to V. We define $F^*X = \bigsqcup_x (F^*X)_x$ and let F^*P be the induced array $F^*X \Rightarrow V$.

Fourthly, suppose given a finite family of κ-to-finite arrays $\{P_i \colon X_i \Rightarrow U\}_{1 \le i \le n}$. For each family $\{x_i\}_{1 \le i \le n}$ with $x_i \in X_i$ for each i, consider a local κ-prelimit of the (finite) diagram consisting of U, all the objects x_i, and all the morphisms $(p_i)_{x_i,u}$. We write $\bigwedge_i P_i \colon \bigwedge_i X_i \Rightarrow U$ for the disjoint union of these local κ-prelimits over all families $\{x_i\}_{1 \le i \le n}$, with its induced array to U.

Finally, suppose given arrays $P \colon X \Rightarrow \{u, v\}$ and $Q \colon Y \Rightarrow \{v, w\}$. For each $x \in X$ and $y \in Y$, let $R^{xy} \colon Z^{xy} \Rightarrow \{x, y\}$ be a local κ-prelimit of the cospan $x \xrightarrow{p_{xv}} v \xleftarrow{q_{yv}} y$. Let $Z = \bigsqcup Z^{xy}$, with induced functional arrays $R \colon Z \Rightarrow X$ and $S \colon Z \Rightarrow Y$, and define $T \colon Z \Rightarrow \{u, w\}$ by $t_{zu} = p_{r(z),u} r_z$ and $t_{zw} = q_{s(z),w} s_z$. We write $P \times_v Q$ for T.

Of course, all of these "operations" depend on a choice of local κ-prelimits, but the result is unique up to local equivalence in the sense of Definition. 3.7.

5.2. Definition. *Let $\mathbf{C}$ be a κ-ary site, and X a κ-ary family of objects of $\mathbf{C}$. A* κ***-ary congruence on*** X *consists of*

(i) *For each $x_1, x_2 \in X$, a κ-sourced array $\Phi(x_1, x_2) \colon \Phi[x_1, x_2] \Rightarrow \{x_1, x_2\}$.*
(ii) *For each $x \in X$ we have $\Delta_x \preceq \Phi(x, x)$.*
(iii) *For each $x_1, x_2 \in X$ we have $\sigma \circ \Phi(x_1, x_2) \preceq \Phi(x_2, x_1)$.*
(iv) *For each $x_1, x_2, x_3 \in X$ we have $\Phi(x_1, x_2) \times_{x_2} \Phi(x_2, x_3) \preceq \Phi(x_1, x_3)$.*

We say Φ is ***strict*** *if each array $\Phi(x_1, x_2)$ is a monic cone.*

5.3. Remark. If Φ is a κ-ary congruence and we replace each $\Phi(x_1, x_2)$ by a locally equivalent κ-sourced array $\Psi(x_1, x_2)$, then Ψ is again a κ-ary congruence on the same underlying family X, since all the operations used in Definition. 5.2 respect $\preceq$. In this case we say that Φ and Ψ are **equivalent** congruences. By Remark. 3.9, if covering families are strong-epic, then any two equivalent *strict* congruences are in fact isomorphic.

5.4. Lemma. *Let X and Y be κ-ary families of objects in a κ-ary site.*

(i) There is a κ-ary congruence Δ_X on X with

$$\Delta_X(x_1, x_2) = \begin{cases} \Delta_x & x_1 = x_2 \\ \emptyset & x_1 \neq x_2. \end{cases}$$

(ii) If $F\colon X \Rightarrow Y$ is a functional array and Ψ is a κ-ary congruence on Y, we have a κ-ary congruence $F^*\Psi$ on X defined by $(F^*\Psi)(x_1, x_2) = \{f_{x_1}, f_{x_2}\}^*(\Psi(x_1, x_2))$.

(iii) If $\{\Phi_i\}_{1\le i\le n}$ is a finite family of κ-ary congruences on X, we have a κ-ary congruence $\bigwedge_i \Phi_i$ on X defined by $(\bigwedge_i \Phi_i)(x_1, x_2) = \bigwedge_i(\Phi_i(x_1, x_2))$.

Proof. This is mostly straightforward verification. The only possibly non-obvious fact is that Δ_X is κ-ary; its indexing set is the subsingleton $[\![x_1 = x_2]\!]$ from Remark. 2.5. ∎

5.5. Definition. *A **kernel** of a κ-to-finite array $P\colon X \Rightarrow U$ in a κ-ary site is a κ-ary congruence Φ on X such that each $\Phi(x_1, x_2)$ is a local κ-prelimit of the diagram*

$$P|^{x_1} : x_1 \Rightarrow U \Leftarrow x_2 : P|^{x_2}.$$

*A **strict kernel** is one constructed using actual limits rather than local prelimits.*

The condition to be a kernel is equivalent to saying that Φ can be expressed as

$$\bigwedge_{u\in U} (P|_u)^* \Delta_{\{u\}}.$$

This implies, in particular, that a kernel *is* a κ-ary congruence.

Any two kernels of the same array are equivalent in the sense of Remark. 5.3. Conversely, two equivalent congruences are kernels of exactly the same arrays. If Φ is a kernel of $P\colon X \Rightarrow U$, then its defining property is that for any $a\colon v \to x_1$ and $b\colon v \to x_2$ such that $p_{x_1,u}a = p_{x_2,u}b$ for all $u \in U$, we have $\{a, b\} \preceq \Phi(x_1, x_2)$.

5.6. Lemma. *Suppose covering families in $\mathbf{C}$ are strong-epic. Then if a strict congruence is a kernel of P, it is also a strict kernel of P.*

Proof. By Remark. 3.9. ∎

We may regard a congruence as a diagram consisting of all the objects and morphisms occurring in X and all the $\Phi(x_1, x_2)$. In particular, we can talk about *colimits* of congruences, which we also call *quotients*.

5.7. Lemma. *If covering families in $\mathbf{C}$ are epic, then an effective-epic κ-ary cocone is the colimit of any of its kernels.*

Proof. Let $P\colon X \Rightarrow y$ be effective-epic and Φ a kernel of it. It suffices to show that if F is a cocone under Φ, then $f_{x_1}a = f_{x_2}b$ for any $x_1, x_2 \in X$ and $\{a, b\}\colon u \Rightarrow \{x_1, x_2\}$ such that $p_{x_1}a = p_{x_2}b$. But by construction of Φ, we have a covering family $Q\colon V \Rightarrow u$ such that $\{a, b\}Q \le \Phi(x_1, x_2)$, hence $f_{x_1}aQ = f_{x_2}bQ$. Since Q is epic, the claim follows. ∎

The following lemma and theorem are slight generalizations of results in [Str84] (see also [Kel91, Lemma 2.1]).

5.8. Lemma. *Let* $\mathbf{C}$ *be a* κ*-ary site in which covering families are epic, and let* $P\colon V \Rightarrow u$ *be a* κ*-ary cocone and* $Q\colon u \Rightarrow W$ *a cone in* $\mathbf{C}$*.*

(a) If Q *is monic, any (strict) kernel of* P *is also a (strict) kernel of* QP *and vice versa.*
(b) Conversely, if P *is* κ*-universally epic, and some kernel of* P *is also a kernel of* QP*, then* Q *is monic.*

Proof. Part (a) is easy: if Q is monic, then an array $R\colon Z \Rightarrow \{v_1, v_2\}$ satisfies $p_{v_1}R|_{v_1} = p_{v_2}R|_{v_2}$ if and only if it satisfies $Qp_{v_1}R|_{v_1} = Qp_{v_2}R|_{v_2}$.

For part (b), let $a, b\colon x \to u$ be given with $Qa = Qb$. Let $R\colon Y \Rightarrow x$ be a local κ-pre-pullback of P along a, and for each $y \in Y$ let $S^y\colon Z^y \Rightarrow y$ be a local κ-pre-pullback of P along br_y. Since P is κ-universally epic, R and each S^y are epic.

Define $Z = \bigsqcup Z^y$, $S = \bigsqcup S^y$, and $T = RS\colon Z \Rightarrow x$. Then T is epic, and $aT \le P$ and $bT \le P$. Thus, we have $aT = PG$ and $bT = PH$ for some functional arrays $G, H\colon Z \Rightarrow V$. But since $Qa = Qb$, we have $QPG = QaR = QbR = QPH$. Thus for each $z \in Z$, we have $\{g_z, h_z\} \preceq \Phi(g(z), h(z))$, where Φ is a kernel of QP. But since Φ is also a kernel of P, we have $Pg_zF = Ph_zF$ for some covering family F of z. Since covering families are epic, $Pg_z = Ph_z$ for all z, hence $PG = PH$. Thus $aT = bT$, whence $a = b$ as T is epic. ∎

5.9. Theorem. *For any category* $\mathbf{C}$*, the following are equivalent.*

(i) $\mathbf{C}$ *is a* κ*-ary site whose covering families are strong-epic, and any* κ*-to-finite array* $R\colon V \Rightarrow W$ *factors as* QP*, where* $P\colon V \Rightarrow u$ *is covering and* $Q\colon u \Rightarrow W$ *is monic.*
(ii) $\mathbf{C}$ *is a subcanonical* κ*-ary site, and any kernel of a* κ*-to-finite array is also a kernel of some covering cocone.*
(iii) $\mathbf{C}$ *has finite limits, and any* κ*-ary cocone* $R\colon V \Rightarrow w$ *factors as* qP*, where* $P\colon V \Rightarrow u$ *is universally extremal-epic and* $q\colon u \to w$ *is monic.*
(iv) $\mathbf{C}$ *has finite limits, and the strict kernel of any* κ*-ary cocone is also the strict kernel of some universally effective-epic cocone.*
(v) $\mathbf{C}$ *is a regular category (in the ordinary sense) and has pullback-stable unions of* κ*-small families of subobjects.*

Moreover, they imply

(vi) Every κ*-ary extremal-epic cocone is universally effective-epic*

and in (i) and (ii) the topology is automatically the κ*-canonical one.*

Proof. We will prove

$$\begin{array}{ccccccccc} \text{(i)} & \Rightarrow & \text{(iii)} & \Longrightarrow & \text{(iv)} & \Longrightarrow & \text{(ii)} & \Rightarrow & \text{(i)} \\ & & \Updownarrow & \searrow & & & & & \\ & & \text{(v)} & & \text{(vi).} & & & & \end{array}$$

For (i)⇒(iii), it suffices to construct finite limits. Given a finite diagram $\mathbf{g}\colon \mathbf{D} \to \mathbf{C}$, let T be a κ-prelimit of $\mathbf{g}$, and apply (i) to the array $T\colon U \Rightarrow \mathbf{g}(\mathbf{D})$. The resulting array

Q is a cone over $\mathbf{g}$ since P is epic, and a local κ-prelimit since P is a covering family. Hence, by Remark. 3.9 it is a limiting cone.

Now supposing (iii), since $1 \in \kappa$, $\mathbf{C}$ has pullback-stable (extremal-epi, mono) factorizations. This is one definition of a regular category in the ordinary sense, so $\mathbf{C}$ is regular. Moreover, applying the factorization of (iii) to a κ-small family of subobjects supplies a pullback-stable union; hence (iii)⇒(v).

Conversely, supposing (v) and given a κ-small cocone $F: V \Rightarrow w$, we can first factor each f_v using the (extremal-epi, mono) factorization in a regular category, and then take the union of the resulting κ-small family of subobjects of w. This gives the desired factorization, so that (v)⇒(iii).

Now we assume (iii) and prove (vi). Firstly, if R is extremal-epic, then it factors as $R = qP$ where P is universally extremal-epic and q is monic; hence q is an isomorphism and so R, like P, is universally extremal-epic. Thus it suffices to show that every κ-small extremal-epic family is effective-epic.

Let $P: V \Rightarrow u$ be a κ-ary extremal-epic cocone with a kernel Φ, and let $R: V \Rightarrow w$ be a cocone under Φ. Then the induced cocone $(P, R): V \Rightarrow u \times w$ factors as $(a, b)P'$, where $P': V \Rightarrow t$ is (universally) extremal-epic and $(a, b): t \to u \times w$ is monic. Since P factors through a, a must be extremal-epic.

Now since Φ is a kernel of P and R is a cocone under Φ, it follows that Φ is also a kernel of (P, R). And since (a, b) is monic, by Lemma. 5.8(a) Φ is also a kernel of P'. But $aP' = P$ and P' is universally extremal-epic, hence universally epic; so by Lemma. 5.8(b), a is monic. Since it is both extremal-epic and monic, it must be an isomorphism, and then the composite ba^{-1} provides a factorization of R through P. This factorization is unique, since P is extremal-epic and hence epic. Thus (vi) holds.

Now assuming (iii), hence also (vi), we prove (iv). If Φ is a strict kernel of R, then write $R = qP$ with q monic and P universally extremal-epic. By Lemma. 5.8(a) Φ is also a strict kernel of P, and by (vi) P is universally effective-epic, so (iv) holds.

Now suppose (iv) and give $\mathbf{C}$ its κ-canonical topology. Let Φ be a kernel of a κ-to-finite array $R: V \Rightarrow W$, and let Ψ be the strict kernel of R. Then Ψ is also the strict kernel of the induced cocone $V \Rightarrow \prod_{w \in W} w$, so by (iv), Ψ is the strict kernel (hence a kernel) of some universally effective-epic cocone, which is covering in the κ-canonical topology. Since Φ and Ψ are equivalent congruences, Φ is also a kernel of this cocone; thus (iv)⇒(ii).

To complete the circle, assume (ii), and suppose given a κ-to-finite array R. Let Φ be a kernel of R, and let P be universally effective-epic and have Φ also as its kernel. Then by Lemma. 5.7, P is the colimit of Φ, so R factors through it as QP, and by Lemma. 5.8(b) Q is monic; thus (i) holds.

Finally, we show that the topology in (i) and (ii) must be κ-canonical. It is subcanonical by definition in (ii) and by (vi) in (i), so it suffices to show that any universally effective-epic cocone is covering. But by (i) such a cocone factors as a covering family followed by a monomorphism, and since effective-epic families are extremal-epic, the monic must be an isomorphism. ∎

5.10. Definition. *We say that a category* $\mathbf{C}$ *is* κ***-ary regular****, or* κ***-ary coherent****, if* *it satisfies the equivalent conditions of Theorem. 5.9.*

By Theorem. 5.9(v), κ-ary regularity generalizes a number of more common definitions.

- $\mathbf{C}$ is $\{1\}$-ary regular iff it is regular in the usual sense.
- $\mathbf{C}$ is $\{0, 1\}$-ary regular iff it is regular and has a strict initial object.
- $\mathbf{C}$ is ω-ary regular iff it is coherent.
- $\mathbf{C}$ is ω_1-ary regular iff it is countably-coherent, a.k.a. σ-coherent.
- $\mathbf{C}$ is $\mathfrak{K}$-ary regular iff it is infinitary-coherent (a.k.a. geometric, although sometimes geometric categories are also required to be well-powered).

By Theorem. 5.9(vi), in a κ-ary regular category, a κ-ary cocone is extremal-epic if and only if it is strong-epic, if and only if it is effective-epic, and in all cases it is automatically universally so. These κ-ary cocones form the κ-canonical topology on a κ-ary regular category, which we may also call the **κ-regular** or **κ-coherent topology**.

Similarly, a functor between κ-ary regular categories is a morphism of sites (relative to the κ-regular topologies) just when it preserves finite limits and κ-small extremal-epic families; we call such a functor **κ-ary regular**. We write REG_κ for the full sub-2-category of SITE_κ whose objects are the κ-ary regular categories with their κ-canonical topologies.

5.11. Remark. Let LEX denote the 2-category of finitely complete categories and finitely continuous functors, and WLEX that of categories with weak finite limits and weak-finite-limit–preserving functors (i.e. representably flat functors). Equipping such categories with their trivial κ-ary topologies, we can regard LEX and WLEX as full sub-2-categories of SITE_κ for any κ. However, the embedding $\mathrm{REG}_\kappa \hookrightarrow \mathrm{SITE}_\kappa$ does *not* factor through the embeddings of LEX or WLEX. This is why our universal property for regular and exact completions will look simpler than that of [CV98].

5.12. Remark. Our notion of κ-ary regular category is a different "infinitary generalization" of regularity than that considered in [HT96, Lac99]; the latter instead adds to (unary) regularity the existence and compatibility of κ-ary *products*.

5.13. Remark. Following [Joh02, A3.2.7], by a **locally κ-ary regular category** we will mean a category $\mathbf{C}$ with finite *connected* limits and stable image factorizations for arrays with κ-ary domain and *nonempty* finite codomain. Note that the construction of kernels of such arrays requires only connected finite (local κ-pre-)limits. An analogue of Theorem. 5.9 is true in this case, although in the absence of finite products, in (iii) and (iv) we need to assert at least factorization for κ-to-binary arrays as well as for cocones.

5.14. Corollary. *In a κ-ary regular category (with its κ-regular topology):*
(i) Every κ-to-finite array is locally equivalent to a monic cone.
(ii) Every κ-ary congruence is equivalent to a strict one. ∎

Furthermore, in a κ-ary regular category with its κ-regular topology, for monic cones $Q\colon x \Rightarrow U$ and $R\colon y \Rightarrow U$, we have $Q \preceq R$ if and only if $Q \le R$. Thus, in this case a strict κ-ary congruence on X consists of

(i) Monic spans $x_1 \leftarrow \Phi(x_1, x_2) \to x_2$.
(ii) Each diagonal span $x \leftarrow x \to x$ factors through $x \leftarrow \Phi(x, x) \to x$.
(iii) Each $x_1 \leftarrow \Phi(x_1, x_2) \to x_2$ is (isomorphic to) the reversal of $x_2 \leftarrow \Phi(x_2, x_1) \to x_1$.
(iv) Each $x_1 \leftarrow \Phi(x_1, x_2) \times_{x_2} \Phi(x_2, x_3) \to x_3$ factors through $x_1 \leftarrow \Phi(x_1, x_3) \to x_3$.

When X is a singleton, this is just an *equivalence relation* in $\mathbf{C}$ in the usual sense.

5.15. Theorem. *For a category* $\mathbf{C}$*, the following are equivalent.*
(i) $\mathbf{C}$ *is a subcanonical* κ*-ary site in which any* κ*-ary congruence is a kernel of some covering cocone.*
(ii) $\mathbf{C}$ *has finite limits, and every strict* κ*-ary congruence is a strict kernel of some universally effective-epic cocone.*
(iii) $\mathbf{C}$ *is* κ*-ary regular, and every strict* κ*-ary congruence is a strict kernel of some* κ*-ary cocone.*
(iv) $\mathbf{C}$ *is (Barr-)exact in the ordinary sense (a.k.a. effective regular), and has disjoint and universal coproducts of* κ*-small families of objects.*
Moreover, in (i) the topology is automatically the κ*-canonical one.*

Proof. By Theorem. 5.9(ii), condition (i) implies $\mathbf{C}$ is κ-ary regular, hence has finite limits. Lemma. 5.6 then implies the rest of (ii). Theorem. 5.9(iv) immediately implies (ii)$\Leftrightarrow$(iii). In this case, by Corollary. 5.14 any κ-ary congruence is equivalent to a strict one, so since equivalent congruences are kernels of the same cocones, (i) holds.

Now we assume (iii) and prove (iv). Since a singleton strict congruence is an equivalence relation, (iii) implies that $\mathbf{C}$ is exact in the ordinary sense. Moreover, for any κ-ary family of objects X, the congruence Δ_X from Lemma. 5.4 is equivalent to a strict one, and this strict congruence is a kernel of a universally effective-epic cocone just when X has a disjoint and universal coproduct. This shows (iii)$\Rightarrow$(iv).

Conversely, it is well-known that (iv) implies Theorem. 5.9(v), so that $\mathbf{C}$ is κ-ary regular. Moreover, if $\mathbf{C}$ satisfies (iv), then any strict κ-ary congruence Φ on X gives rise to an internal equivalence relation on $\sum_{x\in X} x$ by taking coproducts of the $\Phi(x_1, x_2)$. The quotient of this equivalence relation then admits a universally effective-epic cocone from X of which Φ is the strict kernel. Thus, (iv)$\Rightarrow$(iii). ∎

5.16. Definition. *We say that a category* $\mathbf{C}$ *is* ***κ-ary exact****, or a* ***κ-ary pretopos****, if it satisfies the equivalent conditions of Theorem. 5.15.*

As before, Theorem. 5.15(iv) implies that κ-ary exactness generalizes a number of more common definitions.

- $\mathbf{C}$ is $\{1\}$-ary exact iff it is exact in the usual sense.
- $\mathbf{C}$ is $\{0, 1\}$-ary exact iff it is exact and has a strict initial object.
- $\mathbf{C}$ is ω-ary exact iff it is a pretopos.

- **C** is ω_1-ary exact iff it is a σ-pretopos.
- **C** is $\mathfrak{K}$-ary exact iff it is an infinitary-pretopos[2] (again, minus the occasional requirement of well-poweredness). This means it satisfies all the conditions of Giraud's theorem except possibly the existence of a generating set.

Let $\mathrm{EX}_\kappa \subset \mathrm{REG}_\kappa$ be the full sub-2-category consisting of the κ-ary exact categories.

5.17. Remark. If $\omega \in \kappa$, then any κ-ary exact category admits all κ-small colimits. The proof is well-known: it has κ-small coproducts, and $\omega \in \kappa$ implies that any coequalizer generates an equivalence relation with the same quotient; hence it also has all coequalizers.

Similarly, in this case a functor between κ-ary exact categories is κ-ary regular if and only if it preserves finite limits and κ-small colimits. In particular, a functor between Grothendieck toposes is $\mathfrak{K}$-ary regular if and only if it preserves finite limits and small colimits, i.e. if and only if it is the left-adjoint part of a geometric morphism. Hence $\mathrm{EX}_\mathfrak{K}$ contains, as a full sub-2-category, the opposite of the usual 2-category of Grothendieck toposes.

5.18. Remark. We call a category **locally κ-ary exact** if it is locally κ-ary regular and every κ-ary strict congruence is a kernel. An analogue of Theorem. 5.15 is then true.

6. Framed allegories

We now recall the basic structure used in the "relational" construction of the exact completion of a regular category from [FS90] (see also [Joh02, §A3]), which will be the basis for our construction of the κ-ary exact completion.

6.1. Definition. *A κ-**ary allegory** is a 2-category $\mathcal{A}$ such that*

(i) Each hom-category of $\mathcal{A}$ is a poset with binary meets and κ-ary joins;
(ii) Composition and binary meets preserve κ-ary joins in each variable;
(iii) $\mathcal{A}$ has a contravariant identity-on-objects involution $(-)^\circ \colon \mathcal{A}^{\mathrm{op}} \to \mathcal{A}$; and
(iv) The modular law $\psi\phi \wedge \chi \le (\psi \wedge \chi\phi^\circ)\phi$ holds.

As with categories, all allegories will be be moderate, but not necessarily locally small. The basic example consists of binary relations in a κ-ary regular category. This will be a special case of our more general construction for κ-ary sites in §6.9.

6.2. Remark. Our $\{1\}$-ary allegories are the *allegories* of [FS90, Joh02]. Similarly, our ω-ary allegories and $\mathfrak{K}$-ary allegories are called *distributive* and *locally complete* allegories, respectively, in [FS90]. The *union allegories* and *geometric allegories* of [Joh02] are almost the same, but do not require binary meets to distribute over κ-ary joins (this is automatic for tabular allegories).

[2] Also called an ∞-pretopos, but we avoid that term due to potential confusion with the very different "∞-toposes" of [Lur09b].

Note that $(-)^\circ$ reverses the direction of morphisms, but preserves the ordering on hom-posets. Thus the modular law is equivalent to its dual $\psi\phi \wedge \chi \leq \psi(\phi \wedge \psi^\circ\chi)$. We follow [Joh02] by writing morphisms in an allegory as $\phi\colon x \nrightarrow y$.

A morphism in an allegory is called a **map** if it has a right adjoint. In an allegory, we write a map as $f\colon x \rightsquigarrow y$. For completeness, we reproduce the following basic proofs.

6.3. Lemma.
(i) The right adjoint of a map $f\colon x \rightsquigarrow y$ is necessarily f°.
(ii) The maps in an allegory $\mathcal{A}$ are discretely ordered (i.e. $f \leq g$ implies $f = g$).

Proof. For (i), since $f \dashv \phi$ implies $\phi^\circ \dashv f^\circ$, ϕ° is a map, say g. Then $1_x = 1_x \wedge g^\circ f \leq (f^\circ \wedge g^\circ)f$ by the modular law, while $f(f^\circ \wedge g^\circ) \leq fg^\circ \leq 1_y$. Thus $f \dashv (f^\circ \wedge g^\circ)$ also. So $f^\circ \wedge g^\circ = g^\circ$, i.e. $g^\circ \leq f^\circ$, and by symmetry $f^\circ \leq g^\circ$ and so $f^\circ = g^\circ$.

For (ii), if $f \leq g$ then $f^\circ \leq g^\circ$, so $g \leq gf^\circ f \leq gg^\circ f \leq f$ and hence $f = g$. ∎

We denote by $\mathrm{Map}(\mathcal{A})$ the category whose morphisms are the maps in $\mathcal{A}$.

6.4. Remark. Undoubtedly, the most obscure part of Definition. 6.1 is the modular law. There are similar structures, such as the *bicategories of relations* of [CW87], which replace this with a more familiar-looking "Frobenius" condition. In fact, a bicategory of relations is the same as a "unital and pretabular" allegory [Wal09, T+11]. Unfortunately, the allegories we need are not unital or pretabular, but they do satisfy a weaker condition which also allows us to rephrase the modular law as a Frobenius condition.

On the one hand, notice that if f is a map, then for any ϕ, χ we have

$$\psi f \wedge \chi \leq \psi f \wedge \chi f^\circ f = (\psi \wedge \chi f^\circ)f$$

since precomposition with f, being a right adjoint, preserves meets. Thus, when $\phi = f$ is a map, the modular law is automatic.

On the other hand, if $\phi = f^\circ$ is the right adjoint of a map f, we have

$$(\psi \wedge \chi f)f^\circ \leq (\psi f^\circ f \wedge \chi f)f^\circ = (\psi f^\circ \wedge \chi)ff^\circ \leq \psi f^\circ \wedge \chi \tag{6.5}$$

which is the *reverse* of the modular law. Moreover, asking that (6.5) be an equality is a familiar form of "Frobenius law" for the adjunction $(- \circ f^\circ) \dashv (- \circ f)$.

6.6. Lemma. *Suppose $\mathcal{A}$ is a 2-category satisfying 6.1(i)–(iii) and also the following.*
(a) If a morphism f in $\mathcal{A}$ has a right adjoint, then that adjoint is f°.
(b) For any map f in $\mathcal{A}$, the inequality (6.5) is an equality.
(c) Every morphism $\phi\colon x \nrightarrow y$ in $\mathcal{A}$ can be written as $\phi = \bigvee_u g_u f_u^\circ$, for some κ-ary cocones $F\colon U \Rightarrow x$ and $G\colon U \Rightarrow y$ consisting of maps.
Then $\mathcal{A}$ satisfies the modular law, hence is a κ-ary allegory.

Proof. Given $\phi\colon x \nrightarrow y$, write $\phi = \bigvee_u g_u f_u^\circ$ as in (c). Then for any ψ, χ we have

$$\begin{aligned}\psi\phi \wedge \chi = \psi\left(\bigvee_u g_u f_u^\circ\right) \wedge \chi = \bigvee_u \left(\psi g_u f_u^\circ \wedge \chi\right)\\ = \bigvee_u \left(\psi g_u \wedge \chi f_u\right) f_u^\circ \le \bigvee_u \left(\psi \wedge \chi f_u g_u^\circ\right) g_u f_u^\circ \le \bigvee_{u,u'} \left(\psi \wedge \chi f_u g_u^\circ\right) g_{u'} f_{u'}^\circ\\ = \left(\psi \wedge \chi\left(\bigvee_u f_u g_u^\circ\right)\right)\left(\bigvee_{u'} g_{u'} f_{u'}^\circ\right) = (\psi \wedge \chi\phi^\circ)\phi\end{aligned}$$

which is the modular law for ϕ. ∎

As observed in [Joh02], it is technically even possible to omit $(-)^\circ$ from the structure, since it is determined by Lemma. 6.6(a) and (c). But it seems difficult to ensure that an operation defined in this way is well-defined and functorial, and in all naturally-occurring examples the involution is easy to define directly. However, this observation does imply that a 2-functor between κ-ary allegories of this sort which preserves local κ-ary joins must automatically preserve the involution.

6.7. Definition. *A **κ-ary allegory functor** is a 2-functor preserving the involution and the binary meets and κ-ary joins in the hom-posets. An **allegory transformation** is an* oplax *natural transformation (i.e. we have $\alpha_y \circ F(\phi) \le G(\phi) \circ \alpha_x$ for $\phi\colon x \nrightarrow y$) whose components α_x are all maps.*

Since maps in an allegory are discretely ordered, an allegory transformation is strictly natural on maps ($\alpha_y \circ F(f) = G(f) \circ \alpha_x$). Similarly, there are no "modifications" between allegory transformations. We write ALL_κ for the 2-category of κ-ary allegories.

The central classical theorem about allegories is that the "binary relations" construction induces an equivalence from REG_κ to the 2-category of "unital and tabular" κ-ary allegories. One can then identify those allegories that correspond to exact categories and construct the exact completion in the world of allegories. We aim to proceed analogously, but for this we need a class of allegories which correspond to κ-ary sites in the same way that unital tabular ones correspond to regular categories. This requires generalizing the allegory concept slightly (essentially to allow for non-subcanonical topologies; see Proposition. 7.9).

6.8. Definition. *A **framed κ-ary allegory** $\mathbb{A}$ consists of a κ-ary allegory $\mathcal{A}$, a category* $\mathrm{TMap}(\mathbb{A})$*, and a bijective-on-objects functor $J\colon \mathrm{TMap}(\mathbb{A}) \to \mathrm{Map}(\mathcal{A})$.*

A framed allegory can be thought of as:

- a proarrow equipment [Woo82] whose proarrows form an allegory, or
- a framed bicategory [Shu08] whose horizontal bicategory is an allegory, or
- an $\mathscr{F}$-category [LS12] whose loose morphisms form an allegory, and where every tight morphism has a loose right adjoint.

Following the terminology of [LS12], we call the morphisms of $\mathrm{TMap}(\mathbb{A})$ **tight maps**, and write them as $f\colon x \to y$. We refer to maps in the underlying allegory $\mathcal{A}$ as **loose maps**. The functor J takes each tight map $f\colon x \to y$ to a loose map which we write as $f_\bullet\colon x \rightsquigarrow y$. We also write $f^\bullet = (f_\bullet)^\circ\colon y \nrightarrow x$ for the right adjoint of the underlying loose map of a tight map f. Note that a given loose map may have more than one "tightening".

We call $\mathbb{A}$ **chordate** if $J\colon \mathrm{TMap}(\mathbb{A}) \to \mathrm{Map}(\mathcal{A})$ is an isomorphism (i.e. every loose map has a unique tightening), and **subchordate** if J is faithful (i.e. a loose map has at most one tightening). This does not quite match the terminology of [LS12], where "chordate" means that every *morphism* is tight, but it is "as chordate as a framed allegory can get" since we require all tight morphisms to be maps. Evidently, chordate framed allegories can be identified with ordinary allegories.

A **framed κ-ary allegory functor** $\mathbb{A} \to \mathbb{B}$ consists of a κ-ary allegory functor $\mathcal{A} \to \mathcal{B}$, together with a functor $\mathrm{TMap}(\mathbb{A}) \to \mathrm{TMap}(\mathbb{B})$ making the following square commute:

$$\begin{tikzcd}
\mathrm{TMap}(\mathbb{A}) \arrow[r] \arrow[d, "J"'] & \mathrm{TMap}(\mathbb{B}) \arrow[d, "J"] \\
\mathrm{Map}(\mathcal{A}) \arrow[r] & \mathrm{Map}(\mathcal{B}).
\end{tikzcd}$$

Similarly, a **framed κ-ary allegory transformation** consists of a κ-ary allegory transformation and a compatible natural transformation on tight maps. We obtain a 2-category FALL_κ of framed κ-ary allegories, with a full inclusion $\mathrm{ALL}_\kappa \hookrightarrow \mathrm{FALL}_\kappa$ onto the chordate ones. This has a left 2-adjoint which forgets about the tight maps; after composing it with the inclusion we call this the **chordate reflection**. Similarly, there is a **subchordate reflection** which declares two tight maps to be equal whenever their underlying loose maps are.

6.9. RELATIONS IN κ-ARY SITES To simplify matters, we consider first the framed allegories that correspond to *locally* κ-ary sites, then add conditions to characterize the κ-ary ones. Thus, let $\mathbf{C}$ be a locally κ-ary site. Recall this means that it is weakly κ-ary and has finite *connected* local κ-prelimits. We define a framed allegory $\mathbb{R}\mathsf{el}_\kappa(\mathbf{C})$ as follows.

The objects of $\mathbb{R}\mathsf{el}_\kappa(\mathbf{C})$ are those of $\mathbf{C}$. The hom-poset $\mathbb{R}\mathsf{el}_\kappa(\mathbf{C})(x,y)$ of its underlying allegory is the poset reflection of the preorder of κ-sourced arrays over $\{x,y\}$, with the relation $\preceq$ of local refinement from Definition. 3.7. This has binary meets by the construction $\bigwedge_i$ from §5.1, and κ-ary joins by taking disjoint unions of domains of arrays. Composition is the operation $P \times_v Q$ from §5.1, and the involution is obvious.

Finally, we define $\mathrm{TMap}(\mathbb{R}\mathsf{el}_\kappa(\mathbf{C})) = \mathbf{C}$. For each $f\colon x \to y$ in $\mathbf{C}$, we have a κ-ary array $\{1_x, f\}\colon \{x\} \to \{x,y\}$, which has a right adjoint $\{f, 1_x\}\colon \{x\} \to \{y,x\}$, giving a functor $J\colon \mathbf{C} \to \mathrm{Map}(\mathbb{R}\mathsf{el}_\kappa(\mathbf{C}))$. It is easy to prove that composition and binary meets preserve κ-ary joins and that the modular law holds (perhaps by using Lemma. 6.6).

6.10. EXAMPLES. Suppose $\mathbf{C}$ is a locally κ-ary regular category with its κ-ary regular topology. By Corollary. 5.14, the underlying allegory of $\mathbb{R}\mathsf{el}_\kappa(\mathbf{C})$ is isomorphic to the usual allegory of relations in $\mathbf{C}$. It is well-known that $\mathbf{C}$ can be identified with the category of maps in this allegory, so $\mathbb{R}\mathsf{el}_\kappa(\mathbf{C})$ is chordate (see also Proposition. 7.9).

On the other hand, if $\mathbf{C}$ is a small $\mathfrak{K}$-ary site, the underlying allegory of $\mathbb{R}\mathsf{el}_{\mathfrak{K}}(\mathbf{C})$ is the bicategory used in [Wal82] to construct $\mathrm{Sh}(\mathbf{C})$. Finally, if $\mathbf{C}$ has pullbacks and a trivial unary topology, the underlying allegory of $\mathbb{R}\mathsf{el}_{\{1\}}(\mathbf{C})$ is the allegory used in [Car95, §7] and [Joh02, A3.3.8] to construct the regular completion of $\mathbf{C}$.

We now aim to isolate the essential properties of $\mathbb{R}\mathsf{el}_\kappa(\mathbf{C})$. Recall the notions of matrix, array, and sparse array from §2. Of course, arrays in an allegory inherit a pointwise ordering. Moreover, if $\Phi\colon X \overline{\looparrowright} Y$ is a matrix in a κ-ary allegory such that each set ϕ_{xy} is κ-small, then it has a join $\bigvee \Phi$ which is an array $X \overline{\looparrowright} Y$. In particular, if $\Phi\colon X \overline{\looparrowright} Y$ and $\Psi\colon Y \overline{\looparrowright} Z$ are arrays in a κ-ary allegory and Y is κ-ary, then the composite matrix $\Psi\Phi$ satisfies this hypothesis, hence has a join $\bigvee(\Psi\Phi)$. Similarly, $1_X\colon X \overline{\looparrowright} X$ also always satisfies this condition when X is κ-ary.

We first observe that we can recover the topology of $\mathbf{C}$ from $\mathbb{R}\mathsf{el}_\kappa(\mathbf{C})$.

6.11. LEMMA. *Let $P\colon V \Rightarrow u$ be a κ-ary cocone in a locally κ-ary site $\mathbf{C}$. Then P is a covering family if and only if $\bigvee(P_\bullet P^\bullet) = 1_u$ in $\mathbb{R}\mathsf{el}_\kappa(\mathbf{C})$.*

PROOF. The inequality $\bigvee(P_\bullet P^\bullet) \le 1_u$ is always true since $P_\bullet$ consists of maps. By definition of $\mathbb{R}\mathsf{el}_\kappa(\mathbf{C})$, the other inequality $1_u \le \bigvee(P_\bullet P^\bullet)$ asserts that u admits a covering family factoring through P, which is equivalent to P being covering. ∎

6.12. DEFINITION. *A κ-ary framed allegory $\mathbb{A}$ is* ***weakly κ-tabular*** *if*

(i) Every morphism ϕ can be written as $\phi = \bigvee(F_\bullet G^\bullet)$, where F and G are κ-ary cocones of tight maps.

(ii) If $f, g\colon x \rightrightarrows y$ are parallel tight maps such that $f_\bullet = g_\bullet$, then there exists a κ-ary cocone $P\colon U \Rightarrow x$ of tight maps such that $fP = gP$ and $\bigvee(P_\bullet P^\bullet) = 1_x$.

6.13. PROPOSITION. *$\mathbb{R}\mathsf{el}_\kappa(\mathbf{C})$ is weakly κ-tabular, for any locally κ-ary site $\mathbf{C}$.*

PROOF. By definition, a morphism $\phi\colon x \looparrowright y$ in $\mathbb{R}\mathsf{el}_\kappa(\mathbf{C})$ is an array $H\colon Z \Rightarrow \{x, y\}$. It is easy to see that then $\phi = \bigvee((H|_y)_\bullet (H|_x)^\bullet)$, showing (i).

For (ii), if we have $f, g\colon x \rightrightarrows y$ with $f_\bullet = g_\bullet$, then $f_\bullet \le g_\bullet$, and so by definition $\{1_x, f\} \preceq \{1_x, g\}$. Thus, we have a covering family $P\colon V \Rightarrow x$ and a cocone $H\colon V \Rightarrow x$ such that $1_x P = 1_x H$ and $fP = gH$. Hence $H = P$ and thus $fP = gP$. ∎

We refer to F and G as in (i) as a **weak κ-tabulation** of ϕ. Note that (i) implies that any array of morphisms $\Phi\colon U \overline{\looparrowright} V$ can be written as $\Phi = \bigvee(F_\bullet G^\bullet)$, where F and G are functional arrays of tight maps. If U and V are κ-ary, then the common source of F and G will also be κ-ary.

We now aim to prove that any weakly κ-tabular $\mathbb{A}$ is of the form $\mathbb{R}\mathsf{el}_\kappa(\mathbf{C})$. Lemma. 6.11 suggests the following definition.

6.14. DEFINITION. *A κ-ary cocone $P\colon V \Rightarrow u$ of tight maps in a κ-ary framed allegory is **covering** if $\bigvee(P_\bullet P^\bullet) = 1_u$.*

In the categorified context of [Str81a], such families were called *cauchy dense*. The following fact is useful for recognizing covering families.

6.15. LEMMA. *Suppose $\Phi\colon x \overline{\looparrowright} V$ is a κ-ary cone in a κ-ary allegory such that $\Phi = \bigvee(FG^\circ)$, where $G\colon W \Rightarrow x$ is a κ-ary cocone of maps and $F\colon W \Rightarrow V$ is a functional array of maps. Then $1_x \le \bigvee(\Phi^\circ\Phi)$ if and only if $1_x = \bigvee(GG^\circ)$.*

PROOF. If $1_x = \bigvee(GG^\circ)$, then

$$1_x = \bigvee(GG^\circ) = \bigvee_w \left(g_w g_w^\circ\right) \le \bigvee_w \left(g_w f_w^\circ f_w g_w^\circ\right) \le \bigvee_{\substack{w,w'\\ f(w)=f(w')}} \left(g_{w'} f_{w'}^\circ f_w g_w^\circ\right)$$
$$= \bigvee_v \left(\bigvee_{f(w')=v} (g_{w'} f_{w'}^\circ)\right)\left(\bigvee_{f(w)=v} (f_w g_w^\circ)\right) = \bigvee_v (\phi_v^\circ \phi_v) = \bigvee(\Phi^\circ\Phi).$$

Conversely, suppose $1_x \le \bigvee(\Phi^\circ\Phi)$. Then using the modular law, we have

$$\begin{aligned}
1_x = 1_x \wedge \bigvee(\Phi^\circ\Phi) &= \bigvee_v \left(1_x \wedge \phi_v^\circ \phi_v\right) \\
&= \bigvee_v \left(1_x \wedge \left(\bigvee_{f(w')=v} (g_{w'} f_{w'}^\circ)\right)\left(\bigvee_{f(w)=v} (f_w g_w^\circ)\right)\right) \\
&= \bigvee_{\substack{w,w'\\ f(w)=f(w')}} \left(1_x \wedge g_{w'} f_{w'}^\circ f_w g_w^\circ\right) \\
&\le \bigvee_{\substack{w,w'\\ f(w)=f(w')}} \left(g_w \wedge g_{w'} f_{w'}^\circ f_w\right) g_w^\circ \le \bigvee_w \left(g_w g_w^\circ\right) = \bigvee(GG^\circ). \quad \blacksquare
\end{aligned}$$

6.16. COROLLARY. *If $\Phi\colon X \overline{\looparrowright} V$ is a κ-targeted array in a κ-ary framed allegory and $\Phi = \bigvee(F_\bullet G^\bullet)$, with F and G κ-ary functional arrays of tight maps, then $\bigvee 1_X \le \bigvee(\Phi^\circ\Phi)$ if and only if G is a covering family.* $\blacksquare$

6.17. THEOREM. *For any weakly κ-tabular $\mathbb{A}$, the covering families as defined above form a weakly κ-ary topology on* $\mathrm{TMap}(\mathbb{A})$.

PROOF. Firstly, it is clear that $\{1_u\colon u \to u\}$ is covering.

Secondly, if $P\colon V \Rightarrow u$ is covering, and for each $v \in V$ we have a covering family $Q_v\colon W_v \Rightarrow v$, let $Q\colon W \Rightarrow V$ denote $\bigsqcup Q_v$. Then $\bigvee(Q_\bullet Q^\bullet) = \bigvee 1_V$, so we have

$$1_u = \bigvee(P_\bullet P^\bullet) = \bigvee\left(P_\bullet \circ \bigvee(Q_\bullet Q^\bullet) \circ P^\bullet\right) = \bigvee\left((PQ)_\bullet \circ (PQ)^\bullet\right)$$

so PQ is also covering.

Thirdly, if $P\colon V \Rightarrow u$ is covering and $P \le Q$, say $P = QF$ for some functional array F, then

$$1_u = \bigvee (P_\bullet P^\bullet) \le \bigvee (P_\bullet F_\bullet F^\bullet P^\bullet) = \bigvee (Q_\bullet Q^\bullet)$$

since $F_\bullet$ consists of maps. The reverse inclusion is always true, so Q is also covering.

Fourthly, let $P\colon V \Rightarrow u$ be a covering family and $f\colon x \to u$ a morphism in $\mathbf{C}$; thus f is a tight map and P a cocone of tight maps with $\bigvee(P_\bullet P^\bullet) = 1_u$. Then by weak κ-tabularity, we can write $P^\bullet f_\bullet = \bigvee(F_\bullet G^\bullet)$ for $G\colon Y \Rightarrow x$ a κ-ary cocone of tight maps and $F\colon Y \Rightarrow V$ a functional array of tight maps. Then we have

$$1_x \le f^\bullet f_\bullet = \bigvee (f^\bullet P_\bullet P^\bullet f_\bullet) = \bigvee \Big((P^\bullet f_\bullet)^\circ (P^\bullet f_\bullet) \Big)$$

so by applying Corollary. 6.16 to $P^\bullet f_\bullet$, we conclude that G is covering.

Now the assumption $P^\bullet f_\bullet = \bigvee(F_\bullet G^\bullet)$ implies, in particular, that for any $y \in Y$ we have $(f_y)_\bullet (g_y)^\bullet \le (p_{f(y)})^\bullet f_\bullet$. Since $(p_{f(y)})_\bullet \dashv (p_{f(y)})^\bullet$ and $(g_y)_\bullet \dashv (g_y)^\bullet$, by the mates correspondence for adjunctions this is equivalent to $f_\bullet (g_y)_\bullet \le (p_{f(y)})_\bullet (f_y)_\bullet$. But since maps are discretely ordered, this is equivalent to $f_\bullet (g_y)_\bullet = (p_{f(y)})_\bullet (f_y)_\bullet$. As this holds for all $y \in Y$, we have $f_\bullet G_\bullet = P_\bullet F_\bullet$.

Therefore, by Definition. 6.12(ii) there is a covering family $Q\colon Z \Rightarrow Y$ with $fGQ = PFQ$, hence $f(GQ) \le P$. We have already proven that covering families compose, so GQ is covering. ∎

Henceforth, we will always consider $\mathrm{TMap}(\mathbb{A})$ as a weakly κ-ary site with this topology. Next, we characterize local κ-prelimits in $\mathrm{TMap}(\mathbb{A})$ in terms of $\mathbb{A}$, following [FS90, §2.14].

6.18. Lemma. *In a weakly κ-tabular $\mathbb{A}$, a commuting diagram of arrays of tight maps*

$$\begin{tikzcd} U \arrow[r, Rightarrow, "A"] \arrow[d, Rightarrow, "B"'] & x \arrow[d, "f"] \\ y \arrow[r, "g"'] & z \end{tikzcd}$$

such that $g^\bullet f_\bullet = \bigvee(B_\bullet A^\bullet)$ is a local κ-pre-pullback in $\mathrm{TMap}(\mathbb{A})$*. Therefore,* $\mathrm{TMap}(\mathbb{A})$ *has local κ-pre-pullbacks.*

Note that the given condition holds in $\mathbb{R}\mathsf{el}_\kappa(\mathbf{C})$ for any local κ-pre-pullback in $\mathbf{C}$.

Proof. Suppose $h\colon v \to x$ and $k\colon v \to y$ with $fh = gk$; then

$$\bigvee (k^\bullet B_\bullet A^\bullet h_\bullet) = k^\bullet g^\bullet f_\bullet h_\bullet = h^\bullet f^\bullet f_\bullet h_\bullet \ge 1_v.$$

Define $\Theta = A^\bullet h_\bullet \wedge B^\bullet k_\bullet \colon v \;\overline{\looparrowright}\; U$. Since $\mathbb{A}$ is weakly κ-tabular, we can find a κ-ary cocone $P\colon W \Rightarrow v$ and a functional array $S\colon W \Rightarrow U$ such that $\bigvee(S_\bullet P^\bullet) = \Theta$. Using twice the modular law and the fact that meets distribute over joins, we have

$$\begin{aligned}
\bigvee(\Theta^\circ\Theta) &= \bigvee\left(\Theta^\circ(A^\bullet h_\bullet \wedge \Theta)\right)\\
&\geq \left(\bigvee(\Theta^\circ A^\bullet h_\bullet)\right) \wedge 1_v\\
&= \left(\bigvee((h^\bullet A_\bullet \wedge k^\bullet B_\bullet)A^\bullet h_\bullet)\right) \wedge 1_v\\
&\geq 1_v \wedge \left(\bigvee(k^\bullet B_\bullet A^\bullet h_\bullet)\right) \wedge 1_v\\
&= 1_v
\end{aligned}$$

(using the calculation above in the last step). Thus, by Corollary. 6.16, P is covering. Now by definition of S and P, we have

$$\bigvee(A_\bullet S_\bullet P^\bullet) = \bigvee(A_\bullet(A^\bullet h_\bullet \wedge B^\bullet k_\bullet)) \leq \bigvee(A_\bullet A^\bullet h_\bullet) \leq h_\bullet$$

and therefore (since maps are discretely ordered) $A_\bullet S_\bullet = h_\bullet P_\bullet$. Similarly, $B_\bullet S_\bullet = k_\bullet P_\bullet$. Thus, by the second half of weak κ-tabularity, we can find a further covering family Q such that $ASQ = hPQ$ and $BSQ = kPQ$. Since PQ is covering and SQ is functional, this gives the desired factorization.

Finally, to show that $\mathrm{TMap}(\mathbb{A})$ has local κ-pre-pullbacks, it suffices to show that for any f, g there exist A, B as above. By the first half of weak κ-tabularity, we can find C, D with domain V, say, such that $g^\bullet f_\bullet = \bigvee(D_\bullet C^\bullet)$. Then $(d_v)_\bullet(c_v)^\bullet \leq g^\bullet f_\bullet$ for any v, whence $g_\bullet(d_v)_\bullet \leq f_\bullet(c_v)_\bullet$, i.e. $g_\bullet D_\bullet \leq f_\bullet C_\bullet$. Since maps are discretely ordered, $g_\bullet D_\bullet = f_\bullet C_\bullet$, so by the second half of weak κ-tabularity we can find a covering family $R\colon W \Rightarrow V$ such that $gDR = fCR$. And since $\bigvee(R_\bullet R^\bullet) = 1$, we have $g^\bullet f_\bullet = \bigvee(D_\bullet C^\bullet) = \bigvee(D_\bullet R_\bullet R^\bullet C^\bullet)$, so defining $A = CR$ and $B = DR$ suffices. ■

6.19. Lemma. *If $\phi\colon x \looparrowright x$ in a κ-ary allegory satisfies $\phi \leq 1_x$, and $\phi = \bigvee(FG^\circ)$ is a weak κ-tabulation of ϕ, then $F = G$.*

Proof. Denote by U the domain of F, G. Then for any $u \in U$ we have

$$f_u \leq (f_u g_u^\circ g_u) \leq \bigvee(FG^\circ g_u) = \phi g_u \leq g_u$$

and consequently $f_u = g_u$ since maps are discretely ordered. ■

6.20. Lemma. *If $f, g\colon x \rightrightarrows y$ are tight maps in a weakly κ-tabular $\mathbb{A}$, and $E\colon U \Rightarrow x$ is a κ-ary cocone of tight maps such that $fE = gE$ and*

$$\bigvee\left(E_\bullet E^\bullet\right) = 1_x \wedge (f^\bullet \wedge g^\bullet)(f_\bullet \wedge g_\bullet),$$

then E is a local κ-pre-equalizer of f and g in $\mathrm{TMap}(\mathbb{A})$*. Therefore,* $\mathrm{TMap}(\mathbb{A})$ *has local κ-pre-equalizers.*

Proof. Suppose $fh = gh$ for a tight map $h\colon v \to x$. By weak κ-tabularity, we have a κ-ary cocone $P\colon W \Rightarrow v$ and a functional array $S\colon W \Rightarrow U$ such that $\bigvee(S_\bullet P^\bullet) = E^\bullet h_\bullet$. Then using the modular law frequently, we have

$$\begin{aligned}
\bigvee(h^\bullet E_\bullet E^\bullet h_\bullet) &= h^\bullet \Big(\bigvee(E_\bullet E^\bullet)\Big)\, h_\bullet \\
&= h^\bullet\Big(1_x \wedge \big(f^\bullet \wedge g^\bullet\big)\big(f_\bullet \wedge g_\bullet\big)\Big)h_\bullet \\
&\ge h^\bullet\Big(h_\bullet h^\bullet \wedge \big(f^\bullet \wedge g^\bullet\big)\big(f_\bullet \wedge g_\bullet\big)\Big)h_\bullet \\
&\ge h^\bullet\Big(h_\bullet \wedge \big(f^\bullet \wedge g^\bullet\big)\big(f_\bullet \wedge g_\bullet\big)h_\bullet\Big) \\
&\ge 1_v \wedge h^\bullet\big(f^\bullet \wedge g^\bullet\big)\big(f_\bullet \wedge g_\bullet\big)h_\bullet \\
&\ge 1_v \wedge h^\bullet\big(h_\bullet h^\bullet f^\bullet \wedge g^\bullet\big)\big(f_\bullet h_\bullet h^\bullet \wedge g_\bullet\big)h_\bullet \\
&\ge 1_v \wedge \big(h^\bullet f^\bullet \wedge h^\bullet g^\bullet\big)\big(f_\bullet h_\bullet \wedge g_\bullet h_\bullet\big) \\
&= 1_v \wedge \big(h^\bullet f^\bullet\big)\big(f_\bullet h_\bullet\big) \\
&= 1_v.
\end{aligned}$$

Therefore, by Corollary. 6.16, P is a covering family.

Now, $\bigvee(S_\bullet P^\bullet) = E^\bullet h_\bullet$ implies $(s_w)_\bullet(p_w)^\bullet \le (e_{s(w)})^\bullet h_\bullet$ for all w, hence $E_\bullet S_\bullet \le h_\bullet P_\bullet$. Since maps are discretely ordered, we have $E_\bullet S_\bullet = h_\bullet P_\bullet$, and so by the second half of weak κ-tabularity we have a further covering family Q such that $ESQ = hPQ$. Since PQ is covering and SQ is functional, this gives the desired factorization.

Finally, to show that $\mathrm{TMap}(\mathbb{A})$ has local κ-pre-equalizers, it suffices to show that for any f, g there exists an E as above. By the first half of weak κ-tabularity, we can write

$$\bigvee\big(A_\bullet B^\bullet\big) = 1_x \wedge (f^\bullet \wedge g^\bullet)(f_\bullet \wedge g_\bullet)$$

for some A and B, but then by Lemma. 6.19, we must have $A = B$. Then for any u, we have $(a_u)_\bullet(a_u)^\bullet \le f^\bullet g_\bullet$, hence $f_\bullet(a_u)_\bullet \le g_\bullet(a_u)_\bullet$, whence $f_\bullet A_\bullet = g_\bullet A_\bullet$ since maps are discretely ordered. As usual, we can then find a covering family R such that $fAR = gAR$, and also $\bigvee(A_\bullet A^\bullet) = \bigvee(A_\bullet R_\bullet R^\bullet A^\bullet)$ since R is covering; thus defining $E = AR$ suffices. ∎

Now we can prove the main theorem relating framed allegories to sites.

6.21. Theorem. *A κ-ary framed allegory is of the form $\mathbb{R}\mathsf{el}_\kappa(\mathbf{C})$ for a locally κ-ary site $\mathbf{C}$ if and only if it is weakly κ-tabular.*

Proof. Only "if" remains to be proven. Thus, suppose $\mathbb{A}$ is weakly κ-tabular, and define $\mathbf{C} = \mathrm{TMap}(\mathbb{A})$. By Lemmas 6.17, 6.18, and 6.20 and Proposition. 3.15, $\mathbf{C}$ is a locally κ-ary site, so it remains to show $\mathbb{A} \cong \mathbb{R}\mathsf{el}_\kappa(\mathbf{C})$.

Define $\mathbf{f}\colon \mathbb{R}\mathsf{el}_\kappa(\mathbf{C}) \to \mathbb{A}$ to be the identity on objects and tight maps, and to take a κ-sourced array $P\colon U \Rightarrow \{x, y\}$ (regarded as a morphism $x \nrightarrow y$ in $\mathbb{R}\mathsf{el}_\kappa(\mathbf{C})$) to the morphism $\bigvee\left((P|_y)_\bullet(P|_x)^\bullet\right) : x \nrightarrow y$ in $\mathbb{A}$. Our construction of local κ-pre-pullbacks in

TMap($\mathbb{A}$) implies easily that this operation preserves composition, and it certainly preserves identities. Moreover, if $Q\colon V \Rightarrow U$ is a covering family, then by definition $\bigvee(Q_\bullet Q^\bullet)$ is the identity, so $\mathbf{f}(P) = \mathbf{f}(PQ)$. This implies that $\mathbf{f}$ preserves the ordering on hom-posets, hence defines a 2-functor.

Now we claim that $\mathbf{f}(P) \le \mathbf{f}(Q)$ implies $P \preceq Q$ (i.e. $P \le Q$ in $\mathbb{R}\mathsf{el}_\kappa(\mathbf{C})$). Write U for the domain of P and V for the domain of Q, and define $F = P|_x$, $G = P|_y$, $H = Q|_x$, and $K = Q|_y$; thus $\mathbf{f}(P) \le \mathbf{f}(Q)$ means $\bigvee(G_\bullet F^\bullet) \le \bigvee(K_\bullet H^\bullet)$. Fix some $u \in U$; then by assumption $(g_u)_\bullet (f_u)^\bullet \le \bigvee(K_\bullet H^\bullet)$, hence

$$1_u \le \bigvee \Big((g_u)^\bullet K_\bullet H^\bullet (f_u)_\bullet \Big).$$

Let $\Phi\colon u \looparrowright V$ denote the cone $H^\bullet(f_u)_\bullet \wedge K^\bullet(g_u)_\bullet$. By weak κ-tabularity, we can write $\Phi = \bigvee(S_\bullet R^\bullet)$ for a κ-ary cocone $R\colon W \Rightarrow u$ and a functional array $S\colon W \Rightarrow V$. Moreover, using the modular law twice, we have

$$\begin{aligned}
\bigvee(\Phi^\circ\Phi) &= \bigvee \Big(\Phi^\circ\big(H^\bullet(f_u)_\bullet \wedge \Phi\big)\Big) \\
&\ge \Big(\bigvee(\Phi^\circ H^\bullet(f_u)_\bullet)\Big) \wedge 1_u \\
&= \Big(\bigvee \big((f_u)^\bullet H_\bullet \wedge (g_u)^\bullet K_\bullet\big) H^\bullet(f_u)_\bullet\Big) \wedge 1_u \\
&\ge 1_u \wedge \Big(\bigvee \big((g_u)^\bullet K_\bullet H^\bullet (f_u)_\bullet\big)\Big) \wedge 1_u \\
&= 1_u.
\end{aligned}$$

Thus by Corollary. 6.16, R is covering. By definition, we also have for any $w \in W$

$$(s_w)_\bullet (r_w)^\bullet \le (h_{s(w)})^\bullet (f_{r(w)})_\bullet.$$

Hence $H_\bullet S_\bullet \le F_\bullet R_\bullet$, whence $H_\bullet S_\bullet = F_\bullet R_\bullet$ as maps are discretely ordered. As usual, by the second half of weak κ-tabularity we can then find a further covering family R' such that $HSR' = FRR'$. Repeating this process for K and G, we obtain a covering family R'' such that $KSR'' = GRR''$. By passing to a common refinement of R' and R'', we can conclude that $P \preceq Q$.

Thus, $\mathbf{f}$ is an embedding on hom-posets. But by the first half of weak κ-tabularity, it is also surjective on hom-posets; hence (since it is bijective on objects) it is an isomorphism of 2-categories. Since it evidently preserves the involution and the finite meets and κ-ary joins in hom-posets, this makes it an isomorphism of κ-ary framed allegories. ∎

We can also make this functorial. Recall the notion of *pre-morphism of sites* from Remark. 4.18. Since a pre-morphism of sites preserves $\preceq$ and all the prelimit constructions used in defining $\mathbb{R}\mathsf{el}_\kappa$, it induces a κ-ary framed allegory functor $\mathbb{R}\mathsf{el}_\kappa(\mathbf{f})\colon \mathbb{R}\mathsf{el}_\kappa(\mathbf{C}) \to \mathbb{R}\mathsf{el}_\kappa(\mathbf{D})$. This operation easily extends to a 2-functor $\mathbb{R}\mathsf{el}_\kappa\colon \mathrm{LSITE}_\kappa \to \mathrm{FALL}_\kappa^{\mathrm{wt}}$, where $\mathrm{FALL}_\kappa^{\mathrm{wt}}$ denotes the full sub-2-category of FALL_κ determined by the weakly κ-tabular framed κ-ary allegories.

6.22. THEOREM. *The 2-functor* $\mathbb{R}\mathsf{el}_\kappa\colon \mathrm{LSITE}_\kappa \to \mathrm{FALL}_\kappa^{\mathrm{wt}}$ *is a 2-equivalence.*

PROOF. Let $\mathbf{C}$ and $\mathbf{D}$ be locally κ-ary sites; we must show that

$$\mathbb{R}\mathsf{el}_\kappa\colon \mathrm{LSITE}_\kappa(\mathbf{C},\mathbf{D}) \to \mathrm{FALL}_\kappa(\mathbb{R}\mathsf{el}_\kappa(\mathbf{C}),\mathbb{R}\mathsf{el}_\kappa(\mathbf{D})) \tag{6.23}$$

is an isomorphism of categories. Firstly, since a natural transformation in LSITE_κ is literally still present as the tight part of its induced framed allegory transformation, (6.23) is faithful. Secondly, since a framed allegory transformation is determined uniquely by its tight part, which is a natural transformation and hence automatically a 2-cell in LSITE_κ (since we have imposed no restrictions on these), (6.23) is full. Thirdly, since a pre-morphism of sites is literally still present as the tight part of its induced framed allegory functor, (6.23) is injective on objects. Thus, it remains to show it is surjective on objects.

Let $\mathbf{f}\colon \mathbb{R}\mathsf{el}_\kappa(\mathbf{C}) \to \mathbb{R}\mathsf{el}_\kappa(\mathbf{D})$ be a κ-ary framed allegory functor. Its underlying tight part is a functor

$$\mathbf{C} = \mathrm{TMap}(\mathbb{R}\mathsf{el}_\kappa(\mathbf{C})) \to \mathrm{TMap}(\mathbb{R}\mathsf{el}_\kappa(\mathbf{D})) = \mathbf{D}.$$

By Remark. 4.18 and Lemmas 6.11, 6.18 and 6.20, this functor is a pre-morphism of sites; thus it remains only to show that the framed allegory functor it induces coincides with $\mathbf{f}$. But every morphism of $\mathbb{R}\mathsf{el}_\kappa(\mathbf{C})$ can be written as a κ-ary join $\bigvee(F_\bullet G^\bullet)$ for some κ-ary cocones F and G in $\mathbf{C}$. Thus, since $\mathbf{f}$ preserves J, $(-)^\circ$, and $\bigvee$, its action on all morphisms is determined by its action on tight maps. ∎

It now remains only to characterize those weakly κ-tabular framed allegories $\mathbb{A}$ for which $\mathrm{TMap}(\mathbb{A})$ is a κ-ary site.

6.24. DEFINITION. *An allegory has* ***local maxima*** *if each hom-poset has a top element.*

6.25. LEMMA. *A κ-sourced array $P\colon W \Rightarrow \{u,v\}$ in a locally κ-ary site $\mathbf{C}$ is a local κ-pre-product of u and v if and only if $P|_u$ and $P|_v$ form a weak κ-tabulation of a top element $u \looparrowright v$ in $\mathbb{R}\mathsf{el}_\kappa(\mathbf{C})$.*

PROOF. Essentially by definition. ∎

6.26. DEFINITION. *A* ***weak κ-unit*** *in a κ-ary allegory $\mathcal{A}$ is a κ-ary family of objects U such that for every object x, there is a cone $\Phi\colon x \overline{\looparrowright} U$ such that $1_x \le \bigvee(\Phi^\circ\Phi)$.*

6.27. LEMMA. *A κ-ary family U of objects in a locally κ-ary site is a local κ-pre-terminal-object if and only if it is a weak κ-unit in $\mathbb{R}\mathsf{el}_\kappa(\mathbf{C})$.*

PROOF. By Corollary. 6.16 and weak κ-tabularity, U is a weak κ-unit in $\mathbb{R}\mathsf{el}_\kappa(\mathbf{C})$ if and only if for every x, there exists a κ-ary cocone $G\colon W \Rightarrow x$ and a functional array $F\colon W \Rightarrow U$ in $\mathbf{C}$ such that $\bigvee(G_\bullet G^\bullet) = 1_x$. But by Lemma. 6.11, $\bigvee(G_\bullet G^\bullet) = 1_x$ just says that G is covering, so this is precisely the definition of when U is a local κ-pre-terminal-object. ∎

Let $\mathrm{FALL}_\kappa^{\mathrm{wt}\times}$ denote the locally full sub-2-category of FALL_κ determined by

- weakly κ-tabular κ-ary framed allegories with local maxima and weak κ-units, and
- κ-ary framed allegory functors that preserve local maxima and weak κ-units.

6.28. Theorem. *The 2-functor $\mathbb{R}\mathsf{el}_\kappa$ is a 2-equivalence from SITE_κ to $\mathrm{FALL}_\kappa^{\mathrm{wt}\times}$.*

Proof. Lemmas 6.25 and 6.27 show that a locally κ-ary site $\mathbf{C}$ is κ-ary exactly when $\mathbb{R}\mathsf{el}_\kappa(\mathbf{C})$ lies in $\mathrm{FALL}_\kappa^{\mathrm{wt}\times}$. Similarly, the morphisms in $\mathrm{FALL}_\kappa^{\mathrm{wt}\times}$ are chosen precisely to be those corresponding to morphisms of sites. ∎

7. The exact completion

We now identify those framed allegories that correspond to κ-ary exact categories, using the notion of *collage of a congruence*, an allegorical analogue of the *quotients* of congruences we used in §5. In this section we consider only "unframed" collages.

7.1. Definition. *A κ**-ary congruence** in a κ-ary allegory $\mathcal{A}$ is a κ-ary family of objects X in $\mathcal{A}$ with an array $\Phi\colon X \overline{\looparrowright} X$ such that $\bigvee 1_X \le \Phi$, $\bigvee(\Phi\Phi) \le \Phi$, and $\Phi^\circ \le \Phi$.*

It follows that the last two inequalities are actually equalities: $\bigvee(\Phi\Phi) = \Phi$ and $\Phi^\circ = \Phi$.

7.2. Example. A κ-ary congruence in $\mathbb{R}\mathsf{el}_\kappa(\mathbf{C})$ is precisely an equivalence class of κ-ary congruences in $\mathbf{C}$ as in Definition. 5.2.

7.3. Lemma. *Let X and Y be κ-ary families of objects in a κ-ary site.*

(i) $\bigvee 1_X\colon X \overline{\looparrowright} X$ is a κ-ary congruence.
(ii) For any κ-ary congruence $\Phi\colon Y \overline{\looparrowright} Y$ and functional array $F\colon X \Rightarrow Y$ of maps, $F^\circ\Phi F\colon X \overline{\looparrowright} X$ is a κ-ary congruence.
(iii) For any finite family $\{\Phi_i\colon X \overline{\looparrowright} X\}_{1\le i\le n}$ of κ-ary congruences, $\bigwedge_i \Phi_i$ is a κ-ary congruence. ∎

This should be compared with Lemma. 5.4. In particular, the kernel of a κ-to-finite array $P\colon X \Rightarrow U$ in a κ-ary site can be constructed as $\bigwedge_{u\in U}(P|_u)^\bullet(P|_u)_\bullet$ in $\mathbb{R}\mathsf{el}_\kappa(\mathbf{C})$.

7.4. Definition. *Let $\Phi\colon X \overline{\looparrowright} X$ be a κ-ary congruence in a κ-ary allegory $\mathcal{A}$. A **collage** of Φ is a lax colimit of Φ regarded as a diagram in the 2-category $\mathcal{A}$.*

Explicitly, a collage of Φ is an object w with a cocone $\Psi\colon X \overline{\looparrowright} w$ such that

(a) $\bigvee(\Psi\Phi) \le \Psi$;
(b) For any other object z and cocone $\Theta\colon X \overline{\looparrowright} z$ such that $\bigvee(\Theta\Phi) \le \Theta$, there is a morphism $\chi\colon w \looparrowright z$ such that $\chi\Psi = \Theta$; and
(c) Given morphisms $\chi, \chi'\colon w \looparrowright z$ such that $\chi\Psi \le \chi'\Psi$, we have $\chi \le \chi'$.

Note that (c) implies uniqueness of χ in (b). Also, since $\bigvee 1_X \le \Phi$, we have $\Theta \le \bigvee(\Theta\Phi)$ for any $\Theta\colon X \overline{\looparrowright} z$; hence $\bigvee(\Theta\Phi) \le \Theta$ is equivalent to $\bigvee(\Theta\Phi) = \Theta$.

7.5. Remark. A unary congruence is precisely a *symmetric monad* in the 2-category $\mathcal{A}$, while its collage is a *Kleisli object*.

The following lemma is essentially a special case of parts of [Str81a, Prop. 2.2].

7.6. Lemma. *If* $\Psi\colon X \overline{\looparrowright} w$ *is a collage of* Φ*, then:*

(i) Each component $\psi_x\colon x \looparrowright w$ *is a map.*
(ii) We have $\Phi = \Psi^\circ\Psi$ *and* $1_w = \bigvee(\Psi\Psi^\circ)$.
(iii) The cone $\Psi^\circ\colon w \overline{\looparrowright} X$ *exhibits* w *as the lax limit of* Φ.
(iv) A morphism $\xi\colon w \looparrowright z$ *is a map if and only if* $\xi\Psi$ *is composed of maps.*

Proof. For any $x \in X$, consider the cocone $\Phi|_x\colon X \overline{\looparrowright} x$. Since Φ is a congruence, we have $\bigvee(\Phi|_x \circ \Phi) \le \Phi|_x$, hence a unique induced morphism $\chi_x\colon w \to x$ such that $\chi_x\psi_{x'} = \phi_{x'x}$ for all x'. In particular, $1_x \le \phi_{xx} = \chi_x\psi_x$. On the other hand, since

$$\psi_x\chi_x\psi_{x'} = \psi_x\phi_{x'x} \le \psi_{x'} = 1_x\psi_{x'}$$

we have $\psi_x\chi_x \le 1_w$. Hence $\psi_x \dashv \chi_x$, so ψ_x is a map and $\chi_x = \psi_x^\circ$; thus (i) holds.

Now the equality $\psi_x^\circ\psi_{x'} = \chi_x\psi_{x'} = \phi_{x'x}$ is exactly the first part of (ii). For the second part, since Ψ is composed of maps, we have $\bigvee(\Psi\Psi^\circ) \le 1_w$. But furthermore, for any x we have $\psi_x \le \psi_x\psi_x^\circ\psi_x$, hence $\psi_x \le \bigvee(\Psi\Psi^\circ)\psi_x$. Since w is a collage, this implies $1_w \le \bigvee(\Psi\Psi^\circ)$; thus (ii) holds.

(iii) is clear since $(-)^\circ$ is a contravariant equivalence, so it remains to prove (iv). "Only if" follows from (i), so let $\xi\colon w \looparrowright z$ be such that each $\xi\psi_x$ is a map. Then

$$\psi_x \le \psi_x\psi_x^\circ\xi^\circ\xi\psi_x \le \xi^\circ\xi\psi_x$$

for all x, hence since Ψ is a collage we have $1_w \le \xi^\circ\xi$. On the other hand, using (ii) we have $\xi\xi^\circ = \bigvee \xi\Psi\Psi^\circ\xi \le 1_z$, so ξ is a map. ■

7.7. Lemma. *A cocone* $\Psi\colon X \overline{\looparrowright} w$ *is a collage of* $\Phi\colon X \overline{\looparrowright} X$ *if and only if* $\Phi = \Psi^\circ\Psi$ *and* $1_w = \bigvee(\Psi\Psi^\circ)$.

Proof. "Only if" is just Lemma. 7.6(ii), so suppose $\Phi = \Psi^\circ\Psi$ and $1_w = \bigvee(\Psi\Psi^\circ)$. Then $1_x \le \phi_{xx} \le \psi_x^\circ\psi_x$, while $\psi_x\psi_x^\circ \le 1_w$, so Ψ is composed of maps. Thus, $\phi_{x'x} \le \psi_{x'}^\circ\psi_x$ implies $\psi_{x'}\phi_{x'x} \le \psi_x$, i.e. $\bigvee(\Psi\Phi) \le \Psi$. Now for any $\Theta\colon X \overline{\looparrowright} z$ with $\bigvee(\Theta\Phi) = \Theta$, let $\chi = \bigvee(\Theta\Psi^\circ)\colon w \looparrowright z$; then

$$\chi\Psi = \bigvee(\Theta\Psi^\circ\Psi) = \bigvee(\Theta\Phi) = \Theta.$$

Finally, given $\chi, \chi'\colon w \looparrowright z$ with $\chi\Psi \le \chi'\Psi$, we have

$$\chi = \bigvee(\chi\Psi\Psi^\circ) \le \bigvee(\chi'\Psi\Psi^\circ) = \chi'.$$

Thus, Ψ is a collage of Φ. ■

In particular, any κ-ary allegory functor preserves collages of κ-ary congruences. We are now ready to characterize properties of $\mathbf{C}$ in terms of $\mathbb{R}\mathsf{el}_\kappa(\mathbf{C})$.

7.8. Proposition. *For a locally κ-ary site $\mathbf{C}$, the following are equivalent.*
(i) All covering families in $\mathbf{C}$ are epic.
(ii) $\mathbb{R}\mathsf{el}_\kappa(\mathbf{C})$ is subchordate.

Proof. Since $\mathbb{R}\mathsf{el}_\kappa(\mathbf{C})$ is weakly κ-tabular, if $f_\bullet = g_\bullet$ then $fP = gP$ for a covering family P. Thus, if covering families are epic, $f = g$. Conversely, if $fP = gP$, then

$$f_\bullet = \bigvee f_\bullet P_\bullet P^\bullet = \bigvee g_\bullet P_\bullet P^\bullet = g_\bullet.$$

Thus, if $\mathbb{R}\mathsf{el}_\kappa(\mathbf{C})$ is subchordate, then $f = g$; hence covering families are epic. ∎

7.9. Proposition. *For a locally κ-ary site $\mathbf{C}$, the following are equivalent.*
(i) $\mathbf{C}$ is subcanonical.
(ii) $\mathbb{R}\mathsf{el}_\kappa(\mathbf{C})$ is chordate.

Proof. Suppose $\mathbf{C}$ is subcanonical. By Proposition. 7.8, $\mathbb{R}\mathsf{el}_\kappa(\mathbf{C})$ is subchordate, so it will suffice to show that every loose map in $\mathbb{R}\mathsf{el}_\kappa(\mathbf{C})$ can be tightened. Let $f\colon x \rightsquigarrow y$ be a loose map, and let $f = \bigvee(G_\bullet P^\bullet)$ be a weak κ-tabulation of it, for cocones $G\colon U \Rightarrow y$ and $P\colon U \Rightarrow x$ in $\mathbf{C}$. Then P is covering by Corollary. 6.16. Define $\Phi = P^\bullet P_\bullet$; then Φ is a congruence, and $P_\bullet$ is a collage of Φ by Lemma. 7.7.

Now $f = \bigvee(G_\bullet P^\bullet)$ implies $(g_u)_\bullet p_u^\bullet \le f$ for all $u \in U$, hence $(g_u)_\bullet \le f(p_u)_\bullet$ and so $G_\bullet = fP_\bullet$ since maps are discretely ordered. Thus, we have

$$\bigvee(G_\bullet\Phi) = \bigvee(G_\bullet P^\bullet P_\bullet) = fP_\bullet = G_\bullet$$

so $G_\bullet$ is the cocone which induces f by the universal property of the collage $P_\bullet$.

Choose weak κ-tabulations to obtain functional arrays A and B such that $\Phi = \bigvee(A_\bullet B^\bullet)$. Then by construction of $\mathbb{R}\mathsf{el}_\kappa(\mathbf{C})$, A and B form a kernel of P in $\mathbf{C}$, and by the above we have $\bigvee(G_\bullet A_\bullet B^\bullet) \le G_\bullet$, hence $G_\bullet A_\bullet = G_\bullet B_\bullet$ since maps are discretely ordered. Since covering families in $\mathbf{C}$ are epic, by Proposition. 7.8 we have $GA = GB$. And since $\mathbf{C}$ is subcanonical, by Lemma. 5.7 P is a colimit of A and B. Thus, there is a unique map $h\colon x \to y$ in $\mathbf{C}$ with $hP = G$. Since f is unique such that $fP_\bullet = G_\bullet$, we have $f = h_\bullet$. Thus, $\mathbb{R}\mathsf{el}_\kappa(\mathbf{C})$ is chordate.

Conversely, suppose $\mathbb{R}\mathsf{el}_\kappa(\mathbf{C})$ is chordate, and let $P\colon U \Rightarrow x$ be a covering family in $\mathbf{C}$. As before, define $\Phi = P^\bullet P_\bullet$ and $\Phi = \bigvee(A_\bullet B^\bullet)$; then A and B are a kernel of P, and $P_\bullet$ is a collage of Φ. It will suffice to show that P is a colimit of A and B, so let $G\colon U \Rightarrow y$ satisfy $GA = GB$. Then $G_\bullet \ge \bigvee(G_\bullet B_\bullet B^\bullet) = \bigvee(G_\bullet A_\bullet B^\bullet) = \bigvee(G_\bullet \Phi)$, so G induces a unique (loose) map $f\colon x \rightsquigarrow y$ such that $fP_\bullet = G_\bullet$. Since $\mathbb{R}\mathsf{el}_\kappa(\mathbf{C})$ is chordate, f admits a unique tightening, which says precisely that G factors uniquely through P. Thus, P is effective-epic, so $\mathbf{C}$ is subcanonical. ∎

7.10. Theorem. *For a κ-ary site $\mathbf{C}$, the following are equivalent.*

(i) $\mathbf{C}$ *is κ-ary exact with its κ-canonical topology.*

(ii) $\mathbb{R}\mathsf{el}_\kappa(\mathbf{C})$ *is chordate and has collages of κ-ary congruences.*

Proof. By Theorem. 5.15(i), condition (i) is equivalent to $\mathbf{C}$ being subcanonical and any κ-ary congruence being a kernel of some covering cocone. By Proposition. 7.9, the former is equivalent to $\mathbb{R}\mathsf{el}_\kappa(\mathbf{C})$ being chordate. And by Lemma. 7.7 and the identification of congruences and kernels in $\mathbf{C}$ and $\mathbb{R}\mathsf{el}_\kappa(\mathbf{C})$, the latter is equivalent to $\mathbb{R}\mathsf{el}_\kappa(\mathbf{C})$ having collages of κ-ary congruences. ■

Thus, to show that EX_κ is reflective in SITE_κ, by Theorem. 6.28 it suffices to show that framed allegories satisfying 7.10(ii) are reflective in $\mathrm{FALL}_\kappa^{\mathrm{wt}\times}$. We know that chordate framed allegories are reflective in FALL_κ, and the chordate reflection clearly takes $\mathrm{FALL}_\kappa^{\mathrm{wt}\times}$ into itself, so it remains to freely add collages. This can be done very explicitly, but we prefer a more abstract approach, for reasons that will become clear in §8.

Let $\mathbf{SUP}_\kappa$ denote the category of moderate **κ-cocomplete posets** (those with κ-ary joins). There is a tensor product on $\mathbf{SUP}_\kappa$ which represents "bilinear" maps: functions that preserve κ-ary joins in each variable. A $\mathbf{SUP}_\kappa$-enriched category is precisely a 2-category whose hom-categories are posets with κ-ary joins that are preserved by composition. In particular, every κ-ary allegory is such. We use the same notation and terminology for morphisms and maps in $\mathbf{SUP}_\kappa$-categories as we do in allegories.

7.11. Remark. Of course, $\mathbf{SUP}_\kappa$ is very large, so this is an exception to our general rule that all categories are moderate. We will only consider $\mathbf{SUP}_\kappa$-enriched categories that are moderate (but not necessarily, of course, locally small).

If we omit the condition $\Phi^\circ \le \Phi$ from the definition of a κ-ary congruence, we obtain a notion which makes sense in any $\mathbf{SUP}_\kappa$-category.

7.12. Definition. *A* ***κ-ary directed congruence*** *in a* $\mathbf{SUP}_\kappa$*-category is a κ-ary family of objects X with an array $\Phi\colon X \mathrel{\overline{\looparrowright}} X$ such that $\bigvee 1_X \le \Phi$ and $\bigvee(\Phi\Phi) \le \Phi$.*

7.13. Remark. In [Wal81, Wal82, BCSW83, CKW87] directed congruences in $\mathcal{A}$ are called *$\mathcal{A}$-categories*, but we prefer a different terminology to minimize confusion with (for instance) $\mathbf{SUP}_\kappa$-categories.

A **collage** of a directed congruence is, as before, its lax colimit; this can be expressed as a certain $\mathbf{SUP}_\kappa$-weighted colimit.

7.14. Lemma. *If $\Phi\colon X \mathrel{\overline{\looparrowright}} X$ is a κ-ary directed congruence in a* $\mathbf{SUP}_\kappa$*-category, then a cocone $\Psi\colon X \mathrel{\overline{\looparrowright}} w$ is a collage of Φ if and only if Ψ is composed of maps and for $\Psi^*\colon w \mathrel{\overline{\looparrowright}} X$ its cone of right adjoints, we have $\Phi = \Psi^*\Psi$ and $1_w = \bigvee(\Psi\Psi^*)$.*

Proof. The proofs of Lemmas 7.6(i) and 7.7 apply with trivial modifications. ■

7.15. Corollary. *Any* $\mathbf{SUP}_\kappa$*-functor preserves collages of directed congruences.* ■

That is, collages of directed congruences are **absolute colimits** for $\mathbf{SUP}_\kappa$ (see [Str83a]).

7.16. Corollary. *If $\Psi\colon X \overline{\looparrowright} w$ is a collage of a directed congruence Φ in a $\mathbf{SUP}_\kappa$-category, then $\Psi^*\colon w \overline{\looparrowright} X$ is a lax limit of Φ.*

Proof. Apply one direction of Lemma. 7.14 in $\mathcal{A}$ and the other direction in $\mathcal{A}^{\mathrm{op}}$. ■

7.17. Corollary. *If u and v are collages of directed congruences $\Phi\colon X \overline{\looparrowright} X$ to $\Theta\colon Y \overline{\looparrowright} Y$, respectively, then $\mathcal{A}(u,v)$ is isomorphic to the poset of arrays $\Psi\colon X \overline{\looparrowright} Y$ such that $\bigvee(\Theta\Psi) \le \Psi$ and $\bigvee(\Psi\Phi) \le \Psi$, with pointwise ordering. If $\mathcal{A}$ is additionally a κ-ary allegory, then the involution $\mathcal{A}(u,v) \to \mathcal{A}(v,u)$ is identified with the induced involution taking arrays $X \overline{\looparrowright} Y$ to arrays $Y \overline{\looparrowright} X$. And if w is the collage of a third directed congruence $\Xi\colon Z \overline{\looparrowright} Z$, then the composite of $\Psi\colon X \overline{\looparrowright} Y$ representing a morphism $u \looparrowright v$ with $\Omega\colon Y \overline{\looparrowright} Z$ representing a morphism $v \looparrowright w$ is represented by $\bigvee(\Omega\Psi)\colon X \overline{\looparrowright} Z$.*

Proof. The first sentence follows because collages are both lax colimits and lax limits, and the second because the involution is functorial. The third follows from the construction of χ in the proof of Lemma. 7.7. ■

Note that the inequalities $\bigvee(\Theta\Psi) \le \Psi$ and $\bigvee(\Psi\Phi) \le \Psi$ are automatically equalities.

We would now like to construct the free cocompletion of a κ-ary allegory with respect to collages of congruences, and general enriched category theory suggests that this should be the closure of $\mathcal{A}$ in $[\mathcal{A}^{\mathrm{op}}, \mathbf{SUP}_\kappa]$ under collages of congruences. In general, such a closure must be constructed by (transfinite) iteration, but in the case of collages it stops after one step, due to the following fact.

7.18. Lemma. *Let X and Y be κ-ary families of objects in a $\mathbf{SUP}_\kappa$-category $\mathcal{A}$, let $F\colon X \Rightarrow Y$ be a functional array of maps, and let $G\colon Y \Rightarrow z$ be a cocone of maps. Suppose G is a collage of some κ-ary directed congruence $\Psi\colon Y \overline{\looparrowright} Y$, and that for each $y \in Y$, the cocone $F|_y\colon X|_y \Rightarrow y$ is a collage of some κ-ary directed congruence $\Phi_y\colon X|_y \overline{\looparrowright} X|_y$. Then $GF\colon X \Rightarrow z$ is a collage of $F^*\Psi F$.*

Proof. On the one hand, we have

$$(GF)^*(GF) = F^*G^*GF = F^*\Psi F,$$

since G is a collage of Ψ. On the other hand, we have

$$\bigvee (GF)(GF)^* = \bigvee GFF^*G^* \le \bigvee GG^* \le 1_z$$

since F and G are both covering. Since GF is clearly composed of maps, by Lemma. 7.14 it is a collage of $F^*\Psi F$. ■

Therefore, let us define $\mathrm{Mod}_\kappa(\mathcal{A})$ to be the full sub-$\mathbf{SUP}_\kappa$-category of $[\mathcal{A}^{\mathrm{op}}, \mathbf{SUP}_\kappa]$ determined by the collages of (the images of) congruences in $\mathcal{A}$, and write $\mathfrak{y}\colon \mathcal{A} \to \mathrm{Mod}_\kappa(\mathcal{A})$ for the restricted Yoneda embedding. We can describe $\mathrm{Mod}_\kappa(\mathcal{A})$ more explicitly as follows.

7.19. LEMMA. *Let $\mathcal{A}$ be a κ-ary allegory; then $\mathrm{Mod}_\kappa(\mathcal{A})$ is equivalent to the following $\mathbf{SUP}_\kappa$-category.*

- *Its objects are κ-ary congruences in $\mathcal{A}$.*
- *Its morphisms from $\Phi\colon X \overline{\looparrowright} X$ to $\Theta\colon Y \overline{\looparrowright} Y$ are arrays $\Psi\colon X \overline{\looparrowright} Y$ such that $\bigvee(\Theta\Psi) \le \Psi$ and $\bigvee(\Psi\Phi) \le \Psi$. (This is equivalent to $\bigvee(\Theta\Psi) = \Psi$ and $\bigvee(\Psi\Phi) = \Psi$, and also to $\bigvee(\Theta\Psi\Phi) = \Psi$.)*
- *The ordering on such arrays is pointwise.*
- *The composition of Ψ and Ψ' is $\bigvee(\Psi'\Psi)$, and the identity of Φ is Φ itself.*

Under this equivalence, the functor $\mathfrak{y}$ takes an object x to the corresponding singleton congruence $\Delta_{\{x\}}$.

PROOF. Up to equivalence, we may certainly take the objects to be the κ-ary congruences themselves rather than their collages in $[\mathcal{A}^{\mathrm{op}}, \mathbf{SUP}_\kappa]$. The identification of the morphisms between these collages in $[\mathcal{A}^{\mathrm{op}}, \mathbf{SUP}_\kappa]$, along with their composition, follows from Corollary. 7.17 and fully-faithfulness of the Yoneda embedding. ∎

Unfortunately, since colimits of congruences (as opposed to directed congruences) are not determined by a class of $\mathbf{SUP}_\kappa$-weights, we cannot directly apply a general theorem such as [Kel82, 5.35] to deduce the universal property of $\mathrm{Mod}_\kappa(\mathcal{A})$. However, essentially the same proofs apply.

7.20. LEMMA. *If $\mathcal{A}$ is a κ-ary allegory, then:*

(i) $\mathrm{Mod}_\kappa(\mathcal{A})$ *is a κ-ary allegory.*
(ii) $\mathfrak{y}\colon \mathcal{A} \to \mathrm{Mod}_\kappa(\mathcal{A})$ *is a κ-ary allegory functor.*
(iii) All κ-ary congruences in $\mathrm{Mod}_\kappa(\mathcal{A})$ *have collages.*
(iv) If $\mathcal{B}$ is a κ-ary allegory with collages of all κ-ary congruences, then

$$\mathrm{ALL}_\kappa(\mathrm{Mod}_\kappa(\mathcal{A}), \mathcal{B}) \xrightarrow{(-\circ\mathfrak{y})} \mathrm{ALL}_\kappa(\mathcal{A}, \mathcal{B})$$

is an equivalence of categories.

PROOF. Binary meets and an involution for $\mathrm{Mod}_\kappa(\mathcal{A})$ are defined pointwise. The modular law follows from that in $\mathcal{A}$, since composition and meets in $\mathcal{A}$ distribute over joins. Thus property (i) holds, and (ii) is clear by definition.

Now by Lemma. 7.18, the collage in $[\mathcal{A}^{\mathrm{op}}, \mathbf{SUP}_\kappa]$ of any κ-ary congruence in $\mathrm{Mod}_\kappa(\mathcal{A})$ is also the collage of some κ-ary directed congruence in $\mathcal{A}$. This directed congruence is easily verified to be a congruence, so its collage also lies in $\mathrm{Mod}_\kappa(\mathcal{A})$; thus (iii) holds.

Finally, for (iv), suppose $\mathcal{B}$ has collages of κ-ary congruences. By Corollary. 7.17, the meets and involutions between collages in $\mathcal{B}$ of congruences in $\mathcal{A}$ are determined by meets and involutions in the image of $\mathcal{A}$. Thus, a $\mathbf{SUP}_\kappa$-functor $\mathrm{Mod}_\kappa(\mathcal{A}) \to \mathcal{B}$ is a κ-ary allegory functor if and only if its composition with $\mathfrak{y}$ is.

Now by [Kel82, 4.99], since $\mathfrak{y}$ is fully faithful, $(- \circ \mathfrak{y})$ induces an equivalence

$$\mathrm{ALL}_\kappa(\mathrm{Mod}_\kappa(\mathcal{A}), \mathcal{B})^\ell \xrightarrow{(-\circ\mathfrak{y})} \mathrm{ALL}_\kappa(\mathcal{A}, \mathcal{B})'$$

whose domain is the full subcategory of $\mathrm{ALL}_\kappa(\mathrm{Mod}_\kappa(\mathcal{A}), \mathcal{B})$ determined by the functors that are (pointwise) left Kan extensions of their restrictions to $\mathcal{A}$, and whose codomain is the full subcategory of $\mathrm{ALL}_\kappa(\mathcal{A}, \mathcal{B})$ determined by the functors which admit (pointwise) left Kan extensions along $\mathfrak{y}$.

Now since $\mathrm{Mod}_\kappa(\mathcal{A})$ is a full sub-$\mathbf{SUP}_\kappa$-category of $[\mathcal{A}^{\mathrm{op}}, \mathbf{SUP}_\kappa]$, the inclusion $\mathfrak{y}$ is $\mathbf{SUP}_\kappa$-dense. Thus, since collages of (directed) congruences are absolute $\mathbf{SUP}_\kappa$-colimits, by [Kel82, 5.29] we have $\mathrm{ALL}_\kappa(\mathrm{Mod}_\kappa(\mathcal{A}), \mathcal{B})^\ell = \mathrm{ALL}_\kappa(\mathrm{Mod}_\kappa(\mathcal{A}), \mathcal{B})$. Finally, since $\mathcal{B}$ has collages of κ-ary congruences and every object of $\mathrm{Mod}_\kappa(\mathcal{A})$ is an absolute colimit (a collage) of a κ-ary congruence in $\mathcal{A}$, the same proof as for [Kel82, 4.98] proves that $\mathrm{ALL}_\kappa(\mathcal{A}, \mathcal{B})' = \mathrm{ALL}_\kappa(\mathcal{A}, \mathcal{B})$. ∎

7.21. Theorem. *The 2-category of κ-ary allegories that have collages of all κ-ary congruences is reflective in* ALL_κ. ∎

By "reflective" here we mean the inclusion has a left biadjoint, not a strict 2-adjoint. The same is true in the following, which is the central theorem of the paper.

7.22. Theorem. *The inclusion* $\mathrm{EX}_\kappa \hookrightarrow \mathrm{SITE}_\kappa$ *is reflective.*

Proof. Let $\mathrm{ALL}_\kappa^{\mathrm{wt}\times}$ denote the full sub-2-category of $\mathrm{FALL}_\kappa^{\mathrm{wt}\times}$ on the chordate objects. By the remarks after Theorem. 7.10, it suffices to show that Mod_κ takes $\mathrm{ALL}_\kappa^{\mathrm{wt}\times}$ into itself. Thus, suppose $\mathcal{A} \in \mathrm{ALL}_\kappa^{\mathrm{wt}\times}$.

In the chordate case, the second part of weak κ-tabularity is automatic. For the first part, let $\Phi\colon X \overline{\looparrowright} X$ and $\Theta\colon Y \overline{\looparrowright} Y$ be congruences in $\mathcal{A}$, and $\Psi\colon \Phi \looparrowright \Theta$ a morphism in $\mathrm{Mod}_\kappa(\mathcal{A})$. Thus Ψ is an array $X \overline{\looparrowright} Y$ in $\mathcal{A}$ with $\bigvee(\Theta\Psi) = \Psi$ and $\bigvee(\Psi\Phi) = \Psi$. By assumption, for each $x \in X$ and $y \in Y$ we have a weak κ-tabulation $\psi_{xy} = \bigvee(G^{xy}(F^{xy})^\circ)$ in $\mathcal{A}$, for κ-ary cocones of maps $F^{xy}\colon U^{xy} \Rightarrow x$ and $G^{xy}\colon U^{xy} \Rightarrow y$. Define $U = \bigsqcup_{x,y} U^{xy}$, with induced functional arrays $F\colon U \Rightarrow X$ and $G\colon U \Rightarrow Y$. Then $\bigvee(\Phi F)\colon \mathfrak{y}(U) \Rightarrow \Phi$ and $\bigvee(\Theta G)\colon \mathfrak{y}(U) \Rightarrow \Theta$ are cocones of maps in $\mathrm{Mod}_\kappa(\mathcal{A})$, and we have

$$\bigvee((\Theta G)(\Phi F)^\circ) = \bigvee\left(\Theta \circ \bigvee(GF^\circ) \circ \Phi\right) = \bigvee(\Theta\Psi\Phi) = \Psi.$$

Hence $\bigvee(\Phi F)$ and $\bigvee(\Theta G)$ are a weak κ-tabulation of Ψ.

Next, suppose U is a weak κ-unit in $\mathcal{A}$. Given any κ-ary congruence $\Phi\colon X \overline{\looparrowright} X$, for each $x \in X$ we have $\Psi^x\colon x \overline{\looparrowright} U$ with $1_x \le \bigvee((\Psi^x)^\circ\Psi^x)$. Putting these together, we obtain an array $\Psi\colon X \overline{\looparrowright} U$ with $\bigvee 1_X \le \bigvee(\Psi^\circ\Psi)$. Therefore,

$$\Phi = \bigvee(\Phi^\circ\Phi) \le \bigvee(\Phi^\circ\Psi^\circ\Psi\Phi) = \bigvee\left(\bigvee(\Psi\Phi)^\circ \circ \bigvee(\Psi\Phi)\right),$$

so $\bigvee(\Psi\Phi)\colon \Phi \Rightarrow \mathfrak{y}(U)$ is a cone in $\mathrm{Mod}_\kappa(\mathcal{A})$ exhibiting $\mathfrak{y}(U)$ as a weak κ-unit.

Finally, if $\mathcal{A}$ has local maxima, then so does $\mathrm{Mod}_\kappa(\mathcal{A})$; they are arrays consisting entirely of local maxima in $\mathcal{A}$. Moreover, if $\mathbf{f}\colon \mathcal{A} \to \mathcal{B}$ preserves these structures, then clearly so does $\mathrm{Mod}_\kappa(\mathbf{f})$. ∎

We denote this left biadjoint by $\mathrm{Ex}_\kappa \colon \mathrm{SITE}_\kappa \to \mathrm{EX}_\kappa$.

7.23. Remark. If we ignore products and terminal objects, the same proof shows that locally κ-ary exact categories are reflective in LSITE_κ. We also write $\mathrm{Ex}_\kappa(\mathbf{C})$ for this reflection; if $\mathbf{C}$ is κ-ary the notation is unambiguous.

By definition, the objects of $\mathrm{Ex}_\kappa(\mathbf{C})$ are the κ-ary congruences in $\mathbf{C}$. Its morphisms are less easy to describe explicitly at the moment; see §8 for an alternative description. However, our current definition is sufficient to identify our construction of the exact completion with many of those existing in the literature.

7.24. Example. As in Examples. 6.10, if $\mathbf{C}$ is κ-ary regular with its κ-regular topology, then $\mathbb{R}\mathsf{el}_\kappa(\mathbf{C})$ is chordate and can be identified with the usual 2-category of relations $\mathrm{Rel}(\mathbf{C})$ in a regular category. In the case $\kappa = \{1\}$, we can then identify $\mathrm{Mod}_{\{1\}}(\mathrm{Rel}(\mathbf{C}))$ with the splitting of equivalence relations (unary congruences), which is precisely the construction of the *exact completion* of a regular category $\mathbf{C}$ from [FS90, 2.169].

For general κ, we can factor Mod_κ by first passing to an allegory of κ-ary families, then splitting equivalence relations (see Example. 11.13). In this way we obtain (the κ-ary version of) the *pretopos completion* of a coherent category, as described in [FS90, 2.217].

7.25. Example. If $\mathbf{C}$ is a small $\mathfrak{K}$-ary site, we have remarked that $\mathbb{R}\mathsf{el}_{\mathfrak{K}}(\mathbf{C})$ is the bicategory of [Wal82]. In that paper, the topos $\mathrm{Sh}(\mathbf{C})$ of sheaves on $\mathbf{C}$ was identified with the category of "small symmetric Cauchy-complete $\mathbb{R}\mathsf{el}_{\mathfrak{K}}(\mathbf{C})$-categories", and isomorphism classes of functors between them. Now small symmetric $\mathbb{R}\mathsf{el}_{\mathfrak{K}}(\mathbf{C})$-categories are precisely $\mathfrak{K}$-ary congruences in $\mathbb{R}\mathsf{el}_{\mathfrak{K}}(\mathbf{C})$, and the *profunctors* between them can be identified with the loose morphisms in $\mathbb{M}\mathsf{od}_\kappa(\mathbb{R}\mathsf{el}_{\mathfrak{K}}(\mathbf{C}))$. Thus, our $\mathrm{Ex}_{\mathfrak{K}}(\mathbf{C})$ is equivalent to the category of small symmetric $\mathbb{R}\mathsf{el}_{\mathfrak{K}}(\mathbf{C})$-categories and isomorphism classes of left-adjoint profunctors between them. Not every left-adjoint profunctor is induced by a functor, but this is so when the codomain is Cauchy-complete. Moreover, every $\mathbb{R}\mathsf{el}_{\mathfrak{K}}(\mathbf{C})$-category is Morita equivalent (that is, equivalent by profunctors) to a Cauchy-complete one as in [Str83b]. Thus, our $\mathrm{Ex}_{\mathfrak{K}}(\mathbf{C})$ is also equivalent to $\mathrm{Sh}(\mathbf{C})$. We will reprove and generalize this by a different method in §9.

We can also identify the universal property of $\mathrm{Ex}_\kappa(\mathbf{C})$, as expressed in Theorem. 7.22, with those of other exact completions.

7.26. Example. Since EX_κ is contained in REG_κ as subcategories of SITE_κ, the construction Ex_κ is also left biadjoint to the inclusion $\mathrm{EX}_\kappa \hookrightarrow \mathrm{REG}_\kappa$. In particular, this is the case for $\mathrm{EX}_{\{1\}} \hookrightarrow \mathrm{REG}_{\{1\}}$, which is the usual universal property of the exact completion of a regular category. The case of the (infinitary) pretopos completion of an (infinitary) coherent category is also well-known.

7.27. Example. On the other hand, we have remarked that EX_κ is *not* contained in LEX as a subcategory of SITE_κ, since $\mathrm{LEX} \hookrightarrow \mathrm{SITE}_\kappa$ equips a lex category with its trivial topology, while $\mathrm{EX}_\kappa \hookrightarrow \mathrm{SITE}_\kappa$ equips a κ-ary exact category with its κ-regular topology. Nevertheless, the functor $\mathrm{Ex}_\kappa \colon \mathrm{LEX} \to \mathrm{EX}_\kappa$ is still left biadjoint to the forgetful functor.

This is because if $\mathbf{C}$ has finite limits and a trivial κ-ary topology, while $\mathbf{D}$ is κ-ary exact with its κ-regular topology, then a functor $\mathbf{C} \to \mathbf{D}$ is a morphism of sites precisely when it preserves finite limits (by Proposition. 4.13). Thus, we also reproduce the known universal property of the exact or pretopos completion of a lex category.

7.28. Example. On the third hand, the situation is different for Ex_κ regarded as a functor WLEX $\to$ EX_κ, since the converse part of Proposition. 4.13 fails if the domain does not have true finite limits. Instead, we recover (in the case $\kappa = \{1\}$) the result of [CV98] that when $\mathbf{C}$ has weak finite limits and $\mathbf{D}$ is exact, regular functors $\mathrm{Ex}_\kappa(\mathbf{C}) \to \mathbf{D}$ are naturally identified with *left covering functors* $\mathbf{C} \to \mathbf{D}$ (see Example. 4.7).

We end this section with the following observation.

7.29. Theorem. *Let* $\mathbf{C}$ *be a* κ*-ary site.*

(i) $\mathfrak{y}\colon \mathbf{C} \to \mathrm{Ex}_\kappa(\mathbf{C})$ *is faithful if and only if all covering families in* $\mathbf{C}$ *are epic.*
(ii) $\mathfrak{y}\colon \mathbf{C} \to \mathrm{Ex}_\kappa(\mathbf{C})$ *is fully faithful if and only if* $\mathbf{C}$ *is subcanonical.*
(iii) $\mathfrak{y}\colon \mathbf{C} \to \mathrm{Ex}_\kappa(\mathbf{C})$ *is an equivalence in* SITE_κ *if and only if* $\mathbf{C}$ *is* κ*-ary exact and equipped with its* κ*-canonical topology, and if and only if* $\mathbf{C}$ *is subcanonical and every* κ*-ary congruence is a kernel of some covering cocone.*

Proof. Recall from Proposition. 7.8 that all covering families in $\mathbf{C}$ are epic just when $\mathbb{R}\mathsf{el}_\kappa(\mathbf{C})$ is subchordate. But this is equivalent to faithfulness (on tight maps) of the map from $\mathbb{R}\mathsf{el}_\kappa(\mathbf{C})$ to its chordate reflection. Since $\mathcal{A} \to \mathrm{Mod}_\kappa(\mathcal{A})$ is fully faithful for any κ-ary allegory $\mathcal{A}$, this proves (i). (ii) is analogous using Proposition. 7.9, and (iii) is immediate from the universal property of Ex_κ and Theorem. 5.15. ∎

8. Exact completion with anafunctors

As constructed in §7, the objects of $\mathrm{Ex}_\kappa(\mathbf{C})$ are κ-ary congruences in $\mathbf{C}$, and its morphisms are a sort of "entire functional relations" or "representable profunctors". In this section, we give an equivalent description of these morphisms using a calculus of fractions, i.e. as "anafunctors" in the style of [Rob].

From the perspective of the rest of the paper, the goal of this section is to prove Theorem. 8.22, which will be used to prove Theorem. 9.2. If this were the only point, then Theorem. 8.22 could no doubt be obtained more directly, or the need for it avoided entirely. But I believe that the ideas of this section clarify certain aspects of the theory, and will be useful when we come to categorify it. However, the reader is free to skip ahead to the statement of Theorem. 8.22 on page 153 and then go on to §9.

The central construction of this section is a *framed* version of Mod_κ. Here is where our description of Mod_κ as an enriched cocompletion is most helpful; all we need do is change the enrichment. Let $\mathscr{F}_\kappa$ denote the (very large) category whose objects are functions

$$j\colon A_\tau \to A_\lambda$$

where A_τ is a moderate set and A_λ is a moderate κ-cocomplete poset. Its morphisms are commutative squares consisting of a set-function and a κ-cocontinuous poset map. Then $\mathscr{F}_\kappa$ is complete, cocomplete, and closed symmetric monoidal under the tensor product

$$A_\tau \times B_\tau \to A_\lambda \otimes B_\lambda$$

where $\otimes$ denotes the tensor product in $\mathbf{SUP}_\kappa$.

A $\mathscr{F}_\kappa$-enriched category $\mathbb{A}$ consists of a $\mathbf{SUP}_\kappa$-enriched category $\mathcal{A}$, together with a category $\mathbf{A}$ and a bijective-on-objects functor $J\colon \mathbf{A} \to \mathcal{A}$. In particular, any framed allegory is an $\mathscr{F}_\kappa$-category. As in a framed allegory, we refer to morphisms in $\mathbf{A}$ as **tight**, writing them as $f\colon x \to y$, and denoting by $f_\bullet\colon x \looparrowright y$ the image of f under J. In this generality, $f_\bullet$ is not necessarily a map, but if it is, we denote its right adjoint by $f^\bullet$.

If $\mathbb{A}$ is an $\mathscr{F}_\kappa$-enriched category, by a **directed congruence** in $\mathbb{A}$ we mean a directed congruence, as in Definition. 7.12, in the underlying $\mathbf{SUP}_\kappa$-category of $\mathbb{A}$.

8.1. Definition. *A* ***tight collage*** *of a directed congruence $\Phi\colon X \overline{\looparrowright} X$ in an $\mathscr{F}_\kappa$-category is a cocone of tight morphisms $F\colon X \Rightarrow w$ such that*

(i) $F_\bullet$ is a collage of Φ in the underlying $\mathbf{SUP}_\kappa$-category, and

(ii) For any $\chi\colon w \looparrowright z$, composition with F determines a bijection between

(a) tight morphisms h with $h_\bullet = \chi$ and

(b) tight cocones G with $G_\bullet = \chi F_\bullet$.

Recall (Example. 7.2) that κ-ary congruences in $\mathbb{R}\mathsf{el}_\kappa(\mathbf{C})$ are equivalence classes of κ-ary congruences in $\mathbf{C}$.

8.2. Lemma. *Let $\mathbf{C}$ be a κ-ary site in which covering families are epic, let $\Phi\colon X \overline{\looparrowright} X$ be a κ-ary congruence in $\mathbf{C}$, and let $F\colon X \Rightarrow w$ be a cocone in $\mathbf{C}$ such that $F_\bullet$ is a collage of Φ in the underlying allegory of $\mathbb{R}\mathsf{el}_\kappa(\mathbf{C})$. Then F is a tight collage of Φ in $\mathbb{R}\mathsf{el}_\kappa(\mathbf{C})$ if and only if it is a colimit of Φ in $\mathbf{C}$.*

Proof. First, let $G\colon X \Rightarrow z$ be any cocone in $\mathbf{C}$; we claim that G is a cocone under Φ in $\mathbf{C}$ if and only if $\bigvee G_\bullet\Phi \le G_\bullet$ in $\mathbb{R}\mathsf{el}_\kappa(\mathbf{C})$. If we write $A, B\colon \bigsqcup \Phi[x_1, x_2] \Rightarrow X$ for the two functional arrays that make up Φ, then we have $\Phi = A_\bullet B^\bullet$ in $\mathbb{R}\mathsf{el}_\kappa(\mathbf{C})$. Thus $\bigvee G_\bullet\Phi \le G_\bullet$ is equivalent to $G_\bullet A_\bullet \le G_\bullet B_\bullet$, which is equivalent to $G_\bullet A_\bullet = G_\bullet B_\bullet$ since maps are discretely ordered, and thence to $GA = GB$ since $\mathbb{R}\mathsf{el}_\kappa(\mathbf{C})$ is subchordate. But this is precisely to say that G is a cocone under Φ.

In particular, since $F_\bullet$ is a collage of Φ, we have $\bigvee F_\bullet\Phi \le F_\bullet$, so F is a cocone under Φ. Moreover, a cocone $G\colon X \Rightarrow z$ is under Φ if and only if $G_\bullet$ factors uniquely through $F_\bullet$ by a (loose) map. Thus, any such G factors uniquely through F just when these loose maps are necessarily tight; hence F is a colimit of Φ just when it is a tight collage. ∎

Note that any collage in a chordate framed allegory is automatically tight.

Tight collages can be expressed as $\mathscr{F}_\kappa$-enriched colimits, along the lines of [LS12, Corollary 6.6]. We can therefore hope to construct the free cocompletion of an $\mathscr{F}_\kappa$-enriched category with respect to tight collages of any class of directed congruences.

In order to describe this cocompletion explicitly, we first describe collages in $\mathscr{F}_\kappa$ itself. For clarity, we write $\mathbb{F}_\kappa$ for $\mathscr{F}_\kappa$ regarded as an $\mathscr{F}_\kappa$-category. Its tight morphisms are the morphisms in the ordinary category $\mathscr{F}_\kappa$, while its loose morphisms $A \to B$ are morphisms $A_\lambda \to B_\lambda$ in $\mathbf{SUP}_\kappa$. Thus, a directed congruence $\Phi\colon X \mathrel{\overline{\looparrowright}} X$ in $\mathbb{F}_\kappa$ consists of a κ-ary family X of objects of $\mathscr{F}_\kappa$, together with morphisms $\phi_{xx'}\colon x_\lambda \to (x')_\lambda$ in $\mathbf{SUP}_\kappa$ for all $x, x' \in X$, satisfying $\xi \le \phi_{xx}(\xi)$ and $\phi_{x'x''}(\phi_{xx'}(\xi)) \le \phi_{xx''}(\xi)$ for all $\xi \in x_\lambda$.

8.3. LEMMA. *Let X be a κ-ary family of objects of $\mathscr{F}_\kappa$, and let $\Phi\colon X \mathrel{\overline{\looparrowright}} X$ be a directed congruence in $\mathbb{F}_\kappa$. The tight collage of Φ is the object w of $\mathscr{F}_\kappa$ described as follows.*

- *The κ-cocomplete poset w_λ consists of X-tuples $(\xi_x)_{x\in X}$, where $\xi_x \in x_\lambda$ and for each $x, x' \in X$ we have $\phi_{xx'}(\xi_x) \le \xi_{x'}$.*
- *The set w_τ consists of pairs (z, ζ) where $z \in X$ and $\zeta \in z_\tau$.*
- *The function $w_\tau \to w_\lambda$ sends (z, ζ) to the tuple $(\phi_{zx}(j(\zeta)))_{x\in X}$.*

The tight cocone $F\colon X \mathrel{\overline{\Rightarrow}} w$ is defined by

- *$(f_z)_\tau(\zeta) = (z, \zeta)$ for $z \in X$, $\zeta \in z_\tau$.*
- *$(f_z)_\lambda(\xi) = (\phi_{zx}(\xi))_{x\in X}$ for $z \in X$, $\xi \in z_\lambda$.*

PROOF. Straightforward verification. ∎

Now we need a framed version of Lemma. 7.18.

8.4. LEMMA. *Let X and Y be κ-ary families of objects in an $\mathscr{F}_\kappa$-category $\mathbb{A}$, let $F\colon X \Rightarrow Y$ be a functional array of tight morphisms, and let $G\colon Y \Rightarrow z$ be a cocone of tight morphisms. Suppose G is a tight collage of some κ-ary directed congruence $\Psi\colon Y \mathrel{\overline{\looparrowright}} Y$, and that for each $y \in Y$, the cocone $F|_y\colon X|_y \Rightarrow y$ is a tight collage of some κ-ary directed congruence $\Phi_y\colon X|_y \mathrel{\overline{\looparrowright}} X|_y$. Then $GF\colon X \Rightarrow z$ is a tight collage of $F^\bullet \Psi F_\bullet$.*

PROOF. Note that since tight collages have underlying loose collages, by Lemma. 7.14 if H is a tight collage then $H_\bullet$ is composed of maps. In particular, $F^\bullet$ exists, so that $F^\bullet\Psi F_\bullet$ makes sense. Moreover, by Lemma. 7.18, $(GF)_\bullet$ is a loose collage of $F^\bullet\Psi F_\bullet$, so it remains to check the tight part of the universal property.

Now given $\chi\colon z \looparrowright w$, since G is a tight collage, we have a bijection between tight morphisms $h\colon z \to w$ with $h_\bullet = \chi$ and tight cocones $K\colon Y \Rightarrow w$ with $K_\bullet = \chi G_\bullet$. But for any $y \in Y$, since $F|_y$ is a tight collage, we have a bijection between tight morphisms $k_y\colon y \to w$ with $(k_y)_\bullet = \chi(g_y)_\bullet$ and tight cocones $L_y\colon X|_y \Rightarrow w$ with $(L_y)_\bullet = \chi(g_y)_\bullet(F|_y)_\bullet$. Putting these together, we see that GF is a tight collage as well. ∎

Thus, given a κ-ary framed allegory $\mathbb{A}$, we let $\mathbb{M}\mathsf{od}_\kappa(\mathbb{A})$ denote the full sub-$\mathscr{F}_\kappa$-category of $[\mathbb{A}^{\mathrm{op}}, \mathbb{F}_\kappa]$ determined by the collages of κ-ary congruences in $\mathbb{A}$. We write $\mathfrak{y}\colon \mathbb{A} \to \mathbb{M}\mathsf{od}_\kappa(\mathbb{A})$ for the restricted Yoneda embedding.

8.5. PROPOSITION. *The $\mathscr{F}_\kappa$-category $\mathbb{M}\mathsf{od}_\kappa(\mathbb{A})$ can be described directly as follows.*

- *Its underlying κ-ary allegory is $\mathrm{Mod}_\kappa(\mathcal{A})$, as described in Lemma. 7.19.*
- *A tight morphism from $\Phi\colon X \mathrel{\overline{\looparrowright}} X$ to $\Theta\colon Y \mathrel{\overline{\looparrowright}} Y$ is a functional array $G\colon X \Rightarrow Y$ of tight maps in $\mathbb{A}$ such that $\bigvee(G_\bullet\Phi) \le \Theta G_\bullet$.*

- *For such a G, the underlying loose morphism $\Phi \looparrowright \Theta$ in $\mathbb{M}\mathsf{od}_\kappa(\mathbb{A})$ is the array $\Theta G_\bullet \colon X \overline{\looparrowright} Y$ in $\mathbb{A}$.*

Proof. Write $\widehat{(-)} \colon \mathbb{A} \to [\mathbb{A}^{\mathrm{op}}, \mathbb{F}_\kappa]$ for the $\mathscr{F}_\kappa$-enriched Yoneda embedding of $\mathbb{A}$, and write coll$(-)$ for the collage of a directed congruence. First of all, since the loose parts of colimits in $\mathscr{F}_\kappa$ coincide with colimits in $\mathbf{SUP}_\kappa$, and the analogous cocompletion of $\mathbf{SUP}_\kappa$-categories could have been constructed in the same way using the $\mathbf{SUP}_\kappa$-enriched Yoneda embedding, the loose morphisms must be the same as those described in Lemma. 7.19 for the $\mathbf{SUP}_\kappa$-case.

Now, let $\Phi \colon X \overline{\looparrowright} X$ to $\Theta \colon Y \overline{\looparrowright} Y$ and let $\Psi \colon \mathrm{coll}(\widehat{\Phi}) \looparrowright \mathrm{coll}(\widehat{\Theta})$ be a loose morphism in $[\mathbb{A}^{\mathrm{op}}, \mathbb{F}_\kappa]$. By Lemma. 7.19, Ψ is determined by an array $\Psi \colon X \overline{\looparrowright} Y$ such that $\bigvee(\Theta\Psi) \le \Psi$ and $\bigvee(\Psi\Phi) \le \Psi$. By the universal property of a tight collage, a tightening of Ψ is determined by a tight cocone $G \colon \widehat{X} \Rightarrow \mathrm{coll}(\widehat{\Theta})$ in $[\mathbb{A}^{\mathrm{op}}, \mathbb{F}_\kappa]$ such that $(g_x)_\bullet = \Psi(f_x)_\bullet$ for every x, where $F \colon \widehat{X} \Rightarrow \mathrm{coll}(\widehat{\Phi})$ is the colimiting cocone.

Now each g_x is a tight morphism from $\widehat{x}$ to $\mathrm{coll}(\widehat{\Theta})$. By the enriched Yoneda lemma, this is equivalently an element of $\mathrm{coll}(\widehat{\Theta})(x)_\tau$. And since colimits in $[\mathbb{A}^{\mathrm{op}}, \mathbb{F}_\kappa]$ are pointwise, this is the same as a tight element of the collage of $\widehat{\Theta}(x)$. Finally, by Lemma. 8.3, this is just a choice of some $g(x) \in Y$ and a tight morphism $g_x \colon x \to g(x)$ in $\mathbb{A}$.

Similarly, f_x can be identified with the identity $1_x \colon x \to x$.

By Lemma. 8.3, the loose morphism underlying g_x in $[\mathbb{A}^{\mathrm{op}}, \mathbb{F}_\kappa]$ is determined by the family of composites $(\theta_{g(x),y}(g_x)_\bullet)_{y \in Y}$. Similarly, the loose morphism underlying f_x is determined by the family $(\phi_{x,x'})_{x' \in X}$. Thus, the condition $(g_x)_\bullet = \Psi(f_x)_\bullet$ asks that

$$\theta_{g(x),y}(g_x)_\bullet = \bigvee_{x'} (\psi_{x',y}\phi_{x,x'}),$$

i.e. that $\Theta G_\bullet = \bigvee(\Psi\Phi) = \Psi$. So it suffices to show that $\Theta G_\bullet$ determines a loose morphism $\mathrm{coll}(\widehat{\Phi}) \looparrowright \mathrm{coll}(\widehat{\Theta})$ if and only if $\bigvee(G_\bullet\Phi) \le \Theta G_\bullet$. But the former is equivalent to

$$\bigvee(\Theta\Theta G_\bullet) \le \Theta G_\bullet \quad \text{and} \quad \bigvee(\Theta G_\bullet \Phi) \le \Theta G_\bullet.$$

Since $\bigvee(\Theta\Theta) = \Theta$, the first inequality is automatic, while since additionally $\bigvee 1_Y \le \Theta$, the second is equivalent to $\bigvee(G_\bullet\Phi) \le \Theta G_\bullet$. ∎

8.6. Remark. To avoid confusion, if $G \colon X \Rightarrow Y$ is a functional array of tight maps in $\mathbb{A}$ representing a tight morphism $\Phi \to \Theta$ in $\mathbb{M}\mathsf{od}_\kappa(\mathbb{A})$, we reserve the notation $G_\bullet$ for the underlying array of loose maps in $\mathbb{A}$, and write $G_\blacklozenge$ for the underlying loose morphism $\Phi \rightsquigarrow \Theta$ in $\mathbb{M}\mathsf{od}_\kappa(\mathbb{A})$ (which is represented by $\Theta G_\bullet$ in $\mathbb{A}$).

Composition of tight maps in $\mathbb{M}\mathsf{od}_\kappa(\mathbb{A})$ is just composition of functional arrays in $\mathbb{A}$. Proposition. 8.5 implies that for $G \colon \Phi \to \Theta$ and $H \colon \Theta \to \Psi$, we have $H_\blacklozenge G_\blacklozenge = (HG)_\blacklozenge$. But once we show in Lemma. 8.8 that $\mathbb{M}\mathsf{od}_\kappa(\mathbb{A})$ is a framed allegory, we can deduce this directly. Namely, we have

$$H_\blacklozenge G_\blacklozenge = \bigvee(\Psi H_\bullet \Theta G_\bullet) \le \bigvee(\Psi\Psi H_\bullet G_\bullet) = \Psi(HG)_\bullet = (HG)_\blacklozenge;$$

hence $H_\blacklozenge G_\blacklozenge = (HG)_\blacklozenge$, since both are maps in $\mathbb{M}\mathsf{od}_\kappa(\mathbb{A})$ and maps are discretely ordered.

8.7. Remark. If we regard congruences as a decategorification of the bicategory-enriched categories of [Str81a, CKW87], then loose morphisms between them decategorify *modules* (a.k.a. profunctors). Proposition. 8.5 says that the *tight* maps between them decategorify *functors*, just as framed allegories decategorify proarrow equipments.

For instance, if $\mathcal{A}$ is a monoidal κ-cocomplete poset, regarded as a $\mathscr{F}_\kappa$-enriched category $\mathbb{A}$ with one object and only the identity being tight, then a κ-ary congruence in $\mathbb{A}$ is precisely a symmetric $\mathcal{A}$-enriched category with a κ-small set of objects. In this case, the tight morphisms in $\mathbb{M}\mathsf{od}_\kappa(\mathbb{A})$ are precisely $\mathcal{A}$-enriched functors, while its loose morphisms are $\mathcal{A}$-enriched profunctors.

Similarly, if $\mathbb{A}$ is $\mathbb{R}\mathsf{el}_{\{1\}}(\mathbf{C})$ for regular $\mathbf{C}$, then a unary congruence has a unique representation as an internal equivalence relation. If we view these as a particular sort of internal category, then the tight morphisms in $\mathbb{M}\mathsf{od}_\kappa(\mathbb{A})$ are precisely internal functors, while its loose morphisms are internal profunctors whose defining span is jointly monic.

The general case is analogous, for suitably generalized "functors" and "profunctors".

Unlike the case of $\mathbf{SUP}_\kappa$-enriched categories, tight collages are *not* absolute $\mathscr{F}_\kappa$-colimits. Thus the universal property of $\mathbb{M}\mathsf{od}_\kappa(\mathbb{A})$ differs from that of $\mathrm{Mod}_\kappa(\mathcal{A})$.

8.8. Lemma. *If $\mathbb{A}$ is a κ-ary framed allegory, then:*

(i) $\mathbb{M}\mathsf{od}_\kappa(\mathbb{A})$ *is a κ-ary framed allegory.*

(ii) $\mathfrak{y}\colon \mathbb{A} \to \mathbb{M}\mathsf{od}_\kappa(\mathbb{A})$ *is a κ-ary framed allegory functor.*

(iii) All κ-ary congruences in $\mathbb{M}\mathsf{od}_\kappa(\mathbb{A})$ *have tight collages.*

(iv) If $\mathbb{B}$ is a κ-ary framed allegory with tight collages of κ-ary congruences, then

$$\mathrm{FALL}_\kappa(\mathbb{M}\mathsf{od}_\kappa(\mathbb{A}), \mathbb{B})^\ell \xrightarrow{(-\circ\mathfrak{y})} \mathrm{FALL}_\kappa\big(\mathbb{A}, \mathbb{B}\big)$$

is an equivalence of categories, where the $(-)^\ell$ on the domain denotes the full subcategory determined by the functors which preserve tight collages of κ-ary congruences coming from $\mathbb{A}$.

Proof. Since the underlying $\mathbf{SUP}_\kappa$-category of $\mathbb{M}\mathsf{od}_\kappa(\mathbb{A})$ is $\mathrm{Mod}_\kappa(\mathcal{A})$, condition (i) will follow from Lemma. 7.20(i) as soon as we show that every tight morphism in $\mathbb{M}\mathsf{od}_\kappa(\mathbb{A})$ is a map. But if $G\colon \Phi \to \Theta$ is a tight morphism as in Proposition. 8.5, with underlying loose morphism $G_\blacklozenge = \Theta G_\bullet$, then $G^\bullet\Theta$ defines a loose morphism $\Theta \nrightarrow \Phi$, and we have

$$\bigvee(\Theta G_\bullet G^\bullet \Theta) \le \bigvee(\Theta\Theta) = \Theta \qquad \text{and}$$
$$\bigvee(G^\bullet\Theta\Theta G_\bullet) = G^\bullet\Theta G_\bullet \ge \bigvee(G^\bullet G_\bullet \Phi) \ge \Phi.$$

Thus, $G^\bullet\Theta$ is right adjoint to $\Theta G_\bullet$, so (i) holds. And since an $\mathscr{F}_\kappa$-functor is a κ-ary framed allegory functor just when its underlying $\mathbf{SUP}_\kappa$-functor is a κ-ary allegory functor, (ii) is immediate from Lemma. 7.20(ii).

Now by Lemma. 8.4, the tight collage in $[\mathbb{A}^{\mathrm{op}}, \mathbb{F}_\kappa]$ of any κ-ary congruence in $\mathbb{M}\mathsf{od}_\kappa(\mathbb{A})$ is also the tight collage of a κ-ary directed congruence in $\mathbb{A}$. As in Lemma. 7.20, this directed congruence is actually a congruence, so its collage also lies in $\mathbb{M}\mathsf{od}_\kappa(\mathbb{A})$, yielding (iii).

For (iv), we also argue essentially as in Lemma. 7.20. Note that an $\mathscr{F}_\kappa$-functor $F\colon \mathbb{M}\mathsf{od}_\kappa(\mathbb{A}) \to \mathbb{B}$ is a κ-ary framed allegory functor if and only if $F \circ \mathfrak{y}$ is so. Thus by [Kel82, 4.99], since $\mathfrak{y}$ is fully faithful, $(- \circ \mathfrak{y})$ induces an equivalence

$$\mathrm{FALL}_\kappa(\mathbb{M}\mathsf{od}_\kappa(\mathbb{A}), \mathbb{B})^\ell \xrightarrow{(-\circ\mathfrak{y})} \mathrm{FALL}_\kappa(\mathbb{A}, \mathbb{B})'$$

for any $\mathbb{B}$. Here the domain is the full subcategory of functors that are (pointwise) left Kan extensions of their restrictions to $\mathbb{A}$, and the codomain is the full subcategory of functors which admit (pointwise) left Kan extensions along $\mathfrak{y}$. Since $\mathbb{M}\mathsf{od}_\kappa(\mathbb{A})$ is a full sub-$\mathscr{F}_\kappa$-category of $[\mathbb{A}^{\mathrm{op}}, \mathbb{F}_\kappa]$, the inclusion $\mathfrak{y}$ is $\mathscr{F}_\kappa$-dense. Thus, by [Kel82, 5.29], $\mathrm{FALL}_\kappa(\mathbb{M}\mathsf{od}_\kappa(\mathbb{A}), \mathbb{B})^\ell$ consists of the functors that preserve tight collages of κ-ary congruences coming from $\mathbb{A}$. And since $\mathbb{B}$ has collages of κ-ary congruences, every object of $\mathbb{M}\mathsf{od}_\kappa(\mathbb{A})$ is a colimit (a tight collage) of a κ-ary congruence in $\mathbb{A}$, and these colimits are preserved by maps out of objects of $\mathbb{A}$ (since they are colimits in a presheaf category), the same proof as for [Kel82, 4.98] proves that $\mathrm{FALL}_\kappa(\mathbb{A}, \mathbb{B})' = \mathrm{FALL}_\kappa(\mathbb{A}, \mathbb{B})$. ∎

We write $G^\blacklozenge\colon \Theta \looparrowright \Phi$ for the right adjoint of $G_\blacklozenge$ (represented by $G^\bullet\Theta$ as above). Now we make the following simple observation.

8.9. Proposition. *For a κ-ary framed allegory $\mathbb{A}$, the following are equivalent.*

- *The chordate reflection of $\mathbb{M}\mathsf{od}_\kappa(\mathbb{A})$.*
- Mod_κ *of the chordate reflection of $\mathbb{A}$.*

Proof. This follows immediately from the explicit description in Proposition. 8.5, but we can also show directly that they have the same universal property: they both reflect $\mathbb{A}$ into chordate κ-ary allegories with collages of κ-ary congruences (since every collage in a chordate framed allegory is automatically tight, by Lemma. 7.6(iv)). ∎

Therefore, although in §7 we constructed the κ-ary exact completion of a κ-ary site $\mathbf{C}$ by applying Mod_κ to the chordate reflection of $\mathbb{R}\mathsf{el}_\kappa(\mathbf{C})$, it could equally well be defined as the chordate reflection of $\mathbb{M}\mathsf{od}_\kappa(\mathbb{R}\mathsf{el}_\kappa(\mathbf{C}))$. This is useful because the latter admits an alternative description as a category of fractions.

8.10. Definition. *We say that a tight map $f\colon x \to y$ in a framed allegory is a* ***weak equivalence*** *if $f_\bullet$ is an isomorphism.*

Since $f_\bullet$ has a right adjoint $f^\bullet$, this is equivalent to requiring $1_x = f^\bullet f_\bullet$ and $1_y = f_\bullet f^\bullet$.

8.11. Proposition. *A tight map $G\colon \Phi \to \Theta$ in $\mathbb{M}\mathsf{od}_\kappa(\mathbb{A})$, as in Proposition. 8.5, is a weak equivalence if and only if $\Phi = G^\bullet\Theta G_\bullet$ and $\bigvee 1_Y \le \bigvee(\Theta G_\bullet G^\bullet\Theta)$ in $\mathbb{A}$.*

Proof. Since $G_\blacklozenge = \Theta G_\bullet$ in $\mathbb{A}$, and the identity morphism of Φ in $\mathbb{M}\mathsf{od}_\kappa(\mathbb{A})$ is Φ itself in $\mathbb{A}$, asking that $G^\blacklozenge G_\blacklozenge = 1_\Phi$ in $\mathbb{M}\mathsf{od}_\kappa(\mathbb{A})$ is to ask that $\Phi = \bigvee(G^\bullet\Theta^\circ\Theta G_\bullet)$ in $\mathbb{A}$, which is equivalent to $\Phi = G^\bullet\Theta G_\bullet$.

Similarly, asking that $1_\Theta = G_\blacklozenge G^\blacklozenge$ is to ask that $\Theta = \bigvee(\Theta G_\bullet G^\bullet \Theta)$ in $\mathbb{A}$. Since G consists of maps, we always have $\bigvee(\Theta G_\bullet G^\bullet \Theta) \le \bigvee(\Theta\Theta) = \Theta$, so the content is in $\Theta \le \bigvee(\Theta G_\bullet G^\bullet \Theta)$. But since $\bigvee 1_Y \le \Theta$, this implies $\bigvee 1_Y \le \bigvee(\Theta G_\bullet G^\bullet \Theta)$, while the converse holds since $\bigvee(\Theta\Theta) = \Theta$. ■

8.12. Remark. If we regard tight maps of congruences as "functors" as in Remark. 8.7, Proposition. 8.11 says that that the weak equivalences are those which are "fully faithful" and "essentially surjective".

Recall that a subcategory $\mathbf{W}$ of a category $\mathbf{A}$ which contains all the objects is said to admit a *calculus of right fractions* if

- Given $f\colon y \to z$ in $\mathbf{A}$ and $p\colon x \to z$ in $\mathbf{W}$, there exist $g\colon w \to x$ and $q\colon w \to y$ such that $q \in \mathbf{W}$ and $pg = fq$ (the *right Ore condition*), and
- Given $p\colon y \to z$ in $\mathbf{W}$ and $f, g\colon x \rightrightarrows y$ in $\mathbf{A}$ such that $pf = pg$, there exists $q\colon w \to x$ in $\mathbf{W}$ such that $fq = gq$ (the *right cancellability condition*).

In this case, the localization $\mathbf{A}[\mathbf{W}^{-1}]$ can be constructed as follows: its objects are those of $\mathbf{A}$, and its morphisms from x to y are equivalence classes of spans

$$x \xleftarrow{p} w \xrightarrow{f} y$$

such that $p \in \mathbf{W}$, under the equivalence relation that identifies (p, f) with (p', f') if we have a commutative diagram

$$\begin{tikzcd} & w \arrow[dl, "p"'] \arrow[dr, "f"] & \\ x & z \arrow[u] \arrow[d] & y \\ & w' \arrow[ul, "p'"] \arrow[ur, "f'"'] & \end{tikzcd}$$

with the common composite $z \to x$ in $\mathbf{W}$. The composite of (p, f) with (q, g) is defined to be (pr, gh), where $qh = fr$ and $r \in \mathbf{W}$ (such h and r exist by the right Ore condition).

8.13. Lemma. *In a weakly κ-tabular framed allegory with tight collages of κ-ary congruences, if $f, g\colon x \rightrightarrows y$ are tight maps such that $f_\bullet = g_\bullet$, then there is a weak equivalence $k\colon u \to x$ such that $fk = gk$.*

Proof. By weak κ-tabularity, if $f_\bullet = g_\bullet$ we have $fP = gP$ for a covering family $P\colon V \Rightarrow x$. Let $\Phi = P^\bullet P_\bullet$ be the congruence "kernel" of P, and let $Q\colon V \Rightarrow u$ be a tight collage of Φ. In particular, $Q_\bullet$ is a loose collage of Φ. Since $P_\bullet$ is also a loose collage of Φ (by Lemma. 7.7), there is a unique loose isomorphism $h\colon u \rightsquigarrow x$ with $P_\bullet = hQ_\bullet$. But because $P_\bullet$ admits the tightening P, and Q is a tight collage, h admits a unique tightening $k\colon u \to x$ with $P = kQ$. Since $k_\bullet = h$ is an isomorphism, k is a weak equivalence, and the universal property of Q implies $fk = gk$. ■

8.14. THEOREM. *If $\mathbb{A}$ is weakly κ-tabular with tight collages of κ-ary congruences, then the weak equivalences in* $\mathrm{TMap}(\mathbb{A})$ *admit a calculus of right fractions, and*

$$\mathrm{TMap}(\mathbb{A})[\mathcal{W}^{-1}] \cong \mathrm{Map}(\mathcal{A}).$$

PROOF. The weak equivalences are clearly a subcategory and contain all objects. For the right Ore condition, suppose $f\colon y \to z$ is a tight map and $p\colon x \to z$ a weak equivalence. As in Lemma. 6.18, we can find tight cocones $G\colon U \Rightarrow x$ and $Q\colon U \Rightarrow y$ such that $pG = fQ$ and $p^\bullet f_\bullet = \bigvee(G_\bullet Q^\bullet)$. Since $1_y \le f^\bullet f_\bullet = f^\bullet p_\bullet p^\bullet f_\bullet$, by Corollary. 6.16 Q is covering.

Let $\Phi = Q^\bullet Q_\bullet$, and let $F\colon U \Rightarrow w$ be a tight collage of Φ. By Lemma. 7.7, $Q_\bullet$ is a (loose) collage of Φ, so there is a unique tight map $r\colon w \to y$ such that $rF = Q$, and (by uniqueness of loose collages) r is a weak equivalence. Similarly, we have

$$G_\bullet \Phi = p^\bullet f_\bullet Q_\bullet \Phi \le p^\bullet f_\bullet Q_\bullet = Q_\bullet$$

so there is a unique tight map $h\colon w \to x$ with $hF = G$. Finally, we have $phF = pG = fQ = frF$, so since F is a tight collage, $ph = fr$. This shows the right Ore condition.

For right cancellability, suppose $p\colon y \to z$ is a weak equivalence and $f, g\colon x \rightrightarrows y$ are tight maps with $pf = pg$. Since $p_\bullet$ is an isomorphism, $f_\bullet = g_\bullet$, so we can apply Lemma. 8.13. Thus the weak equivalences admit a calculus of right fractions.

Now since $J\colon \mathrm{TMap}(\mathbb{A}) \to \mathrm{Map}(\mathcal{A})$ inverts weak equivalences, it extends to a functor $\mathrm{TMap}(\mathbb{A})[\mathcal{W}^{-1}] \longrightarrow \mathrm{Map}(\mathcal{A})$, which, like J, takes a span $x \xleftarrow{p} w \xrightarrow{f} y$ (with p a weak equivalence) to $f_\bullet p^\bullet$. Since it is bijective on objects, it suffices to show that it is full and faithful.

For fullness, suppose $\phi\colon x \rightsquigarrow y$ is a loose map in $\mathbb{A}$, and let $\phi = \bigvee(F_\bullet G^\bullet)$ for tight cocones $F\colon Z \Rightarrow y$, $G\colon Z \Rightarrow x$. By Corollary. 6.16, G is covering. Let $\Psi = G^\bullet G_\bullet$ and H be a tight collage of Ψ. Since G is a loose collage of Ψ, we have a weak equivalence r with $G = rH$. Now for any $z \in Z$, we have $(f_z)_\bullet (g_z)^\bullet \le \phi$, hence $(g_z)_\bullet (f_z)^\bullet \le \phi^\circ$ and so

$$\phi(g_z)_\bullet \le \phi(g_z)_\bullet (f_z)^\bullet (f_z)_\bullet \le \phi\phi^\circ (f_z)_\bullet \le (f_z)_\bullet.$$

Thus, $\bigvee(F_\bullet \Psi) = \bigvee(F_\bullet G^\bullet G_\bullet) = \phi G_\bullet \le F_\bullet$, so since H is a tight collage of Ψ, we have a unique $k\colon w \to y$ with $kH = F$. Since tight collages are also loose collages, we have $\phi r_\bullet = k_\bullet$, hence $\phi = k_\bullet r^\bullet$; thus ϕ is in the image of $\mathrm{TMap}(\mathbb{A})[\mathcal{W}^{-1}]$.

For faithfulness, suppose that $x \xleftarrow{p} w \xrightarrow{f} y$ and $x \xleftarrow{q} v \xrightarrow{g} y$ are spans with p, q weak equivalences such that $f_\bullet p^\bullet = g_\bullet q^\bullet$. As in Lemma. 6.18, we can find $R\colon U \Rightarrow w$ and $S\colon U \Rightarrow v$ such that $pR = qS$ and $p^\bullet q_\bullet = \bigvee(R_\bullet S^\bullet)$. Since $p^\bullet q_\bullet$ is an isomorphism, by Corollary. 6.16 R and S are both covering, and $R^\bullet R_\bullet = R^\bullet p^\bullet p_\bullet R_\bullet = S^\bullet q^\bullet q_\bullet S_\bullet = S^\bullet S_\bullet$.

Let $H\colon U \Rightarrow z$ be a tight collage of the congruence $\Phi = R^\bullet R_\bullet = S^\bullet S_\bullet$. Then since R and S are loose collages of Φ, we have weak equivalences $m\colon z \to w$ and $n\colon z \to v$ with $mH = R$ and $nH = S$. Hence $pmH = pR = qS = qnH$, so since H is a tight collage, $pm = qn$ and is a weak equivalence. Now

$$f_\bullet m_\bullet H_\bullet = f_\bullet R_\bullet = f_\bullet p^\bullet p_\bullet R_\bullet = g_\bullet q^\bullet q_\bullet S_\bullet = g_\bullet S_\bullet = g_\bullet n_\bullet H_\bullet,$$

so since $H_\bullet$ is a loose collage, $f_\bullet m_\bullet = g_\bullet n_\bullet$. By Lemma. 8.13, we can find a weak equivalence t with $fmt = gnt$ and (of course) $pmt = qnt$. Thus $(p, f) = (q, g)$ in $\mathrm{TMap}(\mathbb{A})[\mathcal{W}^{-1}]$. ∎

Unfortunately, $\mathbb{M}\mathsf{od}_\kappa(\mathbb{R}\mathsf{el}_\kappa(\mathbf{C}))$ does not quite satisfy the hypotheses of Theorem. 8.14; it may not inherit the second half of weak κ-tabularity from $\mathbb{R}\mathsf{el}_\kappa(\mathbf{C})$. But we can remedy this by considering its subchordate reflection, which is easily seen to inherit all the other relevant properties of $\mathbb{R}\mathsf{el}_\kappa(\mathbf{C})$. This yields a fairly explicit description of $\mathrm{Ex}_\kappa(\mathbf{C})$ as a category of fractions, but we can improve it even further as follows.

8.15. Definition. *Let $\Phi\colon X \overline{\looparrowright} X$ and $\Theta\colon Y \overline{\looparrowright} Y$ be κ-ary congruences in a framed κ-ary allegory $\mathbb{A}$. A tight map $G\colon \Phi \to \Theta$ in $\mathbb{M}\mathsf{od}_\kappa(\mathbb{A})$ is a* ***surjective equivalence*** *if*

(i) $\Phi = G^\bullet \Theta G_\bullet$, and

(ii) $G\colon X \Rightarrow Y$ is a covering family in $\mathbb{A}$.

Note that the condition on a functional array $G\colon X \Rightarrow Y$ to be a tight map $\Phi \to \Theta$ in $\mathbb{M}\mathsf{od}_\kappa(\mathbb{A})$, namely $\bigvee(G_\bullet\Phi) \le \Theta G_\bullet$, is equivalent by adjunction to $\Phi \le G^\bullet\Theta G_\bullet$. Thus, in Definition. 8.15 we do not need to assert that G is a tight map; it follows from (i).

8.16. Lemma. *A surjective equivalence is a weak equivalence in $\mathbb{M}\mathsf{od}_\kappa(\mathbb{A})$.*

Proof. If G is a surjective equivalence, we have

$$\bigvee 1_Y \le \Theta = \bigvee(\Theta\Theta) = \bigvee(\Theta G_\bullet G^\bullet \Theta).$$

So by Proposition. 8.11, G is a weak equivalence. ∎

8.17. Lemma. *If $\mathbb{A}$ is weakly κ-tabular and $G\colon \Phi \to \Theta$ is a weak equivalence in $\mathbb{M}\mathsf{od}_\kappa(\mathbb{A})$, then there exist surjective equivalences $F\colon \Psi \to \Theta$ and $H\colon \Psi \to \Phi$ with $G_\blacklozenge H_\blacklozenge = F_\blacklozenge$.*

Proof. Let $\Theta G_\bullet = \bigvee(F_\bullet H^\bullet)$ for functional arrays of tight maps F and H in $\mathbb{A}$. Since

$$\bigvee 1 \le \bigvee(G^\bullet G_\bullet) \le \bigvee(G^\bullet \Theta^\circ \Theta G_\bullet),$$

by Corollary. 6.16 H is a covering family. But since G is a weak equivalence, we also have $\bigvee 1 \le \bigvee(\Theta G_\bullet G^\bullet \Theta^\circ)$, so F is also a covering family. Let $\Psi = H^\bullet \Phi H_\bullet$; then H becomes by definition a tight map $\Psi \to \Phi$ that is a surjective equivalence.

Now $\bigvee(F_\bullet H^\bullet) = \Theta G_\bullet$ implies, by the mates correspondence, that $H^\bullet G^\bullet \le F^\bullet \Theta$, and hence also $G_\bullet H_\bullet \le \Theta F_\bullet$. Therefore, we have

$$\Psi = H^\bullet \Phi H_\bullet = H^\bullet G^\bullet \Theta G_\bullet H_\bullet \le F^\bullet \Theta\Theta\Theta F_\bullet = F^\bullet \Theta F_\bullet.$$

Hence F is a tight map $\Psi \to \Theta$. Now we also have

$$\bigvee(\Theta G_\bullet \Phi H_\bullet) = \bigvee(F_\bullet H^\bullet \Phi H_\bullet) = \bigvee(F_\bullet \Psi) \le \Theta F_\bullet,$$

so that in $\mathbb{M}\mathsf{od}_\kappa(\mathbb{A})$, we have $G_\blacklozenge H_\blacklozenge \le F_\blacklozenge$, hence $G_\blacklozenge H_\blacklozenge = F_\blacklozenge$. It follows that since G and H are weak equivalences in $\mathbb{M}\mathsf{od}_\kappa(\mathbb{A})$, so is F. Thus, as it is a covering family in $\mathbb{A}$, it is also a surjective equivalence $\Psi \to \Theta$. ∎

8.18. Lemma. *If $F, G \colon \Phi \rightrightarrows \Theta$ are two tight maps in $\mathbb{M}\mathsf{od}_\kappa(\mathbb{A})$, then $F_\blacklozenge = G_\blacklozenge$ if and only if $\bigvee(F_\bullet G^\bullet) \le \Theta$.*

Proof. If $F_\blacklozenge = G_\blacklozenge$, then

$$1 \le F^\blacklozenge G_\blacklozenge = \bigvee(F^\bullet \Theta^\circ \Theta G_\bullet) = F^\bullet \Theta G_\bullet,$$

and hence

$$\bigvee(F_\bullet G^\bullet) \le \bigvee(F_\bullet F^\bullet \Theta G_\bullet G^\bullet) \le \Theta.$$

Conversely, if $F_\bullet G^\bullet \le \Theta$, then

$$F_\blacklozenge = \Theta F_\bullet \le \bigvee(\Theta F_\bullet G^\bullet G_\bullet) \le \bigvee(\Theta\Theta G_\bullet) = \Theta G_\bullet = G_\blacklozenge,$$

and dually. ∎

8.19. Theorem. *If $\mathbb{A}$ is weakly κ-tabular, then the category of loose maps in $\mathbb{M}\mathsf{od}_\kappa(\mathbb{A})$ can be described as follows.*

(i) Its objects are κ-ary congruences in $\mathbb{A}$.

(ii) For congruences $\Phi \colon X \mathrel{\overline{\looparrowright}} X$ and $\Theta \colon Y \mathrel{\overline{\looparrowright}} Y$, a morphism $\Phi \to \Theta$ is represented by a span $X \overset{P}{\Leftarrow} W \overset{F}{\Rightarrow} Y$ of functional arrays of tight maps in $\mathbb{A}$, such that P is a covering family and $P^\bullet \Phi P_\bullet \le F^\bullet \Theta F_\bullet$.

(iii) Two such spans represent the same morphism $\Phi \to \Theta$ if there is a diagram of functional arrays of tight maps in $\mathbb{A}$:

$$\begin{tikzcd}
& W \arrow[dl, Rightarrow, "P"'] \arrow[dr, Rightarrow, "F"] & \\
X & U \arrow[u, Rightarrow, "S"'] \arrow[d, Rightarrow, "S'"] & Y \\
& W' \arrow[ul, Rightarrow, "P'"] \arrow[ur, Rightarrow, "F'"'] &
\end{tikzcd} \tag{8.20}$$

in which the left-hand quadrilateral commutes (but not necessarily the other one), $PS = P'S'$ is a covering family, and $\bigvee\big((FS)_\bullet (F'S')^\bullet\big) \le \Theta$.

Proof. By Theorem. 8.14, it suffices to consider the category of fractions of the tight maps in the subchordate reflection of $\mathbb{M}\mathsf{od}_\kappa(\mathbb{A})$, so a morphism $\Phi \to \Theta$ can be represented by a span $\Phi \overset{P}{\leftarrow} \Psi \overset{F}{\rightarrow} \Theta$ where P and F are tight maps in $\mathbb{M}\mathsf{od}_\kappa(\mathbb{A})$ and P is a weak equivalence. By Lemma. 8.17, we may assume P is a surjective equivalence. Hence, $P \colon W \Rightarrow X$ is a covering family and $\Psi = P^\bullet \Phi P_\bullet$, so F being a tight map in $\mathbb{M}\mathsf{od}_\kappa(\mathbb{A})$ is equivalent to $P^\bullet \Phi P_\bullet \le F^\bullet \Theta F_\bullet$. This gives (ii).

Now, by the construction of a category of fractions, two such spans represent the same morphism just when we have a diagram of tight functional arrays

$$\begin{tikzcd}
& W \arrow[dl, Rightarrow, "P"'] \arrow[dr, Rightarrow, "F"] & \\
X & Z \arrow[u, Rightarrow, "H"'] \arrow[d, Rightarrow, "H'"] & Y \\
& W' \arrow[ul, Rightarrow, "P'"] \arrow[ur, Rightarrow, "F'"'] &
\end{tikzcd} \tag{8.21}$$

(not necessarily commutative) and a congruence $\Psi\colon Z \overline{\looparrowright} Z$ such that

(1) H and H' are tight maps $\Psi \to P^\bullet \Phi P_\bullet$ and $\Psi \to (P')^\bullet \Phi (P')_\bullet$, respectively;
(2) $(PH)_\blacklozenge = (P'H')_\blacklozenge$;
(3) PH is a weak equivalence $\Psi \to \Phi$ (hence so is $P'H'$); and
(4) $(FH)_\blacklozenge = (F'H')_\blacklozenge$.

If we are given (8.20), then we define $Z = U$, $H = S$, $H' = S'$, and

$$\Psi = (PS)^\bullet \Phi (PS)_\bullet = (P'S')^\bullet \Phi (P'S')_\bullet.$$

Then (1)–(3) are immediate, while (4) follows from Lemma. 8.18.

Conversely, suppose given (8.21) satisfying (1)–(4). Then by Lemma. 8.17, we can find a surjective equivalence $Q\colon \Psi' \to \Psi$ such that PHQ is a surjective equivalence. Thus we have three covering families P, P', and PHQ of X.

Since $\mathrm{TMap}(\mathbb{A})$ is a locally κ-ary site, there is a covering family $R\colon U \Rightarrow X$ and functional arrays S, S', and S'' such that $R = PS = P'S' = PHQS''$. Thus we have (8.20) in which the left-hand quadrilateral commutes. Let $\Psi'' = R^\bullet \Phi R_\bullet$; then S is a tight map $\Psi'' \to P^\bullet \Phi P_\bullet$ and S' is a tight map $\Psi'' \to (P')^\bullet \Phi (P')_\bullet$, and we calculate

$$\begin{aligned} F_\blacklozenge S_\blacklozenge &= F_\blacklozenge (P^\bullet \Phi P_\bullet) S_\bullet \\ &= F_\blacklozenge P^\bullet \Phi P_\bullet H_\bullet Q_\bullet (S'')_\bullet \\ &= F_\blacklozenge H_\blacklozenge Q_\bullet (S'')_\bullet \\ &= (F')_\blacklozenge (H')_\blacklozenge Q_\bullet (S'')_\bullet \\ &= (F')_\blacklozenge (P')^\bullet \Phi (P')_\bullet (H')_\bullet Q_\bullet (S'')_\bullet \\ &= (F')_\blacklozenge (P')^\bullet \Phi P_\bullet H_\bullet Q_\bullet (S'')_\bullet \\ &= (F')_\blacklozenge (P')^\bullet \Phi (P')_\bullet (S')_\bullet \\ &= (F')_\blacklozenge (S')_\blacklozenge. \end{aligned}$$

By Lemma. 8.18, this is equivalent to $\bigvee \big((FS)_\bullet (F'S')^\bullet\big) \le \Theta$, so (iii) holds. ∎

8.22. Theorem. *The exact completion of a κ-ary site $\mathbf{C}$ can be described as follows.*

(i) Its objects are κ-ary congruences Φ in $\mathbf{C}$.

(ii) Each morphism $\Phi \to \Psi$ is represented by a span of functional arrays $X \overset{P}{\Leftarrow} W \overset{F}{\Rightarrow} Y$ such that P is a covering family and $P^*\Phi \preceq F^*\Psi$.

(iii) Two such spans (P, F) and (Q, G) determine the same morphism $\Phi \to \Psi$ if there is a covering family $R\colon U \Rightarrow X$ and functional arrays H and K such that $R = PH = QK$ and for all $u \in U$, $\{f_{h(u)} h_u, g_{k(u)} k_u\} \le \Psi(fh(u), gk(u))$.

Proof. This is a direct translation of Theorem. 8.19, except that in (iii) we assert $\le$ rather than $\preceq$. However, this can easily be obtained by passing to an extra covering family. ∎

This description of $\mathrm{Ex}_\kappa(\mathbf{C})$ is a decategorification of the bicategory of *internal anafunctors*, as described in [Bar06, Rob]. This approach to exact completion does not seem as popular as the relational one, but one case of it appears in the literature.

8.23. Example. If $\mathbf{C}$ has weak finite limits and the trivial unary topology, then a unary congruence in $\mathbf{C}$ reduces precisely to a *pseudo-equivalence relation* as defined in [CV98, Def. 6]. The tight maps in $\mathbb{M}\mathsf{od}_{\{1\}}(\mathbb{R}\mathsf{el}_{\{1\}}(\mathbf{C}))$ similarly reduce to morphisms of pseudo-equivalence relations, and every surjective equivalence has a section. This implies that every span as in Theorem. 8.22(ii) is equivalent to one where P is the identity, and likewise in (iii) we may assume R is an identity. Thus, in this case the above construction of $\mathrm{Ex}_{\{1\}}(\mathbf{C})$ yields precisely the exact completion of $\mathbf{C}$ as constructed in [CV98, Def. 14].

9. Exact completion and sheaves

Suppose that $\mathbf{C}$ is a *small* κ-ary site. Since the category $\mathrm{Sh}(\mathbf{C})$ of (small) sheaves on $\mathbf{C}$ is $\mathfrak{K}$-ary exact, hence also κ-ary exact, and the sheafified Yoneda embedding $\mathbf{y}\colon \mathbf{C} \to \mathrm{Sh}(\mathbf{C})$ is a morphism of sites, we have an induced κ-ary regular functor

$$\widetilde{\mathbf{y}}\colon \mathrm{Ex}_\kappa(\mathbf{C}) \to \mathrm{Sh}(\mathbf{C}).$$

9.1. Lemma. *Given $\mathcal{F} \in \mathrm{Sh}(\mathbf{C})$, consider the following statements.*

(i) $\mathcal{F}$ is in the image of $\widetilde{\mathbf{y}}$.

(ii) $\mathcal{F}$ is the colimit, in $\mathrm{Sh}(\mathbf{C})$, of the $\mathbf{y}$-image of some κ-ary congruence in $\mathbf{C}$.

(iii) $\mathcal{F}$ is the colimit in $\mathrm{Sh}(\mathbf{C})$ of a κ-small diagram of sheafified representables.

Then (i)⇔(ii) always, while (ii)⇒(iii) if $2 \in \kappa$, and (iii)⇒(ii) if $\omega \in \kappa$.

Proof. Firstly, every object of $\mathrm{Ex}_\kappa(\mathbf{C})$ is the colimit of some κ-ary congruence in $\mathbf{C}$, and $\widetilde{\mathbf{y}}$ preserves such colimits. This immediately gives (i)⇒(ii). Conversely, if (ii) holds, say $\mathcal{F} \cong \operatorname{colim} \mathbf{y}(\Phi)$, then $\mathfrak{y}(\Phi)$ has a colimit in $\mathrm{Ex}_\kappa(\mathbf{C})$ and this colimit is preserved by $\widetilde{\mathbf{y}}$, hence its image is isomorphic to $\mathcal{F}$; thus (ii)⇒(i).

If $2 \in \kappa$, then κ-small sets are closed under binary coproducts, and thus a κ-ary congruence is a κ-small diagram; hence (ii)⇒(iii). Conversely, suppose that $\omega \in \kappa$ and that $\mathcal{F}$ satisfies (iii). Then we can present $\mathcal{F}$ as a coequalizer

$$\textstyle\sum \mathbf{y}(Y) \underset{\sum T}{\overset{\sum S}{\rightrightarrows}} \sum \mathbf{y}(X) \longrightarrow \mathcal{F}.$$

where X and Y are κ-ary families of objects of $\mathbf{C}$ and $S, T\colon Y \Rightarrow X$ are functional arrays. We define a congruence Φ on X as follows. Given $x, x' \in X$, consider zigzags of the form

$$\begin{tikzcd} & y_1 \arrow[dl] \arrow[dr] & & y_2 \arrow[dl] \arrow[dr] & & y_n \arrow[dl] \arrow[dr] & \\ x = x_0 & & x_1 & & \cdots & & x_n = x', \end{tikzcd}$$

in which each span $x_{i-1} \leftarrow y_i \rightarrow x_i$ is either (s_{y_i}, t_{y_i}) or (t_{y_i}, s_{y_i}). For each n, there are a κ-small number of such zigzags, so since $\omega \in \kappa$ there are overall a κ-small number of them. Since $\mathbf{C}$ is a κ-ary site, each zigzag has a local κ-prelimit; let $\Phi(x, x')$ be the disjoint union of one local κ-prelimit of each zigzag. Then Φ is a κ-ary congruence, and the colimit of $\mathbf{y}(\Phi)$ is also $\mathcal{F}$. Thus, (iii)$\Rightarrow$(ii) when $\omega \in \kappa$. ■

We write $\mathrm{Sh}_\kappa(\mathbf{C})$ for the full image of $\mathrm{Ex}_\kappa(\mathbf{C})$ in $\mathrm{Sh}(\mathbf{C})$.

9.2. Theorem. *The functor* $\widetilde{\mathbf{y}} \colon \mathrm{Ex}_\kappa(\mathbf{C}) \to \mathrm{Sh}_\kappa(\mathbf{C})$ *is an equivalence of categories.*

Proof. It remains only to show that it is fully faithful. Moreover, since every object of $\mathrm{Ex}_\kappa(\mathbf{C})$ is the colimit of some diagram in $\mathbf{C}$ (specifically, a κ-ary congruence), and these colimits are preserved by $\widetilde{\mathbf{y}}$, it suffices to prove that

$$\mathrm{Ex}_\kappa(\mathbf{C})\Big(\mathfrak{y}(y), \Phi\Big) \longrightarrow \mathrm{Sh}_\kappa(\mathbf{C})\Big(\mathbf{y}(y), \widetilde{\mathbf{y}}(\Phi)\Big) \cong \widetilde{\mathbf{y}}(\Phi)(y)$$

is a bijection, for any object y and κ-ary congruence $\Phi \colon X \looparrowright X$ in $\mathbf{C}$.

Now by Theorem. 8.22, a morphism $\mathfrak{y}(y) \to \Phi$ in $\mathrm{Ex}_\kappa(\mathbf{C})$ is determined by a covering family $P \colon W \Rightarrow y$ and a functional array $F \colon W \Rightarrow X$ such that the kernel of P locally refines $F^*\Phi$. More concretely, this is a covering $P \colon W \Rightarrow y$ and for each $w \in W$, a morphism $f_w \colon w \to f(w)$ for some $f(w) \in X$, such that for any $a \colon u \to w_1$ and $b \colon u \to w_2$ with $p_{w_1} a = p_{w_2} b$, there is a covering family $Q \colon V \Rightarrow u$ such that $\{f_{w_1} a q_v, f_{w_2} b q_v\}$ factors through $\Phi(f(w_1), f(w_2))$ for all $v \in V$. Two such collections define the same morphism $\mathfrak{y}(y) \to \Phi$ if there is a covering $R \colon Z \Rightarrow y$, which refines both P and P' as $R = PS = P'S'$, such that $\{f_{s(z)} s_z, f'_{s'(z)} s'_z\}$ factors through $\Phi(fs(z), f's'(z))$ for any $z \in Z$.

On the other hand, let $\mathcal{F}$ denote the colimit of the congruence $\mathbf{y}(\Phi)$ in the presheaf category $[\mathbf{C}^{\mathrm{op}}, \mathbf{Set}]$. Then

$$\mathcal{F}(w) = \coprod_{x \in X} \mathbf{C}(w, x) \Big/ \sim,$$

where the equivalence relation $\sim$ relates $\alpha_1 \colon w \to x_1$ and $\alpha_2 \colon w \to x_2$ if $\{\alpha_1, \alpha_2\}$ factors through $\Phi(x_1, x_2)$. Then $\widetilde{\mathbf{y}}(\Phi)$ is the sheafification of $\mathcal{F}$. Using the 1-step construction of sheafification via hypercoverings (e.g. [DHI04, Prop. 7.9] or [Lur09b, §6.5.3]), we can describe this as follows. An element of $\widetilde{\mathbf{y}}(\Phi)(y)$ is determined by a covering family $P \colon W \Rightarrow y$ together with for each $w \in W$, an element $f_w \in \mathcal{F}(w)$, such that for any $a \colon u \to w_1$ and $b \colon u \to w_2$ with $p_{w_1} a = p_{w_2} b$, there is a covering family $Q \colon V \Rightarrow u$ such that $(q_v)^* a^*(f_{w_1}) = (q_v)^* b^*(f_{w_2})$ for all $v \in V$. Two such collections of data define the same element of $\widetilde{\mathbf{y}}(\Phi)$ if there is a covering family $R \colon Z \Rightarrow y$, which refines both P and P' as $R = PS = P'S'$, and such that for any $z \in Z$ we have $s_z^*(f_{s(z)}) = (s')_z^*(f'_{s'(z)})$.

These descriptions are essentially identical. The only difference is that there is a bit more identification at first in an element of $\widetilde{\mathbf{y}}(\Phi)$ (the elements f_w are only specified up to Φ to begin with), but this disappears after we quotient by the full equivalence relations. ■

9.3. Example. Suppose $\mathbf{C}$ has a trivial unary topology. Then $\mathrm{Sh}(\mathbf{C}) = [\mathbf{C}^{\mathrm{op}}, \mathbf{Set}]$. And if $X_1 \rightrightarrows X_0$ is a unary congruence in $\mathbf{C}$, i.e. a pseudo-equivalence relation, then the image of $\mathbf{y}(X_1) \to \mathbf{y}(X_0) \times \mathbf{y}(X_0)$ is an equivalence relation on $\mathbf{y}(X_0)$ whose quotient is the colimit of $\mathbf{y}(X_1 \rightrightarrows X_0)$. Thus, every presheaf in the image of $\widetilde{\mathbf{y}}$ admits a surjection from a representable, such that the kernel of the surjection also admits a surjection from a representable. Conversely, given a presheaf with this property, the two resulting representables give a pseudo-equivalence relation. Thus we reproduce the characterization of the exact completion of a weakly lex category from [HT96] in terms of presheaves.

9.4. Example. If $\mathbf{C}$ is a (unary) regular category with its regular unary topology, then we have seen that every unary congruence in $\mathbf{C}$ is equivalent to an internal equivalence relation. Equivalent congruences have isomorphic colimits in $\mathrm{Sh}(\mathbf{C})$, so a sheaf on $\mathbf{C}$ lies in the image of $\widetilde{\mathbf{y}}$ just when it is the quotient of an equivalence relation in $\mathbf{C}$. Thus we also reproduce the characterization of the exact completion of a regular category from [Lac99] in terms of sheaves.

Finally, the following example is important enough to call a theorem.

9.5. Theorem. *If $\mathbf{C}$ is a small $\mathfrak{K}$-ary site, then $\mathrm{Ex}_{\mathfrak{K}}(\mathbf{C}) \simeq \mathrm{Sh}(\mathbf{C})$.* ■

Combining this with Theorem. 7.22, we obtain a new proof of the classical theorem that for a small site $\mathbf{C}$ and a Grothendieck topos $\mathbf{E}$, geometric morphisms $\mathbf{E} \to \mathrm{Sh}(\mathbf{C})$ are equivalent to (what we call) morphisms of $\mathfrak{K}$-ary sites $\mathbf{C} \to \mathbf{E}$.

9.6. Small sheaves on large sites Now suppose that $\mathbf{C}$ is a *large* (but moderate) $\mathfrak{K}$-ary site. We write $\mathbf{SET}$ for the very large category of moderate sets. Similarly, we write $\mathrm{SH}(\mathbf{C})$ for the very large category of $\mathbf{SET}$-valued sheaves on $\mathbf{C}$. As before, we can show:

9.7. Proposition. *For any $\mathfrak{K}$-ary site $\mathbf{C}$, we have a full embedding $\widetilde{\mathbf{y}} \colon \mathrm{Ex}_{\mathfrak{K}}(\mathbf{C}) \hookrightarrow \mathrm{SH}(\mathbf{C})$, whose image consists of those $\mathbf{SET}$-valued sheaves which are colimits in $\mathrm{SH}(\mathbf{C})$ of small diagrams in $\mathbf{C}$.*

By a **small sheaf** we mean a sheaf in the image of $\widetilde{\mathbf{y}}$, or equivalently an object of $\mathrm{Ex}_{\mathfrak{K}}(\mathbf{C})$.

9.8. Example. If $\mathbf{C}$ is a moderate category with finite $\mathfrak{K}$-prelimits and a trivial $\mathfrak{K}$-ary topology, then $\mathrm{Ex}_{\mathfrak{K}}(\mathbf{C})$ is equivalent to the category $\mathcal{P}(\mathbf{C})$ of *small presheaves* on $\mathbf{C}$ in the sense of [DL07]: the colimits in $[\mathbf{C}^{\mathrm{op}}, \mathbf{SET}]$ of small diagrams in $\mathbf{C}$. Thus, under these hypotheses, $\mathcal{P}(\mathbf{C})$ is $\mathfrak{K}$-ary exact, and in particular has finite limits. This last conclusion is the finitary version of one direction of a theorem of [DL07]; we will also deduce the converse in Example. 11.4.

9.9. Remark. While a small presheaf on a locally small category necessarily takes values in small sets (since colimits in $[\mathbf{C}^{\mathrm{op}}, \mathbf{SET}]$ are pointwise), the same is not true of a small sheaf. One virtue of our approach is that we have defined $\mathrm{Ex}_{\mathfrak{K}}(\mathbf{C})$, for a large $\mathfrak{K}$-ary site $\mathbf{C}$, without needing the whole very-large category $\mathrm{SH}(\mathbf{C})$.

When $\mathbf{C}$ is large, $\mathrm{Ex}_{\mathfrak{K}}(\mathbf{C})$ is not, in general, a Grothendieck topos: it lacks a small generating set. However, we have shown that it is an infinitary-pretopos. Conversely, Theorem. 7.29(iii) implies that any infinitary-pretopos is equivalent to $\mathrm{Ex}_{\mathfrak{K}}(\mathbf{C})$ for some $\mathfrak{K}$-ary site $\mathbf{C}$, namely the infinitary-pretopos itself with its $\mathfrak{K}$-canonical topology. Thus we have a "purely size-free" version of Giraud's theorem: a category is an infinitary-pretopos if and only if it is the $\mathfrak{K}$-ary exact completion of a $\mathfrak{K}$-ary site. (This viewpoint also shows that there is really nothing special in this about the case $\kappa = \mathfrak{K}$.)

Moreover, Theorem. 7.22 implies that the $\mathfrak{K}$-ary exact completion also satisfies a suitable version of Diaconescu's theorem: for any $\mathfrak{K}$-ary site $\mathbf{C}$ and infinitary-pretopos $\mathbf{D}$, functors $\mathrm{Ex}_{\mathfrak{K}}(\mathbf{C}) \to \mathbf{D}$ which preserve finite limits and small colimits are naturally equivalent to morphisms of $\mathfrak{K}$-ary sites $\mathbf{C} \to \mathbf{D}$. It is natural to think of such functors between infinitary pretoposes as "the inverse image parts of geometric morphisms," although in the absence of smallness hypotheses, we have no adjoint functor theorem to ensure the existence of a "direct image part." In particular, if $\mathbf{C}$ is the syntactic category of a "large geometric theory," then $\mathrm{Ex}_{\mathfrak{K}}(\mathbf{C})$ might naturally be considered the "classifying (pre)topos" of that theory.

9.10. Remark. In our presentation, the objects of $\mathrm{Ex}_{\mathfrak{K}}(\mathbf{C})$ are transparently seen as "objects of $\mathbf{C}$ glued together." This makes it obvious, for instance, how to express schemes as objects of $\mathrm{Ex}_{\mathfrak{K}}(\mathbf{Ring}^{\mathrm{op}})$. Namely, let X be a family of rings covering a scheme S by open affines, and for each $x_1, x_2 \in X$ let $\Phi(x_1, x_2)$ be an open affine cover of $\mathrm{Spec}(x_1) \cap \mathrm{Spec}(x_2) \subseteq S$. Then Φ is a $\mathfrak{K}$-ary congruence which presents S as a small sheaf on $\mathbf{Ring}^{\mathrm{op}}$.

It should be possible to axiomatize further "open map structure" on a $\mathfrak{K}$-ary site, along the lines of [JM94] and [Lur09a], enabling the identification of a general class of "schemes" in $\mathrm{Ex}_{\mathfrak{K}}(\mathbf{C})$ as the congruences where gluing happens only along "open subspaces."

10. Postulated and lex colimits

In this section we consider how our notion of exact completion is related to the *postulated colimits* of [Koc89] and the *lex colimits* of [GL12]. We will also obtain cocompletions of sites with respect to weaker exactness properties, generalizing those of [GL12].

Let $\mathbf{C}$ be a (moderate) category, $\mathbf{SET}$ the category of moderate sets, and $\mathbf{y}\colon \mathbf{C} \to [\mathbf{C}^{\mathrm{op}}, \mathbf{SET}]$ the Yoneda embedding. By a **realization** of a presheaf $\mathcal{J}\colon \mathbf{C}^{\mathrm{op}} \to \mathbf{SET}$ we will mean a morphism $\mathcal{J} \to \mathbf{y}(z)$ into a representable through which any map from $\mathcal{J}$ into a representable factors uniquely. In other words, $\mathcal{J} \to \mathbf{y}(z)$ is a reflection of $\mathcal{J}$ into representables.

For instance, for any functor $\mathbf{g}\colon \mathbf{D} \to \mathbf{C}$, colimits of $\mathbf{g}$ are equivalent to realizations of $\mathrm{colim}(\mathbf{y} \circ \mathbf{g})$. More generally, colimits of $\mathbf{g}$ weighted by $\mathcal{I}\colon \mathbf{D}^{\mathrm{op}} \to \mathbf{SET}$ are equivalent to realizations of $\mathrm{Lan}_{\mathbf{g}^{\mathrm{op}}} \mathcal{I}$.

10.1. Definition. *Let $\mathbf{C}$ be a κ-ary site. A presheaf $\mathcal{J}\colon \mathbf{C}^{\mathrm{op}} \to \mathbf{SET}$ is κ-**admissible** if there exists a κ-ary congruence Φ in $\mathbf{C}$ such that $\mathcal{J} \cong \mathrm{colim}\, \mathbf{y}(\Phi)$ in $[\mathbf{C}^{\mathrm{op}}, \mathbf{SET}]$.*

Since a κ-ary congruence is a small diagram, any κ-admissible weight must be a **small presheaf**, i.e. a small colimit of representables. If $2 \in \kappa$, then a κ-ary congruence is a κ-small diagram, and so any κ-admissible weight is a **κ-small presheaf**, i.e. a κ-small colimit of representables. Conversely, the construction used for Lemma. 9.1(iii)⇒(ii) gives:

10.2. Lemma. *If $\omega \in \kappa$, then any κ-small presheaf on $\mathbf{C}$ is κ-admissible.* ■

In particular, every presheaf on a small site is $\mathfrak{K}$-admissible.

10.3. Example. Let $\mathbf{C}$ have finite limits and let $R \rightrightarrows X$ be an internal equivalence relation in $\mathbf{C}$. Then the quotient of $\mathbf{y}(R) \rightrightarrows \mathbf{y}(X)$ in $[\mathbf{C}^{\mathrm{op}}, \mathbf{SET}]$ is $\{1\}$-admissible. A realization of this presheaf is a quotient of $R \rightrightarrows X$.

10.4. Example. Given a κ-ary family X of objects in $\mathbf{C}$, the coproduct $\sum \mathbf{y}(X)$ in $[\mathbf{C}^{\mathrm{op}}, \mathbf{SET}]$ is κ-small. Moreover, it is κ-admissible regardless of what κ may be, since it is the colimit of the discrete congruence Δ_X. Of course, realizations of $\sum \mathbf{y}(X)$ are coproducts of X in $\mathbf{C}$.

10.5. Example. Suppose $\mathbf{C}$ has finite limits and that we have a span $y \hookleftarrow x \to z$ in which $x \hookrightarrow y$ is monic. If $\mathcal{J}$ denotes the pushout of $\mathbf{y}(y) \hookleftarrow \mathbf{y}(x) \to \mathbf{y}(z)$, then a realization of $\mathcal{J}$ is a pushout of the given span in $\mathbf{C}$. This $\mathcal{J}$ is ω-small, but as $\omega \notin \omega$, Lemma. 10.2 does not apply. Nevertheless, $\mathcal{J}$ is still ω-admissible. For if we define $X = \{y, z\}$, we have a congruence Φ on X where:

- $\Phi(z, z) = \Delta_z$,
- $\Phi(y, z)$ and $\Phi(z, y)$ are the given span and its opposite, and
- $\Phi(y, y)$ consists of Δ_y together with the span $y \leftarrow x \times_z x \to y$.

The fact that $x \hookrightarrow y$ is monic makes this a congruence, and it is clearly ω-ary. It is easy to check that the colimit of $\mathbf{y}(\Phi)$ is $\mathcal{J}$.

If $\mathcal{J}$ is κ-admissible, then a congruence Φ with $\operatorname{colim} \mathbf{y}(\Phi) \cong \mathcal{J}$ is of course not uniquely determined. However, it is determined up to a suitable sort of equivalence.

10.6. Lemma. *Suppose Φ and Ψ are κ-ary congruences in a κ-ary site on families X and Y, with $\operatorname{colim} \mathbf{y}(\Phi) \cong \operatorname{colim} \mathbf{y}(\Psi)$. Then Φ and Ψ are isomorphic in $\mathrm{Ex}_\kappa(\mathbf{C})$.*

Proof. By Theorem. 9.2, $\mathrm{Ex}_\kappa(\mathbf{C})$ is equivalent to a subcategory of $\mathrm{SH}(\mathbf{C})$, via a functor $\widetilde{\mathbf{y}}$ which preserves colimits of κ-ary congruences and extends the sheafified Yoneda embedding of $\mathbf{C}$. Since sheafification preserves colimits, $\operatorname{colim} \mathbf{y}(\Phi) \cong \operatorname{colim} \mathbf{y}(\Psi)$ implies $\widetilde{\mathbf{y}}(\Phi) \cong \widetilde{\mathbf{y}}(\Psi)$, hence $\Phi \cong \Psi$ in $\mathrm{Ex}_\kappa(\mathbf{C})$. ■

Since a collage of Φ in $\mathbb{R}\mathsf{el}_\kappa(\mathbf{C})$ is equivalently an object of $\mathbf{C}$ together with an isomorphism to Φ in $\mathrm{Ex}_\kappa(\mathbf{C})$, it follows that under the hypotheses of Lemma. 10.6, giving a collage of Φ is equivalent to giving a collage of Ψ. This justifies considering the following definition as a property of $\mathcal{J}$ alone.

10.7. Definition. *Let* $\mathbf{C}$ *be a* κ*-ary site and* $\mathcal{J}\colon \mathbf{C}^{\mathrm{op}} \to \mathbf{SET}$ *be* κ*-admissible. A morphism* $\mathcal{J} \to \mathbf{y}(z)$ *is* ***postulated*** *if for some (hence any)* κ*-ary congruence* $\Phi\colon X \overline{\looparrowright} X$ *with an isomorphism* $\operatorname{colim}\mathbf{y}(\Phi) \cong \mathcal{J}$*, the induced cocone* $F\colon X \Rightarrow z$ *has the property that* $F_\bullet$ *is a (loose) collage of* Φ *in* $\mathbb{R}\mathsf{el}_\kappa(\mathbf{C})$*.*

Postulated colimits are defined in [Koc89] for conical colimits in a site with finite limits, using the internal logic. The notion was reformulated in [GL12, Prop. 6.5], also assuming finite limits, in terms of a presentation of $\mathcal{J}$ as a coequalizer:

$$\sum \mathbf{y}(Y) \rightrightarrows \sum \mathbf{y}(X) \to \mathcal{J}. \tag{10.8}$$

In these definitions, there are two conditions on $\mathcal{J} \to \mathbf{y}(z)$ to be postulated:

(i) The induced family $X \Rightarrow z$ is covering.

(ii) For each $x, x' \in X$, the induced family of maps into $x \times_z x'$ out of pullbacks of zigzags built out of spans with vertices in Y (as in the proof of Lemma. 9.1) is covering.

If (10.8) exhibits $\mathcal{J}$ as the colimit of a congruence Φ, then the pullback of any such zigzag factors locally through $\Phi(x, x')$. Hence, condition (ii) above is equivalent to asking that $x \times_z x'$ be covered by $\Phi(x, x')$ — which in turn is equivalent to saying that the kernel of $X \Rightarrow z$ is equivalent to Φ. Thus, in this case the above two conditions reduce precisely to the two conditions in Lemma. 7.7 characterizing z as a collage of Φ.

Conversely, for an arbitrary (10.8), the congruence constructed in Lemma. 9.1 is built out of zigzags, so Kock's notion of postulatedness reduces to being its collage. Therefore, modulo κ-admissibility, our notion of postulatedness coincides with Kock's.

We remark in passing that [Koc89, Prop. 1.1] now follows easily.

10.9. Proposition. *In a subcanonical site, any postulated morphism is a realization.*

Proof. According to our definition, a postulated morphism $\mathcal{J} \to \mathbf{y}(z)$ exhibits z as the collage of some congruence Φ with $\operatorname{colim}\mathbf{y}(\Phi) \cong \mathcal{J}$. But if $\mathbf{C}$ is subcanonical, then $\mathbb{R}\mathsf{el}_\kappa(\mathbf{C})$ is chordate by Proposition. 7.9, so any collage is a tight collage. Thus, by Lemma. 8.2, any such collage is also a colimit of Φ, hence a realization of $\mathcal{J}$. ∎

Thus, in a subcanonical site we may refer to a postulated morphism $\mathcal{J} \to \mathbf{y}(z)$ as a **postulated realization**.

In [GL12], Garner and Lack define a *lex-weight* to be a functor $\mathcal{J}\colon \mathbf{D}^{\mathrm{op}} \to \mathbf{Set}$ where $\mathbf{D}$ is small and has finite limits, and a $\mathcal{J}$-weighted *lex-colimit* in a category $\mathbf{C}$ with finite limits to be a $\mathcal{J}$-weighted colimit of a finite-limit-preserving functor $\mathbf{g}\colon \mathbf{D} \to \mathbf{C}$. Let $\mathscr{J}$ be a class of lex-weights; by [GL12, 6.4], the following definition is equivalent to theirs.

10.10. Definition. *A category* $\mathbf{C}$ *is* $\mathscr{J}$***-exact*** *if it has finite limits, and there exists a subcanonical topology on* $\mathbf{C}$ *such that for any* $\mathcal{J}\colon \mathbf{D}^{\mathrm{op}} \to \mathbf{Set}$ *in* $\mathscr{J}$ *and* $\mathbf{g}\colon \mathbf{D} \to \mathbf{C}$ *preserving finite limits, the presheaf* $\operatorname{Lan}_{\mathbf{g}^{\mathrm{op}}} \mathcal{J}$ *on* $\mathbf{C}$ *has a postulated realization.*

Now let κ be an arity class and suppose that each $\mathcal{J}\colon \mathbf{D}^{\mathrm{op}} \to \mathbf{Set}$ in $\mathscr{J}$ is κ-admissible for the trivial topology on $\mathbf{D}$. For each $\mathcal{J} \in \mathscr{J}$, let $\Phi_{\mathcal{J}}\colon X_{\mathcal{J}} \overline{\looparrowright} X_{\mathcal{J}}$ be a κ-ary congruence in $\mathbf{D}$ with $\operatorname{colim} \mathbf{y}(\Phi_{\mathcal{J}}) \cong \mathcal{J}$. Note that if $\mathbf{g}\colon \mathbf{D} \to \mathbf{C}$ preserves finite limits, then it is a morphism of sites for any topology on $\mathbf{C}$, and hence $\mathbf{g}(\Phi_{\mathcal{J}})$ is a congruence in $\mathbf{C}$.

Thus, $\mathbf{C}$ is $\mathscr{J}$-exact if and only if it admits some subcanonical topology such that for any $\mathcal{J}\colon \mathbf{D}^{\mathrm{op}} \to \mathbf{Set}$ in $\mathscr{J}$ and $\mathbf{g}\colon \mathbf{D} \to \mathbf{C}$ preserving finite limits, this congruence $\mathbf{g}(\Phi_{\mathcal{J}})$ has a collage. (In particular, any κ-ary exact category is $\mathcal{J}$-exact.) This is equivalent to asking that there exist a cocone $\mathbf{g}(X_{\mathcal{J}}) \Rightarrow \mathbf{y}(z_{\mathcal{J},\mathbf{g}})$ under $\mathbf{g}(\Phi_{\mathcal{J}})$ which is covering, and such that all the induced cocones $\mathbf{g}(\Phi(x,x')) \Rightarrow \mathbf{g}(x) \times_z \mathbf{g}(x')$ are also covering.

In particular, if $\mathbf{C}$ is $\mathscr{J}$-exact, then the cocones $\mathbf{g}(X_{\mathcal{J}}) \Rightarrow \mathbf{y}(z_{\mathcal{J},\mathbf{g}})$ and $\mathbf{g}(\Phi(x,x')) \Rightarrow \mathbf{g}(x) \times_z \mathbf{g}(x')$ are all universally effective-epic, so the topology that they generate is subcanonical. Since these cocones are κ-ary, the topology they generate is weakly κ-ary, hence κ-ary (since $\mathbf{C}$ has finite limits). We call it the **$\mathscr{J}$-exact topology** on $\mathbf{C}$. It makes an implicit appearance in [GL12, §7], where its category of sheaves is characterized directly as those presheaves which send "$\mathscr{J}^*$-lex colimits" to limits in $\mathbf{Set}$.

10.11. Definition. *A topology on a $\mathscr{J}$-exact category is **$\mathscr{J}$-superexact** if it contains the $\mathscr{J}$-exact topology.*

Thus, the topologies on $\mathbf{C}$ which exhibit it as $\mathscr{J}$-exact as in Definition. 10.10 are precisely the subcanonical and $\mathscr{J}$-superexact ones. For instance, the κ-canonical topology on a κ-ary exact category is $\mathscr{J}$-superexact. The prefix "super-" in "$\mathscr{J}$-superexact" is intended to dualize the prefix "sub-" in "subcanonical".

10.12. Remark. For many familiar classes $\mathscr{J}$ of lex-weights, the $\mathscr{J}$-exact topology is already generated by the coprojections $\mathbf{g}(X_{\mathcal{J}}) \Rightarrow z_{\mathcal{J},\mathbf{g}}$. This is related to the remark in [GL12] that $\mathscr{J}$-lex-cocompleteness often, but not always, coincides with "$\mathscr{J}^*$-lex-cocompleteness". In particular, the example in [GL12, §5.11] also gives a class $\mathscr{J}$ for which the $\mathscr{J}$-exact topology is not generated merely by these coprojections.

Recall from [GL12] that a functor $\mathbf{f}$ between $\mathscr{J}$-exact categories is called **$\mathscr{J}$-exact** if it preserves finite limits and $\mathscr{J}$-lex-colimits.

10.13. Lemma. *If $\mathbf{C}$ and $\mathbf{D}$ are $\mathscr{J}$-superexact subcanonical κ-ary sites, then any morphism of sites $\mathbf{f}\colon \mathbf{C} \to \mathbf{D}$ is $\mathscr{J}$-exact. The converse holds if $\mathbf{C}$ has the $\mathscr{J}$-exact topology.*

Proof. By Corollary. 4.14, $\mathbf{f}\colon \mathbf{C} \to \mathbf{D}$ is a morphism of sites if and only if it preserves finite limits and covering families. On the other hand, since $\mathbf{D}$ is $\mathscr{J}$-superexact, $\mathbf{f}$ is $\mathscr{J}$-exact if and only if it preserves finite limits and also the covering families that exhibit $\mathscr{J}$-lex-colimits as collages. If $\mathbf{f}$ is a morphism of sites then it clearly does this. Conversely, since the $\mathscr{J}$-exact topology is generated by these covering families, if $\mathbf{f}$ preserves them then it preserves all covering families in that topology. ∎

Let $\mathrm{SITE}_\kappa^{\mathscr{J}}$ denote the full sub-2-category of SITE_κ determined by the $\mathscr{J}$-exact categories equipped with $\mathscr{J}$-superexact subcanonical κ-ary topologies. Let $\mathrm{EX}_\kappa^{\mathscr{J}}$ denote its full sub-2-category determined by the $\mathscr{J}$-exact topologies.

By Lemma. 10.13, the morphisms in $\mathrm{EX}_\kappa^{\mathscr{J}}$ are the $\mathscr{J}$-exact functors, so it is equivalent to the category $\mathscr{J}$-**EX** of [GL12]. In particular, $\mathrm{EX}_\kappa^{\mathscr{J}}$ is independent of κ, as long as $\mathscr{J}$ is κ-admissible (and every $\mathscr{J}$ is $\mathfrak{K}$-admissible).

10.14. Theorem. *The inclusion* $\mathrm{SITE}_\kappa^{\mathscr{J}} \hookrightarrow \mathrm{SITE}_\kappa$ *is reflective.*

Proof. By applying chordate reflection to framed allegories first, we may assume all our sites are subcanonical. Let $\mathrm{Ex}_\kappa^{\mathscr{J}}(\mathbf{C})$ denote the smallest full subcategory of $\mathrm{Ex}_\kappa(\mathbf{C})$ which contains $\mathbf{C}$ and is closed under finite limits and $\mathscr{J}$-lex-colimits, and let $\mathfrak{y}_\mathbf{C}^{\mathscr{J}} : \mathbf{C} \to \mathrm{Ex}_\kappa^{\mathscr{J}}(\mathbf{C})$ be the inclusion. Since $\mathrm{Ex}_\kappa^{\mathscr{J}}(\mathbf{C})$ is closed under finite limits in $\mathrm{Ex}_\kappa(\mathbf{C})$, the κ-canonical topology of $\mathrm{Ex}_\kappa(\mathbf{C})$ restricts to it, making it a κ-ary site such that both $\mathfrak{y}_\mathbf{C}^{\mathscr{J}}$ and the inclusion $\mathrm{Ex}_\kappa^{\mathscr{J}}(\mathbf{C}) \to \mathrm{Ex}_\kappa(\mathbf{C})$ are morphisms of sites. Moreover, for any $\mathcal{J} : \mathbf{D}^{\mathrm{op}} \to \mathbf{Set}$ in $\mathscr{J}$ and any $\mathbf{g} : \mathbf{D} \to \mathrm{Ex}_\kappa(\mathbf{C})$ preserving finite limits, a postulated realization of $\mathrm{Lan}_{\mathbf{g}^{\mathrm{op}}} \mathcal{J}$ is in particular a $\mathscr{J}$-lex-colimit. Thus $\mathrm{Ex}_\kappa^{\mathscr{J}}(\mathbf{C})$ is closed under these, hence inherits $\mathscr{J}$-exactness from $\mathrm{Ex}_\kappa(\mathbf{C})$.

Now suppose $\mathbf{D}$ is a $\mathscr{J}$-superexact subcanonical κ-ary site with finite limits. Then $\mathrm{Ex}_\kappa(\mathbf{D})$ is κ-ary exact, hence also $\mathscr{J}$-superexact and subcanonical. Moreover, $\mathfrak{y}_\mathbf{D} : \mathbf{D} \to \mathrm{Ex}_\kappa(\mathbf{D})$ is $\mathscr{J}$-exact by Lemma. 10.13, which is to say that $\mathbf{D}$ is closed in $\mathrm{Ex}_\kappa(\mathbf{D})$ under finite limits and $\mathscr{J}$-lex-colimits. We must show that

$$(- \circ \mathfrak{y}_\mathbf{C}^{\mathscr{J}}) : \mathrm{SITE}_\kappa(\mathrm{Ex}_\kappa^{\mathscr{J}}(\mathbf{C}), \mathbf{D}) \longrightarrow \mathrm{SITE}_\kappa(\mathbf{C}, \mathbf{D}) \tag{10.15}$$

is an equivalence. Firstly, any morphism of sites $\mathbf{f} : \mathbf{C} \to \mathbf{D}$ induces a morphism of sites $\mathrm{Ex}_\kappa(\mathbf{f}) : \mathrm{Ex}_\kappa(\mathbf{C}) \to \mathrm{Ex}_\kappa(\mathbf{D})$. Since $\mathrm{Ex}_\kappa(f)$ is $\mathscr{J}$-exact, $(\mathrm{Ex}_\kappa(\mathbf{f}))^{-1}(\mathbf{D}) \subseteq \mathrm{Ex}_\kappa(\mathbf{C})$ is closed under finite limits and $\mathscr{J}$-lex-colimits. Therefore, it contains $\mathrm{Ex}_\kappa^{\mathscr{J}}(\mathbf{C})$, which is to say that the composite morphism of sites $\mathrm{Ex}_\kappa^{\mathscr{J}}(\mathbf{C}) \hookrightarrow \mathrm{Ex}_\kappa(\mathbf{C}) \xrightarrow{\mathrm{Ex}_\kappa(f)} \mathrm{Ex}_\kappa(\mathbf{D})$ corestricts to $\mathbf{D}$. Since $\mathfrak{y}_\mathbf{D} : \mathbf{D} \to \mathrm{Ex}_\kappa(\mathbf{D})$ is fully faithful and creates both finite limits and covering families, the corestriction is again a morphism of sites (this might be regarded as a trivial special case of Theorem. 11.10 combined with Corollary. 11.6 from the next section). Thus (10.15) is essentially surjective.

Secondly, suppose $\mathbf{f}_1, \mathbf{f}_2 : \mathrm{Ex}_\kappa^{\mathscr{J}}(\mathbf{C}) \rightrightarrows \mathbf{D}$ are morphisms of sites and $\alpha : \mathbf{f}_1 \circ \mathfrak{y}_\mathbf{C}^{\mathscr{J}} \to \mathbf{f}_2 \circ \mathfrak{y}_\mathbf{C}^{\mathscr{J}}$ is a transformation. Then we have an induced transformation between functors $\mathrm{Ex}_\kappa(\mathbf{C}) \rightrightarrows \mathrm{Ex}_\kappa(\mathbf{D})$. Since $\mathbf{f}_1$ and $\mathbf{f}_2$ map $\mathrm{Ex}_\kappa^{\mathscr{J}}(\mathbf{C})$ into $\mathbf{D}$ and the inclusion $\mathbf{D} \hookrightarrow \mathrm{Ex}_\kappa(\mathbf{D})$ is fully faithful, this transformation restricts and corestricts to a transformation $\mathbf{f}_1 \to \mathbf{f}_2$, which in turn also restricts to α. Hence (10.15) is full.

Finally, every object of $\mathrm{Ex}_\kappa^{\mathscr{J}}(\mathbf{C})$ is a colimit of objects of $\mathbf{C}$ (a collage of a congruence), and morphisms of sites $\mathrm{Ex}_\kappa^{\mathscr{J}}(\mathbf{C}) \to \mathbf{D}$ preserve these colimits. Thus, a transformation between such functors is determined by its restriction to $\mathbf{C}$; hence (10.15) is faithful. ∎

10.16. Example. For any κ, there is a $\mathscr{J}$ for which $\mathscr{J}$-exact categories are exactly κ-ary exact categories. In fact, there are many. One is the class of *all* small lex-weights. Another is the union of [GL12, §5.2] with the κ-analogue of [GL12, §5.3].

For such a $\mathscr{J}$, the $\mathscr{J}$-exact topology on a κ-ary exact category is the κ-canonical one, so it is the *only* $\mathscr{J}$-superexact subcanonical κ-ary topology. Thus $\mathrm{SITE}_\kappa^{\mathscr{J}} = \mathrm{EX}_\kappa^{\mathscr{J}} = \mathrm{EX}_\kappa$ and hence $\mathrm{Ex}_\kappa^{\mathscr{J}} = \mathrm{Ex}_\kappa$.

The most-studied case of $\mathrm{Ex}_\kappa^{\mathscr{J}}$ other than Ex_κ itself is the (κ-ary) *regular completion.* By an evident κ-ary generalization of [GL12, §5.6], there exists a $\mathscr{J}$ such that $\mathscr{J}$-exact categories coincide with κ-ary regular categories. As with κ-ary exact categories, for such a $\mathscr{J}$ the only subcanonical $\mathscr{J}$-superexact κ-ary topology on a κ-ary regular category is the κ-canonical one, so we have $\mathrm{SITE}_\kappa^{\mathscr{J}} = \mathrm{EX}_\kappa^{\mathscr{J}} = \mathrm{REG}_\kappa$. Thus we deduce:

10.17. THEOREM. *The inclusion* $\mathrm{REG}_\kappa \hookrightarrow \mathrm{SITE}_\kappa$ *is reflective.* ∎

Let $\mathrm{Reg}_\kappa(\mathbf{C})$ denote this left biadjoint, the *κ-ary regular completion*. We can describe it more explicitly:

10.18. LEMMA. *The following are equivalent for* $\Phi \in \mathrm{Ex}_\kappa(\mathbf{C})$*:*

(i) Φ *lies in* $\mathrm{Reg}_\kappa(\mathbf{C})$*.*

(ii) Φ *is a kernel of some κ-to-finite array in* $\mathbf{C}$*.*

(iii) Φ *admits a monomorphism to some finite product of objects of* $\mathbf{C}$*.*

PROOF. It is easy to see that the second two conditions are equivalent, and that they define a subcategory of $\mathrm{Ex}_\kappa(\mathbf{C})$ which is κ-ary regular and hence contains $\mathrm{Reg}_\kappa(\mathbf{C})$. The converse containment follows since $\mathrm{Reg}_\kappa(\mathbf{C})$ contains $\mathbf{C}$ and is closed under quotients of kernels of κ-to-finite arrays. ∎

We can now identify $\mathrm{Reg}_\kappa(\mathbf{C})$ with various regular completions in the literature.

10.19. EXAMPLE. When $\kappa = \{1\}$ and $\mathbf{C}$ has (weak) finite limits and a trivial unary topology, the above description and universal property of $\mathrm{Reg}_{\{1\}}(C)$ are equivalent to those of the *regular completion* of a category with (weak) finite limits, as in [CV98]. If we instead identify $\mathrm{Ex}_{\{1\}}(\mathbf{C})$ with a subcategory of $\mathrm{Sh}_{\{1\}}(\mathbf{C})$, as in §9, we obtain the characterization of the regular completion from [HT96].

10.20. EXAMPLE. Suppose $\mathbf{C}$ has finite limits and a pullback-stable factorization system $(\mathcal{E},\mathcal{M})$ which is *proper* (i.e. $\mathcal{E}$ consists of epis and $\mathcal{M}$ of monos). As in Example. 3.25, $\mathcal{E}$ is then a unary topology on $\mathbf{C}$. Since $\mathbf{C}$ has products, any unary kernel in $\mathbf{C}$ is the kernel of a single morphism of $\mathbf{C}$. Moreover, if Φ is the kernel of $f\colon x \to z$ and we factor f as $x \xrightarrow{e} y \xrightarrow{m} z$ with $m \in \mathcal{M}$ and $e \in \mathcal{E}$, then Φ is also the kernel of e since m is monic. But since $e \in \mathcal{E}$ is a cover, the induced tight morphism $e\colon \Phi \to \mathfrak{y}(y)$ in $\mathbb{R}\mathsf{el}_{\{1\}}(\mathbf{C})$ is a weak equivalence, so that $\Phi \cong \mathfrak{y}(y)$ in $\mathrm{Ex}_{\{1\}}(\mathbf{C})$. Therefore, we can identify $\mathrm{Reg}_{\{1\}}(\mathbf{C})$ with the full subcategory of $\mathrm{Ex}_{\{1\}}(\mathbf{C})$ determined by objects of the form $\mathfrak{y}(x)$.

Moreover, because we have $(\mathcal{E},\mathcal{M})$-factorizations, any relation $\mathfrak{y}(x) \nrightarrow \mathfrak{y}(y)$ between such congruences is equivalent to an $\mathcal{M}$-morphism $z \to x \times y$. Thus, the full subcategory of $\mathrm{Mod}_{\{1\}}(\mathbf{C})$ on the objects of $\mathrm{Reg}_{\{1\}}(\mathbf{C})$ is precisely the bicategory of relations defined in [Kel91], and hence $\mathrm{Reg}_{\{1\}}(\mathbf{C})$ is precisely the *regular reflection* of $\mathbf{C}$ as defined there. Its universal property is likewise the same: for regular $\mathbf{D}$, regular functors $\mathrm{Reg}_{\{1\}}(\mathbf{C}) \to \mathbf{D}$ correspond to functors $\mathbf{C} \to \mathbf{D}$ preserving finite limits (hence taking $\mathcal{M}$-morphisms, which are monic, to monics) and taking $\mathcal{E}$-morphisms to regular epis.

10.21. EXAMPLE. Let $\mathbf{E}$ be (unary) regular, $\mathbf{C}$ finitely complete, and $\mathbf{f}\colon \mathbf{E} \to \mathbf{C}$ finitely continuous. By Example. 3.22, the $\mathbf{f}$-images of all regular epis in $\mathbf{E}$ generate a smallest unary topology on $\mathbf{C}$, such that $\mathbf{f}$ becomes a morphism of sites. Moreover, by Proposition. 4.13, if $\mathbf{D}$ is regular, then a functor $\mathbf{g}\colon \mathbf{C} \to \mathbf{D}$ is a morphism of sites if and only if it preserves finite limits and the composite $\mathbf{gf}$ is a regular functor. Therefore, the unary regular completion $\mathrm{Reg}_{\{1\}}(\mathbf{C})$ of $\mathbf{C}$ with this topology must be the *relative regular completion* of [Hof04]. Similarly, $\mathrm{Ex}_{\{1\}}(\mathbf{C})$ is the relative exact completion.

Higher-ary regular completions are not well-studied, but one example is worth noting.

10.22. Example. Let $\mathbf{C}_{\mathbb{T}}^{\mathrm{reg}}$ be the syntactic category of a regular theory $\mathbb{T}$, which is unary regular. We can also regard it as an ω-ary site, with ω-ary topology generated by its canonical unary topology. Then its ω-ary regular completion can be identified with the syntactic category of $\mathbb{T}$ regarded as a coherent theory. Similarly, the $\mathfrak{K}$-ary regular completion of $\mathbf{C}_{\mathbb{T}}^{\mathrm{reg}}$ is the syntactic category of $\mathbb{T}$ regarded as a geometric theory, and likewise if we started with a coherent theory and its coherent syntactic category.

We should mention a few other examples of $\mathscr{J}$-superexact completion.

10.23. Example. Generalizing [GL12, §5.3], for any κ we have a class $\mathscr{J}$ for which $\mathscr{J}$-exactness coincides with *κ-ary lextensivity* [CLW93], i.e. having finite limits and disjoint stable κ-ary coproducts. In Example. 10.4 we observed that this $\mathscr{J}$ is always κ-admissible, so any κ-ary site has a **κ-superextensive** completion. If $\mathbf{C}$ is a trivial κ-ary site with finite limits, its κ-superextensive completion is its free κ-ary coproduct completion $\mathrm{Fam}_\kappa(\mathbf{C})$.

10.24. Example. In [GL12, §5.7] is described a (singleton) class $\mathscr{J}$ for which $\mathscr{J}$-exactness coincides with *adhesivity* as in [LS04]. In Example. 10.5 we saw that this class is ω-admissible, so any ω-ary site has a **superadhesive** completion.

We now intend to generalize the relative exact completions of [GL12, §7]. Following [GL12], we write $\mathscr{J}_1 \le \mathscr{J}_2$ if every $\mathscr{J}_2$-exact category or functor is also $\mathscr{J}_1$-exact.

10.25. Lemma. *If $\mathscr{J}_1 \le \mathscr{J}_2$ and both are κ-admissible, then any subcanonical $\mathscr{J}_2$-superexact κ-ary topology is also $\mathscr{J}_1$-superexact.*

Proof. Suppose $\mathbf{C}$ is a subcanonical $\mathscr{J}_2$-superexact κ-ary site. Then $\mathbf{C}$ is a $\mathscr{J}_2$-exact category (and hence in particular has finite limits). By assumption, $\mathbf{C}$ is also $\mathscr{J}_1$-exact. But $\mathrm{Ex}_\kappa(\mathbf{C})$ is also $\mathscr{J}_2$-exact, and by Lemma. 10.13, the embedding $\mathfrak{y}\colon \mathbf{C} \hookrightarrow \mathrm{Ex}_\kappa(\mathbf{C})$ is $\mathscr{J}_2$-exact; hence it is also $\mathscr{J}_1$-exact. In other words, $\mathbf{C}$ is closed in $\mathrm{Ex}_\kappa(\mathbf{C})$ under $\mathscr{J}_1$-lex-colimits. But $\mathscr{J}_1$-lex-colimits in $\mathrm{Ex}_\kappa(\mathbf{C})$ are collages of congruences, and $\mathfrak{y}$ reflects collages of all congruences. Thus, $\mathbf{C}$ is $\mathscr{J}_1$-superexact. ∎

Therefore, we have $\mathrm{SITE}_\kappa^{\mathscr{J}_2} \subseteq \mathrm{SITE}_\kappa^{\mathscr{J}_1}$ as subcategories of SITE_κ. Restricting the domain of $\mathrm{Ex}_\kappa^{\mathscr{J}_2}$, we obtain:

10.26. Corollary. *If $\mathscr{J}_1 \le \mathscr{J}_2$, then the inclusion $\mathrm{SITE}_\kappa^{\mathscr{J}_2} \hookrightarrow \mathrm{SITE}_\kappa^{\mathscr{J}_1}$ is reflective.* ∎

This is close to [GL12, Theorem 7.7], but not quite there, since for general $\mathscr{J}$ we have $\mathrm{SITE}_\kappa^{\mathscr{J}} \neq \mathrm{EX}_\kappa^{\mathscr{J}}$. If $\mathscr{J}_1 \le \mathscr{J}_2$ then we have an obvious forgetful functor $\mathrm{EX}_\kappa^{\mathscr{J}_2} \to \mathrm{EX}_\kappa^{\mathscr{J}_1}$, but just as in Examples 7.27 and 7.28, we do not have $\mathrm{EX}_\kappa^{\mathscr{J}_2} \subseteq \mathrm{EX}_\kappa^{\mathscr{J}_1}$ as subcategories of SITE_κ. However, as in Example. 7.27 (but not Example. 7.28) we do have:

10.27. Theorem. *If $\mathscr{J}_1 \le \mathscr{J}_2$, the forgetful functor $\mathrm{EX}_\kappa^{\mathscr{J}_2} \to \mathrm{EX}_\kappa^{\mathscr{J}_1}$ has a left biadjoint.*

Proof. Suppose $\mathbf{C}$ is $\mathscr{J}_1$-exact with its $\mathscr{J}_1$-exact topology, and $\mathbf{D}$ is $\mathscr{J}_2$-exact with its $\mathscr{J}_2$-exact topology. By Lemma. 10.25, $\mathbf{D}$ is also $\mathscr{J}_1$-superexact. Thus, by Lemma. 10.13, morphisms of sites $\mathbf{C} \to \mathbf{D}$ are the same as $\mathscr{J}_1$-exact functors. Therefore, it suffices to show that $\mathrm{Ex}_\kappa^{\mathscr{J}_2}(\mathbf{C})$ lies in $\mathrm{EX}_\kappa^{\mathscr{J}_2}$, i.e. that its topology is the $\mathscr{J}_2$-exact one.

Now the universal property of $\mathrm{Ex}_\kappa^{\mathscr{J}_2}(\mathbf{C})$ says that if $\mathbf{E}$ is any other subcanonical $\mathscr{J}_2$-superexact site, then morphisms of sites $\mathrm{Ex}_\kappa^{\mathscr{J}_2}(\mathbf{C}) \to \mathbf{E}$ are equivalent to morphisms of sites $\mathbf{C} \to \mathbf{E}$. Let $\mathbf{E}$ be the category $\mathrm{Ex}_\kappa^{\mathscr{J}_2}(\mathbf{C})$ with its $\mathscr{J}_2$-exact topology. Then by Lemma. 10.13, the embedding $\mathfrak{y}^{\mathscr{J}_2}$ is a morphism of sites $\mathbf{C} \to \mathbf{E}$, and hence there is a morphism of sites $\mathbf{f}\colon \mathrm{Ex}_\kappa^{\mathscr{J}_2}(\mathbf{C}) \to \mathbf{E}$ such that the composite $\mathbf{f} \circ \mathfrak{y}^{\mathscr{J}_2}$ is isomorphic to $\mathfrak{y}^{\mathscr{J}_2}$. But the identity functor $\mathbf{E} \to \mathrm{Ex}_\kappa^{\mathscr{J}_2}(\mathbf{C})$ is also a morphism of sites by Lemma. 10.13, and so the universal property of $\mathrm{Ex}_\kappa^{\mathscr{J}_2}(\mathbf{C})$ implies that the composite $\mathrm{Ex}_\kappa^{\mathscr{J}_2}(\mathbf{C}) \xrightarrow{\mathbf{f}} \mathbf{E} \xrightarrow{1} \mathrm{Ex}_\kappa^{\mathscr{J}_2}(\mathbf{C})$ is isomorphic to the identity functor. This implies that $\mathbf{f}$ itself is isomorphic to the identity, and hence the topology of $\mathrm{Ex}_\kappa^{\mathscr{J}_2}(\mathbf{C})$ must be the $\mathscr{J}_2$-exact one. ∎

If $\mathbf{C}$ is small and we identify $\mathrm{Ex}_\kappa(\mathbf{C})$ with a subcategory of $\mathrm{Sh}(\mathbf{C})$ as in §9, then we can do the same for these relative exact completions. This is how they are described in [GL12, §7]. However, as in the absolute case, our relative exact completions require no smallness hypotheses.

11. Dense morphisms of sites

The following definition is essentially standard.

11.1. Definition. *Let $\mathbf{D}$ be a site and let $\mathbf{C}$ be a category with a notion of "covering family". We say that a functor $\mathbf{f}\colon \mathbf{C} \to \mathbf{D}$ is **dense** if the following hold.*

(i) P is a covering family in $\mathbf{C}$ if and only if $\mathbf{f}(P)$ is a covering family in $\mathbf{D}$.
(ii) For every $u \in \mathbf{D}$, there exists a covering family $P\colon V \Rightarrow u$ in $\mathbf{D}$ such that each $v \in V$ is in the image of $\mathbf{f}$.
(iii) For every $x, y \in \mathbf{C}$ and $g\colon \mathbf{f}(x) \to \mathbf{f}(y)$ in $\mathbf{D}$, there exists a covering family $P\colon Z \Rightarrow x$ and a cocone $H\colon Z \Rightarrow y$ in $\mathbf{C}$ such that $g \circ \mathbf{f}(P) = \mathbf{f}(H)$.
(iv) For every $x, y \in \mathbf{C}$ and morphisms $h, k\colon x \rightrightarrows y$ in $\mathbf{C}$ such that $\mathbf{f}(h) = \mathbf{f}(k)$, there exists a covering family $P\colon Z \Rightarrow x$ in $\mathbf{C}$ such that $hP = kP$.

Of course, (i) just means the covering families in $\mathbf{C}$ are determined by those of $\mathbf{D}$. Denseness is usually defined only for inclusions of subcategories, in which case (iv) is unnecessary, as is (iii) if the subcategory is full.

11.2. Theorem. *Suppose $\mathbf{D}$ is a weakly κ-ary site and $\mathbf{f}\colon \mathbf{C} \to \mathbf{D}$ is dense. Then:*
(a) $\mathbf{C}$ is a weakly κ-ary site.
(b) $\mathbf{f}$ is a morphism of sites.
(c) If $\mathbf{D}$ is κ-ary, so is $\mathbf{C}$.
(d) For any family X of objects in $\mathbf{C}$, $\mathbf{f}$ induces an equivalence between the preorders of κ-sourced arrays over X and over $\mathbf{f}(X)$ (under the relation $\preceq$).

Proof. We prove (d) first. Since $\mathbf{f}$ preserves covering families, it preserves $\preceq$. Conversely, suppose $S\colon U \Rightarrow X$ and $T\colon V \Rightarrow X$ are arrays in $\mathbf{C}$ with $\mathbf{f}(S) \preceq \mathbf{f}(T)$. Then we have a covering family $P\colon W \Rightarrow \mathbf{f}(U)$ and a functional array $F\colon W \Rightarrow \mathbf{f}(V)$ with $\mathbf{f}(S) \circ P = \mathbf{f}(T) \circ F$. By (ii), we have a covering family $Q\colon \mathbf{f}(Y) \Rightarrow W$, and $\mathbf{f}(S) \circ P \circ Q = \mathbf{f}(T) \circ F \circ Q$. Applying (iii) twice to $P \circ Q$ and $F \circ Q$ and passing to a common refinement, we obtain a covering family $R\colon Z \Rightarrow Y$ and functional arrays $G\colon Z \Rightarrow U$ and $H\colon Z \Rightarrow X$ such that $\mathbf{f}(G) = P \circ Q \circ \mathbf{f}(R)$ and $\mathbf{f}(H) = F \circ Q \circ \mathbf{f}(R)$. Hence,

$$\mathbf{f}(S \circ G) = \mathbf{f}(S) \circ P \circ Q \circ \mathbf{f}(R) = \mathbf{f}(T) \circ F \circ Q \circ \mathbf{f}(R) = \mathbf{f}(T \circ H).$$

Thus, by (iv) we have a covering family $N\colon M \Rightarrow Z$ such that $SGN = THN$. Finally, since $\mathbf{f}(G) = P \circ Q \circ \mathbf{f}(R)$ and P, Q, and R are covering, by (i) G is also covering. Therefore, the equality $SGN = THN$ exhibits $S \preceq T$.

Thus, $\mathbf{f}$ reflects as well as preserves $\preceq$, so for it to be an equivalence of preorders it suffices for it to be essentially surjective. But for any array $S\colon U \Rightarrow \mathbf{f}(X)$, we can find a covering family $P\colon \mathbf{f}(Y) \Rightarrow U$ in $\mathbf{D}$, and a covering family $Q\colon Z \Rightarrow Y$ and an array $T\colon Z \Rightarrow X$ with $\mathbf{f}(T) = S \circ P \circ \mathbf{f}(Q)$. Hence S is locally equivalent to $\mathbf{f}(T)$.

Next we prove (a). The only axiom of a weakly κ-ary site not obviously implied by (i) is pullback-stability. Suppose, therefore, that $P\colon V \Rightarrow u$ is a covering family in $\mathbf{C}$ and $g\colon x \to u$ is a morphism; then we have a covering family $R\colon Z \Rightarrow \mathbf{f}(x)$ with $\mathbf{f}(g) \circ R \le \mathbf{f}(P)$. By (d), R is locally equivalent to $\mathbf{f}(S)$ for some S, and $\mathbf{f}(gS) \preceq \mathbf{f}(P)$ implies $gS \preceq P$. By definition of $\preceq$, we have a covering T with $gST \le P$; hence ST is what we want.

Now we prove (b). Suppose $\mathbf{g}\colon \mathbf{E} \to \mathbf{C}$ is a finite diagram and $T\colon u \Rightarrow \mathbf{fg}(\mathbf{E})$ a cone over $\mathbf{fg}$ in $\mathbf{D}$. Then by (d), T is locally equivalent to $\mathbf{f}(S)$ for some array $S\colon V \Rightarrow \mathbf{g}(\mathbf{E})$ in $\mathbf{C}$. Applying (iv) some finite number of times (once for each morphism in $\mathbf{E}$) and passing to a common refinement, we obtain a covering family $P\colon W \Rightarrow V$ such that SP is an array over $\mathbf{g}$. Then T is locally equivalent to, hence factors locally through, $\mathbf{f}(SP)$.

Finally, we prove (c). Let $\mathbf{g}$, T, S, and P be as in the previous paragraph, and suppose $R\colon x \Rightarrow \mathbf{g}(\mathbf{E})$ is some cone over $\mathbf{g}$ in $\mathbf{C}$. Then $\mathbf{f}(R) \preceq T \preceq \mathbf{f}(SP)$, hence $R \preceq SP$. Thus, SP is a local κ-prelimit of $\mathbf{g}$. ∎

Because of Theorem. 11.2(b), we may speak of a **dense morphism of sites**. Note that (b) fails for the classical notion of "morphism of sites".

11.3. Remark. In particular, if $\mathbf{C}$ is a site which admits a dense morphism of sites $\mathbf{C} \to \mathbf{D}$, where $\mathbf{D}$ is a subcanonical weakly κ-ary site with finite limits, then $\mathbf{C}$ has local κ-prelimits. Thus, the "solution-set condition" of having local κ-prelimits is a *necessary* condition for a site to map densely to a κ-ary exact category. In Theorem. 11.10 we will show that any κ-ary site $\mathbf{C}$ is dense in $\mathrm{Ex}_\kappa(\mathbf{C})$, so that this condition is also *sufficient*. This may be regarded as a generalization of the observation in [CV98, Prop. 4] that any projective cover of an exact category must have weak finite limits (see also Example. 11.9).

11.4. Example. If the category $\mathcal{P}(\mathbf{C})$ of small presheaves on $\mathbf{C}$ is finitely complete, then its $\mathfrak{K}$-canonical topology is $\mathfrak{K}$-ary and induces the trivial $\mathfrak{K}$-ary topology on $\mathbf{C}$, while every small presheaf is covered by a small family of representables. Thus, Theorem. 11.2 implies that $\mathbf{C}$ has finite $\mathfrak{K}$-prelimits. This is the other direction of the theorem mentioned in Example. 9.8.

11.5. Remark. For locally κ-ary sites as in Remark. 3.18, we can consider the analogous notion of a *dense pre-morphism of sites*. The analogue of Theorem. 11.2 then holds.

11.6. Corollary. *Let* $\mathbf{C}_1$*,* $\mathbf{C}_2$*, and* $\mathbf{C}_3$ *be sites and* $\mathbf{f}_1 : \mathbf{C}_1 \to \mathbf{C}_2$ *and* $\mathbf{f}_2 : \mathbf{C}_2 \to \mathbf{C}_3$ *functors such that* $\mathbf{f}_2\mathbf{f}_1$ *is a morphism of sites and* $\mathbf{f}_2$ *is a dense morphism of sites. Then* $\mathbf{f}_1$ *is a morphism of sites.*

Proof. Since $\mathbf{f}_2\mathbf{f}_1$ preserves covering families and $\mathbf{f}_2$ reflects them, $\mathbf{f}_1$ must preserve them. Now suppose $\mathbf{g} : \mathbf{E} \to \mathbf{C}$ is a finite diagram and T a cone over $\mathbf{f}_1\mathbf{g}$ with vertex u. Then $\mathbf{f}_2(T)$ is a cone over $\mathbf{f}_2\mathbf{f}_1\mathbf{g}$ with vertex $\mathbf{f}_2(u)$, so it factors locally through the $\mathbf{f}_2\mathbf{f}_1$-image of some array over $\mathbf{g}$. By Theorem. 11.2(d), this implies that T factors locally through the $\mathbf{f}_1$-image of some array over $\mathbf{g}$; hence $\mathbf{f}_1$ is covering-flat. ∎

We now show that exact completion interacts as expected with dense morphisms of sites. On the one hand, dense functors become equivalences on exact completion.

11.7. Lemma. *If* $\mathbf{f} : \mathbf{C} \to \mathbf{D}$ *is a dense pre-morphism of sites, with* $\mathbf{C}$ *and* $\mathbf{D}$ *locally* κ*-ary, then the induced functor* $\mathbb{R}\mathsf{el}_\kappa(\mathbf{f}) : \mathbb{R}\mathsf{el}_\kappa(\mathbf{C}) \to \mathbb{R}\mathsf{el}_\kappa(\mathbf{D})$ *is 2-fully-faithful (an isomorphism on hom-posets) on underlying allegories. Moreover, every object of* $\mathbb{R}\mathsf{el}_\kappa(\mathbf{D})$ *is the (loose) collage of the image of some congruence in* $\mathbb{R}\mathsf{el}_\kappa(\mathbf{C})$*.*

Proof. The first statement follows immediately from Theorem. 11.2(d). For the second, we observe that any object $u \in \mathbf{D}$ admits a covering family $P : \mathbf{f}(V) \Rightarrow u$. Then $P^\bullet P_\bullet$ is a congruence in $\mathbb{R}\mathsf{el}_\kappa(\mathbf{D})$ whose collage is u, and by 2-fully-faithfulness it is the image of some congruence in $\mathbf{C}$. ∎

11.8. Theorem. *If* $\mathbf{f} : \mathbf{C} \to \mathbf{D}$ *is a dense pre-morphism of locally* κ*-ary sites, then the induced functor* $\mathrm{Ex}_\kappa(\mathbf{f}) : \mathrm{Ex}_\kappa(\mathbf{C}) \to \mathrm{Ex}_\kappa(\mathbf{D})$ *is an equivalence.*

Proof. Since $\mathbb{R}\mathsf{el}_\kappa(\mathbf{f})$ is 2-fully-faithful on allegories, by Lemma. 7.19, so is $\mathbb{R}\mathsf{el}_\kappa(\mathrm{Ex}_\kappa(\mathbf{f}))$. Hence, so is $\mathrm{Ex}_\kappa(\mathbf{f})$. And since every object of $\mathbb{R}\mathsf{el}_\kappa(\mathbf{D})$ is a collage of a congruence in $\mathbb{R}\mathsf{el}_\kappa(\mathbf{C})$, by Lemma. 7.18, so is every object of $\mathbb{R}\mathsf{el}_\kappa(\mathrm{Ex}_\kappa(\mathbf{D}))$. Hence $\mathrm{Ex}_\kappa(\mathbf{f})$ is also essentially surjective. ∎

11.9. Example. We say that an object z of a κ-ary exact category is *κ-ary projective* if every κ-ary extremal-epic cocone with target z contains a split epic. If $\mathbf{C}$ has a trivial κ-ary topology, then every object of the form $\mathfrak{y}(x)$ is κ-ary projective in $\mathrm{Ex}_\kappa(\mathbf{C})$. Moreover, in this case every object of $\mathrm{Ex}_\kappa(\mathbf{C})$ is covered by a κ-ary family of κ-ary projectives, namely the family of objects on which it is a congruence.

On the other hand, if $\mathbf{C}$ is κ-ary exact and every object of $\mathbf{C}$ is covered by a κ-ary family of κ-ary projectives, then the full subcategory $\mathbf{P}$ of κ-ary projectives satisfies 11.1(ii) (and, obviously, (iii)–(iv)), so that $\mathbf{P} \to \mathbf{C}$ is a dense morphism of sites and hence $\mathbf{C} \simeq \mathrm{Ex}_\kappa(\mathbf{C}) \simeq \mathrm{Ex}_\kappa(\mathbf{P})$. Moreover, the induced topology on $\mathbf{P}$ is trivial. Thus, a κ-ary exact category is the κ-ary exact completion of a trivial κ-ary site exactly when every object is covered by a κ-ary family of κ-ary projectives, and in this case the category of κ-ary projectives has κ-prelimits.

For $\kappa = \{1\}$, this was observed in [CV98]. For $\kappa = \mathfrak{K}$, we obtain a characterization of small-presheaf categories of categories with finite $\mathfrak{K}$-prelimits. If we add the additional assumption that there is a *small* generating set of $\mathfrak{K}$-ary projectives, we recover a well-known characterization of presheaf toposes.

On the other hand, every site is dense in its own exact completion.

11.10. THEOREM. *For any (locally) κ-ary site $\mathbf{C}$, the functor $\mathfrak{y}\colon \mathbf{C} \to \mathrm{Ex}_\kappa(\mathbf{C})$ is a dense (pre-)morphism of sites.*

PROOF. For Definition. 11.1(i), since the embedding of a κ-ary allegory in its cocompletion under κ-ary congruences is fully faithful, it reflects as well as preserves the property $\bigvee(P_\bullet P^\bullet) = 1_u$ for a cocone of maps. Since this characterizes covering families in $\mathbf{C}$ and $\mathrm{Ex}_\kappa(\mathbf{C})$, it follows that $\mathfrak{y}$ reflects as well as preserves covering families.

For (ii), since every object of $\mathrm{Ex}_\kappa(\mathbf{C})$ is a quotient of the image of a κ-ary congruence in $\mathbf{C}$, in particular it admits a covering family whose domains are in the image of $\mathfrak{y}$.

For (iii), note that since the chordate reflection of $\mathbb{R}\mathsf{el}_\kappa(\mathbf{C})$ embeds fully faithfully in its collage cocompletion, to give a morphism $g\colon \mathfrak{y}(x) \to \mathfrak{y}(y)$ in $\mathrm{Ex}_\kappa(\mathbf{C})$ is the same as to give a loose map $\phi\colon x \rightsquigarrow y$ in $\mathbb{R}\mathsf{el}_\kappa(\mathbf{C})$. By weak κ-tabularity, for any such ϕ we have $\phi = \bigvee(F_\bullet P^\bullet)$ for $P\colon Z \Rightarrow x$ and $F\colon Z \Rightarrow y$ in $\mathbf{C}$. Since ϕ is a map, by Corollary. 6.16 P is covering, and we have

$$F_\bullet \le \bigvee(F_\bullet P^\bullet P_\bullet) = \phi P_\bullet.$$

Since maps are discretely ordered, $F_\bullet = \phi P_\bullet$, hence $\mathfrak{y}(F) = g \circ \mathfrak{y}(P)$.

Finally, for (iv), if $f, g\colon x \rightrightarrows y$ are morphisms in $\mathbf{C}$ with $\mathfrak{y}(f) = \mathfrak{y}(g)$, then $f_\bullet = g_\bullet$ as loose maps, hence (by the familiar weak κ-tabularity of $\mathbb{R}\mathsf{el}_\kappa(\mathbf{C})$) there is a covering family P with $fP = gP$. ∎

11.11. EXAMPLE. Recall (e.g. from [Joh02, D3.3]) that a Grothendieck topos is called *coherent* if it is the topos of sheaves for the ω-canonical topology on an (ω-ary) pretopos, or equivalently if it has a small, finitely complete, and ω-ary site of definition. Panagis Karazeris has pointed out that in fact, the topos of sheaves on *any* small ω-ary site is coherent. For by Theorem. 11.10, the canonical functor $\mathfrak{y}\colon \mathbf{C} \to \mathrm{Ex}_\omega(\mathbf{C})$ is dense, hence induces an equivalence of sheaf toposes (a.k.a. $\mathfrak{K}$-ary exact completions); but $\mathrm{Ex}_\omega(\mathbf{C})$ is a pretopos with its ω-canonical topology, so its sheaf topos is coherent. Similarly, the topos of sheaves on any unary site is a regular topos.

11.12. EXAMPLE. Let $\mathbf{C}$ be a κ-ary extensive category with a κ-superextensive κ-ary topology, as in Example. 10.23. (Actually, we can be more general here, not requiring $\mathbf{C}$ to have all finite limits; the κ-extensive topology exists on any κ-extensive category.) Then a κ-ary cocone $V \Rightarrow u$ is covering if and only if the single morphism $\sum V \to u$ is covering. Thus, the topology of $\mathbf{C}$ is uniquely determined by its singleton covers, which themselves form a unary topology. Let $\mathbf{C}_{\{1\}}$ and $\mathrm{Ex}_\kappa(\mathbf{C})_{\{1\}}$ denote the categories $\mathbf{C}$ and

$\mathrm{Ex}_\kappa(\mathbf{C})$ equipped with their unary topologies of singleton covers. Note that this topology on $\mathrm{Ex}_\kappa(\mathbf{C})_{\{1\}}$ is the $\{1\}$-canonical one.

Now since κ-ary coproducts are postulated in any κ-superextensive site, they are preserved by $\mathfrak{y}\colon \mathbf{C} \to \mathrm{Ex}_\kappa(\mathbf{C})$. Thus, since we can replace any covering family in $\mathbf{C}$ by a singleton cover, the denseness of $\mathfrak{y}\colon \mathbf{C} \to \mathrm{Ex}_\kappa(\mathbf{C})$ (Theorem. 11.10) implies that it is also dense as a functor $\mathbf{C}_{\{1\}} \to \mathrm{Ex}_\kappa(\mathbf{C})_{\{1\}}$. But since $\mathrm{Ex}_\kappa(\mathbf{C})_{\{1\}}$ is unary exact with its $\{1\}$-canonical topology, it is its own unary exact completion. Thus, by Theorem. 11.8, we have an equivalence $\mathrm{Ex}_{\{1\}}(\mathbf{C}_{\{1\}}) \simeq \mathrm{Ex}_\kappa(\mathbf{C})_{\{1\}}$.

11.13. Example. Let $\mathbf{C}$ be any κ-ary site, and $\mathrm{Fam}_\kappa(\mathbf{C})$ its free completion under κ-ary coproducts. Recall that the objects of $\mathrm{Fam}_\kappa(\mathbf{C})$ are κ-ary families of objects of $\mathbf{C}$ and its morphisms are functional arrays. We remarked after Lemma. 3.6 that $\mathrm{Fam}_\kappa(\mathbf{C})$ inherits a weakly unary topology, whose covers are covering families as in Definition. 3.5.

In fact, it is easy to see that this topology is unary. This generalizes the observation of [Car95, 4.1(ii)] that $\mathrm{Fam}_\kappa(\mathbf{C})$ has finite limits when $\mathbf{C}$ does. Since $\mathrm{Fam}_\kappa(\mathbf{C})$ is κ-extensive by [CLW93, 2.4], this unary topology corresponds to a κ-superextensive κ-ary topology on $\mathrm{Fam}_\kappa(\mathbf{C})$. We notate the two resulting sites as $\mathrm{Fam}_\kappa(\mathbf{C})_{\{1\}}$ and $\mathrm{Fam}_\kappa(\mathbf{C})_\kappa$ respectively. By Example. 11.12, we have $\mathrm{Ex}_{\{1\}}(\mathrm{Fam}_\kappa(\mathbf{C})_{\{1\}}) \simeq \mathrm{Ex}_\kappa(\mathrm{Fam}_\kappa(\mathbf{C})_\kappa)$.

However, it is also easy to verify that the inclusion $\mathbf{C} \hookrightarrow \mathrm{Fam}_\kappa(\mathbf{C})_\kappa$ is dense, so that $\mathrm{Ex}_\kappa(\mathbf{C}) \simeq \mathrm{Ex}_\kappa(\mathrm{Fam}_\kappa(\mathbf{C})_\kappa)$, and hence $\mathrm{Ex}_\kappa(\mathbf{C}) \simeq \mathrm{Ex}_{\{1\}}(\mathrm{Fam}_\kappa(\mathbf{C})_{\{1\}})$. This justifies our earlier comments that the κ-ary exact completion can be obtained as a κ-ary coproduct completion followed by a unary exact completion.

Theorem. 11.10 also implies a generalization of [Joh02, C2.3.8] to arbitrary κ-ary sites (which would not be true for the classical notion of "morphism of sites").

11.14. Proposition. *Let $\mathbf{C}$ and $\mathbf{D}$ be κ-ary sites and $\mathbf{g}\colon \mathrm{Ex}_\kappa(\mathbf{C}) \to \mathrm{Ex}_\kappa(\mathbf{D})$ a morphism of sites. For a functor $\mathbf{f}\colon \mathbf{C} \to \mathbf{D}$, the following are equivalent.*

(i) $\mathbf{f}$ is a morphism of sites and $\mathbf{g} \cong \mathrm{Ex}_\kappa(\mathbf{f})$.

(ii) The following square commutes up to isomorphism:

$$\begin{tikzcd}
\mathbf{C} \arrow[r, "\mathbf{f}"] \arrow[d, "\mathfrak{y}_{\mathbf{C}}"'] & \mathbf{D} \arrow[d, "\mathfrak{y}_{\mathbf{D}}"] \\
\mathrm{Ex}_\kappa(\mathbf{C}) \arrow[r, "\mathbf{g}"'] & \mathrm{Ex}_\kappa(\mathbf{D})
\end{tikzcd}$$

Proof. By naturality of $\mathfrak{y}$, we have (i)⇒(ii). Conversely, if (ii), then $\mathbf{g} \circ \mathfrak{y}_{\mathbf{C}} \cong \mathfrak{y}_{\mathbf{D}} \circ \mathbf{f}$ is a morphism of sites, and since $\mathfrak{y}_{\mathbf{D}}$ is dense by Theorem. 11.10, Corollary. 11.6 implies that $\mathbf{f}$ is a morphism of sites. The isomorphism $\mathbf{g} \cong \mathrm{Ex}_\kappa(\mathbf{f})$ then follows by the universal property of exact completion. ∎

Finally, we can prove Proposition. 4.16.

11.15. Proposition. *For a small category* $\mathbf{C}$ *and a small site* $\mathbf{D}$*, a functor* $\mathbf{f}\colon \mathbf{C} \to \mathbf{D}$ *is covering-flat if and only if the composite*

$$[\mathbf{C}^{\mathrm{op}}, \mathbf{Set}] \xrightarrow{\mathrm{Lan}_{\mathbf{f}}} [\mathbf{D}^{\mathrm{op}}, \mathbf{Set}] \xrightarrow{\mathbf{a}} \mathrm{Sh}(\mathbf{D}) \tag{11.16}$$

preserves finite limits, where $\mathbf{a}$ *denotes sheafification. If* $\mathbf{C}$ *is moreover a site and* $\mathbf{f}$ *a morphism of sites, then*

$$\mathbf{f}^*\colon [\mathbf{D}^{\mathrm{op}}, \mathbf{Set}] \to [\mathbf{C}^{\mathrm{op}}, \mathbf{Set}]$$

takes $\mathrm{Sh}(\mathbf{D})$ *into* $\mathrm{Sh}(\mathbf{C})$*, so* $\mathbf{f}$ *induces a geometric morphism* $\mathrm{Sh}(\mathbf{D}) \to \mathrm{Sh}(\mathbf{C})$*.*

Proof. We regard $\mathbf{D}$ as a $\mathfrak{K}$-ary site, and $\mathbf{C}$ as a $\mathfrak{K}$-ary site with trivial topology. Then by Theorem. 9.5 we have $\mathrm{Ex}_{\mathfrak{K}}(\mathbf{C}) \simeq [\mathbf{C}^{\mathrm{op}}, \mathbf{Set}]$ and $\mathrm{Ex}_{\mathfrak{K}}(\mathbf{D}) \simeq \mathrm{Sh}(\mathbf{D})$.

Moreover, $\mathbf{f}$ is covering-flat just when it is a morphism of sites. By Proposition. 11.14, this holds exactly when there is a morphism of sites $\mathbf{g}$ such that

$$\begin{tikzcd}
\mathbf{C} \arrow[r, "\mathbf{f}"] \arrow[dd, "\mathbf{y}"'] & \mathbf{D} \arrow[d, "\mathbf{y}"] \\
 & {[\mathbf{D}^{\mathrm{op}}, \mathbf{Set}]} \arrow[d, "\mathbf{a}"] \\
{[\mathbf{C}^{\mathrm{op}}, \mathbf{Set}]} \arrow[r, "\mathbf{g}"'] & \mathrm{Sh}(\mathbf{D})
\end{tikzcd}$$

commutes up to isomorphism. Now recall that morphisms of sites between Grothendieck toposes are precisely finite-limit-preserving and cocontinuous functors. Thus, since $[\mathbf{C}^{\mathrm{op}}, \mathbf{Set}]$ is the free cocompletion of $\mathbf{C}$, if $\mathbf{g}$ exists it must be (11.16). Since (11.16) is always cocontinuous, this holds precisely when (11.16) preserves finite limits.

Finally, if $\mathbf{C}$ has instead a nontrivial topology for which $\mathbf{f}$ is a morphism of sites, then $\mathrm{Ex}_{\mathfrak{K}}(\mathbf{f})\colon \mathrm{Sh}(\mathbf{C}) \to \mathrm{Sh}(\mathbf{D})$ is cocontinuous, hence has a right adjoint. It is easy to verify that this right adjoint must be $\mathbf{f}^*$. ∎

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