Theory and Applications of Categories, Vol. 28, No. 13, 2013, pp. 332–370.

CODENSITY AND THE ULTRAFILTER MONAD

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ABSTRACT. Even a functor without an adjoint induces a monad, namely, its codensity monad; this is subject only to the existence of certain limits. We clarify the sense in which codensity monads act as substitutes for monads induced by adjunctions. We also expand on an undeservedly ignored theorem of Kennison and Gildenhuys: that the codensity monad of the inclusion of (finite sets) into (sets) is the ultrafilter monad. This result is analogous to the correspondence between measures and integrals. So, for example, we can speak of integration against an ultrafilter. Using this language, we show that the codensity monad of the inclusion of (finite-dimensional vector spaces) into (vector spaces) is double dualization. From this it follows that compact Hausdorff spaces have a linear analogue: linearly compact vector spaces. Finally, we show that ultra*products* are categorically inevitable: the codensity monad of the inclusion of (finite families of sets) into (families of sets) is the ultraproduct monad.

Now we have at last obtained permission to ventilate the facts. . .

—Arthur Conan Doyle, The Adventure of the Creeping Man (1927)

Introduction

The codensity monad of a functor G can be thought of as the monad induced by G and its left adjoint, even when no such adjoint exists. We explore the remarkable fact that when G is the inclusion of the category of finite sets into the category of all sets, the codensity monad of G is the ultrafilter monad. Thus, the mere notion of finiteness of a set gives rise automatically to the notion of ultrafilter, and so in turn to the notion of compact Hausdorff space.

Many of the results in this paper are known, but not well known. In particular, the characterization of the ultrafilter monad as a codensity monad appeared in the 1971 paper of Kennison and Gildenhuys [19] and the 1976 book of Manes ([34], Exercise 3.2.12(e)), but has not, to my knowledge, appeared anywhere else. Part of the purpose of this paper is simply to ventilate the facts.

Ultrafilters belong to the minimalist world of set theory. There are several concepts in more structured branches of mathematics of which ultrafilters are the set-theoretic shadow:

Partially supported by an EPSRC Advanced Research Fellowship.

Received by the editors 2012-10-09 and, in revised form, 2013-06-05.

Transmitted by Myles Tierney. Published on 2013-07-01.

2010 Mathematics Subject Classification: 18C15, 18A99, 03C20, 43J99.

Key words and phrases: density, codensity, monad, ultrafilter, ultraproduct, integration, compact Hausdorff space, double dual, linearly compact vector space.

Probability measures An ultrafilter is a finitely additive probability measure in which every event has probability either 0 or 1 (Lemma 3.1). The elements of an ultrafilter on a set X are the subsets that occupy 'almost all' of X, and the other subsets of X are to be regarded as 'null', in the sense of measure theory.

Integration operators Ordinary real-valued integration on a measure space (X, μ) is an operation that takes as input a suitable function $f\colon X \longrightarrow \mathbb{R}$ and produces as output an element $\int_X f\,d\mu$ of $\mathbb{R}$. We can integrate against ultrafilters, too. Given an ultrafilter $\mathscr{U}$ on a set X, a set R, and a function $f\colon X \longrightarrow R$ with finite image, we obtain an element $\int_X f\,d\mathscr{U}$ of R; it is the unique element of R whose f-fibre belongs to $\mathscr{U}$.

Averages To integrate a function against a *probability* measure is to take its mean value with respect to that measure. Integrating against an ultrafilter $\mathscr{U}$ is more like taking the mode: if we think of elements of $\mathscr{U}$ as 'large' then $\int_X f\,d\mathscr{U}$ is the unique value of f taken by a large number of elements of X. Ultrafilters are also used to prove results about more sophisticated types of average. For example, a **mean** on a group G is a left invariant finitely additive probability measure defined on all subsets of G; a group is **amenable** if it admits at least one mean. Even to prove the amenability of $\mathbb{Z}$ is nontrivial, and is usually done by choosing a nonprincipal ultrafilter on $\mathbb{N}$ (e.g. [37], Exercise 1.1.2).

Voting systems In an election, each member of a set X of voters chooses one element of a set R of options. A voting system computes from this a single element of R, intended to be some kind of average of the individual choices. In the celebrated theorem of Arrow [2], R has extra structure: it is the set of total orders on a list of candidates. In our structureless context, ultrafilters can be seen as (unfair!) voting systems: when each member of a possibly-infinite set X of voters chooses from a finite set R of options, there is—according to any ultrafilter on X—a single option chosen by almost all voters, and that is the outcome of the election.

Section 1 is a short introduction to ultrafilters. It includes a very simple and little-known characterization of ultrafilters, as follows. A standard lemma states that if $\mathscr{U}$ is an ultrafilter on a set X, then whenever X is partitioned into a finite number of (possibly empty) subsets, exactly one belongs to $\mathscr{U}$. But the converse is also true [13]: any set $\mathscr{U}$ of subsets of X satisfying this condition is an ultrafilter. Indeed, it suffices to require this just for partitions into three subsets.

We also review two characterizations of monads: one of Börger [7]:

the ultrafilter monad is the terminal monad on **Set** *that preserves finite coproducts*

and one of Manes [33]:

the ultrafilter monad is the monad for compact Hausdorff spaces.

Density and codensity are reviewed in Section 2. A functor $G\colon \mathscr{B} \longrightarrow \mathscr{A}$ is either codense or not: yes or no. Finer-grained information can be obtained by calculating the codensity *monad* of G. This is a monad on $\mathscr{A}$, defined subject only to the existence of certain limits, and it is the identity exactly when G is codense. Thus, the codensity monad of a functor measures its failure to be codense.

This prepares us for the codensity theorem of Kennison and Gildenhuys (Section 3): writing **FinSet** for the category of finite sets,

the ultrafilter monad is the codensity monad of the inclusion **FinSet** $\hookrightarrow$ **Set**.

(In particular, since nontrivial ultrafilters exist, **FinSet** is not codense in **Set**.) We actually prove a more general theorem, which has as corollaries both this and an unpublished result of Lawvere.

Writing $\mathbf{T} = (T, \eta, \mu)$ for the codensity monad of **FinSet** $\hookrightarrow$ **Set**, the elements of $T(X)$ can be thought of as integration operators on X, while the ultrafilters on X are thought of as measures on X. The theorem of Kennison and Gildenhuys states that integration operators correspond one-to-one with measures, as in analysis. In general, the notions of integration and codensity monad are bound together tightly. This is one of our major themes.

Integration is most familiar when the integrands take values in some kind of algebraic structure, such as $\mathbb{R}$. In Section 4, we describe integration against an ultrafilter for functions taking values in a rig (semiring). We prove that when the rig R is sufficiently nontrivial, ultrafilters on X correspond one-to-one with integration operators for R-valued functions on X.

To continue, we need to review some further basic results on codensity monads, including their construction as Kan extensions (Section 5). This leads to another characterization:

the ultrafilter monad is the terminal monad on **Set**
that restricts to the identity on **FinSet**.

In Section 6, we justify the opening assertion of this introduction: that the codensity monad of a functor G is a surrogate for the monad induced by G and its left adjoint (which might not exist). For a start, if a left adjoint exists then the two monads are the same. More subtly, any monad on $\mathscr{A}$ induces a functor into $\mathscr{A}$ (the forgetful functor on its category of algebras), and, under a completeness hypothesis, any functor into $\mathscr{A}$ induces a monad on $\mathscr{A}$ (its codensity monad). Theorem 6.5, due to Dubuc [9], states that the two processes are adjoint. From this we deduce:

CptHff *is the codomain of the universal functor from* **FinSet** *to*
a category monadic over **Set**.

(This phrasing is slightly loose; see Corollary 6.7 for the precise statement.) Here **CptHff** is the category of compact Hausdorff spaces.

We have seen that when standard categorical constructions are applied to the inclusions **FinSet** $\hookrightarrow$ **Set**, we obtain the notions of ultrafilter and compact Hausdorff space. In Section 7 we ask what happens when sets are replaced by vector spaces. The answers give us the following table of analogues:

sets	vector spaces
finite sets	finite-dimensional vector spaces
ultrafilters	elements of the double dual
compact Hausdorff spaces	linearly compact vector spaces.

The main results here are that the codensity monad of **FDVect** $\hookrightarrow$ **Vect** is double dualization, and that its algebras are the linearly compact vector spaces (defined below). The close resemblance between the **Set** and **Vect** cases raises the question: can analogous results be proved for other algebraic theories? We leave this open.

It has long been a challenge to synthesize the complementary insights offered by category theory and model theory. For example, model theory allows insights into parts of algebraic geometry where present-day category theory seems to offer little. (This is especially so when it comes to transferring results between fields of positive characteristic and characteristic zero, as exemplified by Ax's model-theoretic proof that every injective endomorphism of a complex algebraic variety is surjective [3].) A small part of this challenge is to find a categorical home for the ultraproduct construction.

Section 8 does this. The theorem of Kennison and Gildenhuys shows that the notion of finiteness of a set leads inevitably to the notion of ultrafilter. Similarly, we show here that the notion of finiteness of a *family* of sets leads inevitably to the notion of ultraproduct. More specifically, we define a category of families of sets, and prove that the codensity monad of the full subcategory of finitely-indexed families is the ultraproduct monad. This theorem (with a different proof) was transmitted to me by the anonymous referee, to whom I am very grateful.

History and related work The concept of density was first isolated in a 1960 paper by Isbell [16], who gave a definition of dense (or in his terminology, left adequate) full subcategory. Ulmer generalized the definition to arbitrary functors, not just inclusions of full subcategories, and introduced the word 'dense' [41]. At about the same time, the codensity monad of a functor was defined by Kock [20] (who gave it its name) and, independently, by Appelgate and Tierney [1] (who concentrated on the dual notion, calling it the model-induced cotriple).

Other early sources on codensity monads are the papers of Linton [27] and Dubuc [9]. (Co)density of functors is covered in Chapter X of Mac Lane's book [31], with codensity monads appearing in the very last exercise. Kelly's book [17] treats (co)dense functors in detail, but omits (co)density (co)monads.

The codensity characterization of the ultrafilter monad seems to have first appeared in the paper [19] of Kennison and Gildenhuys, and is also included as Exercise 3.2.12(e) of Manes's book [34]. (Manes used the term 'algebraic completion' for codensity monad.) It is curious that no result resembling this appears in Isbell's 1960 paper, as even though he did not have the notion of codensity monad available, he performed similar and more set-theoretically sophisticated calculations. However, his paper does not mention ultrafilters. On the other hand, a 2010 paper of Litt, Abel and Kominers [29] proves a result equivalent to a weak form of Kennison and Gildenhuys's theorem, but does not mention codensity.

The integral notation that we use so heavily has been used in similar ways by Kock [22, 23] and Lucyshyn-Wright [30] (and slightly differently by Lawvere and Rosebrugh in Chapter 8 of [25]). In [23], Kock traces the idea back to work of Linton and Wraith.

Richter [36] found a different proof of Theorem 1.7 below, originally due to Börger. Section 3 of Kennison and Gildenhuys [19] may provide some help in answering the question posed at the end of Section 7.

Notation We fix a category **Set** of sets satisfying the axiom of choice. **Top** is the category of all topological spaces and continuous maps, and **CAT** is the category of locally small categories. When X is a set and Y is an object of some category, $[X, Y]$ denotes the X-power of Y, that is, the product of X copies of Y. In particular, when X and Y are sets, $[X, Y]$ is the set $Y^X = \mathbf{Set}(X, Y)$ of maps from X to Y. For categories $\mathscr{A}$ and $\mathscr{B}$, we write $[\mathscr{A}, \mathscr{B}]$ for the category of functors from $\mathscr{A}$ to $\mathscr{B}$. Where necessary, we silently assume that our general categories $\mathscr{A}, \mathscr{B}, \ldots$ are locally small.

1. Ultrafilters

We begin with the standard definitions. Write $P(X)$ for the power set of a set X.

1.1. Definition. *Let X be a set. A **filter** on X is a subset $\mathscr{F}$ of $P(X)$ such that:*

i. $\mathscr{F}$ is upwards closed: if $Z \subseteq Y \subseteq X$ with $Z \in \mathscr{F}$ then $Y \in \mathscr{F}$

ii. $\mathscr{F}$ is closed under finite intersections: $X \in \mathscr{F}$, and if $Y, Z \in \mathscr{F}$ then $Y \cap Z \in \mathscr{F}$.

Filters on X amount to meet-semilattice homomorphisms from $P(X)$ to the two-element totally ordered set $2 = \{0 < 1\}$, with $f \colon P(X) \longrightarrow 2$ corresponding to the filter $f^{-1}(1) \subseteq X$.

It is helpful to view the elements of a filter as the 'large' subsets of X, and their complements as 'small'. Thus, the union of a finite number of small sets is small. An ultrafilter is a filter in which every subset is either large or small, but not both.

1.2. Definition. *Let X be a set. An **ultrafilter** on X is a filter $\mathscr{U}$ such that for all $Y \subseteq X$, either $Y \in \mathscr{U}$ or $X \setminus Y \in \mathscr{U}$, but not both.*

Ultrafilters on X correspond to lattice homomorphisms $P(X) \longrightarrow 2$.

1.3. Example. Let X be a set and $x \in X$. The **principal ultrafilter** on x is the ultrafilter $\mathscr{U}_x = \{Y \subseteq X : x \in Y\}$. Every ultrafilter on a finite set is principal.

The set of filters on X is ordered by inclusion. The largest filter is $P(X)$; every other filter is called **proper**. (What we call proper filters are often just called filters.)

A standard lemma (Proposition 1.1 of [10]) states that the ultrafilters are precisely the maximal proper filters. Zorn's lemma then implies that every proper filter is contained in some ultrafilter. No explicit example of a nonprincipal ultrafilter can be given, since their existence implies a weak form of the axiom of choice. However:

1.4. Example. Let X be an infinite set. The subsets of X with finite complement form a proper filter $\mathscr{F}$ on X. Then $\mathscr{F}$ is contained in some ultrafilter, which cannot be principal. Thus, every infinite set admits at least one nonprincipal ultrafilter.

We will use the following simple characterization of ultrafilters. The equivalence of (i) and (ii) appears to be due to Galvin and Horn [13], whose result nearly implies the equivalence with (iii), too.

1.5. Proposition. [Galvin and Horn] *Let X be a set and $\mathscr{U} \subseteq P(X)$. The following are equivalent:*

i. $\mathscr{U}$ is an ultrafilter

ii. $\mathscr{U}$ satisfies the ***partition condition:*** *for all $n \geq 0$ and partitions*

$$X = Y_1 \amalg \cdots \amalg Y_n$$

of X into n pairwise disjoint (possibly empty) subsets, there is exactly one $i \in \{1, \ldots, n\}$ such that $Y_i \in \mathscr{U}$.

Moreover, for any $N \geq 3$, these conditions are equivalent to:

iii. $\mathscr{U}$ satisfies the partition condition for $n = N$.

Proof. Let $N \geq 3$. The implication (i)$\Rightarrow$(ii) is standard, and (ii)$\Rightarrow$(iii) is trivial. Now assume (iii); we prove (i).

From the partition $X = X \amalg \underbrace{\emptyset \amalg \cdots \amalg \emptyset}_{N-1}$ and the fact that $N \geq 3$, we deduce that $\emptyset \notin \mathscr{U}$ and $X \in \mathscr{U}$. It follows that $\mathscr{U}$ satisfies the partition condition for all $n \leq N$. Taking $n = 2$, this implies that for all $Y \subseteq X$, either $Y \in \mathscr{U}$ or $X \setminus Y \in \mathscr{U}$, but not both. It remains to prove that $\mathscr{U}$ is upwards closed and closed under binary intersections.

For upwards closure, let $Z \subseteq Y \subseteq X$ with $Z \in \mathscr{U}$. We have

$$X = Z \amalg (Y \setminus Z) \amalg (X \setminus Y)$$

with $Z \in \mathscr{U}$, so $X \setminus Y \notin \mathscr{U}$. Hence $Y \in \mathscr{U}$.

To prove closure under binary intersections, first note that if $Y_1, Y_2 \in \mathscr{U}$ then $Y_1 \cap Y_2 \neq \emptyset$: for if $Y_1 \cap Y_2 = \emptyset$ then $Y_1 \subseteq X \setminus Y_2$, so $X \setminus Y_2 \in \mathscr{U}$ by upwards closure, so $Y_2 \notin \mathscr{U}$, a contradiction. Now let $Y, Z \in \mathscr{U}$ and consider the partition

$$X = (Y \cap Z) \amalg (Y \setminus Z) \amalg (X \setminus Y).$$

Exactly one of these three subsets, say S, is in $\mathscr{U}$. But $S, Y \in \mathscr{U}$, so $S \cap Y \neq \emptyset$, so $S \neq X \setminus Y$; similarly, $S \neq Y \setminus Z$. Hence $S = Y \cap Z$, as required. ∎

Perhaps the most striking part of this result is:

1.6. COROLLARY. *Let X be a set and $\mathscr{U}$ a set of subsets of X such that whenever X is expressed as a disjoint union of three subsets, exactly one belongs to $\mathscr{U}$. Then $\mathscr{U}$ is an ultrafilter.* ■

The number three cannot be lowered to two: consider a three-element set X and the set $\mathscr{U}$ of subsets with at least two elements.

Given a map of sets $f\colon X \longrightarrow X'$ and a filter $\mathscr{F}$ on X, there is an induced filter

$$f_*\mathscr{F} = \{Y' \subseteq X' : f^{-1}Y' \in \mathscr{F}\}$$

on X'. If $\mathscr{F}$ is an ultrafilter then so is $f_*\mathscr{F}$. This defines a functor

$$U\colon \mathbf{Set} \longrightarrow \mathbf{Set}$$

in which $U(X)$ is the set of ultrafilters on X.

In fact, U carries the structure of a monad, $\mathbf{U}$. The unit map $X \longrightarrow U(X)$ sends $x \in X$ to the principal ultrafilter $\mathscr{U}_x$. We will avoid writing down the multiplication explicitly. (The contravariant power set functor P from **Set** to **Set** is self-adjoint on the right, and therefore induces a monad PP on **Set**; it contains $\mathbf{U}$ as a submonad.) What excuses us from this duty is the following powerful pair of results, both due to Börger [7].

1.7. THEOREM. [Börger] *The ultrafilter endofunctor U is terminal among all endofunctors of* **Set** *that preserve finite coproducts.*

PROOF. (Sketch.) Given a finite-coproduct-preserving endofunctor S of **Set**, the unique natural transformation $\alpha\colon S \longrightarrow U$ is described as follows: for each set X and element $\sigma \in S(X)$,

$$\alpha_X(\sigma) = \{Y \subseteq X : \sigma \in \mathrm{im}(S(Y \hookrightarrow X))\}.$$

For details, see Theorem 2.1 of [7]. ■

1.8. COROLLARY. [Börger] *The ultrafilter endofunctor U has a unique monad structure. With this structure, it is terminal among all finite-coproduct-preserving monads on* **Set**.

PROOF. (Corollary 2.3 of [7].) Since $U \circ U$ and the identity preserve finite coproducts, there are unique natural transformations $U \circ U \longrightarrow U$ and $1 \longrightarrow U$. The monad axioms follow by terminality of the endofunctor U, as does terminality of the monad. ■

There is also a topological description of the ultrafilter monad. As shown by Manes [33], it is the monad induced by the forgetful functor **CptHff** $\longrightarrow$ **Set** and its left adjoint. In particular, the Stone–Čech compactification of a discrete space is the set of ultrafilters on it.

2. Codensity

Here we review the definitions of codense functor and codensity monad. The dual notion, density, has historically been more prominent, so we begin our review there.

As shown by Kan, any functor F from a small category $\mathscr{A}$ to a cocomplete category $\mathscr{B}$ induces an adjunction

$$\begin{tikzcd} \mathscr{B} \arrow[r, shift left, "\mathrm{Hom}(F,-)"] \arrow[r, phantom, "\top"] & {[\mathscr{A}^{\mathrm{op}}, \mathbf{Set}]} \arrow[l, shift left, "-\otimes F"] \end{tikzcd}$$

where $(\mathrm{Hom}(F,B))(A) = \mathscr{B}(F(A),B)$. A famous example is the functor $F\colon \Delta \longrightarrow \mathbf{Top}$ assigning to each nonempty finite ordinal $[n]$ the topological n-simplex Δ^n. Then $\mathrm{Hom}(F,-)$ sends a topological space to its singular simplicial set, and $-\otimes F$ sends a simplicial set to its geometric realization.

Another example gives an abstract explanation of the concept of sheaf ([32], Section II.6). Let X be a topological space, with poset $\mathbf{O}(X)$ of open subsets. Define $F\colon \mathbf{O}(X) \longrightarrow \mathbf{Top}/X$ by $F(W) = (W \hookrightarrow X)$. This induces an adjunction between presheaves on X and spaces over X, and, like any adjunction, it restricts canonically to an equivalence between full subcategories. Here, these are the categories of sheaves on X and étale bundles over X. The induced monad on the category of presheaves is sheafification.

In general, F is **dense** if the right adjoint $\mathrm{Hom}(F,-)$ is full and faithful, or equivalently if the counit is an isomorphism. For the counit to be an isomorphism means that every object of $\mathscr{B}$ is a colimit of objects of the form $F(A)$ ($A \in \mathscr{A}$) in a canonical way; for example, the Yoneda embedding $\mathscr{A} \longrightarrow [\mathscr{A}^{\mathrm{op}}, \mathbf{Set}]$ is dense, so every presheaf is canonically a colimit of representables. More loosely, F is dense if the objects of $\mathscr{B}$ can be effectively probed by mapping into them from objects of the form $F(A)$. In the case of the Yoneda embedding, this is the familiar idea that presheaves can be probed by mapping into them from representables.

Finitely presentable objects provide further important examples. For instance, the embedding $\mathbf{Grp}_{\mathrm{fp}} \hookrightarrow \mathbf{Grp}$ is dense, where $\mathbf{Grp}$ is the category of groups and $\mathbf{Grp}_{\mathrm{fp}}$ is the full subcategory of groups that are finitely presentable. Similarly, $\mathbf{FinSet}$ is dense in $\mathbf{Set}$.

Here we are concerned with *co*density. The general theory is of course formally dual to that of density, but its application to familiar functors seems not to have been so thoroughly explored.

Let $G\colon \mathscr{B} \longrightarrow \mathscr{A}$ be a functor. There is an induced functor

$$\mathrm{Hom}(-,G)\colon \mathscr{A} \longrightarrow [\mathscr{B}, \mathbf{Set}]^{\mathrm{op}}$$

defined by

$$\big(\mathrm{Hom}(A,G)\big)(B) = \mathscr{A}(A, G(B))$$

($A \in \mathscr{A}, B \in \mathscr{B}$). The functor G is **codense** if $\mathrm{Hom}(-,G)$ is full and faithful.

Assume for the rest of this section that $\mathscr{B}$ is **essentially small** (equivalent to a small category) and that $\mathscr{A}$ has small limits. (This assumption will be relaxed in Section 5.) Then $\mathrm{Hom}(-, G)$ has a right adjoint, also denoted by $\mathrm{Hom}(-, G)$:

$$\begin{tikzcd} \mathscr{A} \arrow[r, shift left, "\mathrm{Hom}(-,G)"] \arrow[r, phantom, "\perp"] & {[\mathscr{B}, \mathbf{Set}]^{\mathrm{op}}} \arrow[l, shift left, "\mathrm{Hom}(-,G)"] \end{tikzcd}. \tag{1}$$

This right adjoint can be described as an end or as a limit: for $Y \in [\mathscr{B}, \mathbf{Set}]$,

$$\mathrm{Hom}(Y, G) = \int_{B \in \mathscr{B}} [Y(B), G(B)] = \varprojlim_{B \in \mathscr{B},\ y \in Y(B)} G(B),$$

where the square bracket notation is as defined at the end of the introduction, and the limit is over the category of elements of Y. If $\mathscr{A} = \mathbf{Set}$ then $\mathrm{Hom}(Y, G)$ is the set of natural transformations from Y to G. In any case, the adjointness asserts that

$$\mathscr{A}(A, \mathrm{Hom}(Y, G)) \cong [\mathscr{B}, \mathbf{Set}](Y, \mathrm{Hom}(A, G))$$

naturally in $A \in \mathscr{A}$ and $Y \in [\mathscr{B}, \mathbf{Set}]$.

The adjunction (1) induces a monad $\mathbf{T}^G = (T^G, \eta^G, \mu^G)$ on $\mathscr{A}$, the **codensity monad** of G. Explicitly,

$$T^G(A) = \int_{B \in \mathscr{B}} [\mathscr{A}(A, G(B)),\ G(B)] = \varprojlim_{\substack{B \in \mathscr{B}, \\ f\colon A \longrightarrow G(B)}} G(B)$$

($A \in \mathscr{A}$). As for any adjunction, the left adjoint is full and faithful if and only if the unit is an isomorphism. Thus, G is codense if and only if for each $A \in \mathscr{A}$, the canonical map

$$\eta^G_A \colon A \longrightarrow \int_B [\mathscr{A}(A, G(B)),\ G(B)]$$

is an isomorphism. (Then each object of $\mathscr{A}$ is a limit of objects $G(B)$ in a canonical way.) This happens if and only if the codensity monad of G is isomorphic to the identity. In that sense, the codensity monad of a functor measures its failure to be codense.

In many cases of interest, G is a subcategory inclusion $\mathscr{B} \hookrightarrow \mathscr{A}$. We then transfer epithets, calling $\mathscr{B}$ codense if G is, and writing $\mathbf{T}^{\mathscr{B}}$ instead of $\mathbf{T}^G$.

We continue with the theory of codensity monads in Sections 5 and 6, but we now have all we need to proceed to the result on ultrafilters.

3. Ultrafilters via codensity

Here we give an account of the fact, due to Kennison and Gildenhuys, that the ultrafilter monad is the codensity monad of the subcategory **FinSet** of **Set**. The proof is made more transparent by adopting the language of integration and measure.

First, though, let us see roughly why the result might be true. Write $\mathbf{T} = (T, \eta, \mu)$ for the codensity monad of $\mathbf{FinSet} \hookrightarrow \mathbf{Set}$. Fix a set X. Then

$$T(X) = \int_{B \in \mathbf{FinSet}} [[X, B], B],$$

which is the set of natural transformations

$$\begin{tikzcd} \mathbf{FinSet} \arrow[r, bend left, "{[X,-]}"] \arrow[r, bend right, "\text{inclusion}"'] \arrow[r, phantom, "\Downarrow"] & \mathbf{Set}. \end{tikzcd}$$

An element of $T(X)$ is, therefore, an operation that takes as input a finite set B and a function $X \longrightarrow B$, and returns as output an element of B; and it does so in a way that is natural in B. There is certainly one such operation for each element x of X, namely, evaluation at x. Less obviously, there is one such operation for each ultrafilter $\mathscr{U}$ on X: given $f\colon X \longrightarrow B$ as input, return as output the unique element $b \in B$ such that $f^{-1}(b) \in \mathscr{U}$. (There *is* a unique b with this property, by Proposition 1.5(ii).) For example, if $\mathscr{U}$ is the principal ultrafilter on $x \in X$, this operation is just evaluation at x. It turns out that every element $I \in T(X)$ arises from an ultrafilter, which one recovers from I by taking $B = 2$ and noting that $[[X, 2], 2] \cong PP(X)$. That, in essence, is how we will prove the theorem.

An ultrafilter is a probability measure that paints the world in black and white: everything is either almost surely true or almost surely false. Indeed, an ultrafilter $\mathscr{U}$ on a set X is in particular a subset of $P(X)$, and therefore has a characteristic function $\mu_{\mathscr{U}}\colon P(X) \longrightarrow \{0, 1\}$. On the other hand, a **finitely additive measure** on a set X (or properly speaking, on the algebra of all subsets of X) is a function $\mu\colon P(X) \longrightarrow [0, \infty]$ such that

$$\mu(\emptyset) = 0, \qquad \mu(Y \cup Z) + \mu(Y \cap Z) = \mu(Y) + \mu(Z)$$

for all $Y, Z \subseteq X$. (Equivalently, $\mu(\bigcup_i Y_i) = \sum_i \mu(Y_i)$ for all finite families (Y_i) of pairwise disjoint subsets of X.) We call μ a finitely additive **probability** measure if also $\mu(X) = 1$. The following correspondence has been observed many times.

3.1. Lemma. *Let X be a set. A subset $\mathscr{U}$ of $P(X)$ is an ultrafilter if and only if its characteristic function $\mu_{\mathscr{U}}\colon P(X) \longrightarrow \{0, 1\}$ is a finitely additive probability measure. This defines a bijection between the ultrafilters on X and the finitely additive probability measures on X with values in $\{0, 1\}$.* ■

With every notion of measure comes a notion of integration. Integrating a function with respect to a *probability* measure amounts to taking its average value, and taking averages typically requires some algebraic or order-theoretic structure, which we do not have. Nevertheless, it can be done, as follows.

Let us say that a function between sets is **simple** if its image is finite. (The name is justified in Section 4.) The set of simple functions from one set, X, to another, R, is written as $\mathrm{Simp}(X,R)$; categorically, it is the coend

$$\mathrm{Simp}(X,R) = \int^{B\in\mathbf{FinSet}} \mathbf{Set}(X,B)\times\mathbf{Set}(B,R).$$

The next result states that given an ultrafilter $\mathscr{U}$ on a set X, there is a unique sensible way to define integration of simple functions on X with respect to the measure $\mu_{\mathscr{U}}$. The two conditions defining 'sensible' are that the average value (integral) of a constant function is that constant, and that changing a function on a set of measure zero does not change its integral.

3.2. Proposition. *Let X be a set and $\mathscr{U}$ an ultrafilter on X. Then for each set R, there is a unique map*

$$\int_X - \, d\mathscr{U}\colon\ \mathrm{Simp}(X,R)\longrightarrow R$$

such that

i. *$\int_X r\, d\mathscr{U} = r$ for all $r\in R$, where the integrand is the function with constant value r*

ii. *$\int_X f\, d\mathscr{U} = \int_X g\, d\mathscr{U}$ whenever $f,g\in \mathrm{Simp}(X,R)$ with $\{x\in X : f(x)=g(x)\}\in\mathscr{U}$.*

In analysis, it is customary to write $\int_X f\, d\mu$ for the integral of a function f with respect to (or 'against') a measure μ. Logically, then, we should write our integration operator as $\int_X - \, d\mu_{\mathscr{U}}$. However, we blur the distinction between $\mathscr{U}$ and $\mu_{\mathscr{U}}$, writing $\int_X - \, d\mathscr{U}$ (or just $\int - \, d\mathscr{U}$) instead.

Proof. Let R be a set. For existence, given any $f\in\mathrm{Simp}(X,R)$, simplicity guarantees that there is a unique element $\int_X f\, d\mathscr{U}$ of R such that

$$f^{-1}\Bigl(\int_X f\, d\mathscr{U}\Bigr)\in\mathscr{U}.$$

Condition (i) holds because $X\in\mathscr{U}$. For (ii), let f and g be simple functions such that $\mathrm{Eq}(f,g)=\{x\in X : f(x)=g(x)\}$ belongs to $\mathscr{U}$. We have

$$f^{-1}\Bigl(\int_X f\, d\mathscr{U}\Bigr)\cap \mathrm{Eq}(f,g)\ \subseteq\ g^{-1}\Bigl(\int_X f\, d\mathscr{U}\Bigr),$$

and $f^{-1}\bigl(\int f\, d\mathscr{U}\bigr), \mathrm{Eq}(f,g)\in\mathscr{U}$, so by definition of ultrafilter, $g^{-1}\bigl(\int f\, d\mathscr{U}\bigr)\in\mathscr{U}$. But $\int g\, d\mathscr{U}$ is by definition the unique element r of R such that $g^{-1}(r)\in\mathscr{U}$, so $\int f\, d\mathscr{U} = \int g\, d\mathscr{U}$, as required.

For uniqueness, let $f\in\mathrm{Simp}(X,R)$. Since f is simple, there is a unique $r\in R$ such that $f^{-1}(r)\in\mathscr{U}$. Then $\mathrm{Eq}(f,r)\in\mathscr{U}$, so (i) and (ii) force $\int f\, d\mathscr{U} = r$. ∎

Integration is natural in both the codomain R and the domain pair $(X, \mathscr{U})$:

3.3. Lemma.

i. Let $\mathscr{U}$ be an ultrafilter on a set X. Then integration of simple functions against $\mathscr{U}$ defines a natural transformation

$$\begin{tikzcd} \mathbf{Set} \arrow[r, bend left, "\mathrm{Simp}(X,-)"] \arrow[r, bend right, "\mathrm{id}"'] \arrow[r, phantom, "\Downarrow \int - \, d\mathscr{U}"] & \mathbf{Set}. \end{tikzcd}$$

ii. For any map $X \xrightarrow{p} Y$ of sets and ultrafilter $\mathscr{U}$ on X, the triangle

$$\begin{tikzcd} \mathrm{Simp}(X,-) \arrow[dr, "\int_X - \, d\mathscr{U}"'] & & \mathrm{Simp}(Y,-) \arrow[ll, "- \circ p"'] \arrow[dl, "\int_Y - \, d(p_* \mathscr{U})"] \\ & \mathrm{id} & \end{tikzcd}$$

in $[\mathbf{Set}, \mathbf{Set}]$ commutes.

Proof. For (i), we must prove that for any map $R \xrightarrow{\theta} S$ of finite sets and any function $f \colon X \longrightarrow R$,

$$\theta\left(\int_X f \, d\mathscr{U}\right) = \int_X \theta \circ f \, d\mathscr{U}. \tag{2}$$

Indeed,

$$(\theta \circ f)^{-1}\left(\theta\left(\int_X f \, d\mathscr{U}\right)\right) \supseteq f^{-1}\left(\int_X f \, d\mathscr{U}\right) \in \mathscr{U},$$

so $(\theta \circ f)^{-1}\big(\theta\big(\int f \, d\mathscr{U}\big)\big) \in \mathscr{U}$, and (2) follows.

For (ii), let $R \in \mathbf{FinSet}$ and $g \in \mathrm{Simp}(Y, R)$. We must prove that

$$\int_X (g \circ p) \, d\mathscr{U} = \int_Y g \, d(p_* \mathscr{U}) \tag{3}$$

(the analogue of the classical formula for integration under a change of variable). Indeed,

$$g^{-1}\left(\int_Y g \, d(p_* \mathscr{U})\right) \in p_* \mathscr{U},$$

which by definition of $p_* \mathscr{U}$ means that

$$(g \circ p)^{-1}\left(\int_Y g \, d(p_* \mathscr{U})\right) \in \mathscr{U},$$

giving (3). ∎

For the next few results, we will allow R to vary within a subcategory $\mathscr{B}$ of **FinSet**. (The most important case is $\mathscr{B} = \mathbf{FinSet}$.) Clearly $\mathrm{Simp}(X, B) = [X, B]$ for all $B \in \mathscr{B}$. The notation $\mathbf{T}^{\mathscr{B}}$ will mean the codensity monad of $\mathscr{B} \hookrightarrow \mathbf{Set}$ (not $\mathscr{B} \hookrightarrow \mathbf{FinSet}$). Thus, whenever X is a set, $T^{\mathscr{B}}(X)$ is the set of natural transformations

$$\begin{tikzcd} \mathscr{B} \arrow[r, bend left, "{[X,-]}"] \arrow[r, bend right, "\text{inclusion}"'] \arrow[r, phantom, "\Downarrow"] & \mathbf{Set}. \end{tikzcd}$$

We will regard elements of $T^{\mathscr{B}}(X)$ as integration operators: an element $I \in T^{\mathscr{B}}(X)$ consists of a function $I = I_B \colon [X, B] \longrightarrow B$ for each $B \in \mathscr{B}$, such that

$$\begin{tikzcd} {[X,B]} \arrow[r, "\theta\circ -"] \arrow[d, "I_B"'] & {[X,C]} \arrow[d, "I_C"] \\ B \arrow[r, "\theta"'] & C \end{tikzcd} \tag{4}$$

commutes whenever $B \xrightarrow{\theta} C$ is a map in $\mathscr{B}$.

3.4. Proposition. *Let $\mathscr{B}$ be a subcategory of* **FinSet**. *Then there is a natural transformation $U \longrightarrow T^{\mathscr{B}}$ with components*

$$\begin{array}{ccc} U(X) & \longrightarrow & T^{\mathscr{B}}(X) \\ \mathscr{U} & \longmapsto & \int_X - \, d\mathscr{U} \end{array} \tag{5}$$

*($X \in$ **Set**).*

Proof. Lemma 3.3(i) guarantees that (5) is a well-defined function for each X. Lemma 3.3(ii) tells us that it is natural in X. ∎

The transformation of Proposition 3.4 turns measures (ultrafilters) into integration operators. In analysis, we recover a measure μ from its corresponding integration operator via the equation $\mu(Y) = \int \chi_Y \, d\mu$. To imitate this here, we need some notion of characteristic function, and for that we need $\mathscr{B}$ to contain some set with at least two elements.

So, suppose that we have fixed some set $\Omega \in \mathscr{B}$ and elements $0, 1 \in \Omega$ with $0 \neq 1$. For any set X and $Y \subseteq X$, define $\chi_Y \colon X \longrightarrow \Omega$ by

$$\chi_Y(x) = \begin{cases} 1 & \text{if } x \in Y \\ 0 & \text{otherwise.} \end{cases} \tag{6}$$

Then for any ultrafilter $\mathscr{U}$ on X, we have

$$\int_X \chi_Y \, d\mathscr{U} = \begin{cases} 1 & \text{if } Y \in \mathscr{U} \\ 0 & \text{otherwise.} \end{cases} \tag{7}$$

Hence

$$\mathscr{U} = \left\{ Y \subseteq X : \int_X \chi_Y \, d\mathscr{U} = 1 \right\}. \tag{8}$$

We have thus recovered $\mathscr{U}$ from $\int_X - \, d\mathscr{U}$.

The full theorem is as follows.

3.5. THEOREM. *Let $\mathscr{B}$ be a full subcategory of* **FinSet** *containing at least one set with at least three elements. Then the codensity monad of $\mathscr{B} \hookrightarrow$* **Set** *is isomorphic to the ultrafilter monad.*

PROOF. We show that the natural transformation $U \longrightarrow T^{\mathscr{B}}$ of Proposition 3.4 is a natural isomorphism. Then by Corollary 1.8, it is an isomorphism of monads.

Let X be a set and $I \in T^{\mathscr{B}}(X)$. We must show that there is a unique ultrafilter $\mathscr{U}$ on X such that $I = \int_X - \, d\mathscr{U}$. Choose a set $\Omega \in \mathscr{B}$ with at least two elements, say 0 and 1, and whenever $Y \subseteq X$, define χ_Y as in (6).

Uniqueness follows from (8). For existence, put $\mathscr{U} = \{Y \subseteq X : I(\chi_Y) = 1\}$. Whenever B is a set in $\mathscr{B}$ and $f \colon X \longrightarrow B$ is a function, $I(f)$ is the unique element of B satisfying $f^{-1}(I(f)) \in \mathscr{U}$: for given $b \in B$, we have

$$\begin{aligned} f^{-1}(b) \in \mathscr{U} &\iff I(\chi_{f^{-1}(b)}) = 1 \iff I(\chi_{\{b\}} \circ f) = 1 \iff \chi_{\{b\}}(I(f)) = 1 \\ &\iff b = I(f), \end{aligned}$$

where the penultimate step is by (4). Applying this when B is a set in $\mathscr{B}$ with at least three elements proves that $\mathscr{U}$ is an ultrafilter, by Proposition 1.5(iii). Moreover, since $f^{-1}(I(f)) \in \mathscr{U}$ for any f, we have $I = \int - \, d\mathscr{U}$, as required. ∎

3.6. REMARK. In this proof, we used Börger's Corollary 1.8 as a labour-saving device; it excused us from checking that the constructed isomorphism $U \longrightarrow T^{\mathscr{B}}$ preserves the monad structure. We could also have checked this directly. Remark 7.6 describes a third method.

3.7. REMARK. The condition that $\mathscr{B}$ contains at least one set with at least three elements is sharp. There are $2^3 = 8$ full subcategories $\mathscr{B}$ of **Set** containing only sets of cardinality 0, 1 or 2, and in no case is $\mathbf{T}^{\mathscr{B}}$ isomorphic to the ultrafilter monad. If $2 \notin \mathscr{B}$ then $T^{\mathscr{B}}(X) = 1$ for all nonempty X. If $2 \in \mathscr{B}$ then $T^{\mathscr{B}}(X)$ is canonically isomorphic to the set of all $\mathscr{U} \subseteq P(X)$ satisfying the partition condition of Proposition 1.5 for $n \in \{1, 2\}$. In that case, $U(X) \subseteq T^{\mathscr{B}}(X)$, but by the example after Corollary 1.6, the inclusion is in general strict.

We immediately deduce an important result from [19]:

3.8. COROLLARY. [Kennison and Gildenhuys] *The codensity monad of* **FinSet** $\hookrightarrow$ **Set** *is the ultrafilter monad.* ∎

We can also deduce an unpublished result stated by Lawvere in 2000 [24]. (See also [5].) It does not mention codensity explicitly. Write $\mathbf{End}(B)$ for the endomorphism monoid of a set B, and $\mathbf{Set}^{\mathbf{End}(B)}$ for the category of left $\mathbf{End}(B)$-sets. Given a set X, equip $[X, B]$ with the natural left action by $\mathbf{End}(B)$.

3.9. Corollary. [Lawvere] *Let B be a finite set with at least three elements. Then*

$$\mathbf{Set}^{\mathbf{End}(B)}([X, B], B) \cong U(X)$$

naturally in $X \in \mathbf{Set}$.

Proof. Let $\mathscr{B}$ be the full subcategory of $\mathbf{Set}$ consisting of the single object B. Then $T^{\mathscr{B}}(X) = \mathbf{Set}^{\mathbf{End}(B)}([X, B], B)$, and the result follows from Theorem 3.5. ∎

For example, let 3 denote the three-element set; then an ultrafilter on X amounts to a map $3^X \longrightarrow 3$ respecting the natural action of the 27-element monoid $\mathbf{End}(3)$.

We have exploited the idea that an ultrafilter on a set X is a primitive sort of probability measure on X. But there are monads other than $\mathbf{U}$, in other settings, that assign to a space X some space of measures on X: for instance, there are those of Giry [14] and Lucyshyn-Wright [30]. It may be worth investigating whether they, too, arise canonically as codensity monads.

4. Integration of functions taking values in a rig

Integration of the most familiar kind involves integrands taking values in the ring $\mathbb{R}$ and an integration operator that is $\mathbb{R}$-linear. So far, the codomains of our integrands have been mere sets. However, we can say more when the codomain has algebraic structure. The resulting theory sheds light on the relationship between integration as classically understood and integration against an ultrafilter.

Let R be a rig (semiring). To avoid complications, we take all rigs to be commutative. Since R has elements 0 and 1, we may define the characteristic function $\chi_Y : X \longrightarrow R$ of any subset Y of a set X, as in equation (6).

In analysis, a function on a measure space X is called simple if it is a finite linear combination of characteristic functions of measurable subsets of X. The following lemma justifies our own use of the word.

4.1. Lemma. *A function from a set X to a rig R is simple if and only if it is a finite R-linear combination of characteristic functions of subsets of X.* ∎

Integration against an ultrafilter is automatically linear:

4.2. Lemma. *Let X be a set, $\mathscr{U}$ an ultrafilter on X, and R a rig. Then the map $\int_X - \, d\mathscr{U} : \mathrm{Simp}(X, R) \longrightarrow R$ is R-linear.*

Here, we are implicitly using the notion of a **module** over a rig R, which is an (additive) commutative monoid equipped with an action by R satisfying the evident axioms. In particular, $\mathrm{Simp}(X, R)$ is an R-module with pointwise operations.

PROOF. We have the natural transformation

$$\begin{tikzcd} \mathbf{Set} \arrow[r, bend left, "\mathrm{Simp}(X,-)"] \arrow[r, bend right, "\mathrm{id}"'] \arrow[r, phantom, "\Downarrow \int - \, d\mathscr{U}"] & \mathbf{Set} \end{tikzcd}$$

in which **Set** has finite products and both functors preserve finite products. The theory of R-modules is a finite product theory, so taking internal R-modules throughout gives a natural transformation

$$\begin{tikzcd} R\text{-}\mathbf{Mod} \arrow[r, bend left, "\mathrm{Simp}(X,-)"] \arrow[r, bend right, "\mathrm{id}"'] \arrow[r, phantom, "\Downarrow \int - \, d\mathscr{U}"] & R\text{-}\mathbf{Mod} \end{tikzcd}$$

This new functor $\mathrm{Simp}(X, -)$ sends an R-module M to $\mathrm{Simp}(X, M)$ with the pointwise R-module structure, and $\int - \, d\mathscr{U}$ defines an R-linear map $\mathrm{Simp}(X, M) \longrightarrow M$. Applying this to $M = R$ gives the result. ∎

4.3. PROPOSITION. *Let X be a set, $\mathscr{U}$ an ultrafilter on X, and R a rig. Then $\int_X - \, d\mathscr{U}$ is the unique R-linear map* $\mathrm{Simp}(X, R) \longrightarrow R$ *such that for all $Y \subseteq X$,*

$$\int_X \chi_Y \, d\mathscr{U} = \begin{cases} 1 & \textit{if } Y \in \mathscr{U} \\ 0 & \textit{otherwise} \end{cases}$$

(that is, $\int_X \chi_Y \, d\mathscr{U} = \mu_{\mathscr{U}}(Y)$).

PROOF. We have already shown that $\int_X - \, d\mathscr{U}$ has the desired properties (Lemma 4.2 and equation (7)). Uniqueness follows from Lemma 4.1. ∎

Let X be a set and R a rig. For any ultrafilter $\mathscr{U}$ on X, the R-linear map $\int - \, d\mathscr{U} \colon \mathrm{Simp}(X, R) \longrightarrow R$ has the property that $\int f \, d\mathscr{U}$ always belongs to $\mathrm{im}(f)$. Abstracting, let us define an **R-valued integral on** X to be an R-linear map $I \colon \mathrm{Simp}(X, R) \longrightarrow R$ such that $I(f) \in \mathrm{im}(f)$ for all $f \in \mathrm{Simp}(X, R)$.

Our main result states that an ultrafilter on a set X is essentially the same thing as an R-valued integral on X, as long as the rig R is sufficiently nontrivial.

4.4. THEOREM. *Let R be a rig in which $3 \neq 1$. Then for any set X, there is a canonical bijection*

$$U(X) \xrightarrow{\sim} \{R\text{-valued integrals on } X\},$$

defined by $\mathscr{U} \mapsto \int_X - \, d\mathscr{U}$.

PROOF. Injectivity follows from the equation

$$\mathscr{U} = \left\{ Y \subseteq X : \int_X \chi_Y \, d\mathscr{U} = 1 \right\}$$

($\mathscr{U} \in U(X)$), which is itself a consequence of (7) and the fact that $0 \neq 1$ in R.

For surjectivity, let I be an R-valued integral on X. Put $\mathscr{U} = \{Y \subseteq X : I(\chi_Y) = 1\}$. To show that $\mathscr{U}$ is an ultrafilter, take a partition $X = Y_1 \amalg Y_2 \amalg Y_3$. We have

$$\sum_{i=1}^{3} I(\chi_{Y_i}) = I\biggl(\sum_{i=1}^{3} \chi_{Y_i}\biggr) = I(1) = 1$$

where the '1' in $I(1)$ is the constant function and the last equality follows from the fact that $I(1) \in \mathrm{im}(1)$. On the other hand, $I(\chi_{Y_i}) \in \mathrm{im}(\chi_{Y_i}) \subseteq \{0,1\}$ for each $i \in \{1,2,3\}$, and $0 \neq 1$, $2 \neq 1$, $3 \neq 1$ in R, so $I(\chi_{Y_i}) = 1$ for exactly one value of $i \in \{1,2,3\}$. By Corollary 1.6, $\mathscr{U}$ is an ultrafilter. Finally, $I = \int_X - \, d\mathscr{U}$: for by linearity, it is enough to check this on characteristic functions, and this follows from (7) and the definition of $\mathscr{U}$. ■

5. Codensity monads as Kan extensions

The only ultrafilters on a finite set B are the principal ultrafilters; hence $U(B) \cong B$. We prove that **U** is the universal monad on **Set** with this property. For the proof, we first need to review some standard material on codensity, largely covered in early papers such as [1], [20] and [27].

So far, we have only considered codensity monads for functors whose domain is essentially small and whose codomain is complete. We now relax those hypotheses. An arbitrary functor $G\colon \mathscr{B} \longrightarrow \mathscr{A}$ **has a codensity monad** if for each $A \in \mathscr{A}$, the end

$$\int_{B \in \mathscr{B}} [\mathscr{A}(A, G(B)), G(B)] \tag{9}$$

exists. In that case, we write $T^G(A)$ for this end, so that T^G is a functor $\mathscr{A} \longrightarrow \mathscr{A}$. As the end formula reveals, T^G together with the canonical natural transformation

$$\begin{tikzcd} \mathscr{B} \arrow[r, "G"] \arrow[dr, "G"'] & \mathscr{A} \arrow[d, "T^G"] \arrow[dl, phantom, "\Swarrow \kappa^G"] \\ & \mathscr{A} \end{tikzcd} \tag{10}$$

is the right Kan extension of G along itself.

It will be convenient to phrase the universal property of the Kan extension in the following way. Let $\mathscr{K}(G)$ be the category whose objects are pairs (S, σ) of the type

$$\begin{tikzcd} \mathscr{B} \arrow[r, "G"] \arrow[dr, "G"'] & \mathscr{A} \arrow[d, "S"] \arrow[dl, phantom, "\Swarrow \sigma"] \\ & \mathscr{A} \end{tikzcd}$$

and whose maps $(S', \sigma') \longrightarrow (S, \sigma)$ are natural transformations $\theta \colon S' \longrightarrow S$ such that

$$\begin{tikzcd} \mathscr{B} \arrow[r, "G"] \arrow[dr, "G"'] & \mathscr{A} \arrow[d, bend right, "S"'] \arrow[d, bend left, "S'"] \arrow[d, phantom, "\overset{\Leftarrow}{\theta}"] \arrow[dl, phantom, "\swArrow \sigma"] \\ & \mathscr{A} \end{tikzcd} \quad = \quad \begin{tikzcd} \mathscr{B} \arrow[r, "G"] \arrow[dr, "G"'] & \mathscr{A} \arrow[d, bend left, "S'"] \arrow[dl, phantom, "\swArrow \sigma'"] \\ & \mathscr{A}. \end{tikzcd}$$

The universal property of (T^G, κ^G) is that it is the terminal object of $\mathscr{K}(G)$.

The category $\mathscr{K}(G)$ is monoidal under composition. Being the terminal object of a monoidal category, (T^G, κ^G) has a unique monoid structure. This gives T^G the structure of a monad, the **codensity monad** of G, which we write as $\mathbf{T}^G = (T^G, \eta^G, \mu^G)$. When $\mathscr{B}$ is essentially small and $\mathscr{A}$ is complete, this agrees with the definition in Section 2.

5.1. Example. Let **Ring** be the category of commutative rings, **Field** the full subcategory of fields, and $G \colon \mathbf{Field} \hookrightarrow \mathbf{Ring}$ the inclusion. Since **Field** is not essentially small, it is not instantly clear that G has a codensity monad. We show now that it does.

Let A be a ring. Write $A/\mathbf{Field}$ for the comma category in which an object is a field k together with a homomorphism $A \longrightarrow k$. There is a composite forgetful functor

$$A/\mathbf{Field} \longrightarrow \mathbf{Field} \hookrightarrow \mathbf{Ring},$$

and the end (9), if it exists, is its limit. The connected-components of $A/\mathbf{Field}$ are in natural bijection with the prime ideals of A (by taking kernels). Moreover, each component has an initial object: in the component corresponding to the prime ideal $\mathfrak{p}$, the initial object is the composite homomorphism

$$A \twoheadrightarrow A/\mathfrak{p} \hookrightarrow \mathrm{Frac}(A/\mathfrak{p}),$$

where $\mathrm{Frac}(-)$ means field of fractions. Hence the end (or limit) exists, and it is

$$T^G(A) = \prod_{\mathfrak{p} \in \mathrm{Spec}(A)} \mathrm{Frac}(A/\mathfrak{p}).$$

The unit homomorphism $\eta^G_A \colon A \longrightarrow T^G(A)$ is algebraically significant: its kernel is the nilradical of A, and its image is, therefore, the free reduced ring on A ([35], Section 1.1). In particular, this construction shows that a ring can be embedded into a product of fields if and only if it has no nonzero nilpotents. On the geometric side, $\mathrm{Spec}(T^G(A))$ is the Stone–Čech compactification of the discrete space $\mathrm{Spec}(A)$.

For example,

$$T^G(\mathbb{Z}) = \mathbb{Q} \times \prod_{\text{primes } p>0} \mathbb{Z}/p\mathbb{Z}$$

(the product of one copy each of the prime fields), and for positive integers n,

$$T^G(\mathbb{Z}/n\mathbb{Z}) = \mathbb{Z}/\mathrm{rad}(n)\mathbb{Z}$$

where $\mathrm{rad}(n)$ is the radical of n, that is, the product of its distinct prime factors.

Now consider the case where the functor G is the inclusion of a full subcategory $\mathscr{B} \subseteq \mathscr{A}$. Let us say that a monad $\mathbf{S} = (S, \eta^S, \mu^S)$ on $\mathscr{A}$ **restricts to the identity on** $\mathscr{B}$ if $\eta^S_B \colon B \longrightarrow S(B)$ is an isomorphism for all $B \in \mathscr{B}$, or equivalently if the natural transformation $\eta^S G \colon G \longrightarrow SG$ is an isomorphism. When this is so, $(S, (\eta^S G)^{-1})$ is an object of the monoidal category $\mathscr{K}(G)$, and by a straightforward calculation, $((S, (\eta^S G)^{-1}), \eta^S, \mu^S)$ is a monoid in $\mathscr{K}(G)$. For notational simplicity, we write this monoid as $(\mathbf{S}, (\eta^S G)^{-1})$.

Since G is full and faithful, the natural transformation κ^G is an isomorphism. But

$$\begin{tikzcd} \mathscr{B} \arrow[r, "G"] \arrow[dr, "G"'] & \mathscr{A} \arrow[d, bend right, "T^G"'] \arrow[d, bend left, "1"] \arrow[d, phantom, "\Leftarrow\eta^G"] \arrow[dl, phantom, "\swarrow\kappa^G"] \\ & \mathscr{A} \end{tikzcd} \quad = \quad \begin{tikzcd} \mathscr{B} \arrow[r, "G"] \arrow[dr, "G"'] & \mathscr{A} \arrow[d, bend left, "1"] \arrow[dl, phantom, "\swarrow\mathrm{id}"] \\ & \mathscr{A}, \end{tikzcd}$$

so $\eta^G G$ is an isomorphism; that is, $\mathbf{T}^G$ restricts to the identity on $\mathscr{B}$. (For example, the set of ultrafilters on a finite set B is isomorphic to B.) Note that $\kappa^G = (\eta^G G)^{-1}$. Also, (T^G, κ^G) is the terminal object of $\mathscr{K}(G)$, so $(\mathbf{T}^G, \kappa^G)$ is the terminal monoid in $\mathscr{K}(G)$. The following technical lemma will be useful.

5.2. Lemma. *Let $\mathscr{B}$ be a full subcategory of a category $\mathscr{A}$, such that the inclusion functor $G \colon \mathscr{B} \hookrightarrow \mathscr{A}$ has a codensity monad. Let $\mathbf{S} = (S, \eta^S, \mu^S)$ be a monad on $\mathscr{A}$ restricting to the identity on $\mathscr{B}$. For a natural transformation $\alpha \colon S \longrightarrow T^G$, the following are equivalent:*

i. *α is a map $(\mathbf{S}, (\eta^S G)^{-1}) \longrightarrow (\mathbf{T}^G, \kappa^G)$ of monoids in $\mathscr{K}(G)$*

ii. *α is a map $\mathbf{S} \longrightarrow \mathbf{T}^G$ of monads*

iii. *$\alpha \circ \eta^S = \eta^G$.*

Proof. The implications (i)⇒(ii)⇒(iii) are trivial. Assuming (iii), the fact that $\kappa^G = (\eta^G G)^{-1}$ implies that α is a map $(S, (\eta^S G)^{-1}) \longrightarrow (T^G, \kappa^G)$ in $\mathscr{K}(G)$; and (T^G, κ^G) is terminal in $\mathscr{K}(G)$, so α is the unique map of this type. But also $(\mathbf{T}^G, \kappa^G)$ is the terminal monoid in $\mathscr{K}(G)$, so there is a unique map of monoids $\beta \colon (\mathbf{S}, (\eta^S G)^{-1}) \longrightarrow (\mathbf{T}^G, \kappa^G)$. Then $\alpha = \beta$ by uniqueness of α, giving (i). ∎

Given a monad, it is often possible to find another monad with the same underlying endofunctor and the same unit, but a different multiplication. (For example, consider monads $M \times -$ on **Set**, where M is a monoid.) The same is true of codensity monads in general, since by Proposition 6.1, every monad can be constructed as a codensity monad. However, codensity monads *of full and faithful functors* have the special property that their multiplication is immutable, as follows:

5.3. Proposition. *Let* $G\colon \mathscr{B} \longrightarrow \mathscr{A}$ *be a full and faithful functor that has a codensity monad. Let* $\mathbf{S} = (S, \eta^S, \mu^S)$ *be a monad on* $\mathscr{A}$*. Then:*

i. Any natural isomorphism $\alpha\colon S \longrightarrow T^G$ *satisfying* $\alpha \circ \eta^S = \eta^G$ *is an isomorphism of monads.*

ii. If $S = T^G$ *and* $\eta^S = \eta^G$ *then* $\mu^S = \mu^G$*.*

Proof. We might as well assume that G is the inclusion of a full subcategory. Since $\mathbf{T}^G$ restricts to the identity on $\mathscr{B}$, so does $\mathbf{S}$, under the hypotheses of either (i) or (ii). Lemma 5.2 then gives both parts, taking α to be an isomorphism or the identity, respectively. ■

Lemma 5.2 also implies:

5.4. Proposition. *Let* $\mathscr{B}$ *be a full subcategory of a category* $\mathscr{A}$*, such that the inclusion functor* $G\colon \mathscr{B} \hookrightarrow \mathscr{A}$ *has a codensity monad. Then* $\mathbf{T}^G$ *is the terminal monad on* $\mathscr{A}$ *restricting to the identity on* $\mathscr{B}$*.*

Proof. Let $\mathbf{S} = (S, \eta^S, \mu^S)$ be a monad on $\mathscr{A}$ restricting to the identity on $\mathscr{B}$. Then $(\mathbf{S}, (\eta^S G)^{-1})$ is a monoid in $\mathscr{K}(G)$, and $(\mathbf{T}^G, \kappa^G)$ is the terminal such, so there exists a unique map $(\mathbf{S}, (\eta^S G)^{-1}) \longrightarrow (\mathbf{T}^G, \kappa^G)$ of monoids in $\mathscr{K}(G)$. But by (i)$\Leftrightarrow$(ii) of Lemma 5.2, an equivalent statement is that there exists a unique map $\mathbf{S} \longrightarrow \mathbf{T}^G$ of monads. ■

This gives a further characterization of the ultrafilter monad:

5.5. Theorem. *The ultrafilter monad is the terminal monad on* **Set** *restricting to the identity on* **FinSet***.* ■

To put this result into perspective, note that the *initial* monad on **Set** restricting to the identity on **FinSet** is itself the identity, and that a *finitary* monad on **Set** restricting to the identity on **FinSet** can only be the identity. In this sense, the ultrafilter monad is as far as possible from being finitary.

6. Codensity monads as substitutes for adjunction-induced monads

In the Introduction it was asserted that the codensity monad of a functor G is a substitute for the monad induced by G and its left adjoint, valid in situations where no adjoint exists. The crudest justification is the following theorem, which goes back to the earliest work on codensity monads.

6.1. Proposition. *Let* G *be a functor with a left adjoint,* F*. Then* G *has a codensity monad, which is isomorphic to* GF *with its usual monad structure.*

PROOF. If G is a functor $\mathscr{B} \longrightarrow \mathscr{A}$ then by the Yoneda lemma,

$$GF(A) \cong \int_B [\mathscr{B}(F(A), B),\, G(B)] \cong \int_B [\mathscr{A}(A, G(B)),\, G(B)] = T^G(A).$$

Hence $T^G \cong GF$, and it is straightforward to check that the isomorphism respects the monad structures. ∎

A more subtle justification is provided by the following results, especially Corollary 6.6. Versions of them appeared in Section II.1 of Dubuc [9].

We will need some further notation. Given a category $\mathscr{A}$, write $\mathbf{Mnd}(\mathscr{A})$ for the category of monads on $\mathscr{A}$ and $\mathbf{CAT}/\mathscr{A}$ for the (strict) slice of $\mathbf{CAT}$ over $\mathscr{A}$. For $\mathbf{S} \in \mathbf{Mnd}(\mathscr{A})$, write $U^{\mathbf{S}} \colon \mathscr{A}^{\mathbf{S}} \longrightarrow \mathscr{A}$ for the forgetful functor on the category of $\mathbf{S}$-algebras. The assignment $\mathbf{S} \mapsto (\mathscr{A}^{\mathbf{S}}, U^{\mathbf{S}})$ defines a functor $\mathbf{Alg} \colon \mathbf{Mnd}(\mathscr{A})^{\mathrm{op}} \longrightarrow \mathbf{CAT}/\mathscr{A}$.

Now let $G \colon \mathscr{B} \longrightarrow \mathscr{A}$ be a functor with a codensity monad. There is a functor $K^G \colon \mathscr{B} \longrightarrow \mathscr{A}^{\mathbf{T}^G}$, the **comparison functor** of G, defined by

$$B \longmapsto \left(\begin{array}{c} T^G G(B) \\ \big\downarrow {\scriptstyle \kappa^G_B} \\ G(B) \end{array} \right)$$

(where κ^G is as in (10)). When G has a left adjoint F, this is the usual comparison functor of the monad GF. In any case, the diagram

$$\begin{tikzcd} \mathscr{B} \arrow[r, "K^G"] \arrow[dr, "G"'] & \mathscr{A}^{\mathbf{T}^G} \arrow[d, "U^{\mathbf{T}^G}"] \\ & \mathscr{A} \end{tikzcd} \tag{11}$$

commutes.

6.2. PROPOSITION. [Dubuc] *Let $\mathscr{B} \xrightarrow{G} \mathscr{A}$ be a functor that has a codensity monad. Then*

$$(\mathbf{CAT}/\mathscr{A}) \left(\begin{array}{cc} \mathscr{B} & \mathscr{A}^{\mathbf{S}} \\ \big\downarrow {\scriptstyle G}, & \big\downarrow {\scriptstyle U^{\mathbf{S}}} \\ \mathscr{A} & \mathscr{A} \end{array} \right) \cong \mathbf{Mnd}(\mathscr{A})\,(\mathbf{S}, \mathbf{T}^G)$$

naturally in $\mathbf{S} \in \mathbf{Mnd}(\mathscr{A})$.

PROOF. Diagram (11) states that K^G is a map $(\mathscr{B}, G) \longrightarrow (\mathscr{A}^{\mathbf{T}^G}, U^{\mathbf{T}^G})$ in $\mathbf{CAT}/\mathscr{A}$. Let $\mathbf{S} \in \mathbf{Mnd}(\mathscr{A})$ and let $L \colon (\mathscr{B}, G) \longrightarrow (\mathscr{A}^{\mathbf{S}}, U^{\mathbf{S}})$ be a map in $\mathbf{CAT}/\mathscr{A}$. We show that there is a unique map of monads $\overline{L} \colon \mathbf{S} \longrightarrow \mathbf{T}^G$ satisfying

$$L = \Big((\mathscr{B}, G) \xrightarrow{K^G} (\mathscr{A}^{\mathbf{T}^G}, U^{\mathbf{T}^G}) \xrightarrow{\mathscr{A}^{\overline{L}}} (\mathscr{A}^{\mathbf{S}}, U^{\mathbf{S}}) \Big). \tag{12}$$

For each $B \in \mathscr{B}$, we have an $\mathbf{S}$-algebra $L(B) = \begin{pmatrix} SG(B) \\ \downarrow \lambda_B \\ G(B) \end{pmatrix}$. This defines a natural transformation

$$\begin{tikzcd} \mathscr{B} \arrow[r, "G"] \arrow[dr, "G"'] & \mathscr{A} \arrow[d, "S"] \arrow[dl, phantom, "\swarrow\lambda"] \\ & \mathscr{A}. \end{tikzcd}$$

By the universal property of (T^G, κ^G), there is a unique map $\overline{L}\colon (S, \lambda) \longrightarrow (T^G, \kappa^G)$ in $\mathscr{K}(G)$. The algebra axioms on $L(B)$ imply that $(\mathbf{S}, \lambda)$ is a monoid in $\mathscr{K}(G)$; and since $(\mathbf{T}^G, \kappa^G)$ is the terminal monoid in $\mathscr{K}(G)$, the map $\overline{L}$ is in fact a map of monads $\mathbf{S} \longrightarrow \mathbf{T}^G$. Equation (12) states exactly that $\overline{L}$ is a map $(S, \lambda) \longrightarrow (T^G, \kappa^G)$ in $\mathscr{K}(G)$, so the proof is complete. ∎

6.3. Example. Every object of a sufficiently complete category has an endomorphism monad. Indeed, let $\mathscr{A}$ be a category with small powers, and let $A \in \mathscr{A}$. The functor $A\colon \mathbf{1} \longrightarrow \mathscr{A}$ has a codensity monad, given by $X \mapsto [\mathscr{A}(X, A), A]$. This is the **endomorphism monad $\underline{\mathbf{End}}(A)$** of A [21]. The name is explained by Proposition 6.2, which tells us that for any monad $\mathbf{S}$ on $\mathscr{A}$, the $\mathbf{S}$-algebra structures on A correspond one-to-one with the monad maps $\mathbf{S} \longrightarrow \underline{\mathbf{End}}(A)$.

Proposition 6.2 can be rephrased explicitly as an adjunction. Given a category $\mathscr{A}$, denote by $(\mathbf{CAT}/\mathscr{A})_{\mathrm{CM}}$ the full subcategory of $\mathbf{CAT}/\mathscr{A}$ consisting of those functors into $\mathscr{A}$ that have a codensity monad. Since every monadic functor has a left adjoint and therefore a codensity monad, $\mathbf{Alg}$ determines a functor $\mathbf{Mnd}(\mathscr{A})^{\mathrm{op}} \longrightarrow (\mathbf{CAT}/\mathscr{A})_{\mathrm{CM}}$. On the other hand, $\mathbf{T}^G$ varies contravariantly with G, by either direct construction or Proposition 6.2. Thus, we have a functor

$$\mathbf{T}^\bullet\colon (\mathbf{CAT}/\mathscr{A})_{\mathrm{CM}}^{\mathrm{op}} \longrightarrow \mathbf{Mnd}(\mathscr{A}).$$

6.4. Example. Let $\{2\}$ denote the non-full subcategory of $\mathbf{Set}$ consisting of the two-element set and its identity map. Then the inclusion

$$\begin{pmatrix} \{2\} \\ \big\downarrow \\ \mathbf{Set} \end{pmatrix} \hookrightarrow \begin{pmatrix} \mathbf{FinSet} \\ \big\downarrow \\ \mathbf{Set} \end{pmatrix}$$

in $\mathbf{CAT}/\mathbf{Set}$ is mapped by $\mathbf{T}^\bullet$ to the inclusion $\mathbf{U} \hookrightarrow PP$ of the ultrafilter monad into the double power set monad. (In the notation of Example 6.3, $PP = \underline{\mathbf{End}}(2)$.)

Proposition 6.2 immediately implies that the construction of codensity monads is adjoint to the construction of categories of algebras:

6.5. Theorem. *Let $\mathscr{A}$ be a category. Then* $\mathbf{Alg}$ *and* $\mathbf{T}^\bullet$*, as contravariant functors between* $\mathbf{Mnd}(\mathscr{A})$ *and* $(\mathbf{CAT}/\mathscr{A})_{\mathrm{CM}}$*, are adjoint on the right.* ∎

We can usefully express this in another way still. Recall that the functor $\mathbf{Alg}\colon \mathbf{Mnd}(\mathscr{A})^{\mathrm{op}} \longrightarrow \mathbf{CAT}/\mathscr{A}$ is full and faithful [39]. The image is the full subcategory $(\mathbf{CAT}/\mathscr{A})_{\mathrm{mndc}}$ of $\mathbf{CAT}/\mathscr{A}$ consisting of the monadic functors into $\mathscr{A}$.

6.6. Corollary. *For any category $\mathscr{A}$, the inclusion*

$$(\mathbf{CAT}/\mathscr{A})_{\mathrm{mndc}} \hookrightarrow (\mathbf{CAT}/\mathscr{A})_{\mathrm{CM}}$$

has a left adjoint, given by

$$G \longmapsto \left(\begin{array}{c} \mathscr{A}^{\mathbf{T}^G} \\ \big\downarrow{\scriptstyle U^{\mathbf{T}^G}} \\ \mathscr{A} \end{array} \right).$$

■

In other words, among all functors into $\mathscr{A}$ admitting a codensity monad, the monadic functors form a reflective subcategory. The reflection turns a functor G into the monadic functor corresponding to the codensity monad of G. This is the more subtle sense in which the codensity monad of a functor G is the best approximation to the monad induced by G and its (possibly non-existent) left adjoint.

6.7. Corollary. *In* $\mathbf{CAT}/\mathbf{Set}$, *the initial map from* $(\mathbf{FinSet} \hookrightarrow \mathbf{Set})$ *to a monadic functor is*

$$\left(\begin{array}{c} \mathbf{FinSet} \\ \big\downarrow \\ \mathbf{Set} \end{array} \right) \hookrightarrow \left(\begin{array}{c} \mathbf{CptHff} \\ \big\downarrow \\ \mathbf{Set} \end{array} \right).$$

■

As a footnote, we observe that being codense is, in a sense, the opposite of being monadic. Indeed, if $G\colon \mathscr{B} \longrightarrow \mathscr{A}$ is codense then $\mathscr{A}^{\mathbf{T}^G} \simeq \mathscr{A}$, whereas if G is monadic then $\mathscr{A}^{\mathbf{T}^G} \simeq \mathscr{B}$. More precisely:

6.8. Proposition. *A functor is both codense and monadic if and only if it is an equivalence.*

Proof. An equivalence is certainly codense and monadic. Conversely, for any functor $G\colon \mathscr{B} \longrightarrow \mathscr{A}$ with a codensity monad, diagram (11) states that

$$G = \left(\mathscr{B} \xrightarrow{K^G} \mathscr{A}^{\mathbf{T}^G} \xrightarrow{U^{\mathbf{T}^G}} \mathscr{A} \right).$$

If G is monadic then G has a codensity monad and the comparison functor K^G is an equivalence; on the other hand, if G is codense then $\mathbf{T}^G$ is isomorphic to the identity, so $U^{\mathbf{T}^G}$ is an equivalence. The result follows. ■

7. Double dual vector spaces

In this section we prove that the codensity monad of the inclusion

$$\text{(finite-dimensional vector spaces)} \hookrightarrow \text{(vector spaces)}$$

is double dualization. Much of the proof is analogous to the proof that the codensity monad of **FinSet** $\hookrightarrow$ **Set** is the ultrafilter monad. (See the table in the Introduction.) Nevertheless, aspects of the analogy remain unclear, and finding a common generalization remains an open question.

Fix a field k for the rest of this section. Write **Vect** for the category of k-vector spaces, **FDVect** for the full subcategory of finite-dimensional vector spaces, and $\mathbf{T} = (T, \eta, \mu)$ for the codensity monad of **FDVect** $\hookrightarrow$ **Vect**. The dualization functor $(\ )^*$ is, as a contravariant functor from **Vect** to **Vect**, self-adjoint on the right. This gives the double dualization functor $(\ )^{**}$ the structure of a monad on **Vect**. We prove that $\mathbf{T} \cong (\ )^{**}$.

Pursuing the analogy, we regard elements $\mathscr{U}$ of a double dual space X^{**} as akin to measures on X, and we will define an integral operator $\int_X - \, d\mathscr{U}$. Specifically, let $X \in \mathbf{Vect}$ and $\mathscr{U} \in X^{**}$. We wish to define, for each $B \in \mathbf{FDVect}$, a map

$$\int_X - \, d\mathscr{U} : \mathbf{Vect}(X, B) \longrightarrow B. \tag{13}$$

In the ultrafilter context, integration has the property that $\int_X \chi_Y \, d\mathscr{U} = \mu_{\mathscr{U}}(Y)$ whenever $\mathscr{U}$ is an ultrafilter on a set X and $Y \in P(X)$ (equation (7)). Analogously, we require now that $\int_X \xi \, d\mathscr{U} = \mathscr{U}(\xi)$ whenever $\mathscr{U} \in X^{**}$ and $\xi \in X^*$; that is, when $B = k$, the integration operator (13) is $\mathscr{U}$ itself. Integration should also be natural in B. We show that these two requirements determine $\int_X - \, d\mathscr{U}$ uniquely.

7.1. Proposition. *Let X be a vector space and $\mathscr{U} \in X^{**}$. Let B be a finite-dimensional vector space. Then there is a unique map of sets*

$$\int_X - \, d\mathscr{U} : \mathbf{Vect}(X, B) \longrightarrow B$$

such that for all $\beta \in B^*$, the square

$$\begin{tikzcd} \mathbf{Vect}(X,B) \arrow[r, "\beta \circ -"] \arrow[d, "\int_X - \, d\mathscr{U}"'] & \mathbf{Vect}(X,k) \arrow[d, "\mathscr{U}"] \\ B \arrow[r, "\beta"'] & k \end{tikzcd}$$

commutes. When $B = k$, moreover, $\int_X - \, d\mathscr{U} = \mathscr{U}$.

Proof. The main statement asserts that B has a certain property; but if some vector space isomorphic to B has this property then plainly B does too. So it is enough to prove it when $B = k^n$ for some $n \in \mathbb{N}$.

Write $\mathrm{pr}_1, \ldots, \mathrm{pr}_n \colon k^n \longrightarrow k$ for the projections, and for $f \in \mathbf{Vect}(X, k^n)$, write $f_i = \mathrm{pr}_i \circ f$. For any map of sets $\int_X - \,d\mathscr{U} : \mathbf{Vect}(X, k^n) \longrightarrow k^n$,

$$
\begin{aligned}
& \beta\left(\int_X - \,d\mathscr{U}\right) = \mathscr{U}(\beta \circ -) \text{ for all } \beta \in (k^n)^* \\
\iff \; & \beta\left(\int_X f \,d\mathscr{U}\right) = \mathscr{U}(\beta \circ f) \text{ for all } \beta \in (k^n)^* \text{ and } f \in \mathbf{Vect}(X, k^n) \\
\iff \; & \mathrm{pr}_i\left(\int_X f \,d\mathscr{U}\right) = \mathscr{U}(\mathrm{pr}_i \circ f) \text{ for all } i \in \{1, \ldots, n\} \text{ and } f \in \mathbf{Vect}(X, k^n) \\
\iff \; & \int_X f \,d\mathscr{U} = (\mathscr{U}(f_1), \ldots, \mathscr{U}(f_n)) \text{ for all } f \in \mathbf{Vect}(X, k^n).
\end{aligned} \tag{14}
$$

This proves both existence and uniqueness. The result on $B = k$ also follows. ∎

Equation (14) implies that $\int_X - \,d\mathscr{U}$ is, in fact, linear with respect to the usual vector space structure on $\mathbf{Vect}(X, B)$. (In principle, the notation $\mathbf{Vect}(X, B)$ denotes a mere set.) Thus, a linear map

$$\mathscr{U} : \mathbf{Vect}(X, k) \longrightarrow k$$

gives rise canonically to a linear map

$$\int_X - \,d\mathscr{U} : \mathbf{Vect}(X, B) \longrightarrow B$$

for each finite-dimensional vector space B.

Integration is natural in two ways, as for sets and ultrafilters (Lemma 3.3). Indeed, writing $|\cdot| \colon \mathbf{FDVect} \longrightarrow \mathbf{Set}$ for the underlying set functor, we have the following.

7.2. Lemma.

i. Let X be a vector space and $\mathscr{U} \in X^{**}$. Then integration against $\mathscr{U}$ defines a natural transformation

$$
\begin{tikzcd}
\mathbf{FDVect} \arrow[r, bend left, "\mathbf{Vect}(X,-)"] \arrow[r, bend right, "|\cdot|"'] \arrow[r, phantom, "\Downarrow \int - \,d\mathscr{U}"] & \mathbf{Set}.
\end{tikzcd}
$$

ii. For any map $X \xrightarrow{p} Y$ in $\mathbf{Vect}$ and any $\mathscr{U} \in X^{**}$, the triangle

$$
\begin{tikzcd}
\mathbf{Vect}(X,-) \arrow[dr, "\int_X - \,d\mathscr{U}"'] & & \mathbf{Vect}(Y,-) \arrow[ll, "- \circ p"'] \arrow[dl, "\int_Y - \,d(p^{**}(\mathscr{U}))"] \\
& |\cdot| &
\end{tikzcd}
$$

in $[\mathbf{FDVect}, \mathbf{Set}]$ commutes.

PROOF. For (i), we must prove that for any map $C \xrightarrow{\theta} B$ in **FDVect**, the square

$$\begin{tikzcd}
\mathbf{Vect}(X,C) \arrow[r, "\theta\circ -"] \arrow[d, "\int - d\mathscr{U}"'] & \mathbf{Vect}(X,B) \arrow[d, "\int - d\mathscr{U}"] \\
C \arrow[r, "\theta"'] & B
\end{tikzcd}$$

commutes. Since the points of B are separated by linear functionals, it is enough to prove that the square commutes when followed by any linear $\beta\colon B \longrightarrow k$, and this is a consequence of Proposition 7.1.

For (ii), let $B \in \mathbf{FDVect}$. By the uniqueness part of Proposition 7.1, it is enough to show that for all $\beta \in B^*$, the outside of the diagram

$$\begin{tikzcd}
\mathbf{Vect}(Y,B) \arrow[rr, "\beta\circ -"] \arrow[d, "-\circ p"'] & & \mathbf{Vect}(Y,k) \arrow[dl, "p^*"'] \arrow[dd, "p^{**}(\mathscr{U})"] \\
\mathbf{Vect}(X,B) \arrow[r, "\beta\circ -"] \arrow[d, "\int - d\mathscr{U}"'] & \mathbf{Vect}(X,k) \arrow[dr, "\mathscr{U}"] & \\
B \arrow[rr, "\beta"'] & & k
\end{tikzcd}$$

commutes; and the inner diagrams demonstrate that it does. ■

Now consider the codensity monad $\mathbf{T}$ of $\mathbf{FDVect} \hookrightarrow \mathbf{Vect}$. By definition,

$$T(X) = \int_{B\in\mathbf{FDVect}} [\mathbf{Vect}(X,B), B]$$

$(X \in \mathbf{Vect})$. Thus, an element $I \in T(X)$ is a family

$$\Big(\mathbf{Vect}(X,B) \xrightarrow{I_B} B\Big)_{B\in\mathbf{FDVect}}$$

natural in B. (A priori, each I_B is a mere map of sets, not necessarily linear; but see Lemma 7.4 below.) Since the forgetful functor $\mathbf{Vect} \longrightarrow \mathbf{Set}$ preserves limits, the underlying set of $T(X)$ is just the set of natural transformations

$$\begin{tikzcd}
\mathbf{FDVect} \arrow[r, bend left, "\mathbf{Vect}(X,-)"] \arrow[r, bend right, "|\cdot|"'] \arrow[r, phantom, "\Downarrow"] & \mathbf{Set}.
\end{tikzcd} \tag{15}$$

7.3. PROPOSITION. *There is a natural transformation* $(\ )^{**} \longrightarrow T$ *with components*

$$\begin{array}{ccc} X^{**} & \longrightarrow & T(X) \\ \mathscr{U} & \longmapsto & \int_X - d\mathscr{U} \end{array} \tag{16}$$

($X \in \mathbf{Vect}$*).*

PROOF. Lemma 7.2(i) guarantees that (16) is a well-defined function for each X. The uniqueness part of Proposition 7.1 implies that it is linear for each X. Lemma 7.2(ii) tells us that it is natural in X. ■

We are nearly ready to show that the natural transformation (16) is an isomorphism of monads. But we observed after Proposition 7.1 that integration against an ultrafilter is linear, so if this is isomorphism is to hold, the maps I_B must also be linear. We prove this now.

7.4. Lemma. *Let $X \in \mathbf{Vect}$ and $I \in T(X)$. Then for each $B \in \mathbf{FDVect}$, the map*

$$I_B \colon \mathbf{Vect}(X, B) \longrightarrow B$$

is linear with respect to the usual vector space structure on $\mathbf{Vect}(X, B)$.

Proof. In diagram (15), both categories have finite products and both functors preserve them. Any natural transformation between such functors is automatically monoidal with respect to the product structures. From this it follows that whenever $\theta \colon B_1 \times \cdots \times B_n \longrightarrow B$ is a linear map in $\mathbf{FDVect}$, and whenever $f_i \in \mathbf{Vect}(X, B_i)$ for $i = 1, \ldots, n$, we have

$$I_B(\theta \circ (f_1, \ldots, f_n)) = \theta(I_{B_1}(f_1), \ldots, I_{B_n}(f_n)).$$

Let $B \in \mathbf{FDVect}$. Taking θ to be first $+\colon B \times B \longrightarrow B$, then $c \cdot - \colon B \longrightarrow B$ for each $c \in k$, shows that I_B is linear. ∎

7.5. Theorem. *The codensity monad of $\mathbf{FDVect} \hookrightarrow \mathbf{Vect}$ is isomorphic to the double dualization monad $(\ )^{**}$ on $\mathbf{Vect}$.*

Proof. First we show that the natural transformation $(\ )^{**} \longrightarrow T$ of Proposition 7.3 is a natural isomorphism, then we show that it preserves the monad structure.

Let X be a vector space and $I \in T(X)$. We must show that there is a unique $\mathscr{U} \in X^{**}$ such that $I = \int_X - \, d\mathscr{U}$. Uniqueness is immediate from the last part of Proposition 7.1. For existence, put

$$\mathscr{U} = I_k \colon \mathbf{Vect}(X, k) \longrightarrow k,$$

which by Lemma 7.4 is linear (that is, an element of X^{**}). Naturality of I implies that the square in Proposition 7.1 commutes when $\int_X - \, d\mathscr{U}$ is replaced by I_B, so by the uniqueness part of that proposition, $\int_X - \, d\mathscr{U} = I_B$ for all $B \in \mathbf{FDVect}$.

Next, the isomorphism $(\ )^{**} \longrightarrow T$ respects the monad structures. To prove this, we begin by checking directly that the isomorphism respects the units of the monads: that is, whenever $X \in \mathbf{Vect}$, the triangle

$$\begin{tikzcd} & X \arrow[dl, "\text{unit}"'] \arrow[dr, "\eta_X"] & \\ X^{**} & & T(X) \arrow[ll, "\cong"'] \end{tikzcd}$$

commutes. Let $x \in X$. Then $\eta_X(x) \in T(X)$ has B-component

$$\begin{array}{ccc} \mathbf{Vect}(X, B) & \longrightarrow & B \\ f & \longmapsto & f(x) \end{array}$$

($B \in \mathbf{FDVect}$). In particular, its k-component $\eta_X(x)_k \in X^{**}$ is evaluation of a functional at x, as required.

It now follows from Proposition 5.3(i) that the natural isomorphism $(\ )^{**} \longrightarrow T$ is an isomorphism of monads. ∎

7.6. Remark. The strategy just used to show that the isomorphism is compatible with the monad structures could also have been used in the case of sets and ultrafilters (Theorem 3.5). There we instead used Börger's result that the ultrafilter endofunctor U has a unique monad structure, which itself was deduced from the fact that U is the terminal endofunctor on **Set** preserving finite coproducts.

Results similar to Börger's can also be proved for vector spaces, but they are complicated by the presence of nontrivial endomorphisms of the identity functor on **Vect** (namely, multiplication by any scalar $\neq 1$). These give rise to nontrivial endomorphisms of every nonzero endofunctor of **Vect**. Hence double dualization cannot be the terminal $\oplus$-preserving endofunctor. However, it *is* the terminal $\oplus$-preserving endofunctor S equipped with a natural transformation $1 \longrightarrow S$ whose k-component is an isomorphism. The proof is omitted.

We have already seen that the notion of compact Hausdorff space arises canonically from the notion of finiteness of a set: compact Hausdorff spaces are the algebras for the codensity monad of $\mathbf{FinSet} \hookrightarrow \mathbf{Set}$. What is the linear analogue?

7.7. Definition. *A* ***linearly compact vector space*** *over k is a k-vector space in* **Top** *with the following properties:*

i. the topology is ***linear****: the open affine subspaces form a basis for the topology*

ii. every family of closed affine subspaces with the finite intersection property has nonempty intersection

iii. the topology is Hausdorff.

We write **LCVect** *for the category of linearly compact vector spaces and continuous linear maps.*

For example, a finite-dimensional vector space can be given the structure of a linearly compact vector space in exactly one way: by equipping it with the discrete topology.

Linearly compact vector spaces were introduced by Lefschetz (Chapter II, Definition 27.1 of [26]). A good modern reference is the book of Bergman and Hausknecht [6].

7.8. Theorem. *The category of algebras for the codensity monad of* $\mathbf{FDVect} \hookrightarrow \mathbf{Vect}$ *is equivalent to* **LCVect**, *the category of linearly compact vector spaces.*

Proof. The codensity monad is the double dualization monad, which by definition is the monad obtained from the dualization functor $(\ )^*\colon \mathbf{Vect}^{\mathrm{op}} \longrightarrow \mathbf{Vect}$ and its left adjoint. The dualization functor is, in fact, monadic. A proof can be extracted from Linton's proof that the dualization functor on Banach spaces is monadic [28]. Alternatively, we can use the following direct argument, adapted from a proof by Trimble [40].

We apply the monadicity theorem of Beck. First, $\mathbf{Vect}^{\mathrm{op}}$ has all coequalizers. Second, the dualization functor preserves them: for the object k of the abelian category $\mathbf{Vect}$ is injective, so by Lemma 2.3.4 of [42], the dualization functor is exact. Third, dualization reflects isomorphisms. Indeed, let $f\colon X \longrightarrow Y$ be a linear map such that $f^*\colon Y^* \longrightarrow X^*$ is an isomorphism. Dualizing the exact sequence

$$0 \longrightarrow \ker f \longrightarrow X \xrightarrow{f} Y \longrightarrow \operatorname{coker} f \longrightarrow 0$$

yields another exact sequence, in which the middle map is an isomorphism. Hence $(\ker f)^* \cong 0 \cong (\operatorname{coker} f)^*$. From this it follows that $\ker f \cong 0 \cong \operatorname{coker} f$, so f is an isomorphism, as required.

On the other hand, it was shown by Lefschetz that $\mathbf{Vect}^{\mathrm{op}} \simeq \mathbf{LCVect}$ (Chapter II, number 29 of [26]; or see Proposition 24.8 of [6]). This proves the theorem. ■

A slightly more precise statement can be made. Lefschetz's equivalence $\mathbf{Vect}^{\mathrm{op}} \longrightarrow \mathbf{LCVect}$ sends a vector space X to its dual X^*, suitably topologized. Hence, under the equivalence $\mathbf{Vect}^{\mathbf{T}} \simeq \mathbf{LCVect}$, the forgetful functor $U^{\mathbf{T}}\colon \mathbf{Vect}^{\mathbf{T}} \longrightarrow \mathbf{Vect}$ corresponds to the obvious forgetful functor $\mathbf{LCVect} \longrightarrow \mathbf{Vect}$.

In summary,

sets are to compact Hausdorff spaces
as
vector spaces are to linearly compact vector spaces.

It seems not to be known whether this is part of a larger pattern. Is it the case, for example, that for all algebraic theories, the codensity monad of the inclusion

$$(\text{finitely presentable algebras}) \hookrightarrow (\text{algebras})$$

is equivalent to a suitably-defined category of 'algebraically compact' topological algebras?

8. Ultraproducts

The ultraproduct construction, especially important in model theory, can also be seen as a codensity monad.

Let X be a set, $S = (S_x)_{x\in X}$ a family of sets, and $\mathscr{U}$ an ultrafilter on X. The **ultraproduct** $\prod_{\mathscr{U}} S$ is the colimit of the functor $(\mathscr{U}, \subseteq)^{\mathrm{op}} \longrightarrow \mathbf{Set}$ defined on objects by $H \mapsto \prod_{x\in H} S_x$ and on maps by projection. (See [12] or Section 1.2 of [10]). Explicitly,

$$\prod\nolimits_{\mathscr{U}} S = \Bigl(\sum_{H\in\mathscr{U}} \prod_{x\in H} S_x\Bigr) \Big/ {\sim}$$

where $\sum$ means coproduct and

$$(s_x)_{x\in H} \sim (t_x)_{x\in K} \iff \{x \in H \cap K : s_x = t_x\} \in \mathscr{U}.$$

For a trivial example, if $\mathscr{U}$ is the principal ultrafilter on x then $\prod_{\mathscr{U}} S = S_x$.

Logic texts often assume that all the sets S_x are nonempty [8, 15], in which case the ultraproduct can be described more simply as $(\prod_{x\in X} S_x)/\sim$. The appendix of Barr [4] explains why the present definition is the right one in the general case.

Ultraproducts can also be understood sheaf-theoretically (as in 2.6.2 of [38]). A family $(S_x)_{x\in X}$ of sets amounts to a sheaf S on the discrete space X, with stalks S_x. The unit map $\eta_X : X \longrightarrow U(X)$ embeds the discrete space X into its Stone–Čech compactification, and pushing forward gives a sheaf $(\eta_X)_*S$ on $U(X)$. The stalk of this sheaf over $\mathscr{U}$ is exactly the ultraproduct $\prod_{\mathscr{U}} S$.

Since the category $(\mathscr{U}, \subseteq)^{\mathrm{op}}$ is filtered, the definition of ultraproduct can be generalized from sets to the objects of any other category $\mathscr{E}$ with small products and filtered colimits. Thus, a family $S = (S_x)_{x\in X}$ of objects of $\mathscr{E}$, indexed over a set X, gives rise to a new family $(\prod_{\mathscr{U}} S)_{\mathscr{U}\in U(X)}$ of objects of $\mathscr{E}$.

For the rest of this section, fix a category $\mathscr{E}$ with small products and filtered colimits.

Let $\mathbf{Fam}(\mathscr{E})$ be the category in which an object is a set X together with a family $(S_x)_{x\in X}$ of objects of $\mathscr{E}$, and a map $(S_x)_{x\in X} \longrightarrow (R_y)_{y\in Y}$ is a map of sets $f : X \longrightarrow Y$ together with a map $\phi_x : R_{f(x)} \longrightarrow S_x$ for each $x \in X$. (Note the direction of the last map; this marks a difference from other authors' use of the **Fam** notation.) Let $\mathbf{FinFam}(\mathscr{E})$ be the full subcategory consisting of those families $(S_x)_{x\in X}$ for which the indexing set X is finite.

The main theorem states, essentially, that the codensity monad of the inclusion $\mathbf{FinFam}(\mathscr{E}) \hookrightarrow \mathbf{Fam}(\mathscr{E})$ is given on objects by

$$(S_x)_{x\in X} \longmapsto \Big(\prod\nolimits_{\mathscr{U}} S\Big)_{\mathscr{U}\in U(X)}. \tag{17}$$

So the ultraproduct construction arises naturally from the notion of finiteness of a family of objects.

In particular, the ultraproduct construction determines a monad. This monad, the **ultraproduct monad V** on $\mathbf{Fam}(\mathscr{E})$, was first described by Ellerman [11] and Kennison [18]. We review their definition, then prove that the codensity monad of $\mathbf{FinFam}(\mathscr{E}) \hookrightarrow \mathbf{Fam}(\mathscr{E})$ exists and is isomorphic to $\mathbf{V}$.

Our first task is to define the underlying functor $V : \mathbf{Fam}(\mathscr{E}) \longrightarrow \mathbf{Fam}(\mathscr{E})$. On objects, V is given by (17). Now take a map $(f, \phi) : (S_x)_{x\in X} \longrightarrow (R_y)_{y\in Y}$ in $\mathbf{Fam}(\mathscr{E})$, which by definition consists of maps

$$X \xrightarrow{f} Y, \qquad R_{f(x)} \xrightarrow{\phi_x} S_x \quad (x \in X).$$

Its image under V consists of maps

$$U(X) \xrightarrow{f_*} U(Y), \qquad \prod\nolimits_{f_*\mathscr{U}} R \longrightarrow \prod\nolimits_{\mathscr{U}} S \quad (\mathscr{U} \in U(X)).$$

The first of these maps, f_*, is $U(f)$. The second is the map

$$\varinjlim_{K\in f_*\mathscr{U}} \prod_{y\in K} R_y \longrightarrow \varinjlim_{H\in\mathscr{U}} \prod_{x\in H} S_x \tag{18}$$

whose K-component is the composite

$$\prod_{y\in K} R_y \xrightarrow{\prod_{y\in K}(\phi_x)_{x\in f^{-1}(y)}} \prod_{y\in K}\prod_{x\in f^{-1}(y)} S_x \cong \prod_{x\in f^{-1}K} S_x \xrightarrow{\mathrm{copr}_{f^{-1}K}} \textstyle\prod_{\mathscr{U}} S$$

where copr denotes a coprojection. In the case $\mathscr{E} = \mathbf{Set}$, the map (18) sends the equivalence class of a family $(r_y)_{y\in K}$ to the equivalence class of the family $\big(\phi_x(r_{f(x)})\big)_{x\in f^{-1}K}$.

Next we describe the unit of the ultraproduct monad. Its component at an object $(S_x)_{x\in X}$ consists of maps

$$X \xrightarrow{\eta_X} U(X), \qquad \textstyle\prod_{\eta_X(x)} S \longrightarrow S_x \quad (x \in X).$$

The first map is the unit of the ultrafilter monad $\mathbf{U}$, in which $\eta_X(x)$ is the principal ultrafilter on x. The second map is the canonical isomorphism.

Proposition 5.3 will save us from needing to know the multiplication of the ultraproduct monad.

To prove the main theorem, our first step is to recast the definition of ultraproduct. The usual definition treats an ultrafilter as a collection of subsets; but to connect with the codensity characterization of ultrafilters, we need a definition of ultraproduct that treats ultrafilters as integration operators.

8.1. Lemma. *Let $\mathscr{B}$ be a full subcategory of* **FinSet** *containing at least one set with at least three elements. Let $(S_x)_{x\in X}$ be an object of* $\mathbf{Fam}(\mathscr{E})$*, and let $\mathscr{U} \in U(X)$. Then there is a canonical isomorphism*

$$\textstyle\prod_{\mathscr{U}} S \cong \displaystyle\varinjlim_{\substack{B\in\mathscr{B},\\ f\colon X\longrightarrow B}} \prod_{x\in f^{-1}(\int f\, d\mathscr{U})} S_x \tag{19}$$

where the right-hand side is a colimit over the category of elements of $\mathbf{Set}(X,-)\colon \mathscr{B} \longrightarrow$ **Set**.

Proof. Write $\bigwedge_{\mathscr{U}} S$ for the right-hand side of (19). Thus, whenever $X \xrightarrow{f} B \xrightarrow{g} B'$ with $B, B' \in \mathscr{B}$, we have a commutative triangle

$$\begin{tikzcd}
\displaystyle\prod_{x\in(gf)^{-1}(\int gf\, d\mathscr{U})} S_x \arrow[d, "\mathrm{pr}"'] \arrow[dr, "\mathrm{copr}_{gf}"] & \\
\displaystyle\prod_{x\in f^{-1}(\int f\, d\mathscr{U})} S_x \arrow[r, "\mathrm{copr}_f"'] & \textstyle\bigwedge_{\mathscr{U}} S
\end{tikzcd} \tag{20}$$

where the vertical map is a product projection and the other maps are colimit coprojections.

Define $\theta\colon \prod_{\mathscr{U}} S \longrightarrow \bigwedge_{\mathscr{U}} S$ as follows. For each $H \in \mathscr{U}$, choose some $B \in \mathscr{B}$ and $f\colon X \longrightarrow B$ such that $f^{-1}(\int f\, d\mathscr{U}) = H$; then the H-component of θ is

$$\theta_H = \mathrm{copr}_f\colon \prod_{x \in H} S_x \longrightarrow \bigwedge\nolimits_{\mathscr{U}} S.$$

We have to check (i) that θ_H is well-defined for each individual H, and (ii) that θ_H is natural in H, thus defining a map $\theta\colon \prod_{\mathscr{U}} S \longrightarrow \bigwedge_{\mathscr{U}} S$.

For (i), let $H \in \mathscr{U}$. Choose $\Omega \in \mathscr{B}$ with at least two elements, say 0 and 1, and define characteristic functions by the usual formula (6). There is at least one pair (B, f) such that $f^{-1}(\int f\, d\mathscr{U}) = H$: for example, (Ω, χ_H). Let (B, f) be another such pair. In the triangle (20) with $g = \chi_{\{\int f\, d\mathscr{U}\}}$, we have $gf = \chi_H$, so the vertical map is an identity. Hence $\mathrm{copr}_{\chi_H} = \mathrm{copr}_f$, as required.

For (ii), let $H, H' \in \mathscr{U}$ with $H \subseteq H'$; we must prove the commutativity of

$$\begin{tikzcd}
\prod_{x \in H'} S_x \arrow[d, "\mathrm{pr}"'] \arrow[dr, "\theta_{H'}"] & \\
\prod_{x \in H} S_x \arrow[r, "\theta_H"'] & \bigwedge_{\mathscr{U}} S.
\end{tikzcd} \tag{21}$$

Choose $B \in \mathscr{B}$ with at least three elements, say a, b and c, and define $f\colon X \longrightarrow B$ by

$$f(x) = \begin{cases} a & \text{if } x \in H \\ b & \text{if } x \in H' \setminus H \\ c & \text{if } x \notin H'. \end{cases}$$

Then $\chi_{\{a,b\}} \circ f = \chi_{H'}$, and the commutative triangle (20) with $g = \chi_{\{a,b\}}$ is exactly (21).

Let $\tilde{\theta}\colon \bigwedge_{\mathscr{U}} S \longrightarrow \prod_{\mathscr{U}} S$ be the unique map such that, whenever $B \in \mathscr{B}$ and $f\colon X \longrightarrow B$, the (B, f)-component of $\tilde{\theta}$ is the coprojection

$$\prod_{x \in f^{-1}(\int f\, d\mathscr{U})} S_x \longrightarrow \varinjlim_{H \in \mathscr{U}} \prod_{x \in H} S_x = \prod\nolimits_{\mathscr{U}} S.$$

It is straightforward to check that $\tilde{\theta}$ is a two-sided inverse of θ. ∎

We now turn to the category $\mathbf{Fam}(\mathscr{E})$. Recall that any functor $\Sigma\colon \mathscr{A}^{\mathrm{op}} \longrightarrow \mathbf{CAT}$ has a category of elements (or Grothendieck construction) $\mathbf{E}(\Sigma)$, whose objects are pairs (A, S) with $A \in \mathscr{A}$ and $S \in \Sigma(A)$, and whose maps $(A, S) \longrightarrow (B, R)$ are pairs (f, ϕ) with $f\colon A \longrightarrow B$ and $\phi\colon S \longrightarrow f^*R$. (We write f^* for $\Sigma(f)$.) It comes with a projection functor $\mathrm{pr}\colon \mathbf{E}(\Sigma) \longrightarrow \mathscr{A}$. For example, there is a functor $\Sigma\colon \mathbf{Set}^{\mathrm{op}} \longrightarrow \mathbf{CAT}$ given on objects by

$$\Sigma(X) = \left(\mathscr{E}^X\right)^{\mathrm{op}} \tag{22}$$

and on maps $f\colon X \longrightarrow Y$ by taking $\Sigma(f)$ to be the dual of the reindexing functor $f^*\colon \mathscr{E}^Y \longrightarrow \mathscr{E}^X$. Its category of elements is $\mathbf{Fam}(\mathscr{E})$.

To compute the codensity monad of $\mathbf{FinFam}(\mathscr{E}) \hookrightarrow \mathbf{Fam}(\mathscr{E})$, we will need to know about limits in $\mathbf{Fam}(\mathscr{E})$. (Compare Section 2 of [18].) We work with categories of elements more generally, using the following standard lemma.

8.2. Lemma. *Let $\mathbf{I}$ be a category. Let $\mathscr{A}$ be a category with limits over $\mathbf{I}$. Let $\Sigma\colon \mathscr{A}^{\mathrm{op}} \longrightarrow \mathbf{CAT}$ be a functor such that $\Sigma(A)$ has limits over $\mathbf{I}$ for each object A of $\mathscr{A}$, and f^* preserves limits over $\mathbf{I}$ for each map f in $\mathscr{A}$. Then $\mathbf{E}(\Sigma)$ has limits over $\mathbf{I}$, and the projection $\mathbf{E}(\Sigma) \longrightarrow \mathscr{A}$ preserves them.*

Proof. Let $(A_i, S_i)_{i\in\mathbf{I}}$ be a diagram over $\mathbf{I}$ in $\mathbf{E}(\Sigma)$; thus, $A_i \in \mathscr{A}$ and $S_i \in \Sigma(A_i)$ for each $i \in \mathbf{I}$. Take a limit cone $(A \xrightarrow{p_i} A_i)_{i\in\mathbf{I}}$ in $\mathscr{A}$. We obtain a diagram $\big(p_i^*(S_i)\big)_{i\in\mathbf{I}}$ in $\Sigma(A)$, and its limit in $\Sigma(A)$ is a limit of the original diagram in $\mathbf{E}(\Sigma)$. ∎

Taking the category of elements is a functorial process: given $\Sigma\colon \mathscr{A}^{\mathrm{op}} \longrightarrow \mathbf{CAT}$ and $G\colon \mathscr{B} \longrightarrow \mathscr{A}$, we obtain a commutative square

$$\begin{tikzcd} \mathbf{E}(\Sigma\circ G) \arrow[r, "G'"] \arrow[d, "\mathrm{pr}"'] & \mathbf{E}(\Sigma) \arrow[d, "\mathrm{pr}"] \\ \mathscr{B} \arrow[r, "G"'] & \mathscr{A} \end{tikzcd}$$

where $G'(B,R) = (GB,R)$ whenever $B \in \mathscr{B}$ and $R \in \Sigma(GB)$. For example, if Σ is the functor (22) and G is the inclusion $\mathbf{FinSet} \hookrightarrow \mathbf{Set}$ then G' is the inclusion $\mathbf{FinFam}(\mathscr{E}) \hookrightarrow \mathbf{Fam}(\mathscr{E})$. The next two results will enable us to compute the codensity monad of G'.

8.3. Proposition. *Let G be a functor from an essentially small category $\mathscr{B}$ to a complete category $\mathscr{A}$. Let $\Sigma\colon \mathscr{A}^{\mathrm{op}} \longrightarrow \mathbf{CAT}$ be a functor such that for each object A of $\mathscr{A}$, the category $\Sigma(A)$ is complete, and for each map $f\colon A \longrightarrow A'$ in $\mathscr{A}$, the functor $f^* = \Sigma(f)\colon \Sigma(A') \longrightarrow \Sigma(A)$ has a left adjoint $f_!$.*

Then $G'\colon \mathbf{E}(\Sigma\circ G) \longrightarrow \mathbf{E}(\Sigma)$ has a codensity monad, given at $(A,S) \in \mathbf{E}(\Sigma)$ by

$$T^{G'}(A,S) = \left(T^G(A),\ \varprojlim_{\substack{B\in\mathscr{B},\\ f\colon A\longrightarrow GB}} \mathrm{pr}_f^* f_!(S)\right)$$

where the limit is over the category of elements of $\mathscr{A}(A, G-)\colon \mathscr{B} \longrightarrow \mathbf{Set}$, and $\mathrm{pr}_f\colon T^G(A) \longrightarrow G(B)$ is projection.

Proof. G' has a codensity monad if for each $(A,S) \in \mathbf{E}(\Sigma)$, the limit

$$T^{G'}(A,S) = \varprojlim_{\substack{(B,R)\in\mathbf{E}(\Sigma\circ G),\\ (f,\phi)\colon (A,S)\longrightarrow(GB,R)}} (GB,R)$$

in $\mathbf{E}(\Sigma)$ exists. This is a limit over the category of elements of

$$\mathbf{E}(\Sigma)\big((A,S), G'-\big)\colon \mathbf{E}(\Sigma\circ G) \longrightarrow \mathbf{Set}.$$

An object of this category of elements consists of an object B of $\mathscr{B}$, an object R of $\Sigma(GB)$, a map $f\colon A \longrightarrow GB$ in $\mathscr{A}$, and a map $\phi\colon S \longrightarrow f^*R$ in $\Sigma(A)$ (or equivalently, a map $\bar{\phi}\colon f_!S \longrightarrow R$ in $\Sigma(GB)$). It follows that

$$T^{G'}(A,S) = \varprojlim_{\substack{B\in\mathscr{B},\\ f\colon A\longrightarrow GB}} \varprojlim_{\substack{R\in\Sigma(GB),\\ \bar{\phi}\colon f_!S\longrightarrow R}} (GB,R)$$

provided that the right-hand side exists. Here the outer limit is over the category of elements of $\mathscr{A}(A,G-)\colon \mathscr{B} \longrightarrow \mathbf{Set}$, and the inner limit is over the coslice category $f_!S/\Sigma(GB)$. But the coslice category has an initial object, the identity on $f_!S$, so

$$T^{G'}(A,S) = \varprojlim_{\substack{B\in\mathscr{B},\\ f\colon A\longrightarrow GB}} (GB, f_!S)$$

provided that this limit in $\mathbf{E}(\Sigma)$ exists. By the completeness hypotheses and Lemma 8.2, it does exist, and by the construction of limits given in the proof of that lemma,

$$T^{G'}(A,S) = \left(T^G(A), \varprojlim_{\substack{B\in\mathscr{B},\\ f\colon A\longrightarrow GB}} \mathrm{pr}_f^* f_!S\right),$$

as required. ∎

The proposition describes only the underlying functor of the codensity monad $\mathbf{T}^{G'}$. The component of the unit at an object (A,S) of $\mathbf{E}(\Sigma)$ consists first of a map $A \longrightarrow T^G(A)$, which is just the unit map η_A^G of the codensity monad of G, and then of a map

$$i_S\colon S \longrightarrow \left(\eta_A^G\right)^* \left(\varprojlim_{\substack{B\in\mathscr{B},\\ f\colon A\longrightarrow GB}} \mathrm{pr}_f^* f_!S\right).$$

Since $\left(\eta_A^G\right)^*$ has a left adjoint, the codomain of i_S is

$$\varprojlim_{B,f}\left(\eta_A^G\right)^*\mathrm{pr}_f^* f_!S = \varprojlim_{B,f}\left(\mathrm{pr}_f\circ\eta_A^G\right)^* f_!S = \varprojlim_{B,f} f^*f_!S.$$

Under these isomorphisms, the (B,f)-component of i_S is the unit map $S \longrightarrow f^*f_!S$.

We will need a variant of Proposition 8.3.

8.4. Proposition. *Proposition 8.3 holds under the following alternative hypotheses: $\mathscr{B}$ is now required to have finite limits and G to preserve them, but for each $A \in \mathscr{A}$, the category $\Sigma(A)$ is only required to have* cofiltered *limits.*

Proof. Since $\mathscr{B}$ has finite limits and G preserves them, the category of elements of $\mathscr{A}(A,G-)\colon \mathscr{B} \longrightarrow \mathbf{Set}$ is cofiltered for each $A \in \mathscr{A}$. The proof is now identical to that of Proposition 8.3. ∎

The case $\mathscr{E} = \mathbf{Set}$ of the following theorem is due to the referee (who gave a different proof).

8.5. Theorem. *The inclusion* $\mathbf{FinFam}(\mathscr{E}) \hookrightarrow \mathbf{Fam}(\mathscr{E})$ *has a codensity monad, isomorphic to the ultraproduct monad on* $\mathbf{Fam}(\mathscr{E})$.

Proof. We apply Proposition 8.4 when G is the inclusion $\mathbf{FinSet} \hookrightarrow \mathbf{Set}$ and Σ is the functor of (22). First we verify the hypotheses of that proposition, using our standing assumption that $\mathscr{E}$ is a category with small products and filtered colimits. For each map of sets $f\colon X \longrightarrow X'$, the functor $f^*\colon \mathscr{E}^{X'} \longrightarrow \mathscr{E}^X$ has a right adjoint f_*, given at $S \in \mathscr{E}^X$ by

$$f_*(S) = \left(\prod_{x \in f^{-1}(x')} S_x \right)_{x' \in X'}.$$

Hence $f^*\colon \Sigma(X') \longrightarrow \Sigma(X)$ has a *left* adjoint. The other hypotheses are immediate.

By Proposition 8.4, the inclusion $G'\colon \mathbf{FinFam}(\mathscr{E}) \hookrightarrow \mathbf{Fam}(\mathscr{E})$ has a codensity monad, given at $S \in \mathscr{E}^X$ by

$$T^{G'}(S) = \varinjlim_{\substack{B \in \mathbf{FinSet},\\ f\colon X \longrightarrow B}} \mathrm{pr}_f^* f_*(S)$$

where the colimit is taken in $\mathscr{E}^{T^G(X)} = \Sigma\big(T^G X\big)^{\mathrm{op}}$. By Theorem 3.5, T^G is the ultrafilter monad. For $B \in \mathbf{FinSet}$ and $f\colon X \longrightarrow B$, the projection

$$\mathrm{pr}_f\colon T^G(X) = U(X) \longrightarrow B$$

is $\mathscr{U} \mapsto \int_X f \, d\mathscr{U}$; hence

$$\mathrm{pr}_f^* f_* S = \left(\prod_{x \in f^{-1}(\int f \, d\mathscr{U})} S_x \right)_{\mathscr{U} \in U(X)}.$$

So by Lemma 8.1, $T^{G'}(S)$ is canonically isomorphic to $\big(\prod_{\mathscr{U}} S\big)_{\mathscr{U} \in U(X)} = V(S)$.

This shows that the codensity monad $\mathbf{T}^{G'}$ of $\mathbf{FinFam}(\mathscr{E}) \hookrightarrow \mathbf{Fam}(\mathscr{E})$ exists and has the same underlying functor as the ultraproduct monad $\mathbf{V}$. Using the description of the unit of $\mathbf{T}^{G'}$ given after Proposition 8.3, one can check that their units also agree. It follows from Proposition 5.3 that the monads $\mathbf{T}^{G'}$ and $\mathbf{V}$ are isomorphic. ∎

8.6. Examples.

i. The codensity monad of $\mathbf{FinFam}(\mathbf{Set}) \hookrightarrow \mathbf{Fam}(\mathbf{Set})$ is the ultraproduct monad on $\mathbf{Fam}(\mathbf{Set})$.

ii. The same is true when $\mathbf{Set}$ is replaced by $\mathbf{Ring}$, $\mathbf{Grp}$, or the category $\mathscr{E}$ of algebras for any other finitary algebraic theory. In such categories, small products and filtered colimits are computed as in $\mathbf{Set}$, so ultraproducts are computed as in $\mathbf{Set}$ too.

iii. Take the category $\mathscr{E}$ of structures and homomorphisms for a (finitary) signature, in the sense of model theory. This has products and filtered colimits, both computed as in **Set**, and the codensity monad of $\mathbf{FinFam}(\mathscr{E}) \hookrightarrow \mathbf{Fam}(\mathscr{E})$ is the ultraproduct construction for such structures. It remains to be seen whether Łoś's theorem (the fundamental theorem on ultraproducts) can usefully be understood in this way.

There is an alternative version of Theorem 8.5. Let $\mathscr{B}$ be a full subcategory of **FinSet** containing at least one set with at least three elements, and write $\mathbf{Fam}_{\mathscr{B}}(\mathscr{E})$ for the full subcategory of $\mathbf{Fam}(\mathscr{E})$ consisting of the families $(S_x)_{x \in X}$ with $X \in \mathscr{B}$. Assume that $\mathscr{E}$ has all small colimits (not just filtered colimits). Then the codensity monad of the inclusion $\mathbf{Fam}_{\mathscr{B}}(\mathscr{E}) \hookrightarrow \mathbf{Fam}(\mathscr{E})$ is the ultraproduct monad. The proof is the same as that of Theorem 8.5, but replacing **FinSet** by $\mathscr{B}$ and Proposition 8.4 by Proposition 8.3.

We finish by describing the algebras for the ultraproduct monad, restricting our attention to $\mathscr{E} = \mathbf{Set}$.

Let **Sheaf** be the category in which an object is a topological space X equipped with a sheaf S of sets, and a map $(X, S) \longrightarrow (Y, R)$ is a continuous map $f : X \longrightarrow Y$ together with a map $f^*R \longrightarrow S$ of sheaves on X. It has a full subcategory consisting of the objects (X, S) where X is discrete; this is nothing but $\mathbf{Fam}(\mathbf{Set})$. It also has a full subcategory **CHSheaf** consisting of the objects (X, S) for which X is compact and Hausdorff.

The following corollary was also pointed out by the referee.

8.7. Corollary. *The category of algebras for the codensity monad of* $\mathbf{FinFam}(\mathbf{Set}) \hookrightarrow \mathbf{Fam}(\mathbf{Set})$ *is equivalent to* **CHSheaf**, *the category of sheaves on compact Hausdorff spaces.*

Proof. Theorem 1.4 of Kennison [18] states that **CHSheaf** is the category of algebras for the ultraproduct monad. The result follows from Theorem 8.5. ∎

So the notion of finiteness of a family of sets leads inevitably not only to the notion of ultraproduct, but also to the notion of sheaf on a compact Hausdorff space.

Acknowledgements I thank Ken Brown, Eugenia Cheng, Alastair Craw, Michel Hébert, Thomas Holder, Peter Johnstone, Anders Kock, Uli Krähmer, Kobi Kremnitzer, Steve Lack, Fred Linton, Mark Meckes, Tim Porter, Emily Riehl, Ben Steinberg, Ross Street, Myles Tierney and Boris Zilber. I am especially grateful to Todd Trimble for encouragement, for pointing out a result of Lawvere (Corollary 3.9 above), and for showing me a proof [40] that the algebras for the double dualization monad are the linearly compact vector spaces (Theorem 7.8). Finally, I am enormously grateful to the anonymous referee, who very generously gave me the main theorem of Section 8.

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