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HOMOTOPY THEORIES OF DIAGRAMS

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ABSTRACT. Suppose that S is a space. There is an injective and a projective model structure for the resulting category of spaces with S-action, and both are easily derived. These model structures are special cases of model structures for presheaf-valued diagrams X defined on a fixed presheaf of categories E which is enriched in simplicial sets.

Varying the parameter category object E (or parameter space S) along with the diagrams X up to weak equivalence requires model structures for E-diagrams having weak equivalences defined by homotopy colimits, and a generalization of Thomason's model structure for small categories to a model structure for presheaves of simplicial categories.

Introduction

The work displayed in this paper arose from a preliminary study of the homotopy theory of dynamical systems.

In general, a dynamical system consists of an action

$$X \times S \to X$$

of a parameter space S on a space X. These objects appear most often in the context of manifolds, where S is some kind of time parameter which is a subobject of the real numbers.

In this paper, a "space" is a simplicial set, and we consider dynamical systems within the category of simplicial sets — I say that such objects are S-spaces. A morphism $X \to Y$ of S-spaces is the obvious thing, namely an S-equivariant map between the respective simplicial sets. One could reasonably ask if a framework for homotopy theory exists in some form for S-spaces.

Some preliminary constructions can already be found in the literature, albeit in a different language [6], within the context of local homotopy theory.

On a more down to earth level, say that a map $f : X \to Y$ of S-spaces is a weak equivalence if the underlying simplicial set map is a weak equivalence, a cofibration of S-spaces is a monomorphism, and that a map of S-spaces is an injective fibration if it has the right lifting property with respect to all maps which are simultaneously cofibrations and weak equivalences. "Dually" say that a map $p : Z \to W$ of S-spaces is a projective

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fibration if the underlying simplicial set map is a Kan fibration, and a map of S-spaces is a projective cofibration if it has the left lifting property with respect to all maps which are projective fibrations and weak equivalences. Then it is not hard to show directly that there are two distinct Quillen model structures on S-spaces which reflect these definitions: an injective structure with cofibrations, weak equivalences and injective fibrations, and a projective structure with projective cofibrations, weak equivalences and projective fibrations. This is a start — these model structures are much like the variants of naive equivariant homotopy theory that one encounters for spaces with group actions.

The first step in the translation to the language of [6] is to observe that an S-space is a module

$$X \times F(S) \to X$$

over the free simplicial monoid $F(S)$ associated to S, or equivalently an $F(S)$-diagram in simplicial sets, where $F(S)$ is identified with a simplicial category having one object. The $F(S)$-diagram X is then a very particular case of an E-diagram in simplicial presheaves, where E is a presheaf of categories enriched in simplicial sets. Definitions analogous to those of the previous paragraph can be made relative to the injective model structure on simplicial presheaves, and the corresponding injective and projective model structures for E-diagrams are derived in [6].

Beyond this, Theorem 5.2 of this paper says that there is an infinite list of model structures for the category of E-diagrams having the same weak equivalences, for a fixed presheaf E of categories enriched in simplicial sets. Of these, the projective model structure has the fewest cofibrations and the injective model structure has the most. This result specializes to give an infinite list of model structures for the category of S-spaces, all having the same weak equivalences, and among which the projective and injective model structures described above appear as extremal examples.

But we want more. All homotopy theoretic structures discussed so far are defined relative to either a fixed choice of presheaf of simplicial categories E, or a fixed parameter space S. The point of the remainder of the paper is to display a homotopy theory for E-diagrams or S-spaces in which the parameter objects E or S can vary up to weak equivalence. Further, the presheaves of simplicial categories E which have been considered so far have had simplicially discrete objects, and we want to escape from this assumption.

There are essentially two preparatory steps:

1) develop an appropriate model structure for presheaves of simplicial categories E, and

2) develop a "homotopy colimit" model structure for E-diagrams which is consistent with the interpretation of an E-diagram as a bisimplicial set map $X \to BE$.

The homotopy theory for presheaves of simplicial categories which appears here in Theorem 4.1 is a generalization of Thomason's model structure for small categories [11]. It is Quillen equivalent to an "$\mathrm{sd}^{2,0}$-model structure" on the category of bisimplicial presheaves, in which the weak equivalences are diagonal local weak equivalences, and the cofibrations are generated by taking double subdivisions of ordinary cofibrations in all vertical degrees. This $\mathrm{sd}^{2,0}$-model structure for bisimplicial presheaves is Quillen equivalent to the diagonal model structure of [10], but also gives a setting for the model structure for presheaves of simplicial categories: a morphism $f : C \to D$ of such objects is a weak equivalence if the map $BC \to BD$ of bisimplicial nerves is a diagonal weak equivalence, and $f : C \to D$ is a fibration if the map $BC \to BD$ is an $\mathrm{sd}^{2,0}$-fibration.

The $\mathrm{sd}^{2,0}$-model structure on bisimplicial presheaves is just an example: there are $\mathrm{sd}^{m,n}$-model structures for all $m, n \geq 0$, all of which are Quillen equivalent to the diagonal model structure (Theorem 1.2). This is a generalization of a phenomenon which one finds in simplicial presheaves: there are sd^{n}-model structures for the category of simplicial presheaves, all of which are Quillen equivalent to the injective model structure (Theorem 1.1). The sd^{2}-model structure is the setting for a generalization of Thomason's original result to a model structure for presheaves of categories, which model structure is Quillen equivalent to the injective model structure for simplicial presheaves (Theorem 3.1). Thomason's result depended on a theory of Dwyer maps, which are special types of fully faithful imbeddings of small categories. Dwyer maps are discussed in Section 2 of this paper.

The homotopy colimit model structure for E-diagrams appears in Theorem 5.4. In this theory, the cofibrations are the monomorphisms, and a map $X \to Y$ of E-diagrams is a weak equivalence if it induces a diagonal weak equivalence

$$\underrightarrow{\mathrm{holim}}_E\, X \to \underrightarrow{\mathrm{holim}}_E\, Y$$

of bisimplicial presheaves. Unlike earlier results (Theorem 5.2 and Theorem 5.2) there is no requirement for the object presheaf $\mathrm{Ob}(E)$ to be simplicially discrete — Theorem 5.4 is universal. The associated homotopy category for this model structure is equivalent to homotopy category $\mathrm{Ho}(s^2\,\mathrm{Pre}(\mathcal{C})/BE)$ for the diagonal model structure on bisimplicial presheaves fibred over BE, so that we can now identify E-diagrams with maps of bisimplicial presheaves $Y \to BE$.

Morphisms of diagrams may then be defined as commutative diagrams

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d] & Y \arrow[d] \\ BE \arrow[r, "Bg"'] & BF \end{tikzcd}$$

in bisimplicial presheaves, and we can say that the map (f, g) is a weak equivalence if f is a diagonal equivalence and g is a weak equivalence of presheaves of simplicial categories. The map (f, g) is a cofibration if f is an $sd^{2,0}$-cofibration and g is a cofibration of presheaves of simplicial categories. Finally, the map (f, g) is a fibration if g is a fibration of simplicial category objects and the induced map

$$X \to BC \times_{BD} Y$$

is an $\mathrm{sd}^{2,0}$-fibration of bisimplicial presheaves. The resulting model structure, which is easily derived, appears in Theorem 4.2.

There is an analogous model structure for diagrams defined on ordinary presheaves of categories; the corresponding result is Theorem 3.3.

All of these results for presheaves specialize to set-based results in the realm of ordinary homotopy theory. In that case, there are sd^n-model structures for simplicial sets and $\mathrm{sd}^{m,n}$-models for bisimplicial sets, all of which are models for standard homotopy category. Theorem 4.1 specializes to a model structure for simplicial categories which is induced by the $\mathrm{sd}^{2,0}$-structure for bisimplicial sets. We also have specializations of the various model structures for diagram categories to the categories of A-diagrams for a simplicial category A.

In particular, if the maps $f : X \to Y$ and $g : S \to T$ define a morphism of dynamical systems in the sense that the diagram

$$\begin{tikzcd} X \times S \arrow[r] \arrow[d, "f \times g"'] & X \arrow[d, "f"] \\ Y \times T \arrow[r] & Y \end{tikzcd}$$

commutes, then the pair (f, g) defines a weak equivalence of dynamical systems if and only if the induced commutative diagram

$$\begin{tikzcd} \underrightarrow{\mathrm{holim}}_{\,F(S)} X \arrow[r] \arrow[d] & \underrightarrow{\mathrm{holim}}_{\,F(T)} Y \arrow[d] \\ BF(S) \arrow[r] & BF(T) \end{tikzcd}$$

of bisimplicial sets is a weak equivalence of diagrams. This will certainly be so, for example, if the maps f and g are themselves weak equivalences of simplicial sets, but this is not the whole story.

The main results of this paper are much more general. They apply, for example, to sheaves and presheaves of dynamical systems for arbitrary Grothendieck topologies. Diagrams on a topological or simplicial category are generalized dynamical systems for which the parameter space has more than one object, and we have a clear interpretation of both the absolute and local homotopy theories of these more general objects as well.

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1. Subdivision model structures

We begin by recalling some of the basic features of subdivisions [2], [5].

Every simplicial set X has a poset NX of non-degenerate simplices, ordered by the face relationship. The assignment $X \mapsto NX$ is functorial in X: if $f : X \to Y$ is a simplicial set map and σ is a non-degenerate simplex of X, then $f(\sigma) = s(\tau)$ for some unique iterated degeneracy s and non-degenerate simplex τ of Y, and the assignment $\sigma \mapsto \tau$ defines the functor $f_* : NX \to NY$.

The subdivision $\mathrm{sd}(X)$ of a simplicial set X is defined by the assignment

$$\mathrm{sd}(X) = \varinjlim_{\Delta^n \to X} BN\Delta^n,$$

where the colimit is indexed over the category $\mathbf{\Delta}/X$ of simplices $\Delta^n \to X$. It follows from the definition that there is an isomorphism

$$\mathrm{sd}(\Delta^n) \cong BN\Delta^n$$

which is natural in maps of simplices.

Define a *polyhedral complex* to be a subcomplex $K \subset BP$ of the nerve of a poset P such that the vertices of each non-degenerate simplex x of K are distinct.

There is a natural map $\pi : \mathrm{sd}(X) \to BNX$ for all simplicial sets X, and the map $\pi : \mathrm{sd}(K) \to BNK$ is an isomorphism for all polyhedral complexes K. In effect, if x is a non-degenerate n-simplex of K, then the classifying map $x : \Delta^n \to K$ is a monomorphism, and induces an isomorphism $\Delta^n \cong \langle x \rangle$ onto the subcomplex of K which is generated by x.

Remark: It is useful to compare the present definition of polyhedral complex K with the one given in [5]. The older definition is not precise enough to imply that the map $\pi : \mathrm{sd}(K) \to BNK$ is an isomorphism.

The last vertex map

$$\gamma : BNK = \mathrm{sd}(K) \to K$$

for a polyhedral complex K is defined by sending the simplex $\sigma : \Delta^n \to K$ to the vertex $\sigma(n)$. The map γ is natural in polyhedral complexes K, and it follows that the maps $\gamma : \mathrm{sd}(\Delta^n) \to \Delta^n$ (which are weak equivalences) induce a natural map

$$\gamma_* : \mathrm{sd}(X) \to X \tag{1}$$

for all simplicial sets X. An induction on skeleta shows that this map γ_* is a weak equivalence for all X.

The right adjoint $Y \mapsto \mathrm{Ex}(Y)$ of the subdivision functor sd is defined by setting $\mathrm{Ex}(Y)_n$ to be the collection of simplicial set maps $\mathrm{sd}(\Delta^n) \to Y$. Precomposition with the last vertex maps $\gamma : \mathrm{sd}(\Delta^n) \to \Delta^n$ induces a natural map

$$\gamma^* : Y \to \mathrm{Ex}(Y),$$

which map is a natural weak equivalence [2, III.4.6]. It follows that the functor Ex preserves and reflects weak equivalences of simplicial sets.

As usual, sd^n and Ex^n denote the n-fold iterations of the subdivision and Ex functors, respectively, and the functor sd^n is left adjoint to Ex^n.

There is an isomorphism

$$\varinjlim_{\Delta^p \to \mathrm{sd}^n X} \mathrm{sd}^m \Delta^p \xrightarrow{\cong} \mathrm{sd}^m(\mathrm{sd}^n X) = \mathrm{sd}^{m+n} X \tag{2}$$

which is natural in simplicial sets X, by adjointness.

Suppose that $\mathcal{C}$ is a small Grothendieck site, and let $s\,\mathrm{Pre}(\mathcal{C})$ be the category of simplicial presheaves on $\mathcal{C}$. The category $s\,\mathrm{Pre}(\mathcal{C})$ has a proper closed simplicial model structure [3], [4], for which the cofibrations are the monomorphisms, the weak equivalences are the local weak equivalences, and the fibrations are defined by a right lifting property with respect to trivial fibrations. The fibrations for this model structure are usually called *injective fibrations*. This model structure is cofibrantly generated, with generators consisting of the α-bounded cofibrations and α-bounded trivial cofibrations, where α is an infinite cardinal such that $\alpha > |\,\mathrm{Mor}(\mathcal{C})|$. Suppose henceforth that α is a fixed choice of such a cardinal.

Say that a simplicial presheaf map $p : X \to Y$ is an Ex^n-*fibration* if the induced map $\mathrm{Ex}^n X \to \mathrm{Ex}^n Y$ is an injective fibration. An sd^n-*cofibration* is a map which has the left lifting property with respect to all trivial sd^n-fibrations. Examples of sd^n-cofibrations include all maps $\mathrm{sd}^n A \to \mathrm{sd}^n B$ which are induced by cofibrations $A \to B$ of simplicial presheaves.

1.1. Theorem. *Suppose that $\mathcal{C}$ is a small Grothendieck site.*

1) *The classes of sd^n-fibrations, sd^n-cofibrations and local weak equivalences satisfy the axioms for a proper closed model structure on $s\,\mathrm{Pre}(\mathcal{C})$.*
2) *The adjoint functors*
$$sd^n : s\,\mathrm{Pre}(\mathcal{C}) \leftrightarrows s\,\mathrm{Pre}(\mathcal{C}) : \mathrm{Ex}^n$$
define a Quillen equivalence between the injective model structure and the model structure for simplicial presheaves on the site $\mathcal{C}$ which is given by part 1).

The model structure of Theorem 1.1 is called the sd^n-*model structure* for simplicial presheaves.

Proof. The functor sd^n preserves cofibrations of simplicial sets. The presheaf-level functor sd^n therefore preserves cofibrations of simplicial presheaves. The functor sd^n preserves and reflects local weak equivalences, on account of the natural weak equivalences (1).

It follows that a map $p : X \to Y$ is an sd^n-fibration if and only if it has the right lifting property with respect to all maps $\mathrm{sd}^n A \to \mathrm{sd}^n B$ which are induced by α-bounded trivial cofibrations $A \to B$. Similarly, a map $q : Z \to W$ is a trivial sd^n-fibration if and only if it has the right lifting property with respect to all maps $\mathrm{sd}^n C \to \mathrm{sd}^n D$ which are induced by α-bounded cofibrations $C \to D$.

A small object argument shows that every simplicial presheaf map $f : X \to Y$ has factorizations

$$\begin{tikzcd} & Z \arrow[dr, "p"] & \\ X \arrow[ur, "i"] \arrow[rr, "f"] \arrow[dr, "j"'] & & Y \\ & W \arrow[ur, "q"'] & \end{tikzcd}$$

in which

1) the map p is an sd^n-fibration and i is a trivial sd^n-cofibration which has the left lifting property with respect to all sd^n-fibrations,

2) the map q is a trivial sd^n-fibration and j is an sd^n-cofibration.

The factorization axiom **CM5** is therefore proved. The lifting axiom **CM4** follows in a standard way: every trivial sd^n-cofibration is a retract of a map which has the left lifting property with respect to all sd^n-fibrations, on account of statement 1) above. The other closed model axioms are easily verified.

Suppose that

$$\begin{tikzcd} Z \times_Y X \arrow[r, "f_*"] \arrow[d] & X \arrow[d, "p"] \\ Z \arrow[r, "f"'] & Y \end{tikzcd}$$

is a pullback diagram such that p is an sd^n-fibration and f is a local weak equivalence. Then the induced diagram

$$\begin{tikzcd} \mathrm{Ex}^n(Z \times_Y X) \arrow[r, "f'_*"] \arrow[d] & \mathrm{Ex}^n X \arrow[d, "p'"] \\ \mathrm{Ex}^n Z \arrow[r, "f'"'] & \mathrm{Ex}^n Y \end{tikzcd}$$

is a pullback in which p' is an injective fibration and f' is a local weak equivalence. It follows that the map $f_* : Z \times_Y X \to X$ is a local weak equivalence.

Every sd^n-cofibration is a cofibration (ie. a monomorphism) in the ordinary sense. In effect, this is true of all maps $\mathrm{sd}^n A \to \mathrm{sd}^n B$ which are induced by cofibrations $A \to B$, and the class of maps which are sd^n-cofibrations and monomorphism is closed under pushout, composition and retraction. It follows that all maps $j : X \to W$ in the proof of **CM5** are monomorphisms as well as sd^n-cofibrations. Any sd^n-cofibration is a retract of such a map j, and is therefore a monomorphism.

The left properness of the model structure in the statement of the Proposition is then a consequence of the corresponding statement for the injective model structure on simplicial presheaves.

The subdivision functor sd^n and its right adjoint determine a Quillen adjunction

$$\mathrm{sd}^n : s\,\mathrm{Pre}(\mathcal{C}) \leftrightarrows s\,\mathrm{Pre}(\mathcal{C}) : \mathrm{Ex}^n\,.$$

The functor Ex^n preserves and reflects weak equivalences of simplicial sets. It follows that Ex^n takes pushout diagrams of simplicial sets

$$\begin{tikzcd} A \arrow[r] \arrow[d, "i"'] & C \arrow[d] \\ B \arrow[r] & D \end{tikzcd}$$

with i a cofibration to homotopy cocartesian diagrams. An induction on skeleta shows that the unit map $\eta : X \to \mathrm{Ex}^n \mathrm{sd}^n X$ for the adjunction is a natural weak equivalence for all simplicial sets X. The counit $\epsilon : \mathrm{sd}^n \mathrm{Ex}^n Y \to Y$ is therefore a natural weak equivalence for all simplicial sets Y, since the map $\mathrm{Ex}^n(\epsilon)$ is a weak equivalence.

The maps $\eta : X \to \mathrm{Ex}^n \mathrm{sd}^n X$ and $\epsilon : \mathrm{sd}^n \mathrm{Ex}^n Y \to Y$ are therefore sectionwise, hence local weak equivalences for all simplicial presheaves X and Y. ■

A map $X \to Y$ of bisimplicial sets is said to be a *diagonal weak equivalence* if the induced map $d(X) \to d(Y)$ of diagonal simplicial sets is a weak equivalence. The diagonal weak equivalences are the weak equivalences for the *diagonal model structure* on the category $s^2\mathbf{Set}$ of bisimplicial sets [10]. The cofibrations of the diagonal model structure are the monomorphisms.

The diagonal structure on bisimplicial sets is a special case of a diagonal model structure for the category $s^2\,\mathrm{Pre}(\mathcal{C})$ of bisimplicial presheaves, whose weak equivalences are the diagonal (local) weak equivalences and whose cofibrations are the monomorphisms. The fibrations for this structure on bisimplicial presheaves are defined by a right lifting property with respect to trivial cofibrations, and are called *injective fibrations*.

If K and L are simplicial sets, then the *external product* $K\tilde{\times}L$ is the bisimplicial set with

$$(K\tilde{\times}L)_{p,q} = K_p \times L_q.$$

Suppose that X is a bisimplicial set and write

$$\mathrm{sd}^{p,q} X = \varinjlim_{\Delta^{r,s}\to X} \mathrm{sd}^p \Delta^r \tilde{\times} \mathrm{sd}^q \Delta^s.$$

Here, p or q could be 0, so that, for example

$$\mathrm{sd}^{0,q} X = \varinjlim_{\Delta^{r,s}\to X} \Delta^r \tilde{\times} \mathrm{sd}^q \Delta^s,$$

and there is a natural isomorphism

$$\mathrm{sd}^{0,0} X \cong X.$$

It follows that there is an isomorphism

$$\mathrm{sd}^{p,q}\Delta^{r,s} \cong \mathrm{sd}^p\Delta^r \tilde{\times} \mathrm{sd}^q\Delta^s$$

which is natural in bisimplices. There is, more generally, an isomorphism

$$\mathrm{sd}^{p,q}(K\tilde{\times}L) \cong \mathrm{sd}^p K \tilde{\times} \mathrm{sd}^q L \tag{3}$$

which is natural in simplicial sets K and L.

There is an isomorphism

$$\mathrm{sd}^{r,s}(\mathrm{sd}^{m,n} X) \xrightarrow{\cong} \mathrm{sd}^{r+m,s+n} X \tag{4}$$

which is natural in bisimplicial sets X. To see this, it is enough to display the isomorphism for bisimplices: there are isomorphisms

$$\begin{aligned}\mathrm{sd}^{r,s}(\mathrm{sd}^{m,n}\Delta^{p,q}) &= \varinjlim_{\Delta^k\to\mathrm{sd}^m\Delta^p,\Delta^l\to\mathrm{sd}^n\Delta^q} \mathrm{sd}^r\Delta^k\tilde{\times}\mathrm{sd}^s\Delta^l \\ &\xrightarrow{\cong} \mathrm{sd}^{r+m}\Delta^p\tilde{\times}\mathrm{sd}^{s+n}\Delta^q \\ &\cong \mathrm{sd}^{r+m,s+n}\Delta^{p,q},\end{aligned}$$

all of which are natural in bisimplices. The key points here are the natural isomorphism (2) and the fact that the external product construction $K\tilde{\times}L$ commutes with colimits in the simplicial sets K and L.

Suppose that X is a bisimplicial set, and let $X_n = X_{n,*}$ be the vertical simplicial set in horizontal degree n. Then there is an isomorphism of simplicial sets

$$(\mathrm{sd}^{0,r} X)_n \cong \mathrm{sd}^r(X_n). \tag{5}$$

which is natural in bisimplicial sets. Again, it is enough to prove this for bisimplices, but it's clear that there is an isomorphism of simplicial sets

$$\Delta^p_n \times \mathrm{sd}^r\Delta^q \cong \mathrm{sd}^r(\Delta^p_n\times\Delta^q)$$

since $\Delta^p_n\times\Delta^q$ is a finite disjoint union of copies of Δ^q.

On account of the natural isomorphism(5), the bisimplicial set $\mathrm{sd}^{0,r} X$ can be defined in terms of vertical simplicial sets by

$$(\mathrm{sd}^{0,r} X)_n = \mathrm{sd}^r(X_n). \tag{6}$$

A similar observation obtains for horizontal simplicial sets and $\mathrm{sd}^{s,0} X$. The natural isomorphisms

$$\mathrm{sd}^{r,s} X \cong \mathrm{sd}^{r,0}\mathrm{sd}^{0,s} X \cong \mathrm{sd}^{0,s}\mathrm{sd}^{r,0} X \tag{7}$$

from (4) then lead to easy descriptions of $\mathrm{sd}^{r,s} X$. It follows (use the equivalence (1) twice) that there are natural diagonal equivalences

$$\mathrm{sd}^{r,s} X \cong \mathrm{sd}^{r,0}\mathrm{sd}^{0,s} X \xrightarrow{\simeq} \mathrm{sd}^{0,s} X \xrightarrow{\simeq} X. \tag{8}$$

Suppose that Y is a bisimplicial set, and write $\mathrm{Ex}^{p,q} Y$ for the bisimplicial set with

$$\mathrm{Ex}^{p,q} Y_{r,s} = \hom(\mathrm{sd}^{p,q} \Delta^{r,s}, Y).$$

The resulting functor

$$\mathrm{Ex}^{p,q} : s^2\mathbf{Set} \to s^2\mathbf{Set}$$

on bisimplicial sets is right adjoint to the functor $\mathrm{sd}^{p,q}$.

The natural isomorphisms (4) and (5) together imply that $\mathrm{Ex}^{0,q} Y$ can be defined in terms of vertical simplicial sets by the isomorphism

$$(\mathrm{Ex}^{0,q} Y)_n \cong \mathrm{Ex}^q(Y_n).$$

The object $\mathrm{Ex}^{p,0} Y$ has a similar description in terms of horizontal simplicial sets. It follows that there are natural diagonal equivalences

$$Y \xrightarrow{\simeq} \mathrm{Ex}^{p,0} Y \xrightarrow{\simeq} \mathrm{Ex}^{0,q} \mathrm{Ex}^{p,0} Y \cong \mathrm{Ex}^{p,q} Y \tag{9}$$

in the category of bisimplicial sets.

Again, let $\mathcal{C}$ be a small Grothendieck site, and write $s^2 \mathrm{Pre}(\mathcal{C})$ for the category of bisimplicial presheaves on $\mathcal{C}$.

Say that a map $f : X \to Y$ of bisimplicial presheaves is an $\mathrm{sd}^{p,q}$-*fibration* if the map $f_* : \mathrm{Ex}^{p,q} X \to \mathrm{Ex}^{p,q} Y$ is an injective fibration for the diagonal model structure on bisimplicial presheaves. An $\mathrm{sd}^{p,q}$-*cofibration* is a map which has the left lifting property with respect to all trivial $\mathrm{sd}^{p,q}$-fibrations.

If $A \to B$ is a cofibration of bisimplicial presheaves then the induced map $\mathrm{sd}^{p,q} A \to \mathrm{sd}^{p,q} B$ is an $\mathrm{sd}^{p,q}$-cofibration.

1.2. THEOREM. *Suppose that $\mathcal{C}$ is a small Grothendieck site.*

1) *The category $s^2 \mathrm{Pre}(\mathcal{C})$ of bisimplicial presheaves, together with the diagonal (local) weak equivalences, the $\mathrm{sd}^{p,q}$-fibrations and the $\mathrm{sd}^{p,q}$-cofibrations, satisfies the axioms for a cofibrantly generated proper closed model category.*

2) *The functors*

$$\mathrm{sd}^{p,q} : s^2 \mathrm{Pre}(\mathcal{C}) \leftrightarrows s^2 \mathrm{Pre}(\mathcal{C}) : \mathrm{Ex}^{p,q}$$

define a Quillen equivalence between the diagonal model structure and the model structure for bisimplicial presheaves given by part 1).

The model structure for bisimplicial presheaves of Theorem 1.2 is called the $\mathrm{sd}^{p,q}$-*model structure.*

PROOF. The functor

$$\mathrm{sd}^{p,q} : s^2\mathbf{Set} \to s^2\mathbf{Set}$$

preserves cofibrations for the diagonal model structure on bisimplicial sets, since the maps

$$(\mathrm{sd}^p\,\partial\Delta^n \tilde{\times} \mathrm{sd}^q\,\Delta^m) \cup (\mathrm{sd}^p\,\Delta^n \tilde{\times} \mathrm{sd}^q\,\partial\Delta^m) \to \mathrm{sd}^p\,\Delta^n \tilde{\times} \mathrm{sd}^q\,\Delta^m$$

are cofibrations. The functor

$$\mathrm{sd}^{p,q} : s^2\,\mathrm{Pre}(\mathcal{C}) \to s^2\,\mathrm{Pre}(\mathcal{C})$$

therefore preserves cofibrations. The existence of the natural weak equivalences (8) also implies that the functor $\mathrm{sd}^{p,q}$ preserves and reflects diagonal equivalences.

Write $\mathrm{sd}^{p,q}\,I$ and $\mathrm{sd}^{p,q}\,J$ for maps $\mathrm{sd}^{p,q}\,A \to \mathrm{sd}^{p,q}\,B$ which are induced by maps $A \to B$ in the set I of generators for the trivial cofibrations and the set J of generators for the cofibrations, respectively, for the diagonal model structure on $s^2\,\mathrm{Pre}(\mathcal{C})$. Then a map $p : X \to Y$ is an $\mathrm{sd}^{p,q}$-fibration (respectively trivial $\mathrm{sd}^{p,q}$-fibration) if and only if it has the right lifting property with respect to all morphisms of the set $\mathrm{sd}^{p,q}\,I$ (respectively $\mathrm{sd}^{p,q}\,J$). The factorization axiom **CM5** and the lifting axiom **CM4** follow in the usual way, as in the proof of Theorem 1.1. The remaining closed model axioms are easily verified.

All $\mathrm{sd}^{p,q}$-cofibrations are monomorphisms, since the functor $\mathrm{sd}^{p,q}$ preserves monomorphisms, and so all members of the generating set $\mathrm{sd}^{p,q}\,J$ are cofibrations — see also the proof of Theorem 1.1. Left properness is then a consequence of left properness of the diagonal model structure for bisimplicial presheaves. Right properness follows from the fact that the functor $\mathrm{Ex}^{p,q}$ preserves pullbacks, and reflects weak equivalences on account of the existence of the natural diagonal weak equivalences (9).

For statement 2), the functors

$$\mathrm{sd}^{p,q} : s^2\,\mathrm{Pre}(\mathcal{C}) \leftrightarrows s^2\,\mathrm{Pre}(\mathcal{C}) : \mathrm{Ex}^{p,q} \tag{10}$$

define a Quillen adjunction.

Recall also that the functor $X \mapsto \mathrm{sd}^{0,q}\,X$ is isomorphic to the functor which is defined by applying the functor sd^q to all vertical simplicial sets. It follows from the proof of Theorem 1.1 that the natural maps $\eta : X \to \mathrm{Ex}^{0,q}\,\mathrm{sd}^{0,q}\,X$ and $\epsilon : \mathrm{sd}^{0,q}\,\mathrm{Ex}^{0,q}\,Y \to Y$ are sectionwise diagonal equivalences for all bisimplicial presheaves X and Y. A similar analysis in horizontal simplicial sets shows that the maps $\eta : X \to \mathrm{Ex}^{p,0}\,\mathrm{sd}^{p,0}\,X$ and $\epsilon : \mathrm{sd}^{p,0}\,\mathrm{Ex}^{p,0}\,Y \to Y$ are sectionwise diagonal equivalences for all X and Y. It follows that all composites

$$\mathrm{sd}^{0,q}\,\mathrm{sd}^{p,0}\,\mathrm{Ex}^{p,0}\,\mathrm{Ex}^{0,q}\,Y \xrightarrow{\mathrm{sd}^{0,q}\,\epsilon} \mathrm{sd}^{0,q}\,\mathrm{Ex}^{0,q}\,Y \xrightarrow{\epsilon} Y \text{ and}$$

$$X \xrightarrow{\eta} \mathrm{Ex}^{p,0}\,\mathrm{sd}^{p,0}\,X \xrightarrow{\mathrm{Ex}^{p,0}\,\eta} \mathrm{Ex}^{p,0}\,\mathrm{Ex}^{0,q}\,\mathrm{sd}^{0,q}\,\mathrm{sd}^{p,0}\,X$$

are sectionwise diagonal equivalences. These composites are the counit and unit, respectively, for the adjunction (10) up to natural isomorphism. ∎

2. Dwyer maps

Say that a functor $j : A \to B$ between small categories is a *sieve* if it is an imbedding (on objects and morphisms), and every morphism $b \to a$ of B with $a \in A$ must be in A. Every sieve $j : A \to B$ is, in particular, a fully faithful imbedding of categories.

The functor $j : A \to B$ is a *cosieve* if it is an imbedding, and every morphism $a \to b$ with $a \in A$ must be in A. Cosieves are also fully faithful imbeddings.

A functor $i : A \to B$ between small categories is a *Dwyer map* if

1) i imbeds A as a sieve in B,
2) Let W be the full subcategory of B consisting of objects v admitting a morphism $a \to v$ with $a \in A$. Then the inclusion $j : A \subset W$ has a right adjoint $r : W \to A$ such that $rj = 1_A$ and the unit $a \to rj(a)$ is the identity map.

In this definition, it is enough to assume that j has a right adjoint r. To prove this, note that the counit $\epsilon : jr(x) \to x$ is a terminal object for the category j/x. The imbedding j is full, so that the identity $j(a) \to j(a)$ is terminal in $j/j(a)$ for all $a \in A$. It follows that $\epsilon : jrj(a) \to j(a)$ is an isomorphism for all $a \in A$, and so the unit $\eta : a \to rj(a)$ is an isomorphism, again since j is full. It follows that r can be replaced up to natural isomorphism by a functor r_* such that $r_*j(a) = a$ for $a \in A$ and $r_*(x) = r(x)$ for $x \in W - A$.

It is a consequence of the definition that the inclusion $W \subset B$ is a cosieve.

Suppose that $i : K \subset L$ is an inclusion of polyhedral complexes (so that both are subcomplexes of the nerve of a poset P), and consider the induced functor

$$i_* : N\,\mathrm{sd}(K) \cong NBNK \to NBNL \cong N\,\mathrm{sd}(L).$$

This functor is a monomorphism. An object of $NBNK$ is a strictly increasing string

$$\sigma_0 < \sigma_1 < \cdots < \sigma_k$$

of non-degenerate simplices of K, ordered by the face relationship, and there is a morphism $\sigma' \subset \sigma$ if and only if σ' is a substring of σ.

Say that an object $\tau \in NBNL$ meets K if some face of τ is in $NBNK$, equivalently if some τ_i is a non-degenerate simplex of K. There is a maximum k such that $\tau_k \in K$, and all simplices in $\langle\tau\rangle \cap NBNK$ are faces of the simplex

$$\tau_K : \tau_0 < \cdots < \tau_k.$$

Let $W \subset NBNL$ be the poset of objects of $NBNL$ which meet K. Then $NBNK \subset W$ and the assignment $\tau \mapsto \tau_K$ defines a functor $r : W \to NBNK$. The composite

$$NBNK \overset{j}{\subset} W \xrightarrow{r} NBNK$$

is the identity, and the relations $\tau_K \leq \tau$ define a natural transformation $jr(\tau) \leq \tau$. The functor r is right adjoint to the inclusion j.

The subcategory W is, in other words, the poset of objects $\tau \in NBNL$ for which there is a morphism $\sigma \leq \tau$ with $\sigma \in NBNK$. The imbedding $NBNK \to NBNL$ is also plainly a sieve. We have therefore proved the following:

2.1. Lemma. *Suppose that $i : K \to L$ is an inclusion of simplicial complexes. Then the induced functor $NBNK \to NBNL$ is a Dwyer map.*

We'll need the following:

2.2. Lemma. *Suppose that $i : A \subset X$ is an inclusion of simplicial sets such that*

1) if

$$a = x_0 \to x_1 \to \cdots \to x_n = b$$

is a path of 1-simplices of X with $a, b \in A$, then all 1-simplices in the path are in A, and

2) if $\sigma : \Delta^2 \to X$ is a simplex whose boundary

$$\partial\Delta^2 \subset \Delta^2 \xrightarrow{\sigma} X$$

is in A, then σ is in A.

Then the induced functor $i_* : P(A) \to P(X)$ of path categories is a full imbedding.

Here, the path category functor $P : s\mathbf{Set} \to \mathbf{Cat}$ is defined to be the left adjoint of the nerve functor $B : \mathbf{Cat} \to s\mathbf{Set}$, where $\mathbf{Cat}$ is the category of small categories. See [8].

Proof. Condition 1) implies that all induced functions

$$i_* : P(A)(a, b) \to P(X)(a, b)$$

are surjective, so that the functor i_* is full.

Suppose that $a \in A_0$. Then the 1-simplex $a \xrightarrow{s_0 a} a$ of X is in A. If $\sigma : \Delta^2 \to X$ is a 2-simplex of A such that the vertices $\sigma(0)$ and $\sigma(2)$ are in A, then the boundary of σ is in A by condition 1), so that $\sigma \in A$ by condition 2). It follows that all relations in X between paths of A from a to b are already in A, so that $i_* : P(A)(a, b) \to P(X)(a, b)$ is an injective function.

The functor $i_* : P(A) \to P(X)$ is therefore fully faithful. The functor i_* is injective on objects (vertices), and is therefore a full imbedding. ∎

2.3. Lemma. *Suppose given a pushout diagram*

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "i"'] & C \arrow[d, "i_*"] \\ B \arrow[r] & D \end{tikzcd}$$

in small categories, where i is a sieve (respectively cosieve). Then the induced functor i_* is a sieve (respectively cosieve).

Proof. We'll prove the sieve statement. The cosieve statement has a similar proof.

Form the pushout diagram

$$\begin{tikzcd} BA \arrow[r, "f"] \arrow[d, "i"'] & BC \arrow[d, "i_*"] \\ BB \arrow[r, "f_*"'] & X \end{tikzcd}$$

in simplicial sets, and recall that D is isomorphic to $P(X)$. The the map i_* is a cofibration, and the simplices of X can be identified with either simplices of BC or (disjointly) simplices of $BB - BA$, in all simplicial degrees.

Suppose that $\sigma : \Delta^n \to X$ is a simplex of X. If $\sigma \in BB - BA$ then σ is a string of morphisms

$$b_0 \to b_1 \to \cdots \to b_n$$

of B with $b_n \notin A$ (for otherwise the simplex is in BA since A is a sieve). It follows that $\sigma(n) \notin BC$. Thus, if $\sigma \in X_n$ is a simplex such that $\sigma(n) \in BC$ then $\sigma \in BC$. Further, if

$$x_0 \to x_1 \to \cdots \to x_n$$

is a path of 1-simplices of X with $x_n \in BC$, then all 1-simplices in the string are in BC.

The cofibration $BC \to X$ therefore satisfies the conditions of Lemma 2.2, so that the induced functor

$$C \cong PBC \to P(X) \cong D$$

is fully faithful and is a sieve. ■

2.4. Lemma. *Suppose given a pushout diagram*

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "i"'] & C \arrow[d, "i_*"] \\ B \arrow[r, "f_*"'] & D \end{tikzcd}$$

in small categories, where i is a sieve. Let V be the full subcategory of B on objects outside of A. Then the composite

$$V \subset B \xrightarrow{f_*} D$$

is a full imbedding.

PROOF. Form the pushout diagram

$$\begin{tikzcd} BA \arrow[r, "f"] \arrow[d, "i"'] & BC \arrow[d, "i_*"] \\ BB \arrow[r, "f_*"'] & X \end{tikzcd}$$

in simplicial sets. Recall that if $\sigma : \Delta^n \to X$ is a simplex with $\sigma(n) \in BC$, then $\sigma \in BC$.

Observe that $BV \subset BB - BA \subset X$ in all degrees, so that the composite

$$BV \subset BB \to X$$

is a cofibration. Observe also that the vertices in $BB - BA$ coincide with the objects of V.

If τ is a 2-simplex of X with boundary in BV, then $\tau \in BV$. In effect, $\tau(0) \in X - BC$ so that $\tau \in X - BC = BB - BA$, and τ is a 2-simplex of BB with vertices in BV and $A \to B$ is a sieve. But then $\tau \in BV$ since V is a full subcategory of B.

Take objects v, w of V and identify them with vertices of X. If

$$v = v_0 \to v_1 \to \cdots \to v_n = w$$

is a path of 1-simplices from v to w in X, then all v_i are in $BB - BA$. Otherwise, if some $v_i \in C$, then $v \in BC$ since the functor $C \to P(X)$ is a sieve by Lemma 2.3, while $v \in BV \subset X - BC$. The simplices $v_i \to v_{i+1}$ are not in BC and are therefore defined by morphisms of B. But V is full in B so that all such 1-simplices are in BV.

Finish by applying Lemma 2.2. ■

2.5. LEMMA. *Suppose given a pushout diagram*

$$\begin{tikzcd} A \arrow[r] \arrow[d, "f"'] & C \arrow[d, "f_*"] \\ B \arrow[r] & D \end{tikzcd}$$

in categories with f a Dwyer map. Then f_* is a Dwyer map.

PROOF. The functor f_* is a sieve, by Lemma 2.3. Let W' be the full subcategory of D consisting of objects w admitting a morphism $c \to w$ with $c \in C$.

Form the pushout diagrams

$$\begin{tikzcd} A \arrow[r, "\alpha"] \arrow[d, "j"'] & C \arrow[d, "j_*"] \\ W \arrow[r, "\alpha_*"'] \arrow[d, "i"'] & W' \arrow[d, "i_*"] \\ B \arrow[r] & D \end{tikzcd} \tag{11}$$

where j and i are full imbeddings and $f = i \cdot j$. The right adjoint r of j induces a functor $r_* : W' \to C$ such that $r_* j_* = 1$ and a homotopy $\epsilon : W' \to W'^{\mathbf{1}}$ which satisfies the condition 2) in the definition of Dwyer map, so that the functor j_* has a right adjoint of the correct form.

The functor $i_* : W' \subset D$ is a cosieve, again by Lemma 2.3.

Observe that the functors j_* and i_* are fully faithful imbeddings.

Suppose that v is an object of D and that there is a morphism $c \to v$ for some object $c \in C$. Then $c \in W'$ and W' is a cosieve in D, so that v is an object of W'. It follows that W' is the full subcategory of D consisting of objects v for which there is a morphism $c \to v$ with $c \in C$. ∎

2.6. Lemma. *Suppose given a pushout diagram*

$$\begin{tikzcd} A \arrow[r] \arrow[d, "f"'] & C \arrow[d] \\ B \arrow[r] & D \end{tikzcd}$$

of small categories, where the functor f is a Dwyer map. Then the induced diagram

$$\begin{tikzcd} BA \arrow[r] \arrow[d, "f"'] & BC \arrow[d] \\ BB \arrow[r] & BD \end{tikzcd}$$

of simplicial set maps is homotopy cocartesian.

Proof. Let V' be the full subcategory of D on objects outside of C.

The composite functor $V \to B \to B/A$ is fully faithful, by Lemma 2.4. Form the diagram

$$\begin{tikzcd} V \arrow[r] \arrow[d] & V' \arrow[d] \\ B/A \arrow[r, "\cong"'] & D/C \end{tikzcd}$$

The vertical maps are fully faithful imbeddings, while the map $V \to V'$ is an isomorphism on objects. It follows that $V \to V'$ is an isomorphism of categories.

Every object w of W' is in C or is in the image of the functor $\alpha_* : W \to W'$ of diagram (11). Thus, if w is not in C (and is therefore in $V' \cap W'$), then w is in the image of the functor $\alpha_* : V \cap W \to V' \cap W'$. It follows that the functor $\alpha_* : V \cap W \to V' \cap W'$ is bijective on objects. This functor is also fully faithful, and is therefore an isomorphism.

The square

$$\begin{tikzcd} BA \arrow[r] \arrow[d] & BC \arrow[d] \\ BW \arrow[r] & BW' \end{tikzcd}$$

is homotopy cocartesian since both vertical maps are weak equivalences. The square

$$\begin{tikzcd} B(V \cap W) \arrow[r] \arrow[d] & BW \arrow[d] \\ BV \arrow[r] & BB \end{tikzcd}$$

is a pushout, since the subobjects BW and BV cover BB. The composite square

$$\begin{tikzcd} B(V \cap W) \arrow[r] \arrow[d] & BW \arrow[r] \arrow[d] & BW' \arrow[d] \\ BV \arrow[r] & BB \arrow[r] & BD \end{tikzcd}$$

is isomorphic to the pushout

$$\begin{tikzcd} B(V' \cap W') \arrow[r] \arrow[d] & BW' \arrow[d] \\ BV' \arrow[r] & BD \end{tikzcd}$$

associated to the Dwyer map f_*, so that the square

$$\begin{tikzcd} BW \arrow[r] \arrow[d] & BW' \arrow[d] \\ BB \arrow[r] & BD \end{tikzcd}$$

is a pushout. ∎

3. Presheaves of small categories

Write $\mathrm{Pre}(\mathbf{Cat}(\mathcal{C}))$ for the category of presheaves of small categories on a small Grothendieck site $\mathcal{C}$.

I say that a functor $f : C \to D$ between presheaves of small categories is

a) a *local weak equivalence* if the induced map $BC \to BD$ is a local weak equivalence of simplicial presheaves,

b) a *fibration* if the induced map $\mathrm{Ex}^2 BC \to \mathrm{Ex}^2 BD$ is an injective fibration, equivalently if the map $BC \to BD$ is an sd^2-fibration,

c) a *cofibration* if it has the left lifting property with respect to all trivial fibrations.

The path category functor $P : s\mathbf{Set} \to \mathbf{Cat}$ induces a functor

$$P : s\,\mathrm{Pre}(\mathcal{C}) \to \mathrm{Pre}(\mathbf{Cat}(\mathcal{C}))$$

which is left adjoint to the nerve functor

$$B : \mathrm{Pre}(\mathbf{Cat}(\mathcal{C})) \to s\,\mathrm{Pre}(\mathcal{C}).$$

3.1. THEOREM. *Suppose that $\mathcal{C}$ is a small Grothendieck site.*

1) *With these definitions, the category* $\mathrm{Pre}(\mathbf{Cat}(\mathcal{C}))$ *of presheaves of small categories on $\mathcal{C}$ has the structure of a proper closed model category.*

2) *The adjoint pair*
$$P\,\mathrm{sd}^2 : s\,\mathrm{Pre}(\mathcal{C}) \leftrightarrows \mathrm{Pre}(\mathbf{Cat}(\mathcal{C})) : \mathrm{Ex}^2 B$$
is a Quillen equivalence, for the injective model structure on the simplicial presheaf category $s\,\mathrm{Pre}(\mathcal{C})$.

3) *The adjoint pair*
$$P : s\,\mathrm{Pre}(\mathcal{C}) \leftrightarrows \mathrm{Pre}(\mathbf{Cat}(\mathcal{C})) : B$$
is a Quillen equivalence for the sd^2*-model structure on* $s\,\mathrm{Pre}(\mathcal{C})$.

3.2. REMARK. Theorem 3.1 specializes to the model structure of Thomason [11] for the category **Cat** of small categories. There is a gap in the main line of argument of Thomason's paper [11], in the proof of his Proposition 4.3.

Thomason's Proposition 4.3 is Lemma 2.6 of this paper, which is proved here by a completely different technique.

Thomason's Lemma 5.3 asserts that Dwyer maps are closed under retracts. Cisinski gives a counterexample to this claim in [1], and proves a corrected version of Thomason's Proposition 4.3, based on a modified definition of Dwyer map. Thomason's Proposition 4.3 is a key step in the derivation of his model structure for the category of small categories, so that Cisinski also corrects the proof of Thomason's main result.

The results of this paper do not involve Cisinski's modified definition of Dwyer map.

PROOF. Suppose given a pushout diagram

$$\begin{tikzcd}
\sqcup_i P\,\mathrm{sd}^2(K_i) \arrow[r] \arrow[d] & C \arrow[d, "j"] \\
\sqcup_i P\,\mathrm{sd}^2(L_i) \arrow[r] & D
\end{tikzcd}$$

in **Cat**, where the vertical map on the left is induced by inclusions $K_i \subset L_i$ of finite simplicial complexes. The induced functor

$$P\,\mathrm{sd}^2 K_i \to P\,\mathrm{sd}^2 L_i$$

is isomorphic to the functor

$$NBNK_i \to NBNL_i$$

since there are natural isomorphisms $PBC \cong C$ for all small categories C [8]. It follows from Lemma 2.1 that the functor

$$\bigsqcup_i P\,\mathrm{sd}^2 K_i \to \bigsqcup_i P\,\mathrm{sd}^2 L_i$$

is a Dwyer map, and so Lemma 2.6 implies that the induced square

$$\begin{tikzcd}
\sqcup_i BP\operatorname{sd}^2(K_i) \arrow[r] \arrow[d] & BC \arrow[d, "j_*"] \\
\sqcup_i BP\operatorname{sd}^2(L_i) \arrow[r] & BD
\end{tikzcd}$$

is homotopy cocartesian.

If all inclusions $K_i \subset L_i$ are weak equivalences, then the induced maps

$$\operatorname{sd}^2 K_i \cong BP(\operatorname{sd}^2 K_i) \to BP(\operatorname{sd}^2 L_i) \cong \operatorname{sd}^2 L_i$$

are weak equivalences, and so the map $i_* : BC \to BD$ is a weak equivalence.

If the diagram

$$\begin{tikzcd}
P\operatorname{sd}^2 A \arrow[r] \arrow[d, "i_*"'] & C \arrow[d] \\
P\operatorname{sd}^2 B \arrow[r] & D
\end{tikzcd}$$

is a pushout in **Cat** such that i_* is induced by a cofibration $i : A \to B$ of simplicial sets, then the induced diagram

$$\begin{tikzcd}
BP\operatorname{sd}^2 A \arrow[r] \arrow[d, "i_*"'] & BC \arrow[d] \\
BP\operatorname{sd}^2 B \arrow[r] & BD
\end{tikzcd}$$

is homotopy cocartesian, since the class of all cofibrations i for which this is so includes the maps $\partial\Delta^n \subset \Delta^n$ by Lemma 2.6.

The functor

$$P\operatorname{sd}^2 : s\mathbf{Set} \to \mathbf{Cat}$$

therefore preserves trivial cofibrations as well as cofibrations. This functor $P\operatorname{sd}^2$ also preserves weak equivalences, since every simplicial set is cofibrant.

The functor

$$P\operatorname{sd}^2 : s\operatorname{Pre}(\mathcal{C}) \to \operatorname{Pre}(\mathbf{Cat}(\mathcal{C}))$$

preserves cofibrations, by definition.

All direct image functors f_* associated with geometric morphisms f of Grothendieck topoi commute with the functor $\operatorname{Ex}^2 B$, so that the inverse image functors f^* commute with the sheaf theoretic version of the composite functor $P\operatorname{sd}^2$ up to natural isomorphism. This is so, in particular, for a Boolean localization $\pi : \mathbf{Shv}(\mathcal{B}) \to \mathbf{Shv}(\mathcal{C})$, for which the inverse image functor π^* detects local weak equivalences. It then follows from the corresponding statement for simplicial sets, which is proved in the first paragraphs, that the functor $P\operatorname{sd}^2$ preserves local weak equivalences.

Let I and J be sets of generators for the class of trivial cofibrations and cofibrations for the injective model structure on $s\operatorname{Pre}(\mathcal{C})$ and let $P\operatorname{sd}^2 I$ and $P\operatorname{sd}^2 J$ be the corresponding sets of images in $\mathrm{Pre}(\mathbf{Cat}(\mathcal{C}))$. A functor $p : C \to D$ is a fibration (respectively trivial fibration) of $\mathrm{Pre}(\mathbf{Cat}(\mathcal{C}))$ if and only if it has the right lifting property with respect to all maps of $P\,\mathrm{sd}^2\,I$ (respectively $P\,\mathrm{sd}^2\,J$). The factorization axiom **CM5** is then proved with small object arguments in the standard way.

The lifting axiom **CM4** is proved in the usual way: every trivial cofibration $\alpha : A \to B$ has a factorization

$$\begin{tikzcd} A \arrow[r, "i"] \arrow[dr, "\alpha"'] & C \arrow[d, "p"] \\ & B \end{tikzcd}$$

such that p is a fibration, and i is a trivial cofibration which has the left lifting property with respect to all fibrations. But then p is a trivial fibration, so that i is a retract of α. The remaining closed model axioms are easily verified.

Suppose given a pushout diagram

$$\begin{tikzcd} C \arrow[r] \arrow[d, "i"'] & E \arrow[d] \\ D \arrow[r] & F \end{tikzcd}$$

of presheaves of categories such that i is a cofibration. Then the induced diagram

$$\begin{tikzcd} BC \arrow[r] \arrow[d, "i"'] & BE \arrow[d] \\ BD \arrow[r] & BF \end{tikzcd} \tag{12}$$

is homotopy cocartesian in simplicial presheaves. This is true if i is a generator $P\,\mathrm{sd}^2\,A \to P\,\mathrm{sd}^2\,B$, so it is true for all cofibrations i.

This observation implies the left properness of the model structure on presheaves of categories. Right properness is proved with an adjointness argument and the observation that the functor $\mathrm{Ex}^2\,B$ preserves pullbacks.

For statement 2), it's clear from the definitions that the functors $P\,\mathrm{sd}^2$ and $\mathrm{Ex}^2\,B$ form a Quillen adjunction

$$P\,\mathrm{sd}^2 : s\,\mathrm{Pre}(\mathcal{C}) \leftrightarrows \mathrm{Pre}(\mathbf{Cat}(\mathcal{C})) : \mathrm{Ex}^2\,B.$$

All simplicial set maps

$$\eta : \Delta^n \to \mathrm{Ex}^2\,BP\,\mathrm{sd}^2\,\Delta^n$$

are weak equivalences, since $BP\,\mathrm{sd}^2\,\Delta^n = \mathrm{sd}^2\,\Delta^n$ is contractible. The inclusion $\mathrm{sk}_{n-1}\,X \to \mathrm{sk}_n\,X$ for a simplicial set X induces a homotopy cocartesian diagram

$$\begin{tikzcd} \sqcup_{x\in NX_n} \mathrm{Ex}^2\,BP\,\mathrm{sd}^2\,\partial\Delta^n \arrow[r] \arrow[d] & \mathrm{Ex}^2\,BP\,\mathrm{sd}^2\,\mathrm{sk}_{n-1}\,X \arrow[d] \\ \sqcup_{x\in NX_n} \mathrm{Ex}^2\,BP\,\mathrm{sd}^2\,\Delta^n \arrow[r] & \mathrm{Ex}^2\,BP\,\mathrm{sd}^2\,\mathrm{sk}_n\,X \end{tikzcd}$$

by Lemma 2.6. It follows by induction on n that the map

$$\eta : \mathrm{sk}_n X \to \mathrm{Ex}^2 BP \,\mathrm{sd}^2 \,\mathrm{sk}_n X$$

is a weak equivalence for all $n \geq 0$. The map

$$\eta : X \to \mathrm{Ex}^2 BP \,\mathrm{sd}^2 X \tag{13}$$

is therefore a weak equivalence for all simplicial sets X. It follows from a triangle identity argument that the counit map

$$\epsilon : P \,\mathrm{sd}^2 \,\mathrm{Ex}^2 BC \to C$$

is a weak equivalence of **Cat** for all small categories C.

Statement 3) follows from the proof of statement 2). The adjoint pair

$$P : s\mathbf{Set} \leftrightarrows \mathbf{Cat} : B$$

is a Quillen adjunction for the sd^2-model structure on the simplicial set category. The counit $\epsilon : PBC \to C$ is a natural isomorphism. There is a natural sectionwise weak equivalence $\mathrm{sd}^2 X \to X$ for each simplicial set X, and so $\mathrm{sd}^2 X$ is a cofibrant model for X in the sd^2-model structure. The unit map $\eta : \mathrm{sd}^2 X \to BP \,\mathrm{sd}^2 X$ is a weak equivalence, since the composite

$$X \xrightarrow{\eta} \mathrm{Ex}^2 \,\mathrm{sd}^2 X \xrightarrow{\mathrm{Ex}^2(\eta)} \mathrm{Ex}^2 BP \,\mathrm{sd}^2 X$$

is the weak equivalence (13), the map $\eta : X \to \mathrm{Ex}^2 \,\mathrm{sd}^2 X$ is a weak equivalence, and the functor Ex^2 reflects weak equivalences.

It follows that the map $\eta : Y \to BPY$ is a local weak equivalence for all sd^2-cofibrant simplicial presheaves Y. ■

Write $\mathbf{Dia}(\mathcal{C})$ for the category of whose objects are all simplicial presheaf maps $X \to BC$ such that C is a presheaf of small categories, and whose morphisms are the commutative diagrams

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d] & Y \arrow[d] \\ BC \arrow[r, "g"'] & BD \end{tikzcd}$$

I say that such a map is

- a *local weak equivalence* if both f and g are local weak equivalences,
- a *cofibration* if f is an sd^2-cofibration of simplicial presheaves and the functor $g : C \to D$ is a cofibration of presheaves of categories, and
- a *fibration* if the map $g : C \to D$ is a fibration of $\mathrm{Pre}(\mathbf{Cat}(\mathcal{C}))$, and the induced map

  $$X \to BC \times_{BD} Y$$

  is an sd^2-fibration of simplicial presheaves.

3.3. THEOREM. *With these definitions, the category* $\mathbf{Dia}(\mathcal{C})$ *satisfies the axioms for a closed model category.*

PROOF. The axioms **CM1**, **CM2** and **CM3** are easy to verify. In particular, the category $\mathbf{Dia}(\mathcal{C})$ is complete and cocomplete.

Observe that if $g : C \to D$ is a fibration (respectively trivial fibration) of $\mathrm{Pre}(\mathbf{Cat}(\mathcal{C}))$, then the pullback diagram

$$\begin{tikzcd} BC \times_{BD} Y \arrow[r] \arrow[d] & Y \arrow[d] \\ BC \arrow[r] & BD \end{tikzcd}$$

is a fibration (respectively trivial fibration) of $\mathbf{Dia}(\mathcal{C})$.

Factorize the functor $g : C \to D$ as

$$\begin{tikzcd} C \arrow[r, "j"] \arrow[dr, "g"'] & E \arrow[d, "q"] \\ & D \end{tikzcd}$$

where j is a cofibration and q is a trivial fibration in $\mathbf{Dia}(\mathcal{C})$, and then find a factorization

$$\begin{tikzcd} X \arrow[r, "j'"] \arrow[dr] & Z \arrow[d, "q'"] \\ & BE \times_{BD} Y \end{tikzcd}$$

of the induced map $X \to BE \times_{BD} Y$, where j' is an sd^2-cofibration and q' is a trivial sd^2-fibration of simplicial sets. Then the map (f, g) has a factorization

$$\begin{tikzcd} X \arrow[r, "j'"] \arrow[d] & Z \arrow[r, "q_* q'"] \arrow[d] & Y \arrow[d] \\ BC \arrow[r, "j"'] & BE \arrow[r, "q"'] & BD \end{tikzcd}$$

such that (j', j) is a cofibration and $(q_* q', q)$ is a trivial fibration. The other part of **CM5** has a similar proof.

Suppose given a diagram

$$\begin{tikzcd}[row sep=small, column sep=small] A \arrow[rr] \arrow[dr] \arrow[dd] & & B \arrow[dr] \arrow[dd] & \\ & X \arrow[rr] \arrow[dd] & & Y \arrow[dd] \\ BE \arrow[rr] \arrow[dr] & & BF \arrow[dr] & \\ & BC \arrow[rr] & & BD \end{tikzcd}$$

in which the back face is a cofibration and the front face is a trivial fibration. The lift θ_1 exists in the diagram of functors

$$\begin{tikzcd} E \arrow[r] \arrow[d] & C \arrow[d] \\ F \arrow[r] \arrow[ur, dotted, "\theta_1"] & D \end{tikzcd}$$

and then the lift θ_2 exists in the induced diagram of simplicial set maps

$$\begin{tikzcd} A \arrow[r] \arrow[d] & X \arrow[d] \\ B \arrow[r] \arrow[ur, dotted, "\theta_2"] & BC \times_{BD} Y \end{tikzcd}$$

It follows that every trivial fibration has the right lifting property with respect to all cofibrations. Similarly, every fibration has the right lifting property with respect to all trivial cofibrations, and we have proved the axiom **CM4**. ∎

4. Presheaves of simplicial categories

A *simplicial category* is a simplicial object in the category **Cat** of small categories. Write s**Cat** for the corresponding category of simplicial categories and their morphisms.

Note that we are not, in general, making any assumption in this definition on the simplicial set of objects $\mathrm{Ob}(C)$ of a simplicial category C.

In all that follows, the nerve BC of a simplicial category C will be the bisimplicial set which is defined in vertical degrees n by

$$BC_{*,n} = BC_n. \tag{14}$$

Recall from Section 1 that the subdivision $\mathrm{sd}^{2,0} X$ of a bisimplicial set consists of the simplicial sets $\mathrm{sd}^2 X_{*,n}$ in all vertical degrees n. The functor

$$P : s^2\mathbf{Set} \to s\mathbf{Cat} \tag{15}$$

is defined by applying the path category functor $P : s\mathbf{Set} \to \mathbf{Cat}$ in all vertical degrees: $P(X)$ is the simplicial category with

$$P(X)_n = P(X_{*,n})$$

for a bisimplicial set X.

Suppose again that $\mathcal{C}$ is a small Grothendieck site. Write $s\,\mathrm{Pre}(\mathbf{Cat}(\mathcal{C}))$ for the category of presheaves in simplicial categories, or equivalently for the category of simplicial objects in presheaves of categories. Then the functors P and B of (15) and (14) induce an adjoint pair of functors

$$P : s^2\,\mathrm{Pre}(\mathcal{C}) \leftrightarrows s\,\mathrm{Pre}(\mathbf{Cat}(\mathcal{C})) : B$$

where $s^2\operatorname{Pre}(\mathcal{C})$ is the category of bisimplicial presheaves on the site $\mathcal{C}$.

A map $f : C \to D$ of presheaves of simplicial categories is a *local weak equivalence* if the induced map $BC \to BD$ is a diagonal weak equivalence of bisimplicial presheaves. The map $f : C \to D$ is a *fibration* if the induced map $BC \to BD$ is an $sd^{2,0}$-fibration. Cofibrations in the category $s\operatorname{Pre}(\mathbf{Cat}(\mathcal{C}))$ are defined by a left lifting property with respect to trivial cofibrations.

4.1. Theorem. *Suppose that $\mathcal{C}$ is a small Grothendieck site.*

1) *With these definitions, the category $s\operatorname{Pre}(\mathbf{Cat}(\mathcal{C}))$ of presheaves of simplicial categories on $\mathcal{C}$ has the structure of a proper closed simplicial model category.*

2) *The adjoint pair*

$$P\operatorname{sd}^{2,0} : s^2\operatorname{Pre}(\mathcal{C}) \leftrightarrows s\operatorname{Pre}(\mathbf{Cat}(\mathcal{C})) : \operatorname{Ex}^{2,0} B$$

is a Quillen equivalence, for the diagonal model structure on the bisimplicial presheaf category $s^2\operatorname{Pre}(\mathcal{C})$.

3) *The adjoint pair*

$$P : s^2\operatorname{Pre}(\mathcal{C}) \leftrightarrows s\operatorname{Pre}(\mathbf{Cat}(\mathcal{C})) : B$$

is a Quillen equivalence for the $\operatorname{sd}^{2,0}$-model structure on $s^2\operatorname{Pre}(\mathcal{C})$.

Proof. Suppose given a pushout diagram

$$\begin{tikzcd} \bigsqcup_i P\operatorname{sd}^{2,0}(K_i) \arrow[r] \arrow[d] & C \arrow[d] \\ \bigsqcup_i P\operatorname{sd}^{2,0}(L_i) \arrow[r] & D \end{tikzcd}$$

in $s^2\operatorname{Pre}(\mathcal{C})$, where the vertical map on the left is induced by inclusions $K_i \subset L_i$ of bisimplicial sets such that each $(L_i)_{*,n}$ is a finite simplicial complex. The induced functor

$$P\operatorname{sd}^{2,0} K_i \to P\operatorname{sd}^{2,0} L_i$$

is isomorphic to the map

$$NBNK_i \to NBNL_i$$

which is defined by applying the functor NBN in all vertical degrees. It follows from Lemma 2.1 that the functor

$$\bigsqcup_i P\operatorname{sd}^{2,0} K_i \to \bigsqcup_i P\operatorname{sd}^{2,0} L_i$$

is a Dwyer map in each vertical degree, and so Lemma 2.6 implies that the induced square

$$\begin{tikzcd} \bigsqcup_i BP\,\mathrm{sd}^{2,0}(K_i) \arrow[r] \arrow[d] & BC \arrow[d] \\ \bigsqcup_i BP\,\mathrm{sd}^{2,0}(L_i) \arrow[r] & BD \end{tikzcd}$$

is homotopy cocartesian in all vertical degrees, and is therefore homotopy cocartesian for the diagonal model structure.

Suppose given a pushout square

$$\begin{tikzcd} P\,\mathrm{sd}^{2,0} A \arrow[r] \arrow[d, "i_*"'] & C \arrow[d] \\ P\,\mathrm{sd}^{2,0} B \arrow[r] & D \end{tikzcd}$$

for which the map i_* is induced by a cofibration $i : A \to B$ of bisimplicial sets. Then the induced diagram

$$\begin{tikzcd} BP\,\mathrm{sd}^{2,0} A \arrow[r] \arrow[d, "i_*"'] & BC \arrow[d] \\ BP\,\mathrm{sd}^{2,0} B \arrow[r] & BD \end{tikzcd}$$

is homotopy cocartesian for the diagonal model structure on bisimplicial sets, since this is true for all inclusions $\partial\Delta^{p,q} \subset \Delta^{p,q}$.

The functor

$$P\,\mathrm{sd}^{2,0} : s^2\mathbf{Set} \to s\mathbf{Cat}$$

therefore preserves trivial cofibrations as well as cofibrations.

All direct image functors commute with the composite $\mathrm{Ex}^{2,0} B$, so that inverse image functors commute with the functor $P\,\mathrm{sd}^{2,0}$ up to isomorphism. A Boolean localization argument implies that the functor

$$P\,\mathrm{sd}^{2,0} : s^2\,\mathrm{Pre}(\mathcal{C}) \to s\,\mathrm{Pre}(\mathbf{Cat}(\mathcal{C}))$$

preserves local weak equivalences.

As in the proof of Theorem 1.2, write $P\,\mathrm{sd}^{2,0}\, I$ and $P\,\mathrm{sd}^{2,0}\, J$ for the sets of morphisms of $s\,\mathrm{Pre}(\mathbf{Cat}(\mathcal{C}))$ which are induced by a set I of generators for the trivial cofibrations and a set J of generators for the cofibrations, respectively, for the diagonal model structure on bisimplicial presheaves. Then a map $p : C \to D$ of presheaves of simplicial categories is a fibration (respectively trivial fibration) if and only if it has the right lifting property with respect to all members of the set $P\,\mathrm{sd}^{2,0}\, I$ (respectively $P\,\mathrm{sd}^{2,0}\, J$). The closed axioms **CM5** and **CM4** then follow in a standard way, via small object arguments. The remaining closed model axioms are easily verified.

Suppose given a pushout diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d, "j"'] & C \arrow[d] \\ B \arrow[r] & D \end{tikzcd}$$

in presheaves of simplicial categories, where j is a cofibration. Then the induced diagram

$$\begin{tikzcd} BA \arrow[r] \arrow[d, "j"'] & BC \arrow[d] \\ BB \arrow[r] & BD \end{tikzcd} \tag{16}$$

is homotopy cocartesian in the diagonal model structure for bisimplicial presheaves, since this is true for all morphisms j in the set of generators $P\,\mathrm{sd}^{2,0}\,J$ of the class of cofibrations by the argument above.

The left properness of the model structure is a consequence. Right properness follows from the corresponding property for the diagonal structure on bisimplicial presheaves, via an adjointness argument and the observation that the functor $\mathrm{Ex}^{2,0}\,B$ preserves pullbacks.

For statement 2), it follows from the proof of the corresponding part of Theorem 3.1 that the natural map

$$\eta : X \to \mathrm{Ex}^{2,0}\,BP\,\mathrm{sd}^{2,0}\,X$$

is a weak equivalence in all vertical degrees, and is hence a diagonal weak equivalence for all bisimplicial sets X. A triangle identity argument shows that the counit map

$$\epsilon : P\,\mathrm{sd}^{2,0}\,\mathrm{Ex}^{2,0}\,BC \to C$$

is a weak equivalence of $\mathrm{Pre}(s\mathbf{Cat})(\mathcal{C})$ for all simplicial categories C.

The proof of statement 3) uses the fact that $\mathrm{sd}^{2,0}\,X$ is a cofibrant model for a bisimplicial set X in the $\mathrm{sd}^{2,0}$-model structure on the category of bisimplicial sets, just as for the proof of the corresponding part of Theorem 3.1. It follows that the map $\eta : Y \to BPY$ is a local weak equivalence for all $\mathrm{sd}^{2,0}$-cofibrant bisimplicial presheaves Y. We also know that the counit $\epsilon : PBC \to C$ is an isomorphism for all presheaves of simplicial categories C. ∎

Write $s\mathbf{Dia}(\mathcal{C})$ for the category of whose objects are all bisimplicial presheaf maps $X \to BC$ where C is a simplicial presheaf of categories, and whose morphisms are the commutative diagrams

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d] & Y \arrow[d] \\ BC \arrow[r, "g"'] & BD \end{tikzcd}$$

in bisimplicial presheaves. I say that such a map is

- a *weak equivalence* if both f and g are diagonal local weak equivalences,
- a *cofibration* if f is an $\mathrm{sd}^{2,0}$-cofibration of bisimplicial presheaves and the functor $g : C \to D$ is a cofibration of $s\,\mathrm{Pre}(\mathbf{Cat}(\mathcal{C}))$, and
- a *fibration* if the map $g : C \to D$ is a fibration of $s\,\mathrm{Pre}(\mathbf{Cat}(\mathcal{C}))$, and the map $X \to BC \times_{BD} Y$ is an $\mathrm{sd}^{2,0}$-fibration of bisimplicial presheaves.

4.2. THEOREM. *With these definitions, the category $s\mathbf{Dia}(\mathcal{C})$ satisfies the axioms for a closed model category.*

PROOF. The proof of this result is a word for word transcription into the present context of the proof of Theorem 3.3. ■

5. Diagrams for simplicial categories

Many of the ideas and notational conventions of this section originated in [6].

Suppose that E is a presheaf of simplicial categories on a Grothendieck site $\mathcal{C}$. We interpret E as a category object in simplicial presheaves, with simplicial presheaves $\mathrm{Ob}(E)$ and $\mathrm{Mor}(E)$, source and target maps $s, t : \mathrm{Mor}(E) \to \mathrm{Ob}(E)$, identity map $e : \mathrm{Ob}(E) \to \mathrm{Mor}(E)$ and a law of composition

$$\mathrm{Mor}(E) \times_{\mathrm{Ob}(E)} \mathrm{Mor}(E) \to \mathrm{Mor}(E),$$

all of which satisfy the usual properties.

Suppose that α is an infinite cardinal which is an upper bound for the cardinality of both the set $\mathrm{Mor}(\mathcal{C})$ and the presheaf $\mathrm{Mor}(E)$.

A diagram X on E, or E-diagram in presheaves consists of a simplicial presheaf map $\pi_X : X \to \mathrm{Ob}(E)$ (called the structure map) and an "action map"

$$X \times_{\mathrm{Ob}(E)} \mathrm{Mor}(E) \to X,$$

in simplicial presheaves which again satisfies the usual properties.

Every category A is a simplicial category which is simplicially discrete in objects and morphisms, and an A-diagram (in sets) in the present sense is just a functor $A \to s\mathbf{Set}$ which takes values in simplicial sets.

A morphism $f : X \to Y$ of E-diagrams is a commutative diagram

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "\pi_X"'] & & Y \arrow[dl, "\pi_Y"] \\ & \mathrm{Ob}(E) & \end{tikzcd}$$

of simplicial presheaves which respects action maps. Write $\mathrm{Pre}(\mathcal{C})^E$ for the corresponding category of all E-diagrams in presheaves.

I say that the morphism $f : X \to Y$ of E-diagrams is a *weak equivalence* (respectively *cofibration*) if the underlying map $X \to Y$ is a weak equivalence (respectively cofibration) of simplicial presheaves. An *injective fibration* of E-diagrams is a map which has the right lifting property with respect to all maps which are cofibrations and weak equivalences.

Suppose that α is an infinite cardinal which is an upper bound for the cardinality of both the set $\mathrm{Mor}(\mathcal{C})$ and the presheaf $\mathrm{Mor}(E)$.

The following result is proved in [6]:

5.1. THEOREM. *Suppose that the simplicial presheaf* $\mathrm{Ob}(E)$ *is simplicially discrete. Then with the definitions given above, the cofibrations, weak equivalences and injective fibrations satisfy the axioms for a proper closed simplicial model structure on* $\mathbf{Set}^E$. *This model structure is cofibrantly generated.*

There is a functor

$$\mathrm{Ob} : \mathrm{Pre}(\mathcal{C})^E \to s\,\mathrm{Pre}(\mathcal{C})/\,\mathrm{Ob}(E)$$

which takes a E-diagram X to the structure map $\pi_X : X \to \mathrm{Ob}(E)$. This functor has a left adjoint

$$L : s\,\mathrm{Pre}(\mathcal{C})/\,\mathrm{Ob}(E) \to \mathrm{Pre}(\mathcal{C})^E$$

which takes a simplicial presheaf map $\phi : Y \to \mathrm{Ob}(E)$ to the E-diagram $L(\phi)$ which has structure map

$$Y \times_s \mathrm{Mor}(E) \xrightarrow{pr} \mathrm{Mor}(E) \xrightarrow{t} \mathrm{Ob}(E).$$

The generating cofibrations (respectively trivial cofibrations) for the injective model structure of Theorem 5.1 are the images under the functor L of the α-bounded cofibrations (respectively trivial cofibrations)

$$\begin{tikzcd} A \arrow[rr, "i"] \arrow[dr] & & B \arrow[dl] \\ & \mathrm{Ob}(E) & \end{tikzcd}$$

of $s\,\mathrm{Pre}(\mathcal{C})/\,\mathrm{Ob}(E)$.

The proof of Theorem 5.1 depends on the assertion the functor L preserves local weak equivalences. If the presheaf of simplicial categories E does not have a discrete simplicial presheaf $\mathrm{Ob}(E)$ of objects, then it is no longer clear that this is so.

Say that a map $p : X \to Y$ of E-diagrams is a *projective fibration* if the underlying map $p : X \to Y$ over $\mathrm{Ob}(E)$ is an injective fibration of simplicial presheaves. A *projective cofibration* is a map $A \to B$ of E-diagrams which has the left lifting property with respect to all maps which are injective fibrations and weak equivalences.

If the map

$$\begin{tikzcd} A \arrow[rr, "i"] \arrow[dr] & & B \arrow[dl] \\ & \mathrm{Ob}(E) & \end{tikzcd}$$

is a cofibration over $\mathrm{Ob}(E)$, then the induced map $i_* : L(A) \to L(B)$ is a projective cofibration. It follows by a standard argument that the class of projective cofibrations is generated by the set S_0 of maps of the form i_* with B α-bounded, and that every projective cofibration is a cofibration.

Suppose that S is some set of cofibrations which contains S_0, and let C_S denote the saturation of the set $\overline{S}$ of the set of cofibrations

$$(B_j \times \partial\Delta^n) \cup (A_j \times \Delta^n) \subset B_j \times \Delta^n,$$

where $n \geq 0$ and the cofibrations $A_j \to B_j$ belong to the set S. The members of the class C_S are called S-cofibrations. Say that a map $f : X \to Y$ of E-diagrams is an S-fibration if it has the right lifting property with respect to all maps which are S-cofibrations and weak equivalences.

5.2. Theorem. *Suppose that the simplicial presheaf* $\mathrm{Ob}(E)$ *is simplicially discrete. Then the category* $\mathbf{Set}^E$ *of E-diagrams, together with the S-cofibrations, weak equivalences, and S-fibrations, satisfies the axioms for a proper closed simplicial model structure. This model structure is cofibrantly generated.*

The model structure of Theorem 5.1 is called the *injective* model structure on the category $\mathbf{Set}^E$ of E-diagrams. The case $S = S_0$ of Theorem 5.2 gives the *projective* model structure on $s\mathbf{Set}^E$. All other S-model structures of Theorem 5.2 are *intermediate model structures.*

Proof. The proof of Theorem 5.2 follows the outline of proof of Theorem 2 of [7]. Among the closed model axioms, only the factorization axiom **CM5** needs proof. For that, every map $f : X \to Y$ of E-diagrams has a factorization

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "j"'] & & Y \\ & Z \arrow[ur, "p"'] & \end{tikzcd}$$

where j is a morphism of C_S and p has the right lifting property with respect to all members of C_S. It follows that p is a projective fibration and a weak equivalence, and we have one of the factorizations required for **CM5**.

By Theorem 5.1, the map f can be factored

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "i"'] & & Y \\ & W \arrow[ur, "q"'] & \end{tikzcd}$$

where q is an injective fibration (and hence an S-fibration) and i is a cofibration and a weak equivalence. Then, by the first paragraph, $i = p \cdot j$, where p is an S-fibration and a weak equivalence, and j is an S-cofibration. Then j is also a weak equivalence, and so

$f = (q \cdot p) \cdot j$ factors f as the composite of an S-fibration $q \cdot p$ with a map j which is an S-cofibration and a weak equivalence.

The function complex $\mathbf{hom}(X, Y)$ is standard: its n-simplices are the maps $X \times \Delta^n \to Y$. If $i : A \to B$ is an S-cofibration and $j : K \to L$ is a cofibration of simplicial sets, then the map

$$(A \times L) \cup (B \times K) \to B \times L$$

is an S-cofibration by construction, and it's a weak equivalence if either i or j is a weak equivalence by the corresponding statement for simplicial presheaves.

The set $\overline{S}$ of cofibrations generates the class C_S of S-cofibrations, by construction.

One proves a bounded cofibration condition for the class of cofibrations in $\mathrm{Pre}(\mathcal{C})^E$: given a diagram of cofibrations

$$\begin{tikzcd} & X \arrow[d, "i"] \\ A \arrow[r] & Y \end{tikzcd}$$

such that i is a trivial cofibration and A is an α-bounded subobject of Y, there is an α-bounded subobject $B \subset Y$ with $A \subset B$ such that the map $B \cap X \to B$ is a weak equivalence. The set of generating trivial cofibrations is then found by a solution set argument, as in the proof of Proposition 5 of [7]. ■

Every E-diagram X has a homotopy colimit $\underrightarrow{\mathrm{holim}}_E X$ in bisimplicial presheaves (composed of nerves of translation categories in all vertical degrees and sections), and there is a canonical map

$$\pi : \underrightarrow{\mathrm{holim}}_E X \to BE$$

of bisimplicial presheaves.

To make the notation easier, write $L_h X = \underrightarrow{\mathrm{holim}}_E X$. I say that a map $X \to Y$ of E-diagrams is an *L_h-equivalence* if the induced map $L_h X \to L_h Y$ is a diagonal weak equivalence of bisimplicial presheaves.

5.3. LEMMA. *Suppose that α is an infinite cardinal with $\alpha > |\mathrm{Mor}(E)|$, and suppose given a diagram*

$$\begin{tikzcd} & X \arrow[d, "i"] \\ A \arrow[r, "\gamma", "\subset"'] & Y \end{tikzcd}$$

is E-diagrams where i is a cofibration which is an L_h- equivalence and A is an α-bounded subobject of Y. Then there is an α-bounded subobject D of Y which contains A such that the map $D \cap X \to D$ is an L_h-equivalence.

PROOF. The simplicial presheaf $L_h A$ is an α-bounded subobject of $L_h Y$, and the induced map $i_* : L_h X \to L_h Y$ is a trivial cofibration of bisimplicial presheaves. It follows from the bounded cofibration property for bisimplicial presheaves (Lemma 1 of [10]) that there is an α-bounded subobject B_1 such that we have a diagram of bisimplicial presheaves

$$\begin{tikzcd} & B_1 \cap L_hX \arrow[r] \arrow[d, "\simeq"'] & L_hX \arrow[d] \\ L_hA \arrow[r, "j", "\subset"'] & B_1 \arrow[r, "\subset"'] & L_hY \end{tikzcd}$$

There is an α-bounded subobject A_1 of Y which contains A and such that $B_1 \subset L_hA_1$, because the homotopy colimit functor preserves colimits and monomorphisms.

Repeat this construction inductively, and let $D = \cup_i A_i$. Then the map $L_hD \cap L_hX \to L_hD$ is a filtered colimit of the maps $B_i \cap L_hX \to L_hB_i$ and is also the filtered colimit of the system

$$\begin{tikzcd} L_hA \cap L_hX \arrow[r] \arrow[d] & B_1 \cap L_hX \arrow[r] \arrow[d, "\simeq"] & L_hA_1 \cap L_hX \arrow[r] \arrow[d] & B_2 \cap L_hX \arrow[r] \arrow[d, "\simeq"] & \dots \\ L_hA \arrow[r] & B_1 \arrow[r] & L_hA_1 \arrow[r] & B_2 \arrow[r] & \dots \end{tikzcd}$$

The map $L_hD \cap L_hX \to L_hY$ is therefore a weak equivalence. The homotopy colimit functor preserves pullbacks and filtered colimits, and it follows that the map $D \cap X \to D$ is an L_h-equivalence. ∎

Say that a map $p : X \to Y$ of E-diagrams is an L_h*-fibration* if it has the right lifting property with respect to all cofibrations which are L_h-equivalences.

5.4. THEOREM. *Suppose that $\mathcal{C}$ is a small Grothendieck site, and that E is a presheaf of simplicial categories on $\mathcal{C}$. There is a model structure on the category* $\mathrm{Pre}(\mathcal{C})^E$ *of E-diagrams in* $\mathrm{Pre}(\mathcal{C})$*, for which the cofibrations are the monomorphisms, the weak equivalences are the L_h-equivalences and the fibrations are the L_h-fibrations.*

PROOF. The axioms **CM1**, **CM2** and **CM3** are easily verified. On account of Lemma 5.3, a map is an L_h-fibration if and only if it has the right lifting property with respect to all α-bounded L_h-trivial cofibrations. The homotopy colimit functor preserves pushouts and cofibrations, and then the factorization axiom **CM5** follows from a small object argument.

We have to show that if $p : X \to Y$ is an L_h-fibration and an L_h-equivalence, then it has the right lifting property with respect to all cofibrations. This is also standard: p has a factorization

$$\begin{tikzcd} X \arrow[r, "j"] \arrow[dr, "p"'] & Z \arrow[d, "q"] \\ & Y \end{tikzcd}$$

where q has the right lifting property with respect to all cofibrations and j is a cofibration. Then j is an L_h-equivalence as well as a cofibration so the lift exists in the diagram

$$\begin{tikzcd} X \arrow[r, "1"] \arrow[d, "j"'] & X \arrow[d, "p"] \\ Z \arrow[r, "q"'] \arrow[ur, dotted] & Y \end{tikzcd}$$

and p is a retract of q. ■

Suppose that A is a simplicial category and that $\phi : Z \to BA$ is a bisimplicial set map. Then ϕ consists of simplicial set maps

$$\phi : Z_n = Z_{*,n} \to BA_n$$

in all vertical degrees, and each such map determines simplicial set-valued functor $\mathrm{pb}(Z_n) : A_n \to s\mathbf{Set}$ in the standard way: $\mathrm{pb}(Z_n)_i$ is defined for $i \in A_n$ by the pullback diagram

$$\begin{tikzcd} \mathrm{pb}(Z_n)_i \arrow[r] \arrow[d] & Z_n \arrow[d, "\phi"] \\ B(A_n/i) \arrow[r] & BA_n \end{tikzcd}$$

Define the simplicial set $\mathrm{pb}(Z_n)$ by

$$\mathrm{pb}(Z_n) = \bigsqcup_{i \in \mathrm{Ob}(A_n)} \mathrm{pb}(Z_n)_i$$

Then the simplicial set maps $\mathrm{pb}(Z_n) \to \mathrm{Ob}(A_n)$ define a bisimplicial set map $\mathrm{pb}(Z) \to \mathrm{Ob}(A)$, and we obtain an A-diagram $\mathrm{pb}(Z)$ in simplicial sets. A similar analysis produces a second A-diagram $B(A/?)$ in simplicial sets and a map $\mathrm{pb}(Z) \to B(A/?)$ of such objects.

Write $s\mathbf{Set}^A$ for the category of A-diagrams in simplicial sets.

In general, if the bisimplicial set map $Y \to \mathrm{Ob}(A)$ defines an A-diagram in simplicial sets, then applying the diagonal functor d to Y gives an A-diagram $d(Y) \to \mathrm{Ob}(A)$ in sets. In each vertical degree, $Y_{*,n} \to \mathrm{Ob}(A_n)$ defines an A_n-diagram in simplicial sets, and we can form the homotopy colimit of this diagram and the corresponding simplicial set map $\underrightarrow{\mathrm{holim}}_{A_n} Y_{*,n} \to BA_n$. Letting n vary gives the bisimplicial set map $\underrightarrow{\mathrm{holim}}_A Y \to BA$. There is a natural isomorphism

$$d(\underrightarrow{\mathrm{holim}}_A Y) \cong \underrightarrow{\mathrm{holim}}_A d(Y)$$

of simplicial sets, since both objects are triple diagonals of the same trisimplicial set.

Taking homotopy colimits in vertical degrees n gives a natural commutative diagram of canonically defined bisimplicial set maps

$$\begin{tikzcd} & \underrightarrow{\mathrm{holim}}_A \mathrm{pb}\, Z \arrow[r, "\simeq", "\gamma"'] \arrow[dl, "\alpha_\phi"'] \arrow[d, "\phi_*"] & Z \arrow[d, "\phi"] \\ BA & \underrightarrow{\mathrm{holim}}_A B(A/?) \arrow[l, "\alpha"', "\simeq"] \arrow[r, "\gamma", "\simeq"'] & BA \end{tikzcd}$$

where the displayed weak equivalences are equivalences in all vertical degrees. It follows that the functor which takes a bisimplicial set map $Z \to BA$ to the A-diagram $d(\mathrm{pb}(Z)) \to \mathrm{Ob}(A)$ takes diagonal equivalences over BA to L_h-equivalences of A-diagrams.

We therefore have induced functors

$$L_h : \mathrm{Ho}(\mathbf{Set}^A) \leftrightarrows \mathrm{Ho}(s^2\mathbf{Set}/BA) : d \cdot \mathrm{pb}\,.$$

It also follows that there is a natural weak equivalence

$$d(\gamma\phi_*) \xrightarrow{d(\gamma)} d(\phi)$$

The maps $\alpha, \gamma : \underrightarrow{\mathrm{holim}}_A\, B(A/?) \to BA$ are homotopic (see Remark 16 of [9]), and so the functors

$$\mathrm{Ho}(s\mathbf{Set}/d(\underrightarrow{\mathrm{holim}}_A B(A/?))) \to \mathrm{Ho}(s\mathbf{Set}/d(BA))$$

which are defined by composition with the maps $d(\alpha)$ and $d(\gamma)$ coincide up to natural isomorphism. It follows that there is a natural isomorphism

$$d(L_h d(\mathrm{pb}(\phi))) = d(L_h\,\mathrm{pb}(\phi)) = d(\alpha_\phi) = d(\alpha)d(\phi_*) \cong d(\gamma)d(\phi_*) \xrightarrow[\cong]{d(\gamma)} d(\phi)$$

of functors

$$\mathrm{Ho}(s^2\mathbf{Set}/BA) \to \mathrm{Ho}(s\mathbf{Set}/d(BA))$$

The diagonal functor d induces an equivalence

$$\mathrm{Ho}(s^2\mathbf{Set}/BA) \simeq \mathrm{Ho}(s\mathbf{Set}/d(BA))$$

of homotopy categories (since it's part of a Quillen equivalence — see Proposition 6 of [10]), and so there is a natural isomorphism

$$L_h d(\mathrm{pb}(\phi)) \cong \phi$$

for all objects $\phi : Z \to BA$ of the homotopy category $\mathrm{Ho}(s^2\mathbf{Set}/BA)$.

Generally, if $X : I \to s\mathbf{Set}$ is a diagram in simplicial sets indexed on a small category I, then there is a sectionwise weak equivalence

$$\epsilon : \mathrm{pb}(\underrightarrow{\mathrm{holim}}_I X)_i \xrightarrow{\simeq} X_i$$

which is natural in I-diagrams. Thus, if Y is a A-diagram in sets, then there is a natural map of A_n-diagrams of simplicial sets

$$\epsilon : \mathrm{pb}(\underrightarrow{\mathrm{holim}}_{A_n} Y_n) \xrightarrow{\simeq} Y_n$$

which is a weak equivalence in each vertical degree. It follows that the induced map

$$\epsilon : d(\mathrm{pb}(Y)) \to Y$$

of A-diagrams in sets is a natural L_h-equivalence.

We have therefore proved the following:

5.5. Proposition. *The functors*

$$L_h : \mathrm{Ho}(\mathbf{Set}^A) \leftrightarrows \mathrm{Ho}(s^2\mathbf{Set}/BA) : d \cdot \mathrm{pb}$$

define an equivalence of categories.

The constructions which are involved in the proof of Proposition 5.5 are easily promoted to the presheaf level.

5.6. Corollary. *Suppose that $\mathcal{C}$ is a small Grothendieck site, and that E is a presheaf of simplicial categories on $\mathcal{C}$. Then the functors*

$$L_h : \mathrm{Pre}(\mathcal{C})^E \leftrightarrows s^2\,\mathrm{Pre}(\mathcal{C})/BE : d \cdot \mathrm{pb}$$

induce an equivalence of homotopy categories

$$\mathrm{Ho}(\mathrm{Pre}(\mathcal{C})^E) \simeq \mathrm{Ho}(s^2\,\mathrm{Pre}(\mathcal{C})/BE).$$

The functor $\mathrm{Ex}^{2,0}$ induces a functor

$$\mathrm{Ex}^{2,0} : s^2\,\mathrm{Pre}(\mathcal{C})/BE \to s^2\,\mathrm{Pre}(\mathcal{C})/\,\mathrm{Ex}^{2,0} BE$$

which takes an object $\phi : X \to BE$ to the object $\mathrm{Ex}^{2,0}\phi : \mathrm{Ex}^{2,0} X \to \mathrm{Ex}^{2,0} BE$. This functor has a left adjoint

$$\mathrm{sd}^{2,0} : s^2\,\mathrm{Pre}(\mathcal{C})/\,\mathrm{Ex}^{2,0} BE \to s^2\,\mathrm{Pre}(\mathcal{C})/BE,$$

which takes an object $Y \to \mathrm{Ex}^{2,0} BE$ to the composite

$$\mathrm{sd}^{2,0} Y \to \mathrm{sd}^{2,0}\,\mathrm{Ex}^{2,0} BE \xrightarrow{\epsilon} BE.$$

The slice category $s^2\,\mathrm{Pre}(\mathcal{C})/BE$ has an $\mathrm{sd}^{2,0}$-model structure, for which a morphism

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr] & & Y \arrow[dl] \\ & BE & \end{tikzcd}$$

is a weak equivalence (respectively fibration) if and only if the map $f : X \to Y$ is a diagonal weak equivalence (respectively $\mathrm{sd}^{2,0}$-fibration) of bisimplicial presheaves. The adjoint pair

$$\mathrm{sd}^{2,0} : s^2\,\mathrm{Pre}(\mathcal{C})/\,\mathrm{Ex}^{2,0} BE \leftrightarrows s^2\,\mathrm{Pre}(\mathcal{C})/BE : \mathrm{Ex}^{2,0} \tag{17}$$

is a Quillen adjunction for the usual model structure on $s^2\,\mathrm{Pre}(\mathcal{C})/\,\mathrm{Ex}^{2,0} BE$ and the $\mathrm{sd}^{2,0}$-structure on $s^2\,\mathrm{Pre}(\mathcal{C})/BE$ The unit and counit maps are natural weak equivalences (see the proof of Theorem 1.2), so that the adjunction (17) is a Quillen equivalence. There is diagonal equivalence $\mathrm{Ex}^{2,0} BE \to BE$ which induces a Quillen equivalence

$$s^2\,\mathrm{Pre}(\mathcal{C})/\,\mathrm{Ex}^{2,0} BE \leftrightarrows s^2\,\mathrm{Pre}(\mathcal{C})/BE.$$

It follows that the homotopy categories for the $\mathrm{sd}^{2,0}$-structure and the standard structure on the slice category $s^2\,\mathrm{Pre}(\mathcal{C})/BE$ are equivalent.

The model structure of Theorem 4.2 on the category $s\mathbf{Dia}(\mathcal{C})$ therefore effectively contains the "homotopy colimit" model structure of Theorem 5.4 for the category $\mathrm{Pre}(\mathcal{C})^E$ of E-diagrams in presheaves, for all presheaves of simplicial categories E.

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