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BOUNDED ARCHIMEDEAN ℓ-ALGEBRAS AND GELFAND-NEUMARK-STONE DUALITY

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ABSTRACT. By Gelfand-Neumark duality, the category $\mathbf{C^*Alg}$ of commutative C^*-algebras is dually equivalent to the category of compact Hausdorff spaces, which by Stone duality, is also dually equivalent to the category $\boldsymbol{ubal}$ of uniformly complete bounded Archimedean ℓ-algebras. Consequently, $\mathbf{C^*Alg}$ is equivalent to $\boldsymbol{ubal}$, and this equivalence can be described through complexification.

In this article we study $\boldsymbol{ubal}$ within the larger category $\boldsymbol{bal}$ of bounded Archimedean ℓ-algebras. We show that $\boldsymbol{ubal}$ is the smallest nontrivial reflective subcategory of $\boldsymbol{bal}$, and that $\boldsymbol{ubal}$ consists of exactly those objects in $\boldsymbol{bal}$ that are epicomplete, a fact that includes a categorical formulation of the Stone-Weierstrass theorem for $\boldsymbol{bal}$. It follows that $\boldsymbol{ubal}$ is the unique nontrivial reflective epicomplete subcategory of $\boldsymbol{bal}$. We also show that each nontrivial reflective subcategory of $\boldsymbol{bal}$ is both monoreflective and epireflective, and exhibit two other interesting reflective subcategories of $\boldsymbol{bal}$ involving Gelfand rings and square closed rings.

Dually, we show that Specker $\mathbb{R}$-algebras are precisely the co-epicomplete objects in $\boldsymbol{bal}$. We prove that the category $\boldsymbol{spec}$ of Specker $\mathbb{R}$-algebras is a mono-coreflective subcategory of $\boldsymbol{bal}$ that is co-epireflective in a mono-coreflective subcategory of $\boldsymbol{bal}$ consisting of what we term ℓ-clean rings, a version of clean rings adapted to the order-theoretic setting of $\boldsymbol{bal}$.

We conclude the article by discussing the import of our results in the setting of complex $*$-algebras through complexification.

1. Introduction

Gelfand-Neumark duality [16] between the categories of commutative C^*-algebras and compact Hausdorff spaces gives a representation of a commutative C^*-algebra as the ring $C(X, \mathbb{C})$ of continuous complex-valued functions on a compact Hausdorff space X. As a consequence of this duality, $C(X, \mathbb{C})$ can be characterized by algebraic properties (involving rings with involution) along with analytic properties (involving Banach spaces). Independently, Stone [45] axiomatized the commutative rings $C(X, \mathbb{R})$ of continuous real-valued functions on compact Hausdorff spaces. In contemporary terminology, these rings

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are precisely the lattice-ordered commutative $\mathbb{R}$-algebras that are bounded, Archimedean, and uniformly complete (all these terms are defined in Section 2). Following Banaschewski [10], we call such a ring a *Stone ring*. Stone rings are thus described by algebraic ($\mathbb{R}$-algebras), order-theoretic (lattice-ordered, bounded, Archimedean), and analytic (uniformly complete) properties. Though the axiomatizations of commutative C^*-algebras and Stone rings are quite different, it is not hard to see that the complexification functor provides an equivalence between the categories of Stone rings and commutative C^*-algebras (see Section 7 for details). Thus, either of the dualities, that of Gelfand-Neumark or of Stone, can be deduced from the other.

Our focus in this article is on Stone's duality rather than that of Gelfand and Neumark—that is, we focus on $\mathbb{R}$-algebras rather than $\mathbb{C}$-algebras—although in Section 7 we discuss how our results apply to the complex case. We term this duality by the collective name of Gelfand-Neumark-Stone duality, as it is often done in the literature, and note that a similar duality for the category of uniformly complete Archimedean vector lattices with strong order unit was obtained independently by Krein–Krein [31], Kakutani [28, 29], and Yosida [48].

In addition to intrinsic interest in the real case, one of our motivations is the tradition in general topology of studying properties of a topological space X via the ring $C(X, \mathbb{R})$, as exemplified by Gillman and Jerison in [17], where many topological properties of X are shown to correspond to algebraic and order-theoretic features of $C(X, \mathbb{R})$. This emphasis on algebra and order is in particular what motivates our point of view. In fact, the presence of a natural lattice order on $C(X, \mathbb{R})$ is one feature that distinguishes the real case from the complex case, and many of our main results rely heavily on order-theoretic properties of lattice-ordered algebras. We study a category, which we denote $\boldsymbol{ba\ell}$, that abstracts the essential algebraic and order-theoretic features of $C(X, \mathbb{R})$; namely, the objects in $\boldsymbol{ba\ell}$ are bounded Archimedean ℓ-algebras (that is, lattice-ordered algebras over $\mathbb{R}$) and the morphisms in the category are ℓ-algebra homomorphisms (that is, lattice-ordered $\mathbb{R}$-algebra homomorphisms).

The Stone rings are the uniformly complete rings in $\boldsymbol{ba\ell}$, and a study of $\boldsymbol{ba\ell}$ is useful to individuate what is special about the analytic aspect of Stone rings, i.e., uniform completeness. To this end, we study not only the algebraic and order-theoretic properties of the rings in $\boldsymbol{ba\ell}$, but also the categorical properties of $\boldsymbol{ba\ell}$. For example, in $\boldsymbol{ba\ell}$ monomorphisms are simply 1-1 morphisms, while epimorphisms are not in general surjective. In fact, the Stone-Weierstrass theorem implies that a morphism $\alpha : A \to B$ in $\boldsymbol{ba\ell}$ is an epimorphism iff $\alpha(A)$ is uniformly dense in B. This leads to the observation that uniform completion is a reflector, and we show that the full subcategory of $\boldsymbol{ba\ell}$ consisting of the Stone rings is the smallest nontrivial reflective subcategory of $\boldsymbol{ba\ell}$, and that it is the unique nontrivial epicomplete reflective subcategory of $\boldsymbol{ba\ell}$. Thus, categorical notions encode the analytic features of uniform completion. Moreover, the category $\boldsymbol{ba\ell}$ has other strong features, such as the fact that every nontrivial reflective subcategory of $\boldsymbol{ba\ell}$ is both monoreflective and epireflective. We exhibit two other natural reflective subcategories of $\boldsymbol{ba\ell}$ related to square closure and closure with respect to bounded inversion.

The category $\boldsymbol{ba\ell}$ also has an important mono-coreflective subcategory: the category of those $\mathbb{R}$-algebras that are generated by their idempotents. (Unlike nontrivial reflective subcategories of $\boldsymbol{ba\ell}$, coreflective subcategories need not be mono-coreflective.) In analogy with Conrad's usage [14] of the term Specker ℓ-group, we term these algebras Specker $\mathbb{R}$-algebras. We show that A is a Specker $\mathbb{R}$-algebra iff A is a co-epicomplete object in $\boldsymbol{ba\ell}$, which happens iff A is a von Neumann regular ring in $\boldsymbol{ba\ell}$. The algebraic features of these rings are so strong that they determine a unique order on the ring. Moreover, each $\mathbb{R}$-algebra homomorphism between Specker $\mathbb{R}$-algebras is automatically an ℓ-algebra homomorphism, and thus the category $\boldsymbol{spec}$ of Specker $\mathbb{R}$-algebras and $\mathbb{R}$-algebra homomorphisms is a subcategory of $\boldsymbol{ba\ell}$.

In unraveling the algebraic, order-theoretic, and analytic properties of Stone rings, it is of interest to determine which Stone rings in $\boldsymbol{ba\ell}$ are the uniform completions of Specker $\mathbb{R}$-algebras, for such Stone rings are structurally determined by these basic algebraic objects. These rings turn out to be precisely the clean Stone rings, and correspond dually to Stone spaces (zero-dimensional compact Hausdorff spaces). This leads us to introduce the class of ℓ-clean rings, an ℓ-ring analogue of a clean ring, and to consider the category $\boldsymbol{cba\ell}$ of ℓ-clean rings. This is shown to be a mono-coreflective subcategory of $\boldsymbol{ba\ell}$ which contains $\boldsymbol{spec}$ as a bi-coreflective subcategory. Moreover, $\boldsymbol{spec}$ forms the smallest epi-coreflective subcategory, as well as the unique epi-coreflective co-epicomplete subcategory of $\boldsymbol{cba\ell}$. As with reflective subcategories of $\boldsymbol{ba\ell}$, a full description of co-reflective subcategories of $\boldsymbol{ba\ell}$ remains an interesting open problem. In particular, we do not know whether there is a co-reflective subcategory of $\boldsymbol{ba\ell}$ that plays a role in $\boldsymbol{ba\ell}$ similar to the role of $\boldsymbol{spec}$ in $\boldsymbol{cba\ell}$ (see Question 7.7(2)).

Thus, $\boldsymbol{ba\ell}$ has a rich categorical structure. Reflection serves to distinguish the "richest" objects in the category — the Stone rings, which have important algebraic, order-theoretic, and analytic features. Coreflection, on the other hand, distinguishes within the subcategory $\boldsymbol{cba\ell}$ the "simplest" algebraic objects in the category — the Specker $\mathbb{R}$-algebras. The Specker $\mathbb{R}$-algebras are defined only in terms of algebra, the order is implicit.

If, instead of compact Hausdorff spaces one works with the larger category of completely regular spaces, then $C(X, \mathbb{R})$ is no longer bounded, and the boundedness condition should be dropped from the definition of $\boldsymbol{ba\ell}$. This results in a natural generalization of $\boldsymbol{ba\ell}$, first developed by Henriksen, Isbell, and Johnson [24, 25] under the name of Φ-algebras. Other natural generalizations include the categories of Archimedean vector lattices with weak order unit (see, for example, Luxemburg–Zaanen [34] and Semadini [42]) and Archimedean ℓ-groups with weak order unit, as developed by Conrad, Hager, Ball, Madden, and others (see [5, 6, 7, 8, 14, 21, 34] and the references therein). There is a large body of results for these categories characterizing epimorphisms, as well as epicomplete objects and epicompletions, and to a lesser degree, of co-epicomplete objects and co-epicompletions. In contrast, less attention has been devoted in this direction to the category of Φ-algebras, and more particularly $\boldsymbol{ba\ell}$, and the early foundational papers [24, 25] appear to provide the most direct and in-depth treatment of this category (but see also the survey article [22] and its references). Another aim of this article is to partially fill in this gap by adding to the knowledge of the category $\boldsymbol{bal}$ of bounded Archimedean ℓ-algebras.

Not surprisingly, the Stone-Weierstrass theorem plays a crucial role in formulating our results, as it does in Gelfand-Neumark-Stone duality. Recall that the (real version of) Stone-Weierstrass theorem asserts that if X is a compact Hausdorff space and A is an $\mathbb{R}$-subalgebra of $C(X,\mathbb{R})$ that separates points of X, then A is uniformly dense in $C(X,\mathbb{R})$. For our purposes we require only a weak version of the Stone-Weierstrass theorem, namely, that when A is an ℓ-subalgebra of $C(X,\mathbb{R})$ (rather than an $\mathbb{R}$-subalgebra) that separates points of X, then A is uniformly dense in $C(X,\mathbb{R})$. The proof of this weak version requires only an elementary application of compactness. Moreover, separation of points can be reformulated to state that A is an ℓ-subalgebra of $C(X,\mathbb{R})$ that separates points of X iff the inclusion mapping $A \to C(X,\mathbb{R})$ is an epimorphism. This leads to a convenient version of the Stone-Weierstrass theorem that does not require reference to the category of compact Hausdorff spaces: If $A, B \in \boldsymbol{bal}$ and $\alpha : A \to B$ is monic and epic in $\boldsymbol{bal}$, then $\alpha(A)$ is uniformly dense in B. We show how to deduce Gelfand-Neumark-Stone duality from this version of the Stone-Weierstrass theorem.

In the last section of the article we translate our results to the setting of complex $*$-algebras through complexification of $\mathbb{R}$-algebras. It is not difficult to see that the complexification and self-adjoint functors yield an equivalence between the categories of commutative $\mathbb{R}$-algebras and commutative complex $*$-algebras, and that under this equivalence, the image of $\boldsymbol{bal}$ is exactly the category of commutative complex $*$-algebras whose self-adjoint part is closed under the absolute value. Furthermore, each subcategory of $\boldsymbol{bal}$ is then equivalent to an appropriate subcategory of complex $*$-algebras. We describe this equivalence for several of the subcategories of $\boldsymbol{bal}$ we study in this article, including an equivalence between Stone rings and commutative C^*-algebras, and between Specker $\mathbb{R}$-algebras and what we call $*$-Specker $\mathbb{C}$-algebras.

2. Bounded Archimedean ℓ-algebras

All rings we will consider are assumed to be commutative with 1, and all homomorphisms are assumed to be unital; that is, preserve 1. We start with the following standard definition; see, e.g., Birkhoff [13, Ch. XIII-XVII].

2.1. Definition.

1. *Let A be a ring with a partial order $\leq$. Then A is a lattice-ordered ring, or an ℓ-ring for short, if (i) $(A, \leq)$ is a lattice, (ii) $a \leq b$ implies $a + c \leq b + c$ for each c, and (iii) $0 \leq a, b$ implies $0 \leq ab$.*

2. *An ℓ-ring A is Archimedean if for each $a, b \in A$, whenever $na \leq b$ for each $n \in \mathbb{N}$, then $a \leq 0$.*

3. *An ℓ-ring A is bounded if for each $a \in A$ there is $n \in \mathbb{N}$ such that $a \leq n \cdot 1$ (that is, 1 is a strong order unit).*

4. *An ℓ-ring A is an f-ring if for each $a, b, c \in A$ with $a \wedge b = 0$ and $c \geq 0$, we have $ac \wedge b = 0$.*

5. *An ℓ-ring A has bounded inversion if each $a \in A$ with $1 \leq a$ is invertible in A.*

6. *An ℓ-ring A is an ℓ-algebra if it is an $\mathbb{R}$-algebra and for each $0 \leq a \in A$ and $0 \leq \lambda \in \mathbb{R}$ we have $\lambda a \geq 0$.*

2.2. Remark. We list below several well-known facts that we will use throughout without explicit mention. They can, for example, be found in [13, Ch. XIII-XVII].

1. In each ℓ-ring A we have $a + b = (a \vee b) + (a \wedge b)$ and $(a \vee b) + c = (a + c) \vee (b + c)$.

2. For each $a \in A$, set $a^+ = a \vee 0$ and $a^- = (-a) \vee 0 = -(a \wedge 0)$. Then $a^+, a^- \geq 0$, $a^+ \wedge a^- = 0$, and $a = a^+ - a^-$.

3. For each $a \in A$, define the absolute value of a by $|a| = a \vee (-a)$. Then $|a| = a^+ + a^-$.

4. Each bounded ℓ-ring A is an f-ring, so $a^2 \geq 0$ and $|ab| = |a||b|$ for each $a, b \in A$.

5. Each bounded Archimedean ℓ-ring is commutative.

6. If A is a nonzero ℓ-algebra, then we view $\mathbb{R}$ as an ℓ-subalgebra of A.

For ℓ-algebras A and B, a map $\alpha : A \to B$ is an *ℓ-algebra homomorphism* if α is an $\mathbb{R}$-algebra homomorphism and a lattice homomorphism. It follows that $\alpha(|a|) = |\alpha(a)|$ for each $a \in A$.

2.3. Notation. Let $\boldsymbol{ba\ell}$ denote the category of bounded Archimedean ℓ-algebras and ℓ-algebra homomorphisms.

2.4. Remark. The zero ring trivially belongs to $\boldsymbol{ba\ell}$, and it is easy to see that it is the terminal object in $\boldsymbol{ba\ell}$. On the other hand, since morphisms in $\boldsymbol{ba\ell}$ are unital, $\mathbb{R}$ is the initial object in $\boldsymbol{ba\ell}$. Since most of the results presented in this article hold easily for the zero ring, in the proofs we will often skip the easy verification for the zero ring and will mostly concentrate on the nonzero objects in $\boldsymbol{ba\ell}$.

We discuss some natural examples of bounded Archimedean ℓ-algebras.

2.5. Example.

1. Probably the most natural examples are the $C(X,\mathbb{R})$ for X compact Hausdorff. It is well known that $C(X,\mathbb{R})$ is a commutative ring with 1, and if we equip $C(X,\mathbb{R})$ with componentwise order $\leq$, then $C(X,\mathbb{R})$ is a bounded Archimedean ℓ-algebra. Note that each $C(X,\mathbb{R})$ has bounded inversion. In addition, $C(X,\mathbb{R})$ is uniformly complete, meaning that $C(X,\mathbb{R})$ is complete with respect to the uniform norm on $C(X,\mathbb{R})$ given by
$$\|f\| = \sup\{|f(x)| : x \in X\}.$$

2. For an example of a bounded Archimedean ℓ-algebra without bounded inversion, let $X = [0,1]$. We recall that $f \in C(X,\mathbb{R})$ is a *piecewise polynomial* function if there are closed intervals $F_1,\dots,F_n$ of X and polynomials $g_1,\dots,g_n \in \mathbb{R}[x]$ such that $X = \bigcup F_i$ and $f = g_i$ on F_i. Let $PP(X,\mathbb{R})$ be the set of all piecewise polynomial functions on X. Then it is not hard to verify that $PP(X,\mathbb{R})$ is an ℓ-subalgebra of $C(X,\mathbb{R})$, and so $PP(X,\mathbb{R}) \in \boldsymbol{ba\ell}$. Let
$$f(x) = \begin{cases} 1 & : x \in [0,\frac{1}{2}] \\ x+\frac{1}{2} & : x \in [\frac{1}{2},1]. \end{cases}$$
Clearly $f \in PP(X,\mathbb{R})$ and $f \geq 1$. If there exists $g \in PP(X,\mathbb{R})$ such that $fg = 1$, then $fg - 1 = 0$ on $[\frac{1}{2},1]$. As $g \in PP(X,\mathbb{R})$, it is easy to see that g and hence fg is a polynomial function on an infinite closed interval $[a,b]$ of $[\frac{1}{2},1]$. But $fg-1$ can only have finitely many zeros on $[a,b]$. The obtained contradiction proves that $PP(X,\mathbb{R})$ does not have bounded inversion.

3. Let X be compact Hausdorff. We call $f \in C(X,\mathbb{R})$ *piecewise constant* if there exists a clopen partition $\{P_1,\dots,P_n\}$ of X and $\lambda_i \in \mathbb{R}$ such that $f(x) = \lambda_i$ for each $x \in P_i$. Let $PC(X,\mathbb{R})$ be the set of all piecewise constant functions on X. It is straightforward to see that $PC(X,\mathbb{R})$ is an ℓ-subalgebra of $C(X,\mathbb{R})$, and so $PC(X,\mathbb{R}) \in \boldsymbol{ba\ell}$. In fact, $PC(X,\mathbb{R})$ is the $\mathbb{R}$-subalgebra of $C(X,\mathbb{R})$ generated by the idempotents of $C(X,\mathbb{R})$, which are the characteristic functions of clopen subsets of X. If X is not a Stone space, $PC(X,\mathbb{R})$ may be rather small. For example, $PC([0,1],\mathbb{R})$ is isomorphic to $\mathbb{R}$.

Let $A \in \boldsymbol{ba\ell}$. We recall that an ideal I of A is an *ℓ-ideal* if for all $a,b \in A$, whenever $|a| \leq |b|$ and $b \in I$, then $a \in I$. In other words, ℓ-ideals of A are exactly the ideals of A that are convex. It follows that if $x,y \in I$, then $x \vee y \in I$. Note that ℓ-ideals are the kernels of ℓ-algebra homomorphisms. Moreover, if I is an ℓ-ideal of A, then A/I is a bounded ℓ-algebra, but A/I may fail to be Archimedean. In fact, A/I is Archimedean iff I is an intersection of maximal ℓ-ideals of A. For, if $I = \bigcap M_i$, where M_i are maximal ℓ-ideals of A, then A/I embeds into $\prod A/M_i$. By [25, Cor. 2.7], each A/M_i is isomorphic to $\mathbb{R}$. Therefore, $\prod A/M_i$ and hence A/I is Archimedean. Conversely, if A/I is Archimedean, then the intersection of all maximal ℓ-ideals of A/I is 0 [26, Thm. II.2.11]. But maximal ℓ-ideals of A/I correspond to maximal ℓ-ideals of A containing I. Thus, I is the intersection of all maximal ℓ-ideals of A containing I. Consequently, if $A, B \in \boldsymbol{bal}$ and $\alpha : A \to B$ is a morphism in $\boldsymbol{bal}$, then the kernel of α is an intersection of maximal ℓ-ideals.

2.6. Example. Given a nonzero $A \in \boldsymbol{bal}$ and an ℓ-ideal I of A, this example shows that $\mathbb{R} + I$ is also in $\boldsymbol{bal}$. This will be used in Lemma 6.1 and Theorem 6.2. It is sufficient to show that $\mathbb{R} + I$ is an ℓ-subalgebra of A. It is straightforward to see that $\mathbb{R} + I$ is an $\mathbb{R}$-subalgebra of A. From the relations between $\vee, \wedge$ and $+$ in an ℓ-algebra, it suffices to show $\mathbb{R} + I$ is closed under $\vee$. To see this, let $a = \lambda + x$ and $b = \mu + y$ with $\lambda, \mu \in \mathbb{R}$ and $x, y \in I$. Without loss of generality, we may assume that $\lambda \le \mu$. Then

$$a \vee b = (\lambda + x) \vee (\mu + y) = \mu + [(\lambda - \mu + x) \vee y].$$

Furthermore, since $\lambda \le \mu$, we see that $y \le (\lambda - \mu + x) \vee y \le x \vee y$. Since $x, y \in I$ and I is an ℓ-ideal, $x \vee y \in I$, and so $(\lambda - \mu + x) \vee y \in I$. Thus, $a \vee b \in \mathbb{R} + I$, and hence $\mathbb{R} + I \in \boldsymbol{bal}$.

2.7. Notation. For $A \in \boldsymbol{bal}$, let $\mathrm{Max}(A)$ denote the set of maximal ideals of A, and let X_A denote the set of maximal ℓ-ideals of A.

Let $A \in \boldsymbol{bal}$. Since each maximal ℓ-ideal of A has real residue field ([25, Cor. 2.7]), each maximal ℓ-ideal is a maximal ideal of A, so $X_A \subseteq \mathrm{Max}(A)$. It is well known that $\mathrm{Max}(A)$ can be given the topology, where the closed sets are the sets of the form $Z(I) = \{M \in \mathrm{Max}(A) : I \subseteq M\}$ for some ideal I of A, and that $\mathrm{Max}(A)$ is a subspace of $\mathrm{Spec}(A)$, where $\mathrm{Spec}(A)$ denotes the prime spectrum of A with the Zariski topology. We recall that $\mathrm{Max}(A)$ is a compact T_1-space, but it may not be Hausdorff in general. We view X_A as a subspace of $\mathrm{Max}(A)$; that is, closed sets of X_A are the sets

$$Z_\ell(I) := Z(I) \cap X_A = \{M \in X_A : I \subseteq M\},$$

where I is an ideal of A. As follows from [25, Thm. 2.3(i)], X_A is compact Hausdorff.

Since each maximal ℓ-ideal of A has real residue field, to each element $a \in A$, we may associate a real-valued function $f_a : X_A \to \mathbb{R}$ by $f_a(M) = a + M$. Moreover, since $f_a^{-1}(\lambda, \mu) = Z_\ell((a - \lambda)^+)^c \cap Z_\ell((\mu - a)^+)^c$ for any $\lambda < \mu$ in $\mathbb{R}$, where $(-)^c$ denotes set-theoretic complement, it follows that $f_a \in C(X_A, \mathbb{R})$. Therefore, if we define $\phi_A : A \to C(X_A, \mathbb{R})$ by $\phi_A(a) = f_a$, then ϕ_A is an ℓ-algebra homomorphism. As $\bigcap X_A = 0$ ([26, Thm. II.2.11]), the mapping ϕ_A is 1-1. Collecting together these observations, we arrive at the following well-known theorem (see [25], especially p. 81, and the references therein), which is fundamental in what follows. (The last assertion of the theorem is clear since any ℓ-subalgebra of an object in $\boldsymbol{bal}$ is in $\boldsymbol{bal}$.)

2.8. Theorem. *If $A \in \boldsymbol{bal}$, then X_A is a compact Hausdorff space and $\phi_A : A \to C(X_A, \mathbb{R})$ is a 1-1 morphism in $\boldsymbol{bal}$. Conversely, if A is isomorphic to an ℓ-subalgebra of $C(X, \mathbb{R})$, for X compact Hausdorff, then $A \in \boldsymbol{bal}$. Thus, an ℓ-algebra A is in $\boldsymbol{bal}$ iff A is isomorphic to an ℓ-subalgebra of $C(X, \mathbb{R})$ for some compact Hausdorff space X.*

Let $A, B \in \boldsymbol{bal}$ and let $\alpha : A \to B$ be a morphism in $\boldsymbol{bal}$. Then α is monic iff α is 1-1. This follows from a more general fact concerning bounded Archimedean ℓ-groups [5, Thm. 2.2(a)], but can also be seen directly. If A is the zero ring, then it is easily checked that α is monic iff α is 1-1. So suppose that A is nonzero. Let $\alpha : A \to B$ be monic and $I = \ker(\alpha)$. By Example 2.6, $\mathbb{R} + I$ is an ℓ-subalgebra of A. Let $\beta : \mathbb{R} + I \to A$ be the identity map and let $\gamma : \mathbb{R} + I \to A$ be given by $\gamma(\lambda + b) = \lambda$ for $\lambda \in \mathbb{R}$ and $b \in I$. A short calculation shows that β, γ are morphisms in $\boldsymbol{ba\ell}$, and that $\alpha \circ \beta = \alpha \circ \gamma$. As α is monic, we obtain $\beta = \gamma$. Thus, $I = 0$, and so α is 1-1. On the other hand, epimorphisms in $\boldsymbol{ba\ell}$ may not be onto, and so there exist bimorphisms (monic and epic morphisms) in $\boldsymbol{ba\ell}$ that are not isomorphisms.

Define $\alpha^* : \mathrm{Spec}(B) \to \mathrm{Spec}(A)$ by $\alpha^*(P) = \alpha^{-1}(P)$ for each $P \in \mathrm{Spec}(B)$. It is well known that α^* is continuous. In fact, α^* restricts to a continuous mapping from X_B to X_A. To see this, note that when B is the zero ring, the claim is clear, so suppose that B is nonzero. If $N \in X_B$, then N has real residue field, so $B = \mathbb{R} + N$, and it follows that $A = \mathbb{R} + \alpha^{-1}(N)$. Thus, $\alpha^{-1}(N)$ is a maximal ideal of A. Moreover, since α is an ℓ-algebra homomorphism, $\alpha^{-1}(N)$ is an ℓ-ideal. This shows that $\alpha^{-1}(N) \in X_A$. Therefore, the restriction of $\alpha^* : \mathrm{Spec}(B) \to \mathrm{Spec}(A)$ is a continuous map $\alpha^* : X_B \to X_A$.

2.9. Lemma. *Let $\alpha : A \to B$ be a morphism in $\boldsymbol{ba\ell}$.*

1. *$\alpha : A \to B$ is monic iff $\alpha^* : X_B \to X_A$ is onto.*
2. *$\alpha : A \to B$ is epic iff $\alpha^* : X_B \to X_A$ is 1-1.*
3. *$\alpha : A \to B$ is a bimorphism iff $\alpha^* : X_B \to X_A$ is a homeomorphism.*

Proof. (1) Let α be monic and $M \in X_A$. Let $J = \{b \in B : |b| \leq \alpha(m) \text{ for some } m \in M\}$. We show that J is an ℓ-ideal of B. If $a, b \in J$, then there are $m, n \in M$ with $|a| \leq \alpha(m)$ and $|b| \leq \alpha(n)$. So $|a + b| \leq \alpha(m + n)$. Thus, $a + b \in J$. Furthermore, if $a \in J$ and $c \in B$, then since B is bounded, there is $\lambda \in \mathbb{N}$ with $|c| \leq \lambda$. Therefore, if $|a| \leq \alpha(m)$ for some $m \in M$, then $|ac| \leq \alpha(\lambda m)$, and $\lambda m \in M$. Thus, $ac \in J$, and so J is an ideal of B. The definition of J shows that it is an ℓ-ideal containing $\alpha(M)$, so $M \subseteq \alpha^{-1}(J)$. We show that J is proper. If not, then $1 \in J$, so $1 \leq \alpha(m)$ for some $m \in M$. As α is monic, this implies $1 \leq m$. Since M is an ℓ-ideal, this forces $1 \in M$, a contradiction to M being proper. Therefore, J is a proper ℓ-ideal of B and hence is contained in a maximal ℓ-ideal N of B. Thus, $M \subseteq \alpha^{-1}(N)$, so $M = \alpha^{-1}(N)$, and so α^* is onto. Conversely, let α^* be onto. If α is not monic, then there exists $a \in A$ such that $a \neq 0$ but $\alpha(a) = 0$. Since $\bigcap X_A = 0$, there exists a maximal ℓ-ideal M of A such that $a \notin M$. As α^* is onto, there exists $N \in X_B$ such that $M = \alpha^{-1}(N)$. But then $0 = \alpha(a) \notin N$, a contradiction to N being an ideal. Thus, α is monic.

(2) Let α be epic and let N_1 and N_2 be distinct maximal ℓ-ideals of B. We let $\beta_1 : B \to \mathbb{R}$ and $\beta_2 : B \to \mathbb{R}$ denote the canonical morphisms in $\boldsymbol{ba\ell}$ which have kernels N_1 and N_2, respectively. Then $\beta_1 \neq \beta_2$. As α is epic, there exists $a \in A$ such that $\beta_1(\alpha(a)) \neq \beta_2(\alpha(a))$. Let $M_i = \ker(\beta_i \circ \alpha)$. Then since $M_i \in X_A$, we have $A = \mathbb{R} + M_i$, so that $a = \lambda_1 + m_1$ for some $\lambda_1 \in \mathbb{R}$ and $m_1 \in M_1$. Thus, $0 = \beta_1(\alpha(m_1)) = \beta_1(\alpha(a)) - \lambda_1$. But if $m_1 \in M_2$, then $0 = \beta_2(\alpha(m_2)) = \beta_2(\alpha(a)) - \lambda_1$, which forces $\beta_1(\alpha(a)) = \beta_2(\alpha(a))$,

which is false. Thus, $M_1 \neq M_2$, so $\alpha^*(N_1) \neq \alpha^*(N_2)$. Therefore, $m_1 \notin M_2$, and hence $\alpha(m_1) \in N_1$ but $\alpha(m_1) \notin N_2$. Thus, $\alpha^{-1}(N_1) \neq \alpha^{-1}(N_2)$, and so α^* is 1-1.

Conversely, let α^* be 1-1. To see that α is epic, let $\beta_1, \beta_2 : B \to C$ be morphisms in $\boldsymbol{ba\ell}$ such that $\beta_1 \circ \alpha = \beta_2 \circ \alpha$. If there is $b \in B$ such that $\beta_1(b) \neq \beta_2(b)$, then there is a maximal ℓ-ideal N of C such that $\beta_1(b) - \beta_2(b) \notin N$. Let $\pi : C \to C/N$ be the canonical map, and define $N_i = \ker(\pi \circ \beta_i)$. Then $N_1, N_2 \in X_B$. If $b \in N_1 \cap N_2$, then $\pi(\beta_1(b) - \beta_2(b)) = 0$, a contradiction. Thus, N_1 and N_2 are distinct maximal ideals of B. As α^* is 1-1, there exists $a \in A$ such that $\alpha(a) \in N_1 \setminus N_2$. But then $0 = \pi(\beta_1(\alpha(a))) = \pi(\beta_2(\alpha(a)))$, a contradiction to $\alpha(a) \notin N_2$. Therefore, $\beta_1 = \beta_2$.

(3) Apply (1) and (2). ∎

Let $A \in \boldsymbol{ba\ell}$ and let $\phi_A : A \to C(X_A, \mathbb{R})$ be the mapping defined in Theorem 2.8. It follows from [25, Cor. 2.6] that ${\phi_A}^* : X_{C(X_A,\mathbb{R})} \to X_A$ is a homeomorphism. As an immediate consequence of Lemma 2.9, we then obtain:

2.10. Proposition. $\phi_A : A \to C(X_A, \mathbb{R})$ *is a bimorphism.*

2.11. Remark. Lemma 2.9 and Proposition 2.10 can be viewed as consequences of a more general result concerning commutative bounded Archimedean ℓ-groups [5, Thm. 2.2]. This is because X_A coincides with the notion of the Yosida space of a commutative Archimedean ℓ-group with weak order unit. To see this, we recall (see, e.g., [13, Ch. XIII]) that an ℓ-group is a group G which is a lattice and in which every group translation is order-preserving. An element $e > 0$ of G is a *weak order unit* if $e \wedge |a| = 0$ implies $a = 0$, and the *Yosida space* of an Archimedean ℓ-group G with weak order unit e is the space (with the hull-kernel topology) of ℓ-subgroups that are maximal with respect to not containing e (see, e.g., [5, Sec. 1]). If we view $A \in \boldsymbol{ba\ell}$ as an Archimedean ℓ-group, then as A is bounded, 1 is a strong order unit of A (cf. Definition 2.1(3)), hence a weak order unit. To see then that X_A is the Yosida space of A, let M be an ℓ-subgroup of A maximal with respect to not containing 1, and let $J = \{a \in A : |a| \leq m \text{ for some } m \in M\}$. By an argument similar to that in the proof of Lemma 2.9(1), we see that J is an ℓ-ideal of A. If $1 \in J$, then $1 \leq m$ for some $m \in M$. Since M is convex, this yields $1 \in M$, a contradiction. Thus, J is a proper ℓ-ideal of A, and hence is contained in a maximal ℓ-ideal N of A. Since $1 \notin N$, this forces $M = N$. Consequently, each ℓ-subgroup of A maximal with respect to not containing 1 is a maximal ℓ-ideal, and it follows that X_A is the Yosida space of A.

3. Epicompletion as a reflector in $\boldsymbol{ba\ell}$

Let $A \in \boldsymbol{ba\ell}$. We define the uniform norm on A by

$$\|a\| = \inf\{\lambda \in \mathbb{R} : |a| \leq \lambda\}.$$

This is well-defined because A is bounded, and as A is Archimedean, it follows that $\|\cdot\|$ is a norm on A. Thus, we have the norm topology on A, and it is easy to see that A is a topological ℓ-ring with respect to this topology. Furthermore, if $\alpha : A \to B$ is an ℓ-algebra homomorphism, then it is immediate that $\|\alpha(a)\| \le \|a\|$ for each $a \in A$. Consequently, α is continuous with respect to the norm topologies.

3.1. Definition. *Let $A \in \boldsymbol{ba\ell}$. We call A uniformly complete if the uniform norm on A is complete. Let $\boldsymbol{uba\ell}$ be the full subcategory of $\boldsymbol{ba\ell}$ consisting uniformly complete objects in $\boldsymbol{ba\ell}$. We call objects in $\boldsymbol{uba\ell}$ Stone rings.*

3.2. Remark. Johnstone [27, p. 155] calls Stone rings C^*-algebras. To avoid confusion, we follow Banaschewski [10] in naming uniformly complete objects in $\boldsymbol{ba\ell}$ Stone rings.

The following proposition, which plays a crucial role throughout the remainder of the article, can be viewed as a categorical reformulation of a weak version of the Stone-Weierstrass theorem, upon which the proof depends.

3.3. Proposition. *The following are equivalent for a monomorphism $\alpha : A \to B$ in $\boldsymbol{ba\ell}$.*

1. *α is a bimorphism.*
2. *$\alpha^* : X_B \to X_A$ is a homeomorphism.*
3. *There is a bimorphism $\beta : B \to C(X_A, \mathbb{R})$ such that $\beta \circ \alpha = \phi_A$.*
4. *$\alpha(A)$ separates points of X_B.*
5. *$\alpha(A)$ is uniformly dense in B.*

Proof. (1) $\Leftrightarrow$ (2): This is Lemma 2.9(3).

(2) $\Rightarrow$ (3): The mapping α^* induces an isomorphism $\widetilde{\alpha^*} : C(X_A, \mathbb{R}) \to C(X_B, \mathbb{R})$ in $\boldsymbol{ba\ell}$, given by $\widetilde{\alpha^*}(f) = f \circ \alpha^*$. We thus have the commutative diagram

$$\begin{tikzcd} A \arrow[r, "\alpha"] \arrow[d, "\phi_A"'] & B \arrow[d, "\phi_B"] \\ C(X_A, \mathbb{R}) \arrow[r, "\widetilde{\alpha^*}"'] & C(X_B, \mathbb{R}). \end{tikzcd}$$

Define $\beta : B \to C(X_A, \mathbb{R})$ by $\beta = \widetilde{\alpha^*}^{-1} \circ \phi_B$. By Proposition 2.10, ϕ_B is a bimorphism. Also, since $\widetilde{\alpha^*}^{-1}$ is an isomorphism, it is a bimorphism. Therefore, β, as a composition of bimorphisms, is a bimorphism. Moreover, $\beta \circ \alpha = \widetilde{\alpha^*}^{-1} \circ \phi_B \circ \alpha = \widetilde{\alpha^*}^{-1} \circ \widetilde{\alpha^*} \circ \phi_A = \phi_A$.

(3) $\Rightarrow$ (2): Since $\beta \circ \alpha = \phi_A$, we have $\alpha^* \circ \beta^* = {\phi_A}^*$. By Proposition 2.10, ϕ_A is a bimorphism, and by assumption β is a bimorphism. Therefore, by Lemma 2.9, both β^* and ${\phi_A}^*$ are homeomorphisms. Thus, so is α^*.

$$\begin{tikzcd} A \arrow[r, "\alpha"] \arrow[d, "\phi_A"'] & B \arrow[dl, "\beta"] \\ C(X_A, \mathbb{R}) & \end{tikzcd} \qquad \begin{tikzcd} X_A & X_B \arrow[l, "\alpha^*"'] \\ X_{C(X_A,\mathbb{R})} \arrow[u, "{\phi_A}^*"] \arrow[ur, "\beta^*"'] & \end{tikzcd}$$

(2) $\Rightarrow$ (4): If M and N are distinct maximal ideals in X_B, then by (2), $\alpha^*(M) = \alpha^{-1}(M)$ and $\alpha^*(N) = \alpha^{-1}(N)$ are distinct maximal ideals in X_A, and hence there exists $a \in A$ such that $\alpha(a) \in M \setminus N$.

(4) $\Rightarrow$ (5): Since A separates points of X_A, the Stone-Weierstrass theorem yields (5).

(5) $\Rightarrow$ (1): It suffices to show that α is epic. Suppose that $\beta_1, \beta_2 : B \to C$ are morphisms in $\boldsymbol{ba\ell}$ with $\beta_1 \circ \alpha = \beta_2 \circ \alpha$. Then since β_1 and β_2 are continuous with respect to the norm topology on B and $\alpha(A)$ is uniformly dense in B, it follows that $\beta_1 = \beta_2$. ■

Using Proposition 3.3, we characterize now Stone rings in several ways. Motivated by the terminology in [5, 6, 7, 8], we call $A \in \boldsymbol{ba\ell}$ *epicomplete* if each epimorphism $\alpha : A \to B$ in $\boldsymbol{ba\ell}$ is onto.

3.4. Corollary. *The following are equivalent for $A \in \boldsymbol{ba\ell}$.*

1. *A is a Stone ring.*
2. *Each bimorphism $\alpha : A \to B$ in $\boldsymbol{ba\ell}$ is an isomorphism.*
3. *A is isomorphic to $C(X, \mathbb{R})$ for some compact Hausdorff space X.*
4. *A is epicomplete in $\boldsymbol{ba\ell}$.*

Proof. (1) $\Rightarrow$ (2): Let $\alpha : A \to B$ be a bimorphism. By Proposition 2.10, $\phi_B : B \to C(X_B, \mathbb{R})$ is a bimorphism. Therefore, $\phi_B \circ \alpha : A \to C(X_B, \mathbb{R})$ is a bimorphism. Thus, by Proposition 3.3, A is isomorphic to a uniformly dense ℓ-subalgebra of $C(X_B, \mathbb{R})$. But as A is a Stone ring, A is uniformly complete. This yields that $\phi_B \circ \alpha$ is an isomorphism, and hence α is an isomorphism.

(2) $\Rightarrow$ (3): By Proposition 2.10, the canonical map $\phi_A : A \to C(X_A, \mathbb{R})$ is a bimorphism, and hence by (2), an isomorphism.

(3) $\Rightarrow$ (1): Since $C(X, \mathbb{R})$ is uniformly complete, (1) follows.

(1) $\Rightarrow$ (4): Let A be a Stone ring, let $\alpha : A \to B$ be an epimorphism in $\boldsymbol{ba\ell}$, and let I be the kernel of α. Then I is an intersection of maximal ideals in X_A, and hence is a uniformly closed subset of A. As such, A/I is uniformly complete [34, Thm. 60.4]. Thus, A/I is a Stone ring, and the induced map $A/I \to B$ is a bimorphism. Since we have established already the equivalence of (1) and (2), we conclude that this mapping is an isomorphism, and hence α is onto.

$$\begin{tikzcd} A \arrow[rr] \arrow[dr] & & B \\ & A/I \arrow[ur] & \end{tikzcd}$$

(4) $\Rightarrow$ (2): This is clear. ■

As we see below, Gelfand-Neumark-Stone duality follows quickly from Corollary 3.4. To formulate the duality, we let **KHaus** denote the category of compact Hausdorff spaces and continuous maps. The notations for this category vary in the literature. Different authors denote it by $\mathbf{Comp}_2$, **CompHaus**, and **HComp**. We follow Johnstone's notation **KHausSp** [27], but abbreviate it to **KHaus**. The reader is cautioned not to confuse **KHaus** with the category of Hausdorff k-spaces.

We define $\mathcal{X} : \boldsymbol{ba\ell} \to \mathbf{KHaus}$ as follows. If $A \in \boldsymbol{ba\ell}$, then $\mathcal{X}(A) = X_A$ and if $\alpha : A \to B$ is a morphism in $\boldsymbol{ba\ell}$, then $\mathcal{X}(\alpha) = \alpha^*$. It is elementary to see that $\mathcal{X}$ is a contravariant functor. On the other hand, associating $C(X, \mathbb{R})$ with each $X \in \mathbf{KHaus}$, and $\widetilde{\eta} : C(Y, \mathbb{R}) \to C(X, \mathbb{R})$ with each morphism $\eta : X \to Y$ in **KHaus**, where $\widetilde{\eta}(f) = f \circ \eta$, produces a contravariant functor $C : \mathbf{KHaus} \to \boldsymbol{ba\ell}$. Moreover, for $A \in \boldsymbol{ba\ell}$ and $X \in \mathbf{KHaus}$, we have $\hom_{\boldsymbol{ba\ell}}(A, C(X, \mathbb{R})) \simeq \hom_{\mathbf{KHaus}}(X, X_A)$. Therefore, $\mathcal{X}$ and C define a contravariant adjunction.

3.5. Corollary. [Gelfand-Neumark-Stone duality] *The functor $\mathcal{X} : \boldsymbol{ba\ell} \to \mathbf{KHaus}$, restricted to $\boldsymbol{uba\ell}$, and the functor $C : \mathbf{KHaus} \to \boldsymbol{ba\ell}$ yield a dual equivalence between $\boldsymbol{uba\ell}$ and* **KHaus**.

Proof. The contravariant adjunction $\mathcal{X} : \boldsymbol{ba\ell} \to \mathbf{KHaus}$ and $C : \mathbf{KHaus} \to \boldsymbol{ba\ell}$ restricts to a contravariant adjunction between $\boldsymbol{uba\ell}$ and **KHaus**. By Corollary 3.4, the unit of the contravariant adjunction is an isomorphism. On the other hand, it is easy to see that for each $X \in \mathbf{KHaus}$, we have X is homeomorphic to $X_{C(X,\mathbb{R})}$. Indeed, that $x \mapsto M_x = \{f \in C(X, \mathbb{R}) : f(x) = 0\}$ is a bijection can be found in [25, Cor. 2.6], and that this map is continuous is straightforward. Therefore, the counit of the contravariant adjunction is also an isomorphism. Thus, the contravariant adjunction restricts to a dual equivalence between $\boldsymbol{uba\ell}$ and **KHaus**. ∎

Let $A \in \boldsymbol{ba\ell}$. In analogy with [5, 6, 7, 8], we call $B \in \boldsymbol{ba\ell}$ the *epicompletion* of A if B is epicomplete and there is a bimorphism $\alpha : A \to B$. It is well known that the *uniform completion* of A is the completion of the metric space A with respect to the uniform topology. From Corollary 3.4 it follows that $C(X_A, \mathbb{R})$ is isomorphic to both the epicompletion and the uniform completion of A. Thus, we have a categorical characterization of uniform completion in $\boldsymbol{ba\ell}$:

3.6. Proposition. *For each $A \in \boldsymbol{ba\ell}$, the epicompletion of A is isomorphic to the uniform completion of A.*

We turn next to a characterization of the full subcategory $\boldsymbol{uba\ell}$ in $\boldsymbol{ba\ell}$. We recall that a subcategory $\mathbf{R}$ of $\boldsymbol{ba\ell}$ is a *reflective* subcategory of $\boldsymbol{ba\ell}$ if the inclusion functor $\mathbf{R} \to \boldsymbol{ba\ell}$ has a left adjoint. In general, $\mathbf{R}$ may be neither full nor replete (closed under isomorphisms) in $\boldsymbol{ba\ell}$. We will be interested in reflective full replete subcategories of $\boldsymbol{ba\ell}$. In order to avoid adding the adjective "full replete" to our discussion, we will assume that all reflective subcategories of $\boldsymbol{ba\ell}$ are full replete.

As we saw, $\mathcal{X} : \boldsymbol{ba\ell} \to \mathbf{KHaus}$ and $C : \mathbf{KHaus} \to \boldsymbol{ba\ell}$ define a contravariant adjunction, with C being full and faithful. Consequently, since Stone rings are, up to isomorphism, the algebras $C(X,\mathbb{R})$, for X compact Hausdorff (Corollary 3.4), $C\circ\mathcal{X}$ is a left adjoint to the inclusion functor $\boldsymbol{uba\ell}\to\boldsymbol{ba\ell}$. Thus, $\boldsymbol{uba\ell}$ is a reflective subcategory of $\boldsymbol{ba\ell}$. Note that the reflector is exactly the mapping $\phi_A : A\to C(X_A,\mathbb{R})$ for each $A\in\boldsymbol{ba\ell}$. Since by Proposition 2.10, ϕ_A is a bimorphism, $\boldsymbol{uba\ell}$ is actually a bireflective subcategory of $\boldsymbol{ba\ell}$.

We call a subcategory of $\boldsymbol{ba\ell}$ *trivial* if it consists of the zero ring and its isomorphic copies. If a subcategory of $\boldsymbol{ba\ell}$ is not trivial, then we call it *nontrivial*. Clearly the trivial subcategory is the least reflective subcategory of $\boldsymbol{ba\ell}$.

3.7. LEMMA. *Every nontrivial reflective subcategory of $\boldsymbol{ba\ell}$ is bireflective.*

PROOF. Let $\mathbf{R}$ be a nontrivial reflective subcategory of $\boldsymbol{ba\ell}$. Let $r:\boldsymbol{ba\ell}\to\mathbf{R}$ be the reflector. Then, for each $A\in\boldsymbol{ba\ell}$, there is a morphism $r_A : A\to r(A)$ in $\boldsymbol{ba\ell}$ such that for each $B\in\mathbf{R}$ and each morphism $\alpha : A\to B$ in $\boldsymbol{ba\ell}$, there is a unique morphism $\beta : r(A)\to B$ for which $\beta\circ r_A=\alpha$.

$$\begin{tikzcd} A \arrow[r, "r_A"] \arrow[dr, "\alpha"'] & r(A) \arrow[d, dashed, "\beta"] \\ & B \end{tikzcd}$$

Consider $R := r(\mathbb{R})\in\mathbf{R}$. Since $\mathbb{R}$ is the initial object in $\boldsymbol{ba\ell}$ and reflectors preserve colimits, R is the initial object in $\mathbf{R}$. Since $\mathbf{R}$ is nontrivial and there is no morphism in $\boldsymbol{ba\ell}$ from the zero ring to a nonzero object in $\boldsymbol{ba\ell}$, we see that R is nonzero. As $\hom_{\mathbf{R}}(R,R)$ is a singleton and $\mathbf{R}$ is a full subcategory of $\boldsymbol{ba\ell}$, we also see that $\hom_{\boldsymbol{ba\ell}}(R,R)$ is a singleton. If $M\in X_R$, then we have the canonical map $\pi_M : R\to\mathbb{R}$ with kernel M. Therefore, $r_{\mathbb{R}}\circ\pi_M\in\hom_{\boldsymbol{ba\ell}}(R,R)$, and its kernel is M. If X_R has two points, then we get two maps from R to R in this way, and these maps are different since their kernels are not equal. Thus, X_R is a singleton. Therefore, $C(X_R,\mathbb{R})\cong\mathbb{R}$, and since R embeds in $C(X_R,\mathbb{R})$ and both of these algebras have the same $\mathbb{R}$-vector space dimension, we see that $R\cong\mathbb{R}$.

We now prove $\mathbf{R}$ is monoreflective. Let $A\in\boldsymbol{ba\ell}$. If A is zero, then $r(A)$ is zero, so r_A is trivially monic. Suppose A is nonzero. Then for each $M\in X_A$, there is an onto morphism $\alpha_M : A\to R$ with kernel M, and hence since $\mathbf{R}$ is reflective, there is a morphism $\beta_M : r(A)\to R$ with $\alpha_M=\beta_M\circ r_A$. Let $N=\mathrm{Ker}(\beta_M)$, and note that $N\in X_{r(A)}$. Also, $\beta_M(r_A(M))=\alpha_M(M)=0$, so that $M\subseteq r_A^{-1}(N)$. Since M is maximal, $M=r_A^{-1}(N)$. It follows that $r_A^* : X_{r(A)}\to X_A$ is onto. Therefore, by Lemma 2.9(1), r_A is monic. Consequently, $\mathbf{R}$ is monoreflective. Now since a monoreflective subcategory is necessarily bireflective [1, Prop. 16.3] (our convention that reflective subcategories are full is being used implicitly here), it follows that $\mathbf{R}$ is bireflective. ∎

3.8. THEOREM. *$\boldsymbol{uba\ell}$ is the smallest nontrivial reflective subcategory of $\boldsymbol{ba\ell}$.*

PROOF. As discussed above, $\boldsymbol{uba\ell}$ is a bireflective subcategory of $\boldsymbol{ba\ell}$. Let $\mathbf{B}$ be a nontrivial reflective subcategory of $\boldsymbol{ba\ell}$. We claim that $\boldsymbol{uba\ell}\subseteq\mathbf{B}$. Let C be a Stone ring. By Lemma 3.7, $\mathbf{B}$ is a bireflective subcategory of $\boldsymbol{ba\ell}$. Therefore, there exists a bimorphism

$C \to B$ for some $B \in \mathbf{B}$. By Corollary 3.4, this bimorphism must be an isomorphism. Since $\mathbf{B}$ is full replete, we have $C \in \mathbf{B}$, which proves the theorem. ∎

Theorem 3.8 distinguishes ***ubal*** as the smallest nontrivial reflective subcategory of ***bal***. Restricting to epicomplete subcategories (that is, subcategories whose objects are epicomplete), we obtain uniqueness:

3.9. Corollary. ***ubal*** *is the unique nontrivial reflective epicomplete subcategory of* ***bal***.

Proof. That ***ubal*** is epicomplete and reflective follows from Corollary 3.4 and Theorem 3.8. If $\mathbf{A}$ is an epicomplete subcategory of ***bal***, then by Corollary 3.4, $\mathbf{A} \subseteq$ ***ubal***, and if also $\mathbf{A}$ is nontrivial and reflective, then by Theorem 3.8, ***ubal*** $\subseteq \mathbf{A}$. ∎

4. Some other reflectors in ***bal***

Our purpose in this section is to exhibit two reflectors arising in a natural way in ***bal*** involving Gelfand rings and square closure. It remains an open problem to characterize all reflective subcategories of ***bal***; see Question 7.7(1).

We recall that a commutative ring A with 1 is a *Gelfand ring* if for each $a, b \in A$, whenever $a + b = 1$, there exist $r, s \in A$ such that $(1 + ar)(1 + bs) = 0$. It is well known (see, e.g., [37, Prop. 1.3] and the references therein) that the following conditions are equivalent:

1. A is a Gelfand ring;
2. each prime ideal of A is contained in a unique maximal ideal of A (that is, A is a pm-ring);
3. for each distinct $M, N \in \mathrm{Max}(A)$ there exist $a \notin M$ and $b \notin N$ such that $ab = 0$;
4. $\mathrm{Spec}(A)$ is a normal space.

In particular, it follows that if A is a Gelfand ring, then $\mathrm{Max}(A)$ is Hausdorff, but the converse is not true in general. However, if the Jacobson radical of A is 0, then the converse is also true [15, Prop. 1.2], where we recall that the *Jacobson radical* of a ring A is $J(A) = \bigcap \mathrm{Max}(A)$. Since $\mathrm{Max}(A)$ is always compact, it follows that if $J(A) = 0$, then A is a Gelfand ring iff $\mathrm{Max}(A)$ is compact Hausdorff. Note that if $A \in$ ***bal***, then $J(A) = \bigcap X_A = 0$.

4.1. Proposition. *The following are equivalent for $A \in$* ***bal***.

1. *A has bounded inversion.*
2. $\mathrm{Max}(A) = X_A$.
3. *A is a Gelfand ring.*
4. $\mathrm{Max}(A)$ *is a Hausdorff space.*

PROOF. The equivalence of (1) and (2) can be found in [24, Lem. 1.1]. To see (2) implies (3), let $\mathrm{Max}(A) = X_A$. Then $\mathrm{Max}(A)$ is Hausdorff. Moreover, since for each $A \in \boldsymbol{ba\ell}$ we have $J(A) = 0$, as discussed above, A is a Gelfand ring. That (3) implies (4) also follows from this same discussion. Finally, to see (4) implies (2), let $\mathrm{Max}(A)$ be Hausdorff. Then, as X_A is a compact subset of $\mathrm{Max}(A)$, it is closed. Therefore, $X_A = Z(\bigcap X_A) = Z(0) = \mathrm{Max}(A)$. ∎

Let $\boldsymbol{gba\ell}$ be the full subcategory of $\boldsymbol{ba\ell}$ consisting of Gelfand rings in $\boldsymbol{ba\ell}$. For each $A \in \boldsymbol{ba\ell}$, define $g(A) = A_S$, where A_S is the localization of A at the multiplicatively closed subset $S = \{s \in A : 1 \leq s\}$. Then, as we show in the next proposition, g defines a reflector from $\boldsymbol{ba\ell}$ to $\boldsymbol{gba\ell}$; in particular, $g(A) \in \boldsymbol{gba\ell}$ for each $A \in \boldsymbol{ba\ell}$.

4.2. PROPOSITION. *$\boldsymbol{gba\ell}$ is a reflective subcategory of $\boldsymbol{ba\ell}$ properly containing $\boldsymbol{uba\ell}$.*

PROOF. To see that $\boldsymbol{uba\ell}$ is a proper subcategory of $\boldsymbol{gba\ell}$, note that if $A \in \boldsymbol{uba\ell}$, then by Gelfand-Neumark-Stone duality, A is isomorphic to $C(X, \mathbb{R})$ for some compact Hausdorff space X. Since each $C(X, \mathbb{R})$ has bounded inversion, we conclude that $A \in \boldsymbol{gba\ell}$, hence $\boldsymbol{uba\ell} \subseteq \boldsymbol{gba\ell}$. On the other hand, if X is the one-point compactification of $\mathbb{N}$ and $PC(X, \mathbb{R})$ is defined as in Example 2.5(3), then it has bounded inversion, so by Proposition 4.1, it is a Gelfand ring. It is easy to see that the sequence (f_n) in $PC(X, \mathbb{R})$, where

$$f_n(m) = \begin{cases} 1/(m+1) & \text{if} \quad m \leq n \\ 0 & \text{if} \quad m > n \text{ or } m = \infty \end{cases}$$

converges in $C(X, \mathbb{R})$ to the function f given by

$$f(m) = \begin{cases} 1/(m+1) & \text{if} \quad m \in \mathbb{N} \\ 0 & \text{if} \quad m = \infty. \end{cases}$$

Since $f \notin PC(X, \mathbb{R})$, we see that (f_n) is Cauchy but does not have a limit in $PC(X, \mathbb{R})$. Therefore, $PC(X, \mathbb{R})$ is not uniformly complete, hence is not a Stone ring. Thus, $\boldsymbol{uba\ell}$ is a proper subcategory of $\boldsymbol{gba\ell}$.

To see that $\boldsymbol{gba\ell}$ is a reflective subcategory of $\boldsymbol{ba\ell}$, let $A \in \boldsymbol{ba\ell}$. First we claim that $A_S \in \boldsymbol{ba\ell}$. If $A = 0$, then clearly $A_S = 0 \in \boldsymbol{ba\ell}$. Thus, assume A is nonzero. Note that no element of S is a zero divisor, for if $a \in A$, $s \in S$, and $as = 0$, then $0 = |as| = |a|s$. Since $1 \leq s$, we have $0 \leq |a| \leq |a|s = 0$, which forces $|a| = 0$, so $a = 0$. As a consequence, if $a \in A$, $s \in S$, and $as \geq 0$, then $a \geq 0$; to see this, $(|a| - a)s = |a|s - as = 0$, so $a = |a| \geq 0$. Thus, since S consists of nonzerodivisors of A, the ring A_S can be viewed as a subring of the complete ring of quotients of A [33, Prop. 6]. Also, since A is an f-ring that is *reduced* (i.e., has no nonzero nilpotent elements), the ordering on A extends to an ordering on the complete ring of quotients of A [36, Thm. 2.1]. The ring A_S inherits this ordering, which is given by $a/s \leq b/t$ if $at \leq bs$, and the lattice operations are given by $(a/s) \vee (b/t) = (at \vee bs)/st$ and $(a/s) \wedge (b/t) = (at \wedge bs)/st$. That $A_S \in \boldsymbol{ba\ell}$ is then straightforward. Moreover, A_S has bounded inversion since if $1 \leq a/s$, then $s \leq a$. This forces $a \geq 1$, so $a \in S$. Therefore, a/s is a unit in A_S, and hence by Proposition 4.1, A_S is a Gelfand ring.

Next we claim that $g : \boldsymbol{ba\ell} \to \boldsymbol{gba\ell}$ is a reflector. Let $A \in \boldsymbol{ba\ell}$ and let $\iota : A \to A_S$ be the canonical mapping. Suppose that $\alpha : A \to B$ is a morphism in $\boldsymbol{ba\ell}$ with $B \in \boldsymbol{gba\ell}$. For each $s \in S$, since $1 \leq s$, we have $1 \leq \alpha(s)$. Therefore, as B has bounded inversion (Proposition 4.1), $\alpha(s)$ is a unit in B. Thus, we define $\beta : A_S \to B$ by $\beta(a/s) = \alpha(a)\alpha(s)^{-1}$ for all $a \in A$ and $s \in S$.

$$\begin{tikzcd} A \arrow[rr, hook, "i"] \arrow[dr, "\alpha"'] & & A_S \arrow[dl, dashed, "\beta"] \\ & B & \end{tikzcd}$$

Since it is clear that $\beta \circ \iota = \alpha$, it follows that $\boldsymbol{gba\ell}$ is a reflective subcategory of $\boldsymbol{ba\ell}$. ■

4.3. Remark. Proposition 4.2 is an analogue for $\boldsymbol{ba\ell}$ of a general result due to Schwartz and Madden: The reduced partially ordered rings with bounded inversion form a monoreflective subcategory of the category of reduced partially ordered rings [41, pp. 41–42].

4.4. Remark. Proposition 4.1 and a localization argument show that when $A \in \boldsymbol{ba\ell}$, then $X_{A_S} = \operatorname{Max}(A_S) \cong X_A$. As a result, we obtain that up to isomorphism A is an ℓ-subalgebra of A_S and A_S is an ℓ-subalgebra of $C(X_A, \mathbb{R})$, and that up to homeomorphism, all three ℓ-algebras A, A_S, and $C(X_A, \mathbb{R})$ have X_A as the space of maximal ℓ-ideals. More generally, if $\mathbf{A}$ is a reflective subcategory of $\boldsymbol{ba\ell}$, then by Lemma 3.7, $\mathbf{A}$ is bireflective. Thus, if r denotes the reflector, then there is a bimorphism $A \to r(A)$. By Proposition 3.3, there is a bimorphism $r(A) \to C(X_A, \mathbb{R})$, and so, up to homeomorphism, A, $r(A)$, and $C(X_A, \mathbb{R})$ all have X_A as the space of maximal ℓ-ideals.

As we saw, not every $A \in \boldsymbol{ba\ell}$ is a Gelfand ring, and hence not every $A \in \boldsymbol{ba\ell}$ is a pm-ring. We show that Proposition 4.2 implies a weaker property holds for each $A \in \boldsymbol{ba\ell}$:

4.5. Corollary. *If $A \in \boldsymbol{ba\ell}$, then each prime ideal of A is contained in at most one maximal ℓ-ideal.*

Proof. Let $S = \{a \in A : 1 \leq a\}$. By Proposition 4.2, $A_S \in \boldsymbol{gba\ell}$. Let P be a prime ideal of A that is contained in $M, N \in X_A$. Since M and N are ℓ-ideals, M and N extend to maximal ideals MA_S and NA_S of A_S, both of which contain the prime ideal PA_S. Moreover, they are ℓ-ideals. To see that MA_S is an ℓ-ideal, suppose that $|a/s| \leq |b/t| \in MA_S$ with $b/t \in MA_S$. Then $b \in M$, so $sb \in M$. Furthermore, $|ta| \leq |sb|$. Therefore, $ta \in M$. Thus, $a/s = (ta)/(st) \in MA_S$, which proves that MA_S is an ℓ-ideal. Since A_S is a Gelfand ring, hence a pm-ring, because the maximal ℓ-ideals MA_S and NA_S both contain PA_S, we must have $MA_S = NA_S$, which in turn implies that $M = N$. ■

So far we have worked with two reflective subcategories of $\boldsymbol{ba\ell}$, namely $\boldsymbol{uba\ell}$ and $\boldsymbol{gba\ell}$. There are many more such subcategories; we refer to the book [41], which introduces many reflective subcategories of the larger category of partially ordered rings. Next we produce one more reflective subcategory of $\boldsymbol{ba\ell}$, which was motivated by the example of real closed rings in [41], which satisfy, among other things, that each positive element is a square.

Recall that if $C \in \boldsymbol{ubal}$, then for each $f \in C$ with $f \geq 0$, there is a unique $0 \leq g \in C$ with $f = g^2$. Let $\boldsymbol{scbal}$ (square root closed) be the full subcategory of $\boldsymbol{bal}$ whose objects C satisfy

$$\{a \in C : a \geq 0\} = \{b^2 : b \in C\}. \tag{1}$$

4.6. Proposition. *$\boldsymbol{scbal}$ is a reflective subcategory of $\boldsymbol{bal}$ that properly contains $\boldsymbol{ubal}$ and is incomparable with $\boldsymbol{gbal}$.*

Proof. For each $A \in \boldsymbol{bal}$ we set $\mathrm{sc}(A)$ to be the intersection of all ℓ-subalgebras of $C(X_A, \mathbb{R})$ containing $\phi_A(A)$ which satisfy Equation (1). Because each positive element of $C(X_A, \mathbb{R})$ has a unique positive square root, it follows that $\mathrm{sc}(A) \in \boldsymbol{scbal}$. We first claim that $\mathrm{sc}(A)$ can be constructed as follows. For each ℓ-subalgebra B of $C(X_A, \mathbb{R})$, let B' be the ℓ-subalgebra of $C(X_A, \mathbb{R})$ generated by $B \cup \{\sqrt{b} : b \in B, b \geq 0\}$. It is clear that $B' \in \boldsymbol{bal}$, that $B \subseteq B' \subseteq C(X_A, \mathbb{R})$, and that each $b \in B$ with $b \geq 0$ has a positive square root in B'. We then claim $\mathrm{sc}(A) = \bigcup_{n=0}^{\infty} A_n$, where $A_0 = \phi_A(A)$ and $A_{n+1} = A'_n$ for each $n \geq 0$. Let C be this union. Then $\phi_A(A) \subseteq C$, and it is clear that $C \in \boldsymbol{bal}$. Moreover, if $c \in C$, then $c \in A_n$ for some n. Therefore, $\sqrt{c} \in A_{n+1} \subseteq C$. Thus, C satisfies Equation (1). Consequently, $\mathrm{sc}(A) \subseteq C$. However, since $\phi_A(A) \subseteq \mathrm{sc}(A)$, it follows that $A_1 = \phi_A(A)' \subseteq \mathrm{sc}(A)$. By induction, we see that $C \subseteq \mathrm{sc}(A)$. Thus, $C = \mathrm{sc}(A)$.

To see that $\mathrm{sc} : \boldsymbol{bal} \to \boldsymbol{scbal}$ is a functor, let $\alpha : A \to B$ be a morphism in $\boldsymbol{bal}$. Then there is an induced morphism $\widetilde{\alpha^*} : C(X_A, \mathbb{R}) \to C(X_B, \mathbb{R})$. For notational convenience, we will write $\alpha' = \widetilde{\alpha^*}$.

$$\begin{tikzcd} A \arrow[r, "\alpha"] \arrow[d] & B \arrow[d] \\ C(X_A, \mathbb{R}) \arrow[r, "\alpha'"] & C(X_B, \mathbb{R}) \end{tikzcd}$$

We claim that α' sends $\mathrm{sc}(A)$ into $\mathrm{sc}(B)$. Let $0 \leq x = \phi_A(a) \in \phi_A(A)$. Then $\alpha'(x) = \phi_B(\alpha(a)) \geq 0$ in $\phi_B(B)$. Thus, $\sqrt{\alpha'(x)} \in \mathrm{sc}(B)$. Now, if y is the (positive) square root of x in $C(X_A, \mathbb{R})$, then $y \in \mathrm{sc}(A)$, and $\alpha'(y)^2 = \alpha'(x)$. Furthermore, as $\sqrt{\alpha'(x)} = \alpha'(y)$ is the unique positive element of $C(X_B, \mathbb{R})$ whose square is $\alpha(a)$, we see that $\alpha'(\sqrt{x}) = \sqrt{\alpha'(x)} \in \mathrm{sc}(B)$. Consequently, α' sends $y = \sqrt{x}$ into $\mathrm{sc}(B)$. Since this is true for all nonnegative elements of $\phi_A(A)$, this implies α' sends A_1 into $\mathrm{sc}(B)$. An inductive argument then yields $\alpha'(\mathrm{sc}(A)) \subseteq \mathrm{sc}(B)$. We thus define $\mathrm{sc}(\alpha) = \alpha'|_{\mathrm{sc}(A)}$, and conclude that sc is a functor.

We now show that $\boldsymbol{scbal}$ is a reflective subcategory of $\boldsymbol{bal}$. We let i_A be the map $\phi_A : A \to \mathrm{sc}(A)$ viewed as a map into $\mathrm{sc}(A)$. Suppose that $C \in \boldsymbol{scbal}$ and $\alpha : A \to C$ is a morphism in $\boldsymbol{bal}$. We need to show there is a unique morphism $\beta : \mathrm{sc}(A) \to C$ in $\boldsymbol{bal}$ with $\beta \circ i_A = \alpha$. Now, we have $\mathrm{sc}(\alpha) : \mathrm{sc}(A) \to \mathrm{sc}(C)$ is a morphism in $\boldsymbol{bal}$. Clearly, $\mathrm{sc}(C) = \phi_C(C)$, since $\mathrm{sc}(C)$ is the intersection of all square closed ℓ-subalgebras of $C(X_C, \mathbb{R})$ containing $\phi_C(C)$, and C is square closed. Set $\beta = i_C^{-1} \circ \mathrm{sc}(\alpha)$. Then

$\beta : \mathrm{sc}(A) \to C$ is a morphism in $\boldsymbol{ba\ell}$.

$$\begin{tikzcd}
A \arrow[r, "i_A"] \arrow[d, "\alpha"'] & \mathrm{sc}(A) \arrow[d, "\mathrm{sc}(\alpha)"] \arrow[dl, "\beta"'] \\
C \arrow[r, "i_C"'] & \mathrm{sc}(C)
\end{tikzcd}$$

We need to show $\beta \circ i_A = \alpha$ and that β is unique with respect to this property. First, by the definition of α', we have $\alpha' \circ \phi_A = \phi_C \circ \alpha$. Thus, $\beta \circ i_A = \alpha$, as desired. Now, suppose that $\gamma : \mathrm{sc}(A) \to C$ also satisfies $\gamma \circ i_A = \alpha$. This implies $\beta|_{A_0} = \gamma|_{A_0}$. Let $0 \leq a \in A$. Then $\sqrt{a} \in \mathrm{sc}(A)$, and $\gamma(\sqrt{a}) = \beta(\sqrt{a})$ since both are positive elements whose squares are equal to $\alpha(a)$. Therefore, γ and β agree on A_1. An inductive argument then shows $\gamma = \beta$. Thus, $\boldsymbol{scba\ell}$ is a reflective subcategory of $\boldsymbol{ba\ell}$.

We next show that $\boldsymbol{gba\ell}$ is not a subcategory of $\boldsymbol{scba\ell}$, nor is $\boldsymbol{scba\ell}$ a subcategory of $\boldsymbol{gba\ell}$. For the first statement, let $A = PP([0,1], \mathbb{R})$. To see that the reflection $g(A)$ of A in $\boldsymbol{gba\ell}$ is not in $\boldsymbol{scba\ell}$, consider the nonnegative function $f(x) = x$. If it is the square of an element of $g(A)$, which is a localization of A, then f would be represented on an infinite closed subinterval of $[0, 1]$ as a square of a rational function. Since this is false, $g(A)$ is not square closed. Thus, $\boldsymbol{gba\ell}$ is not a subcategory of $\boldsymbol{scba\ell}$. To see that $\boldsymbol{scba\ell}$ is not a subcategory of $\boldsymbol{gba\ell}$, we need the following claim.

4.7. Claim. For every $A \in \boldsymbol{ba\ell}$, the reflection $\mathrm{sc}(A)$ is an integral extension of $\phi_A(A)$.

Proof of claim. By transitivity of integrality and the construction of $\mathrm{sc}(A)$, it is enough to prove, for each ℓ-subalgebra B of $C(X_A, \mathbb{R})$, that B' is integral over B, where, as above, B' is the ℓ-subalgebra of $C(X_A, \mathbb{R})$ generated by B and all square roots of nonnegative elements of B. Now, let D be the $\mathbb{R}$-subalgebra of $C(X_A, \mathbb{R})$ generated by $B \cup \{\sqrt{b} : 0 \leq b \in B\}$. Then D is integral over B, and B' is the sublattice of $C(X_A, \mathbb{R})$ generated by D ([23, Thm. 3.3]). Thus, all we need to show is if $a, b \in D$, then $a \vee b$ is integral over B. We first point out that if a is integral over B, then so is $a^2 = |a|^2$, and so $|a|$ is also integral over B. Next, as $a \vee b = \frac{1}{2}(a + b) + \frac{1}{2}|a + b|$, if a, b are integral over B, then so is $a \vee b$. This completes the proof of the claim.

Now we produce an example of $C \in \boldsymbol{scba\ell}$ which does not have bounded inversion. With $A = PP([0,1], \mathbb{R})$ as above, and $C = \mathrm{sc}(A)$, it follows from Claim 4.7 that $\mathrm{sc}(A)$ is an integral extension of $\phi_A(A)$. We identify $C([0,1], \mathbb{R})$ and $C(X_A, \mathbb{R})$, and thus work with the inclusions $PP([0,1], \mathbb{R}) \subseteq C \subseteq C([0,1], \mathbb{R})$. Consider $1 + x \in C$ and suppose that $1 + x$ is a unit in C. Since C is integral over A, this implies that $1 + x$ is a unit in A, and so $(1 + x)^{-1}$ is a piecewise polynomial function on $[0, 1]$. Therefore, $(1 + x)^{-1}$ is a polynomial function on an infinite closed subinterval of $[0, 1]$. This is false since the ring of polynomial functions on such an interval is integrally closed in its quotient field. Thus, since $1 \leq 1 + x$, we see that $\mathrm{sc}(A)$ does not have bounded inversion. Consequently, $\boldsymbol{scba\ell}$ is not a subcategory of $\boldsymbol{gba\ell}$. That $\boldsymbol{uba\ell}$ is a proper subcategory of $\boldsymbol{scba\ell}$ is now obvious; the proof of the proposition is complete. ∎

4.8. Remark. The subcategory of $\boldsymbol{ba\ell}$ of real closed rings (cf. [41, p. 135]) is not the same as $\boldsymbol{scba\ell}$, since, by [41, Prop. 12.4], any such algebra has bounded inversion. We chose to work with $\boldsymbol{scba\ell}$ in this proposition since the defining condition is simpler than that for real closed rings.

4.9. Remark. The reflectors $g : \boldsymbol{ba\ell} \to \boldsymbol{gba\ell}$ and $\mathrm{sc} : \boldsymbol{ba\ell} \to \boldsymbol{scba\ell}$ can be used to define reflective subcategories $\mathbf{C}$ and $\mathbf{D}$ of $\boldsymbol{ba\ell}$ such that $\mathbf{C} \subseteq \boldsymbol{gba\ell} \cap \boldsymbol{scba\ell} \subseteq \boldsymbol{gba\ell} \cup \boldsymbol{scba\ell} \subseteq \mathbf{D}$. In fact, our discussion here is a special instance of a more general framework for infima and suprema of reflectors, which is due to Schwartz and Madden [41, Ch. 9]. Let $\mathbf{C} = \boldsymbol{gba\ell} \cap \boldsymbol{scba\ell}$, and let $r : \boldsymbol{ba\ell} \to \boldsymbol{gba\ell} \cap \boldsymbol{scba\ell}$ be defined by $r(A) = g(\mathrm{sc}(A))$ for all $A \in \boldsymbol{ba\ell}$. Then r is well-defined; i.e., its image consists of Gelfand, square root closed rings in $\boldsymbol{ba\ell}$. Also, since g and sc are reflectors, it follows easily that r is a reflector. Next, let $\mathbf{D}$ be the full subcategory of $\boldsymbol{ba\ell}$ consisting of objects $A \in \boldsymbol{ba\ell}$ such that $\phi_A(A) = B \cap C$, where B and C are ℓ-subalgebras of $C(X_A, \mathbb{R})$ with $B \in \boldsymbol{gba\ell}$ and $C \in \boldsymbol{scba\ell}$. Define a functor $t : \boldsymbol{ba\ell} \to \mathbf{D}$ by $t(A) = \mathrm{sc}(A) \cap g(\phi_A(A))$ for all $A \in \boldsymbol{ba\ell}$. It is straightforward, using the fact that $\phi_A(A)$ is uniformly dense in $C(X_A, \mathbb{R})$, to verify that t is a reflector. Thus, we have constructed subcategories $\mathbf{C}$ and $\mathbf{D}$ such that $\mathbf{C} = \boldsymbol{gba\ell} \cap \boldsymbol{scba\ell} \subseteq \boldsymbol{gba\ell} \cup \boldsymbol{scba\ell} \subseteq \mathbf{D}$. Note also that $\boldsymbol{uba\ell}$ is a proper subcategory of $\boldsymbol{gba\ell} \cap \boldsymbol{scba\ell}$, since $g(\mathrm{sc}(A))$, where $A = PP([0,1], \mathbb{R})$, is not a Stone ring. (For example, it does not contain the restriction of the exponential function to $[0,1]$.)

5. Specker $\mathbb{R}$-algebras, ℓ-clean rings, and Baer rings

We have established that $A \in \boldsymbol{ba\ell}$ is a Stone ring iff A is epicomplete, that $\boldsymbol{uba\ell}$ is the unique reflective epicomplete subcategory of $\boldsymbol{ba\ell}$, and described several other natural reflective subcategories of $\boldsymbol{ba\ell}$. Our next goal is to investigate the dual concepts of epi-cocomplete objects and coreflective subcategories of $\boldsymbol{ba\ell}$. For this purpose, we study Specker $\mathbb{R}$-algebras, ℓ-clean rings, and Baer rings in $\boldsymbol{ba\ell}$.

Specker ℓ-groups were first introduced and studied by Conrad [14]. Here we introduce an analogous concept of Specker $\mathbb{R}$-algebra, which is particularly suited for $\boldsymbol{ba\ell}$. A more general concept, encompassing both Specker ℓ-groups and Specker $\mathbb{R}$-algebras, is that of Specker R-algebra, where R is a commutative ring. It is studied in detail in [12]. Before defining Specker $\mathbb{R}$-algebras, we emphasize again that all rings are assumed to be commutative with 1.

5.1. Definition. *Let A be an $\mathbb{R}$-algebra. We call A a Specker $\mathbb{R}$-algebra if it is generated as an $\mathbb{R}$-algebra by its idempotents.*

5.2. Example. Let X be a Stone space. As we pointed out in Example 2.5(3), $PC(X, \mathbb{R})$ is an $\mathbb{R}$-subalgebra of $C(X, \mathbb{R})$ generated by the idempotents of $C(X, \mathbb{R})$. It follows that $PC(X, \mathbb{R})$ is a Specker $\mathbb{R}$-algebra. In Theorem 6.2 we will see that each Specker $\mathbb{R}$-algebra is isomorphic to $PC(X, \mathbb{R})$ for some Stone space X.

If $e_1, \dots, e_n$ are nonzero idempotents in A such that $e_ie_j = 0$ when $i \neq j$, then we call $\{e_1, \dots, e_n\}$ an *orthogonal* set of idempotents. The next two lemmas are routine observations about decompositions in Specker algebras. For lack of a reference, we include their proofs.

5.3. LEMMA. *Let A be an $\mathbb{R}$-algebra. Suppose that $\{e_1, \dots, e_k\}$ and $\{f_1, \dots, f_n\}$ are orthogonal sets of idempotents in A such that $\lambda_1 e_1 + \cdots + \lambda_k e_k = \mu_1 f_1 + \cdots + \mu_n f_n$ for nonzero distinct real numbers $\lambda_1, \dots, \lambda_k$ and $\mu_1, \dots, \mu_n$. Then $\{e_1, \dots, e_k\} = \{f_1, \dots, f_n\}$ and for each i, j we have $\lambda_i = \mu_j$ iff $e_i = f_j$.*

PROOF. Multiplying by $\lambda_1^{-1}e_1$, we have $e_1 = \sum_{i=1}^n \lambda_1^{-1}\mu_i e_1 f_i$. Squaring both sides and using the fact that e_1 is idempotent and $\{f_1, \dots, f_n\}$ is an orthogonal set of idempotents yields $\sum_{i=1}^n \lambda_1^{-1}\mu_i e_1 f_i = e_1 = \sum_{i=1}^n (\lambda_1^{-1}\mu_i)^2 e_1 f_i$. Using again that f_i's are orthogonal, we conclude that for each i, either $e_1 f_i = 0$ or $\lambda_1^{-1}\mu_i = (\lambda_1^{-1}\mu_i)^2$. If $e_1 f_i \neq 0$, then since $\mu_i \neq 0$, it follows that $\mu_i = \lambda_1$. But the μ_i are all distinct, so there exists exactly one i such that $e_1 f_i \neq 0$. After relabeling we assume $e_1 f_1 \neq 0$, and so, from above, we get $e_1 = e_1 f_1$ and $\lambda_1 = \mu_1$. A symmetrical argument shows that $f_1 = e_i f_1$ for some i. Thus, $e_1 = e_1 f_1 = e_1 e_i f_1$, so that necessarily $i = 1$, and hence $e_1 = e_1 f_1 = f_1$. Having established that $e_1 = f_1$ and $\lambda_1 = \mu_1$, we have then that $\sum_{i=2}^k \lambda_i e_i = \sum_{i=2}^n \mu_i f_i$, and by repeating the argument we obtain the lemma. ■

Motivated by the lemma, we say that an element $a \in A$ has a *canonical decomposition* if there exist orthogonal idempotents $e_1, \dots, e_n \in A$ and distinct nonzero real numbers $\lambda_1, \dots, \lambda_n$ such that $a = \lambda_1 e_1 + \cdots + \lambda_n e_n$. By Lemma 5.3, the λ_i and e_i are unique in such a decomposition.

5.4. LEMMA. *If A is a Specker $\mathbb{R}$-algebra, then each nonzero element of A has a canonical decomposition.*

PROOF. Since A is generated as an $\mathbb{R}$-algebra by idempotents, each element of A can be written as an $\mathbb{R}$-linear combination of idempotents. We prove, by induction on n, that any nonzero linear combination of n idempotents has a canonical decomposition. If $n = 1$, then any such linear combination has the form λe with $0 \neq \lambda \in \mathbb{R}$ and $e \in A$ an idempotent. The element λe clearly has a canonical decomposition. Now, suppose the result holds for $n-1$, and let $x = \sum_{i=1}^n \mu_i f_i$ be a linear combination of n idempotents. Set $y = \sum_{i=1}^{n-1} \mu_i f_i$. By induction, y has a canonical decomposition $y = \sum_{i=1}^m \lambda_i e_i$. Then $x = (\sum_{i=1}^m \lambda_i e_i) + \mu_n f_n$. Since $\{e_1, \dots, e_m\}$ is an orthogonal set of idempotents, $\sum_{i=1}^m e_i$ is an idempotent, and so

$$\begin{aligned} x &= \sum_{i=1}^m \lambda_i e_i(1 - f_n) + \sum_{i=1}^m \lambda_i e_i f_n + \mu_n f_n \sum_{i=1}^m e_i + \mu_n f_n \left(1 - \sum_{i=1}^m e_i\right) \\ &= \sum_{i=1}^m \lambda_i e_i(1 - f_n) + \sum_{i=1}^m (\lambda_i + \mu_n) e_i f_n + \mu_n f_n \left(1 - \sum_{i=1}^m e_i\right) \end{aligned}$$

is a linear combination of orthogonal idempotents. We then obtain a canonical decomposition of x by combining terms with the same coefficient. Thus, by induction, the result follows. ■

We recall that a ring A is *von Neumann regular* if for each $x \in A$ there is $y \in A$ with $xyx = x$. It is well known (see, e.g., [19, Thm. 1.16]) that for a commutative ring A the following conditions are equivalent:

1. A is von Neumann regular;
2. A is reduced and each prime ideal of A is maximal;
3. the localization A_M at M is a field for each maximal ideal M of A.

We also recall that a ring A is *clean* if each element of A is the sum of an idempotent and a unit. It is known (see, e.g., [37, Thm. 1.7] and the references therein) that for a commutative ring A the following conditions are equivalent:

1. A is a clean ring;
2. for each distinct $M, N \in \mathrm{Max}(A)$ there exists an idempotent $e \in A$ such that $e \in M$ and $e \notin N$;
3. A is a Gelfand ring and $\mathrm{Max}(A)$ is a zero-dimensional space.

In particular, each clean ring is a Gelfand ring, and if $J(A) = 0$, then A is a clean ring iff $\mathrm{Max}(A)$ is a Stone space. Note that each von Neumann regular ring A is clean. To see this, let M, N be distinct maximal ideals of A. Then there is $x \in M \backslash N$. Because A is von Neumann regular, there is $y \in A$ with $xyx = x$. If $e = xy$, then e is an idempotent, and $e \in M$ since $x \in M$. Moreover, $e \notin N$ since otherwise $x = xe \in N$. Thus, $e \in M \backslash N$, and hence A is clean.

5.5. Proposition. *Every Specker $\mathbb{R}$-algebra A is a bounded Archimedean ℓ-algebra with bounded inversion, and every $\mathbb{R}$-algebra homomorphism between Specker $\mathbb{R}$-algebras is an ℓ-algebra homomorphism. Furthermore, A is von Neumann regular, hence clean.*

Proof. Let A be a Specker $\mathbb{R}$-algebra. As noted in [9, p. 115], it is straightforward to verify that $A \in \boldsymbol{bal}$. Indeed, by Lemma 5.4, we may write each $x \in A$ uniquely in the following canonical form $x = \sum_i \lambda_i e_i$ with $\lambda_i \in \mathbb{R}$ nonzero and $\{e_1, \dots, e_n\}$ an orthogonal set of idempotents. We then can define $\leq$ on A by $0 \leq x$ if each $\lambda_i \geq 0$. That an $\mathbb{R}$-algebra homomorphism between Specker $\mathbb{R}$-algebras is an ℓ-algebra homomorphism follows from [12, Cor. 5.3]. To see that A is a von Neumann regular ring, let $x \in A$, and write $x = \sum_i \lambda_i e_i$ as above. For each i set $\mu_i = \lambda_i^{-1}$. Setting $y = \sum_i \mu_i x_i$, we see immediately that $xyx = x$. Thus, A is von Neumann regular. It follows that A is a clean ring. Finally, to see that A has bounded inversion, let $x \in A$ with $1 \leq x$. Then $x = \sum_i \lambda_i e_i$ with $1 \leq \lambda_i$ for each i. Thus, $\sum_i \lambda_i^{-1} e_i$ is the multiplicative inverse of x in A. ■

By the proposition, every Specker $\mathbb{R}$-algebra is a von Neumann regular ring in $\boldsymbol{ba\ell}$. In Theorem 6.2 we show that the converse is also true, and give a number of other characterizations of Specker $\mathbb{R}$-algebras.

Let $A \in \boldsymbol{ba\ell}$ be a clean ring. Then, as follows from the discussion above, A is a Gelfand ring and $\mathrm{Max}(A)$ is a Stone space, and Proposition 4.1 gives us that X_A is a Stone space. However, there exist $A \in \boldsymbol{ba\ell}$ such that X_A is a Stone space, but A is not a clean ring, as we see in the next example.

5.6. Example. We modify Example 2.5(2). Let X be the Cantor space and let A be the set of all piecewise polynomial functions on $[0, 1]$ restricted to X. Then $A \in \boldsymbol{ba\ell}$. Since A separates points of X, it follows that X_A is homeomorphic to X (see Section 3 for details). On the other hand, the same argument as in Example 2.5(2) shows that A does not have bounded inversion. Therefore, by Proposition 4.1, A is not a Gelfand ring, hence not a clean ring.

In order to characterize those $A \in \boldsymbol{ba\ell}$ for which X_A is a Stone space, we will weaken the concept of a clean ring. Let A be a ring and let X be a subspace of $\mathrm{Max}(A)$. We say that A is *X-clean* if each $a \in A$ can be written in the form $e + v$ with e an idempotent and $v \notin \bigcup X$. Since $\bigcup \mathrm{Max}(A)$ is the set of units of A, we see that A is clean iff it is $\mathrm{Max}(A)$-clean.

5.7. Definition. *We call $A \in \boldsymbol{ba\ell}$ ℓ-clean if A is X_A-clean. Let $\boldsymbol{cba\ell}$ be the full subcategory of $\boldsymbol{ba\ell}$ consisting of ℓ-clean rings.*

Note that if $A \in \boldsymbol{ba\ell}$ is a clean ring, then A is ℓ-clean. However, Example 5.6 and Theorem 5.9 below show the converse is not true in general. In order to characterize the objects in $\boldsymbol{cba\ell}$, we need the following lemma whose analogue for commutative rings with trivial Jacobson radical is well-known (see, e.g., [40, Lem. 2.1]). We omit the proof because it is nearly identical to that for commutative rings. The only difference is to note that for two ℓ-ideals J and K, the ideal $J + K$, as a sum of two ℓ-ideals, is an ℓ-ideal, and hence, if proper, is contained in a maximal ℓ-ideal.

5.8. Lemma. *Let $A \in \boldsymbol{ba\ell}$ and let I be an ideal of A. Then $Z_\ell(I)$ is clopen in X_A iff I is generated by an idempotent $e \in A$. In particular, if $e \in A$ is an idempotent, then $Z_\ell(e)$ is clopen in X_A; and if U is clopen in X_A, then there exists an idempotent $e \in A$ such that $U = Z_\ell(e)$.*

5.9. Theorem. *The following conditions are equivalent for nonzero $A \in \boldsymbol{ba\ell}$.*

1. *A is ℓ-clean.*
2. *Each element $a \in A$ can be written as $a = e + v$, where e is an idempotent and there exists $\varepsilon \in \mathbb{R}$ such that $0 < \varepsilon < |v|$.*
3. *If $M, N \in X_A$ are distinct, then there is an idempotent $e \in A$ with $e \in M \backslash N$.*
4. *X_A is a Stone space.*

PROOF. (1) $\Rightarrow$ (2): Let A be ℓ-clean and let $a \in A$. Then $a = e + v$ for some idempotent e and $v \notin \bigcup X_A$. It is elementary to see that the ℓ-ideal I of A generated by v is $\{b \in A : |b| \leq n|v| \text{ for some } n \in \mathbb{N}\}$. Since v is not contained in any maximal ℓ-ideal, $I = A$, so $1 \leq n|v|$ for some natural number n. Then $1/n \leq |v|$, so (2) holds.

(2) $\Rightarrow$ (3): Let $M, N \in X_A$ be distinct. Then there is $a \in A$ with $a \in M\backslash N$. Since N is a maximal ideal, there is $x \in A$ with $ax - 1 \in N$. By (2), we may write $ax = e + v$ with e an idempotent and $\lambda < |v|$ for some positive $\lambda \in \mathbb{R}$. Note that $v \notin M, N$ since, say, if $v \in M$, then as M is an ℓ-ideal, $|v| \in M$, so $\lambda \in M$, and hence $1 \in M$. Because $ax \in M$ and $v \notin M$, we have $e \notin M$. Now, $ax - 1 = (e - 1) + v$. Since $v \notin N$ but $ax - 1 \in N$, we have $1 - e \notin N$, so $e \in N$. Thus, $e \in N\backslash M$.

(3) $\Rightarrow$ (4): By Lemma 5.8, each clopen of X_A has the form $Z_\ell(e)$ for some idempotent $e \in A$. By (3), the family $\{Z_\ell(e) : e \text{ an idempotent in } A\}$ separates points of X_A. Therefore, since X_A is compact, X_A is a Stone space (see, e.g., [27, Thm. II.4.2]).

(4) $\Rightarrow$ (1): Let $a \in A$. It is clear that $Z_\ell(1 - a) \subseteq Z_\ell(a)^c$. Therefore, as X_A is a Stone space, there is a clopen set U with $Z_\ell(1 - a) \subseteq U \subseteq Z_\ell(a)^c$. By Lemma 5.8, there is an idempotent $e \in A$ with $U = Z_\ell(e)$. Thus, $Z_\ell(1 - a) \subseteq Z_\ell(e) \subseteq Z_\ell(a)^c$. We show that this implies $a - e \notin \bigcup X_A$. For, suppose $a - e \in M$ for some $M \in X_A$. If $e \in M$, then $a \in M$, which is false since $Z_\ell(e) \cap Z_\ell(a) = \varnothing$. If $e \notin M$, then $1 - e \in M$. So, $(a - e) - (1 - e) = a - 1 \in M$. But, this violates $Z_\ell(a - 1) \subseteq Z_\ell(e)$. Consequently, A is ℓ-clean. ■

5.10. COROLLARY. *Let $A \in$ **bal**. Then A is a clean ring iff A is ℓ-clean and has bounded inversion.*

PROOF. Since the statement is clear for the zero ring, we assume that A is nonzero. If A has bounded inversion, then any element v satisfying $0 < \varepsilon < |v|$ is a unit, since $1 \leq \varepsilon^{-1}|v|$, so $\varepsilon^{-1}|v|$ is a unit. Therefore, $|v|$ is a unit, so there is $w \in A$ with $|v|w = 1$. Squaring gives $1 = |v|^2 w^2 = v^2 w^2$, so v is a unit. Thus, if A is ℓ-clean and has bounded inversion, then it is clean. Conversely, if A is clean, then A is ℓ-clean, and A has bounded inversion by Proposition 4.1. ■

From Theorem 5.9 and Corollary 5.10 we deduce Azarpanah's characterization of clean rings of continuous real-valued functions.

5.11. COROLLARY. ([3, Thm. 2.5]) *Let X be a compact Hausdorff space. Then $C(X, \mathbb{R})$ is a clean ring iff X is a Stone space.*

PROOF. Let $A = C(X, \mathbb{R})$. If X is a Stone space, then since X_A is homeomorphic to X, Theorem 5.9 implies that A is ℓ-clean. As A has bounded inversion, by Corollary 5.10, A is a clean ring. Conversely, if A is a clean ring, then by Corollary 5.10, A is ℓ-clean, hence by Theorem 5.9, X_A is a Stone space. Since X and X_A are homeomorphic, it follows that X is a Stone space. ■

Next we show that in $\boldsymbol{cba\ell}$ bimorphisms are determined entirely by their behavior on idempotents. Let $A \in \boldsymbol{ba\ell}$ and let $\mathrm{Id}(A)$ be the set of idempotents of A. It is well known (see, e.g., [27, p. 181]) that $\mathrm{Id}(A)$ is a Boolean algebra under the operations

$$\begin{aligned} e \vee f &= e + f - ef \\ e \wedge f &= ef \\ \neg e &= 1 - e \end{aligned}$$

5.12. Proposition. *Let $A, B \in \boldsymbol{ba\ell}$ and let $\alpha : A \to B$ be a monomorphism in $\boldsymbol{ba\ell}$. Consider the two statements:*

1. *α is a bimorphism.*
2. *α induces an isomorphism of Boolean algebras $\mathrm{Id}(A) \to \mathrm{Id}(B)$.*

Then (1) $\Rightarrow$ (2). Furthermore, if B is ℓ-clean, then (2) $\Rightarrow$ (1).

Proof. (1) $\Rightarrow$ (2): Suppose that $\alpha : A \to B$ is a bimorphism. It suffices to show that $\mathrm{Id}(B) = \alpha(\mathrm{Id}(A))$. Since α is a bimorphism, by Lemma 2.9(3), $\alpha^* : X_B \to X_A$ is a homeomorphism. Take $e \in \mathrm{Id}(B)$. By Lemma 5.8, $Z_\ell(e)$ is clopen in X_B. Consequently, $\alpha^*(Z_\ell(e))$ is clopen in X_A, so there is $f \in \mathrm{Id}(A)$ with $Z_\ell(f) = \alpha^*(Z_\ell(e))$. If $e \neq \alpha(f)$, then there is $M \in X_B$ with $e - \alpha(f) \notin M$. But, the equality $Z_\ell(f) = \alpha^*(Z_\ell(e))$ says that $e \in M$ iff $f \in \alpha^{-1}(M)$, which happens iff $\alpha(f) \in M$. This is a contradiction, so $e = \alpha(f)$. Therefore, $\mathrm{Id}(B) = \alpha(\mathrm{Id}(A))$.

Next, we prove that if B is ℓ-clean, then (2) $\Rightarrow$ (1). Suppose that $\alpha(\mathrm{Id}(A)) = \mathrm{Id}(B)$. Since $\alpha : A \to B$ is monic, by Lemma 2.9(1), $\alpha^* : X_B \to X_A$ is onto. We show that α^* is also 1-1. As B is an ℓ-clean ring, for distinct $M, N \in X_B$, there exists an idempotent in B contained in one but not in the other. This idempotent is in $\alpha(A)$, which yields $\alpha^{-1}(M) \neq \alpha^{-1}(N)$. Therefore, α^* is 1-1, so α^* is a homeomorphism, and hence by Lemma 2.9(3), α is a bimorphism. ∎

In Example 2.5(3), for a compact Hausdorff space X, we saw that $PC(X, \mathbb{R}) \in \boldsymbol{ba\ell}$ and that $PC(X, \mathbb{R})$ is the $\mathbb{R}$-subalgebra of $C(X, \mathbb{R})$ generated by its idempotents.

5.13. Corollary. *Let $A \in \boldsymbol{ba\ell}$. Then $PC(X_A, \mathbb{R})$ is isomorphic to the $\mathbb{R}$-subalgebra of A generated by the idempotents of A.*

Proof. The canonical map $\phi_A : A \to C(X_A, \mathbb{R})$ is a bimorphism by Proposition 2.10. Consequently, by Proposition 5.12, $C(X_A, \mathbb{R})$ and $\phi_A(A)$ have the same idempotents. Thus, the $\mathbb{R}$-subalgebra of $C(X_A, \mathbb{R})$ generated by its idempotents is the same as the $\mathbb{R}$-subalgebra of $\phi_A(A)$ generated by its idempotents. Since the former $\mathbb{R}$-subalgebra is isomorphic to $PC(X_A, \mathbb{R})$, the result follows. ∎

We recall that an ideal I of a ring A is an *annihilator ideal* if there exists an ideal J of A such that $I = \mathrm{ann}(J) := \{a \in A : aJ = 0\}$, and that A is a *Baer ring* if each annihilator ideal of A is a principal ideal generated by an idempotent. If $J(A) = 0$, then A is a Baer ring iff $\mathrm{Max}(A)$ is an extremally disconnected space [39, Thm. 2.7], where we recall that a space is *extremally disconnected* if the closure of each open set is open. It follows that if A is a Gelfand ring and $J(A) = 0$, then A is a Baer ring iff $\mathrm{Max}(A)$ is an extremally disconnected compact Hausdorff space. In particular, if A is Baer Gelfand with $J(A) = 0$, then A is clean.

In Proposition 5.14 we give one more equivalent characterization of a Baer ring A in terms of the Boolean algebra $\mathrm{Id}(A)$ of idempotents of A. If A is a Baer ring, then $\mathrm{Id}(A)$ is a complete Boolean algebra [32, p. 271]. The converse is false; for example, let $A = C(X, \mathbb{R})$, where $X = [0, 1]$. Then A has no nontrivial idempotents, so $\mathrm{Id}(A)$ is the two-element Boolean algebra, hence $\mathrm{Id}(A)$ is complete. But A is not a Baer ring. Note that A is not clean. We show that if A is a clean ring and $J(A) = 0$, then A is a Baer ring iff $\mathrm{Id}(A)$ is a complete Boolean algebra.

We recall [43] that if B is a Boolean algebra, then the space $\mathrm{Max}(B)$ of maximal ideals of B is a Stone space, where the closed sets are the sets of the form $\{M \in \mathrm{Max}(B) : I \subseteq M\}$ with I an arbitrary ideal of B.

5.14. Proposition. *Let A be a clean ring and $J(A) = 0$. Then the following conditions are equivalent:*

1. *A is a Baer ring.*
2. $\mathrm{Max}(A)$ *is an extremally disconnected compact Hausdorff space.*
3. $\mathrm{Id}(A)$ *is a complete Boolean algebra.*

Proof. That (1) is equivalent to (2) follows from the fact that each clean ring is Gelfand, and when $J(A) = 0$, then, as discussed above, A is a Gelfand ring iff $\mathrm{Max}(A)$ is Hausdorff, and A is a Baer ring iff $\mathrm{Max}(A)$ is an extremally disconnected space. To see that (2) is equivalent to (3), observe that the mapping $\mathrm{Max}(A) \to \mathrm{Max}(\mathrm{Id}(A)) : M \mapsto M \cap \mathrm{Id}(A)$ is continuous and onto [2, Sec. 2]. Moreover, since idempotents in a clean ring A separate maximal ideals in A, it follows that this mapping is a homeomorphism. It is well known that $\mathrm{Max}(\mathrm{Id}(A))$ is an extremally disconnected compact Hausdorff space iff $\mathrm{Id}(A)$ is a complete Boolean algebra [44, Thm. 4.7]. Thus, (2) is equivalent to (3). ■

As a corollary, we recover a result of Azarpanah and Karamzadeh:

5.15. Corollary. ([4, Thm. 3.5]) *A compact Hausdorff space X is extremally disconnected iff $C(X, \mathbb{R})$ is a Baer ring.*

Proof. Let $A = C(X, \mathbb{R})$. Then since $\mathrm{Max}(A)$ is homeomorphic to X, Proposition 5.14 implies that A is a Baer ring iff X is an extremally disconnected space. ■

Let $A \in \boldsymbol{ba\ell}$. We show that unlike the case when A is a clean ring, whether A is a Baer ring is determined entirely by X_A. For a topological space X we denote the closure and interior of $S \subseteq X$ by $\overline{S}$ and $\operatorname{int}(S)$, respectively. For lack of a reference, we include the proof of the following standard observations.

5.16. LEMMA. *Let $A \in \boldsymbol{ba\ell}$ and let I be a proper ℓ-ideal of A.*

1. $\operatorname{ann}(I) = \bigcap Z_\ell(I)^c$.
2. $\overline{\operatorname{int}(Z_\ell(I))} = Z_\ell(\operatorname{ann}(\operatorname{ann}(I)))$.
3. *I is an annihilator ideal of A iff I is an intersection of maximal ℓ-ideals and $Z_\ell(I)$ is regular closed in X_A.*

PROOF. (1) Set $K = \bigcap Z_\ell(I)^c = \bigcap\{M \in X_A : I \not\subseteq M\}$. Since $I \operatorname{ann}(I) = 0$, for each $M \in X_A$, either $I \subseteq M$ or $\operatorname{ann}(I) \subseteq M$. Consequently, $\operatorname{ann}(I) \subseteq K$. Conversely, let $M \in X_A$. Then $I \subseteq M$ or $K \subseteq M$. Therefore, $IK \subseteq M$. Thus, $IK = \bigcap X_A = 0$. This implies $K \subseteq \operatorname{ann}(I)$, and so $\operatorname{ann}(I) = K$.

(2) Recall that if $S \subseteq X_A$, then $\overline{S} = Z_\ell(\bigcap S)$. Therefore, (1) implies $\overline{Z_\ell(I)^c} = Z_\ell(\operatorname{ann}(I))$, so $\operatorname{int}(Z_\ell(I)) = Z_\ell(\operatorname{ann}(I))^c$. Thus,

$$\begin{aligned}
\overline{\operatorname{int}(Z_\ell(I))} &= \overline{Z_\ell(\operatorname{ann}(I))^c} \\
&= Z_\ell\left(\bigcap Z_\ell\left(\operatorname{ann}(I)\right)^c\right) && (\text{by } \overline{S} = Z_\ell(\textstyle\bigcap S)) \\
&= Z_\ell(\operatorname{ann}(\operatorname{ann}(I))) && (\text{by } (1)).
\end{aligned}$$

Consequently, $\overline{\operatorname{int}(Z_\ell(I))} = Z_\ell(\operatorname{ann}(\operatorname{ann}(I)))$.

(3) If I is an annihilator ideal, then by (1), I is an intersection of maximal ℓ-ideals. Moreover, since $I = \operatorname{ann}(\operatorname{ann}(I))$, we have $\overline{\operatorname{int}(Z_\ell(I))} = Z_\ell(I)$, and so $Z_\ell(I)$ is regular closed. Conversely, suppose that $Z_\ell(I)$ is regular closed. By (2), $\overline{\operatorname{int}(Z_\ell(I))} = Z_\ell(\operatorname{ann}(\operatorname{ann}(I)))$, so $Z_\ell(I) = Z_\ell(\operatorname{ann}(\operatorname{ann}(I)))$. Thus, since I and $\operatorname{ann}(\operatorname{ann}(I))$ are intersections of maximal ℓ-ideals, we have $I = \operatorname{ann}(\operatorname{ann}(I))$. ■

5.17. THEOREM. *If $A \in \boldsymbol{ba\ell}$, then A is a Baer ring iff X_A is extremally disconnected.*

PROOF. Suppose that X_A is extremally disconnected. We identify A with its image in $C = C(X_A, \mathbb{R})$. Let I be an ideal of A. Since by Proposition 5.14, C is a Baer ring, there exists an idempotent $e \in C$ such that $\operatorname{ann}_C(I) = eC$. Therefore, $\operatorname{ann}_A(I) = A \cap \operatorname{ann}_C(I) = A \cap eC$. It follows then from Corollary 5.13 that $PC(X_A, \mathbb{R}) \subseteq A$, and in particular, A contains every idempotent of $C(X_A, \mathbb{R})$. Thus, $eA \subseteq A \cap eC$. For the reverse inclusion, if $a \in A \cap eC$, then we may write $a = ec$ for some $c \in C$. Therefore, $ea = e^2c = ec = a$. Consequently, $a \in eA$. Thus, $\operatorname{ann}_A(I) = eA$. This proves that A is Baer.

Conversely, let $A \in \boldsymbol{ba\ell}$ be a Baer ring. By Lemma 5.16, a subset F of X_A is regular closed in X_A iff $F = Z_\ell(I)$ for some annihilator ideal I. Since A is a Baer ring, $I = eA$ for some idempotent e. Therefore, $F = Z_\ell(e)$, which is clopen. Thus, X_A is extremally disconnected. ■

5.18. COROLLARY. *If $A \in$ **bal** is a Baer ring, then A is ℓ-clean.*

PROOF. If $A \in$ **bal** is a Baer ring, then by Theorem 5.17, X_A is an extremally disconnected compact Hausdorff space. Therefore, X_A is a Stone space, and so by Theorem 5.9, A is ℓ-clean. ∎

Let **bbal** be the full subcategory of **bal** consisting of Baer rings. By Corollary 5.18, **bbal** is a full subcategory of **cbal**. To see that **bbal** is properly contained in **cbal**, we utilize Gleason covers of compact Hausdorff spaces. We recall that the *Gleason cover* of a compact Hausdorff space X is a pair (Y, π), where Y is an extremely disconnected compact Hausdorff space and $\pi : Y \to X$ is an irreducible map (a continuous map is *irreducible* if it is onto and the image of a proper closed set is not the entire space). By [18, Thms. 2.5, 3.2], the pair (Y, π) is unique up to homeomorphism and is the projective cover of X in **KHaus**. Gleason constructs Y as the Stone space of the (complete) Boolean algebra of regular open subsets of X.

5.19. EXAMPLE. We give an example of $A \in$ **bbal** which does not have bounded inversion and is not square closed. Let X be the Cantor space and let $\pi : Y \to X$ be its Gleason cover. Then Y is an extremally disconnected compact Hausdorff space. Recall from Example 5.6 the ℓ-algebra $PP(X, \mathbb{R})$ of piecewise polynomial functions on X, which we saw does not have bounded inversion. We may view $C(X, \mathbb{R}) \subseteq C(Y, \mathbb{R})$ via the identification $h \mapsto h \circ \pi$. Set

$$A = \left\{ \sum_{i=1}^{n} a_i e_i : a_i \in PP(X, \mathbb{R}), e_i \in \mathrm{Id}(C(Y, \mathbb{R})) \right\}.$$

We claim that A is an ℓ-subalgebra of $C(Y, \mathbb{R})$ which contains $\mathrm{Id}(C(Y, \mathbb{R}))$ and does not have bounded inversion. Given these claims, since then $PC(Y, \mathbb{R}) \subseteq A \subseteq C(Y, \mathbb{R})$, we conclude, by the argument in the proof of Theorem 5.17, that A is Baer. Since it does not have bounded inversion, it cannot be clean.

The definition of S implies that $PC(Y, \mathbb{R}) \subseteq A$ and $PP(X, \mathbb{R}) \subseteq A$. It also follows from the definition that A is closed under addition, multiplication, and scalar multiplication, since the same is true for $PP(X, \mathbb{R})$. Each element of A can be written in the form $\sum_{i=1}^{n} a_i e_i$ where $\{e_1, \ldots, e_n\}$ is an orthogonal set of idempotents. To see that A is closed under $\vee$, let $a, b \in A$. Write $a = \sum_i a_i e_i$ and $b = \sum_j b_j f_j$ with $a_i, b_j \in PP(X, \mathbb{R})$ and $\{e_1, \ldots, e_n\}$ and $\{f_1, \ldots, f_m\}$ each an orthogonal set of idempotents. We may assume the sum of each set is 1 since, if not, we may adjoin $1 - \sum_i e_i$ or $1 - \sum_j f_j$ to the set. We can then write $a = \sum_{i,j} a_i e_i f_j$ and $b = \sum_{i,j} b_j e_i f_j$. Therefore, $a \vee b = \sum_{i,j} (a_i \vee b_j) e_i f_j \in A$. Thus, A is an ℓ-subalgebra of $C(Y, \mathbb{R})$.

Let $a \in PP(X, \mathbb{R})$ with $a \geq 1$ but for which $1/a$ does not exist in $PP(X, \mathbb{R})$; such a exists since $PP(X, \mathbb{R})$ does not have bounded inversion. If A has bounded inversion, then $1/a \in A$, so we may write $a^{-1} = \sum_{i=1}^{n} b_i e_i$ with $b_i \in PP(X, \mathbb{R})$ and $\{e_1, \ldots, e_n\}$ an orthogonal set. Multiplying both sides by $a e_j$ yields $a b_j e_j = e_j$. Consequently, $a b_j = 1$ on $Z_\ell(1 - e_j)$, which is a clopen set in Y. By [18], Y is the Stone space of the Boolean algebra of regular open subsets of X. Therefore, there is a regular open set $U_j \subseteq X$ with $Z_\ell(1 - e_j) = \pi^{-1}(U_j)$. Then $ab_j = 1$ on $\pi^{-1}(U_j)$, so $ab_j = 1$ on U_j. Now, since $\sum_{i=1}^n b_i e_i$ is a unit, it has no zeros, so it must be the case that the $\pi^{-1}(U_j)$ cover Y. Consequently, the U_j cover X. Therefore, at least one, say U_k, is infinite. Then $ab_k = 1$ on an infinite subset of X, which would imply that we have polynomial equation $ab = 1$ on an infinite subset of X. Since this cannot happen, A does not have bounded inversion.

We now show A is not square closed. Consider $a \in PP(X, \mathbb{R})$, defined by $a(x) = x$ for all $x \in X$. It is clear that a is not a square in $PP(X, \mathbb{R})$. Suppose $a = b^2$ for some $b \in A$. We may write $b = \sum_{i=1}^n b_i e_i$ for some $b_i \in PP(X, \mathbb{R})$ with the e_i forming a set of orthogonal idempotents in $C(Y, \mathbb{R})$. By the argument above, we may assume that $\sum_i e_i = 1$. Then $a = b^2 = \sum_{i=1}^n b_i^2 e_i$. Let $C_i = \{y \in Y : e_i(y) = 1\}$. The C_i form a clopen partition of Y, so the closed sets $\pi(C_i)$ cover X. Thus, there is i for which $\pi(C_i)$ is infinite. Moreover, a and b_i^2 agree on C_i. Consequently, a and b_i^2 agree on an infinite subset of X. Since they are polynomial functions, $a = b_i^2$ on X. This is false since a is not a square in $PP(X, \mathbb{R})$. This proves that A is not square closed.

Let $\boldsymbol{cubal}$ be the full subcategory of $\boldsymbol{ubal}$ consisting of clean Stone rings and let $\boldsymbol{bubal}$ be the full subcategory of $\boldsymbol{ubal}$ consisting of Baer Stone rings. Since each $A \in \boldsymbol{ubal}$ has bounded inversion, $\boldsymbol{bubal}$ is a full subcategory of $\boldsymbol{cubal}$. Let also **Stone** and **ED** be the full subcategories of **KHaus** consisting of Stone spaces and extremally disconnected spaces, respectively. Then **ED** is a full subcategory of **Stone**. We conclude this section by pointing out that, by the results of Azarpanah and Karamzadeh stated in Corollaries 5.11 and 5.15, Gelfand-Neumark-Stone duality restricts to a dual equivalence between $\boldsymbol{cubal}$ and **Stone**, which further restricts to a dual equivalence between $\boldsymbol{bubal}$ and **ED**:

5.20. PROPOSITION.

1. *The functors $\mathcal{X}$ and C, restricted to $\boldsymbol{cubal}$ and* **Stone**, *respectively, yield a dual equivalence between $\boldsymbol{cubal}$ and* **Stone**.

2. *The functors $\mathcal{X}$ and C, restricted to $\boldsymbol{bubal}$ and* **ED**, *respectively, yield a dual equivalence between $\boldsymbol{bubal}$ and* **ED**.

6. Epi-cocomplete objects and coreflective subcategories of $\boldsymbol{bal}$

In this section we give several characterizations of Specker $\mathbb{R}$-algebras, including that they are *epi-cocomplete* objects A in $\boldsymbol{bal}$, meaning that each epimorphism $\alpha : B \to A$ in $\boldsymbol{bal}$ is onto. We show that the category of Specker $\mathbb{R}$-algebras is a mono-coreflective subcategory of $\boldsymbol{bal}$, and that it is the smallest epi-coreflective subcategory of $\boldsymbol{cbal}$, which in turn is a mono-coreflective subcategory of $\boldsymbol{bal}$. We start with the following auxiliary lemma.

6.1. LEMMA. *If $A \in \boldsymbol{bal}$ and P is a minimal prime ideal of A, then $\mathbb{R} + P \in \boldsymbol{bal}$ and the inclusion mapping $\mathbb{R} + P \to A$ is a bimorphism.*

PROOF. By [46, p. 196], each minimal prime ideal of A is an ℓ-ideal. Thus, by Example 2.6, $\mathbb{R}+P$ is an ℓ-subalgebra of A. We prove that the inclusion mapping $\mathbb{R}+P \subseteq A$ is a bimorphism. By Proposition 3.3, it suffices to show that the elements of $\mathbb{R}+P$ separate points of X_A. Let $N_1, N_2 \in X_A$ be distinct. By Corollary 4.5, P cannot be contained in both N_1 and N_2. We assume without loss of generality that $P \not\subseteq N_2$. Then $P \cap N_1 \not\subseteq N_2$ since $N_1 \not\subseteq N_2$ and N_2 is prime. Therefore, $(P \cap N_1) + N_2 = A$, so that $n_1 + n_2 = 1$ for some $n_1 \in P \cap N_1$ and $n_2 \in N_2$. Thus, $n_1, n_2 = 1 - n_1 \in \mathbb{R}+P$ with $n_1 \in N_1 \backslash N_2$ and $n_2 \in N_2 \backslash N_1$. This shows that $\mathbb{R}+P$ separates points of X_A. ■

6.2. THEOREM. *The following are equivalent for $A \in$ **bal**.*

1. *A is a Specker $\mathbb{R}$-algebra.*
2. *A is a von Neumann regular ring.*
3. *A is epi-cocomplete.*
4. *Each bimorphism $\alpha : B \to A$ in **bal** is an isomorphism.*
5. *No proper ℓ-subalgebra of A is uniformly dense in A.*
6. *A is isomorphic to $PC(X, \mathbb{R})$ for some Stone space X.*
7. *$A/I \in$ **bal** for each ℓ-ideal I of A.*

PROOF. Since the theorem is trivially true when $A = 0$, we assume throughout the proof that A is nonzero.

(1) $\Rightarrow$ (2): Apply Proposition 5.5.

(2) $\Rightarrow$ (1): Since A is von Neumann regular, it is clean, hence Gelfand, and so by Proposition 4.1, A has bounded inversion. Let B be the $\mathbb{R}$-subalgebra of A generated by $\mathrm{Id}(A)$. Then by Proposition 5.5, $B \in$ **bal**. We verify that B separates points of X_A. Let $N_1, N_2 \in X_A$ be distinct. Since A is clean, X_A has a basis of clopens. Therefore, there exists a clopen set $U \subseteq X_A$ such that $N_1 \in U$ and $N_2 \notin U$. By Lemma 5.8, there is an idempotent $e \in A$ such that $U = Z_\ell(e)$. Thus, $e \in N_1 \backslash N_2$, which proves that B separates points of A. By Proposition 3.3, the inclusion $B \subseteq A$ is then a bimorphism, and so the induced mapping $X_B \to X_A$ is a homeomorphism. To prove that $B = A$, let $a \in A$, and let $I = \{b \in B : ba \in B\}$. We show that $a \in B$ by proving that $I = B$. If not, then there is a maximal ideal M of B containing I. By Proposition 5.5, B has bounded inversion, and by the above argument, so does A. Therefore, by Proposition 4.1, $\mathrm{Max}(B) = X_B$ and $\mathrm{Max}(A) = X_A$. Since, as noted above, the induced mapping $X_B \to X_A$ is a homeomorphism, there exists a unique maximal ideal N of A such that $M = N \cap B$. Let $S = B \backslash M$. Then S is a multiplicatively closed subset of B. It follows immediately from the definition that A_S is von Neumann regular. Since N is the unique maximal ideal of A with $N \cap B = M$, we see that NA_S is the unique maximal ideal of A_S. But, since A_S is von Neumann regular, $A_S = (A_S)_{NA_S}$ is a field. Thus, $NA_S = 0$. Now, since $N \in X_A$, we have $A = \mathbb{R} + N$. Therefore, writing $a = \lambda + n$, with $\lambda \in \mathbb{R}$ and $n \in N$, there exists

$b \in S = B \setminus M$ such that $bn = 0$, so that $ba = b\lambda + bn = b\lambda \in B$. Therefore, $b \in I$. Thus, since $b \notin M$, we see that $I \not\subseteq M$. This proves that I is not contained in any maximal ideal of B, so $I = B$. Therefore, $a \in B$. Since this is true for each $a \in A$, we conclude that $B = A$.

(1) $\Rightarrow$ (3): Let A be a Specker $\mathbb{R}$-algebra and let $\alpha : B \to A$ be an epimorphism in $\boldsymbol{ba\ell}$. Then $\overline{\alpha} : B/\mathrm{Ker}(\alpha) \to A$ is a bimorphism, and the ℓ-algebra $B/\mathrm{Ker}(\alpha)$ is in $\boldsymbol{ba\ell}$ since it is isomorphic to a subalgebra of $A \in \boldsymbol{ba\ell}$. By Proposition 5.12, $\mathrm{Id}(A) = \overline{\alpha}(\mathrm{Id}(B/\mathrm{Ker}(\alpha)))$. Since A is generated over $\mathbb{R}$ by its idempotents, this implies that α is onto.

$$\begin{tikzcd}
B \arrow[rr, "\alpha"] \arrow[dr] & & A \\
& B/\mathrm{Ker}(\alpha) \arrow[ur, "\overline{\alpha}"'] &
\end{tikzcd}$$

(3) $\Rightarrow$ (4): This is clear.

(4) $\Rightarrow$ (2): Let P be prime ideal of A. By Lemma 6.1, the inclusion mapping $\mathbb{R}+P \subseteq A$ is a bimorphism. Thus, by (4), $\mathbb{R} + P = A$, so P is a maximal ideal of A. Therefore, as A is reduced and each prime ideal of A is maximal, A is von Neumann regular.

(1) $\Rightarrow$ (6): Let A be a Specker $\mathbb{R}$-algebra. By Proposition 5.5 and Theorem 5.9, X_A is a Stone space, and by Proposition 5.12, $\mathrm{Id}(\phi_A(A)) = \mathrm{Id}(C(X_A, \mathbb{R}))$. By Corollary 5.13, $PC(X_A, \mathbb{R})$ is the $\mathbb{R}$-subalgebra of $C(X_A, \mathbb{R})$ generated by $\mathrm{Id}(C(X_A, \mathbb{R}))$. Now, as each element of A is a linear combination of idempotents, the equality $\mathrm{Id}(\phi_A(A)) = \mathrm{Id}(C(X_A, \mathbb{R}))$ shows that $\phi_A(A) = PC(X_A, \mathbb{R})$. Thus, ϕ_A is an isomorphism between A and $PC(X_A, \mathbb{R})$.

(6) $\Rightarrow$ (1): Apply Example 5.2.

(4) $\Leftrightarrow$ (5): Apply Proposition 3.3.

(2) $\Rightarrow$ (7): Let $A \in \boldsymbol{ba\ell}$ be von Neumann regular and let I be an ℓ-ideal of A. Then A is a Specker $\mathbb{R}$-algebra by (2) $\Rightarrow$ (1). Thus, A/I is also a Specker $\mathbb{R}$-algebra, because each element is an $\mathbb{R}$-linear combination of cosets $e + I$ for $e \in \mathrm{Id}(A)$, and each of these elements is idempotent in A/I. By Proposition 5.5, $A/I \in \boldsymbol{ba\ell}$ with respect to the partial order described in the proof of the proposition. We claim this is the same partial order as that inherited from A. To see this, let $a \in A$. Then $a + I \geq 0$ for the inherited order if there is $b \in A$ with $b \geq 0$ and $a + I = b + I$. On the other hand, $a + I \geq 0$ for the order in the proof of the proposition if $a + I$ is a linear combination of idempotents of A/I with all positive coefficients. But, $b \geq 0$ implies $b = \sum_i \lambda_i e_i$ for some idempotents e_i and $\lambda_i \geq 0$. Thus, the two partial orders are the same.

(7) $\Rightarrow$ (2): Let P be a minimal prime ideal of A. We recall that by [46, p. 196], each minimal prime ideal of A is an ℓ-ideal, so $A/P \in \boldsymbol{ba\ell}$, and so P is an intersection of maximal ℓ-ideals. However, by Corollary 4.5, each prime ideal of A is contained in at most one maximal ℓ-ideal. This forces P to be maximal. Thus, as A is reduced, it is von Neumann regular. ∎

6.3. Remark. In the literature on ℓ-groups and vector lattices, the Archimedean objects whose images in the category are Archimedean are known as hyperarchimedean or epi-archimedean. Statement (7) of the theorem shows that Specker $\mathbb{R}$-algebras are precisely the hyperarchimedean objects in $\boldsymbol{ba\ell}$. Conrad in [14, Cor. I to Prop. 1.2] shows that a hyperarchimedean vector lattice A with strong order unit is isomorphic to $PC(X, \mathbb{R})$, where X is the Yosida space of A. Statement (6) of Theorem 6.2 is the corresponding statement in the category $\boldsymbol{ba\ell}$. For a corresponding version of the equivalence of (1) and (2) for vector lattices see [35, Lem. 7] and its references.

We remark that the $\mathbb{R}$-subalgebra of A generated by $\mathrm{Id}(A)$ is the largest Specker $\mathbb{R}$-subalgebra of A. The next result gives alternative characterizations of this subalgebra.

6.4. PROPOSITION. *Let $A \in \boldsymbol{ba\ell}$ be nonzero and let B be the $\mathbb{R}$-subalgebra of A generated by* $\mathrm{Id}(A)$. *Then $B = \bigcap_P(\mathbb{R} + P)$, where P ranges over the minimal prime ideals of A, and B is also the intersection of all uniformly dense ℓ-subalgebras of A.*

PROOF. Let $B' := \bigcap_P(\mathbb{R} + P)$, where P ranges over the minimal prime ideals of A. We show that B' is a Specker $\mathbb{R}$-algebra. To prove this it is enough by Theorem 6.2 to show that B' is a von Neumann regular ring. Since B' is reduced and each prime ideal of a commutative ring contains a minimal prime [30, p. 6, Thm. 10], this amounts to showing that each minimal prime ideal of B' is a maximal ideal of B'. Let Q be a minimal prime ideal of B', and let $S = B' \backslash Q$. Since each ideal of A_S is extended from an ideal of A, there exists a minimal prime ideal P of A such that $PA_S \neq A_S$. Thus, it must be that $P \cap B' \subseteq Q$, and since Q is a minimal prime ideal of B', we have in fact that $Q = P \cap B'$. Now $Q \subseteq B' \subseteq \mathbb{R} + P \subseteq A$, and P is the intersection of $\mathbb{R} + P$ with any maximal ideal of A containing P. Since each ℓ-ideal is contained in a maximal ℓ-ideal, there is a maximal ℓ-ideal M of A containing P. Then $M \cap B' = B' \cap P = Q$, and so Q is a maximal ideal of B' since, by Lemma 2.9, maximal ℓ-ideals of A contract to maximal ℓ-ideals of B'. This proves that each minimal prime ideal of B' is a maximal ideal, and hence that B' is a von Neumann regular ring. Therefore, by Theorem 6.2, B' is a Specker $\mathbb{R}$-algebra. Moreover, $\mathrm{Id}(A) \subseteq B'$. For, if e is an idempotent in A, then $e(e-1) = 0$, so that each minimal prime ideal of A contains either e or $e - 1$, and hence $e \in B'$. Thus, $\mathrm{Id}(A) \subseteq B'$, and since B' is generated by its idempotents and $B' \subseteq A$, it follows that B' is the Specker $\mathbb{R}$-algebra generated by $\mathrm{Id}(A)$.

Finally, to complete the proof of the proposition, we show that B' is equal to the intersection of all the uniformly dense ℓ-subalgebras of A. By Proposition 5.12, each uniformly dense ℓ-subalgebra of A contains $\mathrm{Id}(A)$, so contains B'. Thus, B' is contained in the intersection of all uniformly dense ℓ-subalgebras of A. But, by Proposition 3.3 and Lemma 6.1, this intersection is contained in B'. Thus, B' is the intersection of all uniformly dense ℓ-subalgebras of A. ■

Proposition 6.4 allows us to add two more equivalent conditions to Theorem 5.9.

6.5. THEOREM. *Let $A \in \boldsymbol{ba\ell}$ be nonzero. The following conditions are equivalent to the four equivalent conditions of Theorem 5.9.*

(5) *$PC(X_A, \mathbb{R})$ is uniformly dense in $C(X_A, \mathbb{R})$.*

(6) *The intersection of any collection of uniformly dense ℓ-subalgebras of A is uniformly dense in A.*

Proof. (3) $\Rightarrow$ (5): From (3) it is clear that $PC(X_A, \mathbb{R})$ separates points in X_A. Thus, by Proposition 3.3, (5) holds.

(5) $\Rightarrow$ (6): This follows immediately from Corollary 5.13 and Proposition 6.4.

(6) $\Rightarrow$ (4): Let B be the Specker $\mathbb{R}$-subalgebra of A generated by the idempotents of A. By Proposition 6.4, B is uniformly dense in A. Thus, by Proposition 3.3, X_B is homeomorphic to X_A. However, by Theorem 6.2, X_B is a Stone space, so X_A is a Stone space. ■

Let ***spec*** be the category of Specker $\mathbb{R}$-algebras and $\mathbb{R}$-algebra homomorphisms. It follows from Proposition 5.5 that ***spec*** is a full subcategory of ***bal***. We show that ***spec*** and ***cbal*** are mono-coreflective subcategories of ***bal***. For each $A \in$ ***bal***, let $s(A)$ denote the largest Specker $\mathbb{R}$-subalgebra of A; that is, $s(A)$ is the $\mathbb{R}$-subalgebra of A generated by the idempotents of A. Let $s : s(A) \to A$ be the identity map. For each $A \in$ ***bal***, define $\overline{s}(A)$ to be the closure of $s(A)$ in A with respect to the uniform topology. Alternatively, with $\phi_{s(A)} : s(A) \to C(X_{s(A)}, \mathbb{R})$ and $\widetilde{s^*} : C(X_{s(A)}, \mathbb{R}) \to C(X_A, \mathbb{R})$, we have $\overline{s}(A) = \phi_A^{-1}(\widetilde{s^*}(\phi_{s(A)}(s(A))))$.

$$\begin{tikzcd} s(A) \arrow[r, "s"] \arrow[d, "\phi_{s(A)}"'] & A \arrow[d, "\phi_A"] \\ C(X_{s(A)}, \mathbb{R}) \arrow[r, "\widetilde{s^*}"] & C(X_A, \mathbb{R}) \end{tikzcd}$$

6.6. Theorem. ***spec*** *and* ***cbal*** *are mono-coreflective subcategories of* ***bal***.

Proof. To see that ***spec*** is a mono-coreflective subcategory of ***bal***, let $A \in$ ***bal***, $B \in$ ***spec***, and $\alpha : B \to A$ be a morphism in ***bal***. Then $\alpha(\mathrm{Id}(B)) \subseteq \mathrm{Id}(A) \subseteq s(A)$. As B is a Specker $\mathbb{R}$-algebra, each $b \in B$ is a linear combination of idempotents in B. Therefore, $\alpha(B) \subseteq s(A)$, so $\alpha : B \to A$ factors through the inclusion mapping $s(A) \to A$, and hence ***spec*** is a mono-coreflective subcategory of ***bal***.

$$\begin{tikzcd} s(A) \arrow[rr] \arrow[dr] & & A \\ & B \arrow[ur, "\alpha"'] & \end{tikzcd}$$

To see that ***cbal*** is a mono-coreflective subcategory of ***bal***, let $A \in$ ***bal***. We claim first that $\overline{s}(A)$ is ℓ-clean. If A is zero, then so is $\overline{s}(A)$; hence, we may assume that A is nonzero. Let $x \in \overline{s}(A)$. Then $\phi_A(x) = \widetilde{s^*}(f)$ for some $f \in C(X_{s(A)}, \mathbb{R})$. Since $X_{C(X_{s(A)}, \mathbb{R})}$ is homeomorphic to $X_{s(A)}$, by Corollary 5.11, $C(X_{s(A)}, \mathbb{R})$ is clean, so there exists an idempotent e and a unit v in $C(X_{s(A)}, \mathbb{R})$ such that $f = e + v$. By Proposition 5.12, $e = \phi_{s(A)}(e')$ for some idempotent $e' \in s(A) \subseteq A$. Therefore, $\widetilde{s^*}(e) = \phi_A(e')$, which implies that $\widetilde{s^*}(v) = \phi_A(u)$ for some $u \in A$. Because v is a unit in $C(X_{s(A)}, \mathbb{R})$ and $X_{s(A)}$ is compact, $|v|$ is bounded away from 0. Thus, there is $\varepsilon > 0$ with $0 < \varepsilon \le |v|$. Applying $\widetilde{s^*}$ gives $0 < \varepsilon \le |u|$. Since $x = e' + u$ with $e' \in A$ an idempotent, and u satisfies $0 < \varepsilon < |u|$, Theorem 5.9 yields that $\overline{s}(A)$ is ℓ-clean.

Now suppose that $B \in$ ***cbaℓ*** and $\alpha : B \to A$ is a morphism in ***baℓ***. Since B is ℓ-clean, it follows from Theorem 5.9(2) that $B' := \operatorname{Im}\alpha$ is also ℓ-clean. Then since $s(B')$ is generated by idempotents in A, we have $s(B') \subseteq s(A)$, and hence the uniform closure of $s(B')$ in A is contained in $\overline{s}(A)$. But since B' is ℓ-clean, $s(B')$ is uniformly dense in B' by Proposition 3.3 and Proposition 5.12. Consequently, $\alpha : B \to A$ must factor through the inclusion mapping $\overline{s}(A) \to A$, which shows that ***cbaℓ*** is a mono-coreflective subcategory of ***baℓ***. ■

Neither ***spec*** nor ***cbaℓ***, with the above coreflectors, is an epi-coreflective subcategory of ***baℓ***. For example, if $A = C([0,1],\mathbb{R})$, then $s(A) = \overline{s}(A) = \mathbb{R}$, and clearly the morphism $\mathbb{R} \to C([0,1],\mathbb{R})$ is not epic. Note also that ***spec*** is not the smallest mono-coreflective subcategory of ***baℓ*** (it properly contains the trivial subcategory consisting of copies of $\mathbb{R}$), nor is it the largest (it is properly contained in ***cbaℓ***). As with reflective subcategories, we assume that coreflective subcategories are full replete.

This shows inherent nonsymmetry between ***ubaℓ*** and ***spec*** as ***ubaℓ*** is both epireflective and monoreflective in ***baℓ***, and it is the smallest nontrivial reflective subcategory of ***baℓ***. We show that the symmetry is restored if we restrict our attention to the subcategory ***cbaℓ*** of ***baℓ***.

6.7. Corollary. ***spec*** *is the smallest epi-coreflective subcategory of* ***cbaℓ***, *and the unique epi-coreflective epi-cocomplete subcategory of* ***cbaℓ***.

Proof. We know that ***spec*** is a subcategory of ***cbaℓ*** and, for each $A \in$ ***cbaℓ***, the inclusion morphism $s(A) \to \overline{s}(A) = A$ is a bimorphism by Proposition 3.3. Thus, ***spec*** is an epi-coreflective subcategory of ***cbaℓ***. To see that it is the smallest such subcategory, suppose $\mathbf{C}$ is an epi-coreflective subcategory of ***cbaℓ***, and let $c :$ ***cbaℓ*** $\to \mathbf{C}$ be the coreflector. Let $A \in$ ***cbaℓ***. By the statement dual to [1, Prop. 16.3], $\mathbf{C}$ is a bi-coreflecive subcategory of ***cbaℓ***. Therefore, $c(A) \to A$ is a bimorphism, so by Proposition 5.12, $\mathrm{Id}(A) = \mathrm{Id}(c(A))$. Thus, $s(A) \subseteq c(A)$. In particular, if $B \in$ ***spec***, then $B = s(B) \subseteq c(B) \subseteq B$. Consequently, $B = c(B)$, so ***spec*** $\subseteq \mathbf{C}$. This proves that ***spec*** is the smallest epi-coreflective subcategory of ***cbaℓ***.

For the second statement, by Theorem 6.2, ***spec*** is an epi-cocomplete subcategory of ***cbaℓ***. To prove uniqueness, suppose that $\mathbf{C}$ is an epi-coreflective epi-cocomplete subcategory of ***cbaℓ*** with coreflector $c :$ ***cbaℓ*** $\to \mathbf{C}$. By the argument of the previous paragraph, $s(A) \subseteq c(A)$, and $\mathrm{Id}(s(A)) = \mathrm{Id}(c(A))$. Since $c(A)$ is ℓ-clean, Proposition 5.12 shows that the inclusion $s(A) \to c(A)$ is a bimorphism. Because $\mathbf{C}$ is epi-cocomplete, we conclude that $c(A) = s(A)$. Consequently, $\mathbf{C} =$ ***spec***. ■

The results we have obtained easily yield a duality between ***spec*** and **Stone**, and between the full subcategory ***bspec*** of ***spec*** consisting of Baer rings and **ED**.

6.8. Theorem.

1. ***spec*** *is dually equivalent to* **Stone**.
2. ***bspec*** *is dually equivalent to* **ED**.

Proof. (1) By Proposition 5.5, each Specker $\mathbb{R}$-algebra is a clean ring. Thus, by Proposition 5.20, the restriction of $\mathcal{X}$ to ***spec*** is a contravariant functor $\mathcal{X}$: ***spec*** $\to$ **Stone**. It is also clear that PC : **Stone** $\to$ ***spec***, associating $PC(X,\mathbb{R})$ with each $X \in$ **Stone** and $\widetilde{\eta}|_{PC(Y,\mathbb{R})}$ with each $\eta \in \hom_{\mathbf{Stone}}(X,Y)$, is a contravariant functor. Thus, the theorem comes down to verifying that there are natural isomorphisms $A \cong PC(X_A,\mathbb{R})$ and $X \cong X_{PC(X,\mathbb{R})}$. The first isomorphism follows from Corollary 5.13. For the second, observe that $X_{PC(X,\mathbb{R})} = \mathrm{Max}(PC(X,\mathbb{R}))$, which as noted in the proof of Proposition 5.14 is homeomorphic to $\mathrm{Max}(\mathrm{Id}(PC(X,\mathbb{R}))$, and this by Stone duality for Boolean algebras [43] is homeomorphic to X.

(2) First observe that $A \in$ ***spec*** is a Baer ring iff $\mathrm{Id}(A)$ is a complete Boolean algebra. Indeed, by Proposition 5.5, $A \in$ ***bal*** and has bounded inversion. Hence, $J(A) = 0$, and it is sufficient to apply Proposition 5.14. Now apply (1). ■

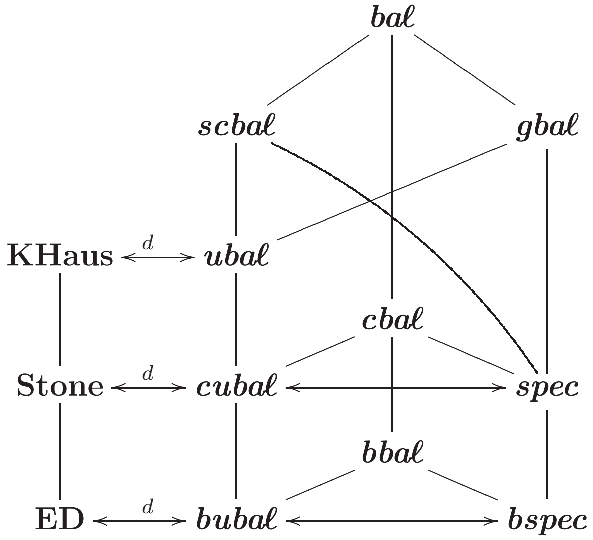
6.9. Remark. While the proof of Theorem 6.8 uses the Stone-Weierstrass theorem, the result can be proved without using Stone-Weierstrass. Indeed, in [12, Thm. 3.8], we prove that ***spec*** is equivalent to the category **BA** of Boolean algebras. By Stone duality for Boolean algebras, **BA** is dually equivalent to **Stone**. Combining these two results yields a proof of Theorem 6.8 that does not use Stone-Weierstrass.

By Proposition 5.20 and Theorem 6.8, ***spec*** is equivalent to ***cubal*** and ***bspec*** is equivalent to ***bubal***. An explicit construction of the functors establishing these equivalences is as follows. The functor ***spec*** $\to$ ***cubal*** (resp. ***bspec*** $\to$ ***bubal***) associates with each $A \in$ ***spec*** the epicompletion (equivalently, the uniform completion) of A, and the functor ***cubal*** $\to$ ***spec*** (resp. ***bubal*** $\to$ ***bspec***) associates with each $A \in$ ***cubal*** the ℓ-subalgebra of A generated by $\mathrm{Id}(A)$. Thus, Theorem 6.8 can be thought of as an "economic" version of Gelfand-Neumark-Stone duality for clean and Baer cases.

We point out that ***cubal*** = ***cbal*** $\cap$ ***ubal***, and that the missing symmetry between ***spec*** and ***ubal*** is restored between ***spec*** and ***cubal***.

We conclude this section with the following diagram which lists the categories we have studied in this article. The line segments without arrows represent category inclusions, and the arrows represent category equivalences or dual equivalences when the arrow has a d superscript. Most of the inclusions (and non-inclusions) are clear; we point out the ones which are not. Each $A \in$ ***spec*** is also in ***scbal***; for, take $a \in A$ positive. We may write $a = \sum_i \lambda_i e_i$ with the $\lambda_i > 0$ in $\mathbb{R}$ and the $e_i \in \mathrm{Id}(A)$ orthogonal. Then $a = \left(\sum_i \sqrt{\lambda_i} e_i\right)^2$. Thus, A is square closed. We also pointed out in Proposition 5.5 that Specker algebras have bounded inversion, hence are Gelfand. As follows from the proof of Proposition 4.2, Specker algebras are not always uniformly complete. To see that ***bspec*** is not contained in ***ubal***, observe that $PC(\beta(\mathbb{N}),\mathbb{R})$ is not uniformly complete since it is a dense proper ℓ-subalgebra of $C(\beta(\mathbb{N}),\mathbb{R})$. Since $\beta(\mathbb{N}) \in$ **ED**, this shows ***bspec*** is not contained in ***ubal***. Finally, Example 5.19 shows that ***bbal*** is contained in neither ***gbal*** nor ***scbal***, and

Proposition 4.6 shows that $\boldsymbol{scba\ell}$ and $\boldsymbol{gba\ell}$ are incomparable.



7. The Complexification of $\boldsymbol{ba\ell}$

In this section we discuss briefly complexification of the category $\boldsymbol{ba\ell}$ in order to indicate how our results give a context for some questions regarding complex $*$-algebras. We continue to assume all rings are commutative with 1.

7.1. *There is a category equivalence between the category of* $\mathbb{R}$*-algebras with* $\mathbb{R}$*-algebra homomorphisms and the category of complex* $*$*-algebras with* $*$*-algebra homomorphisms, and hence for each subcategory of* $\mathbb{R}$*-algebras, complexification yields an equivalent subcategory of complex* $*$*-algebras.* Recall (see, e.g., [38, Def. 9.1.1]) that a complex $*$-algebra is a $\mathbb{C}$-algebra B with an involution $*$ satisfying $(\lambda b)^* = \overline{\lambda} b^*$ for each $\lambda \in \mathbb{C}$ and $b \in B$. If A is an $\mathbb{R}$-algebra, then we can form the *complexification* $B := A \otimes_{\mathbb{R}} \mathbb{C}$, which is a $\mathbb{C}$-algebra. Each tensor $a \otimes (\lambda + \mu i)$ of B has the form $(\lambda a \otimes 1) + (\mu a \otimes i)$. Thus, each element of B can be written in the form $(a \otimes 1) + (b \otimes i)$ for some $a, b \in A$. By identifying $a \in A$ with $a \otimes 1 \in B$ and $1 \otimes i$ with i, we then view $B = \{a + bi : a, b \in A\}$. There is an involution $*$ on B given by $(a + bi)^* = a - bi$, which then makes B into a complex $*$-algebra. If $\alpha : A \to A'$ is an $\mathbb{R}$-algebra homomorphism, then $\alpha \otimes \mathrm{Id}_{\mathbb{C}} : A \otimes_{\mathbb{R}} \mathbb{C} \to A' \otimes_{\mathbb{R}} \mathbb{C}$ is a $*$-homomorphism (i.e., a $\mathbb{C}$-algebra homomorphism compatible with the involution). We thus have a functor from the category of $\mathbb{R}$-algebras and $\mathbb{R}$-algebra homomorphisms to the category of complex $*$-algebras with $*$-algebra homomorphisms.

Going backwards, if B is a complex $*$-algebra, we associate to B the $\mathbb{R}$-algebra of self-adjoint elements of B, and to a $*$-algebra homomorphism $\beta : B \to B'$ its restriction to the self-adjoint part of B. To see that these functors yield an equivalence between the categories of $\mathbb{R}$-algebras and complex $*$-algebras, if A is an $\mathbb{R}$-algebra and $B = A \otimes_{\mathbb{R}} \mathbb{C}$ is its complexification, then the self-adjoint subalgebra of B is $\{a + bi : (a + bi)^* = a + bi\}$, which is equal to A under our identification of A as an $\mathbb{R}$-subalgebra of B. Conversely, if B is a complex $*$-algebra and A is its self-adjoint subalgebra, then for each $x \in B$, we have

$$x = \left(\frac{x+x^*}{2}\right) + \left(\frac{x-x^*}{2i}\right) i,$$

which represents x in the form $a+bi$ with $a, b \in A$. Thus, B is isomorphic to the complexification of A.

7.2. *The complexification of* $\boldsymbol{\mathit{ba\ell}}$ *is the category of complex* $*$*-algebras* B *whose self-adjoint subalgebra is closed under absolute value.* Let $\mathcal{B}$ be the complexification of $\boldsymbol{\mathit{ba\ell}}$. Thus, a complex $*$-algebra B is in $\mathcal{B}$ iff its self-adjoint subalgebra A is in $\boldsymbol{\mathit{ba\ell}}$. Any $B \in \mathcal{B}$ is then a $*$-subalgebra of $C(X, \mathbb{C})$ for some $X \in \mathbf{KHaus}$. We claim that a $*$-subalgebra B of $C(X, \mathbb{C})$ is the complexification of some $A \in \boldsymbol{\mathit{ba\ell}}$ iff B is closed under the operation $x \mapsto |x + x^*|$. If $x = a + bi$, then $|x + x^*| = |2a|$, so B is closed under this operation iff its self-adjoint subalgebra A is closed under the absolute value of $C(X, \mathbb{C})$. Since $a \vee b = \frac{1}{2}(a + b + |b - a|)$, $a \wedge b = \frac{1}{2}(a + b - |b - a|)$, and $|a| = a \vee (-a)$, it follows that $A \in \boldsymbol{\mathit{ba\ell}}$ iff A is closed under absolute value. So, a complex $*$-algebra B is in $\mathcal{B}$ iff B is closed under the operation $x \mapsto |x + x^*|$, iff the self-adjoint subalgebra of B is closed under absolute value.

7.3. *The complexification of* $\boldsymbol{\mathit{uba\ell}}$ *is the category* $\mathbf{C^*Alg}$ *of commutative* C^**-algebras.* We show, by restricting the equivalence between $\boldsymbol{\mathit{ba\ell}}$ and $\mathcal{B}$, that $\boldsymbol{\mathit{uba\ell}}$ is equivalent to $\mathbf{C^*Alg}$. First, let $A \in \boldsymbol{\mathit{uba\ell}}$. If we define a norm on the complexification B of A by $\|a + bi\| = \sqrt{\|a^2 + b^2\|}$, then a routine argument shows that B is complete and that for all $x \in B$, $\|xx^*\| = \|x\|^2$. Thus, B is a commutative C^*-algebra. Conversely, if B is a commutative C^*-algebra, then its self-adjoint subalgebra is complete, since the involution is continuous in the norm topology, and so is in $\boldsymbol{\mathit{uba\ell}}$. This pointfree equivalence between $\boldsymbol{\mathit{uba\ell}}$ and $\mathbf{C^*Alg}$ implies, as was pointed out in the introduction, that Stone duality follows from Gelfand-Neumark duality and vice-versa.

7.4. *The complexification of the category of uniformly complete Baer rings in* $\boldsymbol{\mathit{ba\ell}}$ *is the category of commutative* AW^**-algebras.* Recall (see, e.g., [11, Def. 4.1]) that a $*$-ring B is said to be a *Baer* $*$*-ring* if each annihilator in B is generated by a self-adjoint idempotent (i.e., a projection; see, e.g., [11, Def. 1.2]). An AW^*-algebra is a C^*-algebra which is a Baer $*$-ring ([11, Def. 4.2]). If $X \in \mathbf{KHaus}$, then $C(X, \mathbb{C})$ is a $*$-Baer ring iff $X \in \mathbf{ED}$ ([11, Thm. 7.1]), and $C(X, \mathbb{R})$ is Baer iff $X \in \mathbf{ED}$ (Corollary 5.15). Therefore, complexification yields an equivalence between $\boldsymbol{\mathit{buba\ell}}$ and the category of commutative AW^*-algebras.

7.5. *The complexification of* $\boldsymbol{\mathit{spec}}$ *is the category of complex* $*$*-algebras generated as* $\mathbb{C}$*-algebras by projections.* If $A \in \boldsymbol{\mathit{spec}}$, then each element of A is an $\mathbb{R}$-linear combination of idempotents. Let B be the complexification of A. Then idempotents of A are projections in B. As each element of B has the form $a + bi$ with $a, b \in A$, we see that each element is a $\mathbb{C}$-linear combination of projections. Conversely, suppose that B is a $*$-algebra in which each element is a linear combination of projections. Let $x \in B$ with $x^* = x$. Since there is a unique representation $x = \sum_i \lambda_i e_i$, where the e_i form a set of pairwise disjoint projections and the λ_i are distinct, $x = x^* = \sum_i \overline{\lambda_i} e_i$. Because this representation is unique, we see that $\overline{\lambda_i} = \lambda_i$ for each i, so each $\lambda_i \in \mathbb{R}$. Therefore, the self-adjoint subalgebra A of B is generated over $\mathbb{R}$ by idempotents, so $A \in \boldsymbol{spec}$. Thus, the complexification of $\boldsymbol{spec}$ is the category of $*$-Specker $\mathbb{C}$-algebras; that is, the complex $*$-algebras generated as $\mathbb{C}$-algebras by projections.

7.6. *Define $A \in \boldsymbol{ba\ell}$ to be Pythagorean if for each $a, b \in A$, there is $c \in A$ with $a^2 + b^2 = c^2$. The complexification of the subcategory $\boldsymbol{pba\ell}$ of $\boldsymbol{ba\ell}$ consisting of Pythagorean objects in $\boldsymbol{ba\ell}$ is the subcategory of $\mathcal{B}$ consisting of those objects in $\mathcal{B}$ that are closed under the absolute value.* Recall from **7.2** that, since $\mathcal{B}$ is the complexification of $\boldsymbol{ba\ell}$, each $B \in \mathcal{B}$ is a subalgebra of $C(X, \mathbb{C})$ for some $X \in \mathbf{KHaus}$. We consider those B which are closed under the absolute value of $C(X, \mathbb{C})$. If $h \in C(X, \mathbb{C})$ and we write $h = f + gi$ with $f, g \in C(X, \mathbb{R})$, then $|h| = \sqrt{f^2 + g^2}$. Therefore, if A is the self-adjoint subalgebra of B and B is closed under the absolute value, then $\sqrt{f^2 + g^2} \in A$ for each $f, g \in A$. It follows that the complexification of $\boldsymbol{pba\ell}$ is the subcategory of $\mathcal{B}$ consisting of those objects in $\mathcal{B}$ that are closed under the absolute value. It is easy to see that $\boldsymbol{scba\ell}$ is a subcategory of $\boldsymbol{pba\ell}$. Furthermore, a small modification of Proposition 4.6 shows that $\boldsymbol{pba\ell}$ is a reflective subcategory of $\boldsymbol{ba\ell}$.

We conclude by mentioning several interesting open problems.

7.7. Question.

1. Describe all reflective subcategories of $\boldsymbol{ba\ell}$. By Theorem 3.8, $\boldsymbol{uba\ell}$ is the smallest nontrivial reflective subcategory of $\boldsymbol{ba\ell}$. Does there exist a largest proper such subcategory? In [20], Hager describes all the monoreflective subcategories of $\mathbf{W}$ that are closed under homomorphic images, where $\mathbf{W}$ is the category of Archimedean ℓ-groups with weak order unit. Hager parameterizes the set (*a fortiori* it is a set) of all such subcategories by the subsets of $C(\mathbb{R}^\omega, \mathbb{R})$ [20, pp. 166-169]. Can Hager's method be adapted to parameterize all the reflective subcategories of $\boldsymbol{ba\ell}$? (Recall that by Lemma 3.7, nontrivial reflective subcategories of $\boldsymbol{ba\ell}$ are monoreflective.)

2. Is $\boldsymbol{scba\ell}$ a proper subcategory of $\boldsymbol{pba\ell}$? We conjecture that the Pythagorean reflection of $PP([0,1], \mathbb{R})$ is not square closed, which would give an affirmative answer to the question.

3. What is the complexification of $\boldsymbol{gba\ell}$? In particular, is there a suitable notion of $*$-Gelfand algebra such that the complexification of $\boldsymbol{gba\ell}$ is the category consisting of those subalgebras of commutative C^*-algebras that are $*$-Gelfand?

4. We recall that a $*$-algebra B is $*$-clean if each element of B is the sum of a unit and a projection (see, e.g., [47, Def. 1]). It is not hard to show that $A \in \boldsymbol{ba\ell}$ is clean iff the complexification of A is $*$-clean. What is the complex analogue of $A \in \boldsymbol{ba\ell}$ being ℓ-clean?

5. Describe all coreflective subcategories of $\boldsymbol{ba\ell}$. In particular, does there exist a proper bi-coreflective subcategory of $\boldsymbol{ba\ell}$? This question is motivated by the goal of finding a canonical choice of uniformly dense ℓ-subalgebra B of each $A \in \boldsymbol{ba\ell}$. Corollary 6.7 shows that in $\boldsymbol{cba\ell}$, Specker $\mathbb{R}$-algebras play this role. On the other hand, the piecewise polynomials in $C([0,1])$ serve as a canonical choice of uniformly dense ℓ-subalgebra for a specific ring not in $\boldsymbol{cba\ell}$. Both of these examples are "free" constructions, and it would be interesting to determine whether a similar free construction can be given for any ring in $\boldsymbol{ba\ell}$. Determining whether there exists a proper bi-coreflector for $\boldsymbol{ba\ell}$ would be a step in this direction.

6. In Theorem 6.6, it is shown that $\boldsymbol{cba\ell}$ is a mono-coreflective subcategory of $\boldsymbol{ba\ell}$ that contains $\boldsymbol{spec}$. Is $\boldsymbol{cba\ell}$ the largest proper mono-coreflective subcategory of $\boldsymbol{ba\ell}$? If so, then it follows that Question (5) has a negative answer. For if as in (5), there exists a proper bi-coreflective subcategory $\mathbf{C}$ of $\boldsymbol{ba\ell}$, and $\boldsymbol{cba\ell}$ is the largest proper mono-coreflective subcategory of $\boldsymbol{ba\ell}$, then every object of $\mathbf{C}$ is an ℓ-clean ring, and the bi-coreflectivity of $\mathbf{C}$ forces then every $A \in \boldsymbol{ba\ell}$ to be ℓ-clean, a contradiction.

7. By Lemma 3.7, each nontrivial reflective subcategory of $\boldsymbol{ba\ell}$ is bireflective. On the other hand, not every nontrivial coreflective subcategory of $\boldsymbol{ba\ell}$ is bi-coreflective. Is each nontrivial coreflective subcategory of $\boldsymbol{cba\ell}$ bi-coreflective?

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