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DUALITY FOR DISTRIBUTIVE SPACES

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ABSTRACT. The main source of inspiration for the present paper is the work of R. Rosebrugh and R.J. Wood on constructively completely distributive lattices where the authors elegantly employ the concepts of adjunction and module. Both notions (suitably adapted) are available in topology too, which permits us to investigate topological, metric and other kinds of spaces in a similar spirit. We introduce here the notion of distributive space and algebraic space and show in particular that the category of distributive spaces and colimit preserving maps is dually equivalent to the idempotent split completion of a category of spaces and convergence relations between them. We explain the connection of this result to the well-known duality between topological spaces and frames, and deduce further duality theorems.

Contents

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Introduction

The work presented in this paper grew out of a naive comparison of the well-known adjunctions

$$\begin{tikzcd} \mathsf{Ord} \arrow[r, shift left] \arrow[r, phantom, "\perp"] & \mathsf{CCD}^{\mathrm{op}} \arrow[l, shift left] \end{tikzcd} \qquad \text{and} \qquad \begin{tikzcd} \mathsf{Top} \arrow[r, shift left] \arrow[r, phantom, "\perp"] & \mathsf{Frm}^{\mathrm{op}} \arrow[l, shift left] \end{tikzcd}$$

between the category Ord of ordered sets and monotone maps and the dual of the category CCD of (constructively) completely distributive lattices and left and right-adjoint monotone maps on one side, and the category Top of topological spaces and continuous maps and the dual of the category Frm of frames and frame homomorphisms on the other. Here the functor $\mathsf{Ord} \to \mathsf{CCD}^{\mathrm{op}}$ can be constructed by sending an ordered set X to the ordered set $\mathrm{Up}(X) \simeq \mathsf{Ord}(X, 2)$ of all up-sets of X, and the functor $\mathsf{Top} \to \mathsf{Frm}^{\mathrm{op}}$ takes a topological space X to the frame $\mathcal{O}X \simeq \mathsf{Top}(X, 2)$ of opens of X where 2 denotes the Sierpiński space. Since $(-)^{\mathrm{op}} : \mathsf{Ord} \to \mathsf{Ord}$ is an equivalence, in the first adjunction we can equivalently consider the functor $\mathsf{Ord} \to \mathsf{CCD}^{\mathrm{op}}$ which sends X to the ordered set $\mathrm{Down}(X) \simeq \mathsf{Ord}(X^{\mathrm{op}}, 2)$ of all down-sets of X; but for topological spaces this construction does not seem to make sense since it is not clear what X^{op} means. However, in our recent study of "spaces as categories" we introduced a candidate for dual space (see [Clementino and Hofmann, 2009a]) which in several results took the role of the dual ordered set. Therefore we ask in this paper about the construction $X \mapsto \mathsf{Top}(X^{\mathrm{op}}, 2)$, and the answer leads to a scenario which appears to be even closer to the Ord-case than the "usual" dual adjunction with frames.

As it is well known, the dual adjunction between Ord and CCD described above restricts to a dual equivalence between Ord and the full subcategory TAL of CCD defined by the totally algebraic lattices. This equivalence is actually the restriction of a larger one: in [Rosebrugh and Wood, 1994] it is shown that the category $\mathsf{CCD}_{\mathrm{sup}}$ of constructively completely distributive lattices and suprema preserving maps is equivalent to the idempotent split completion of the category Rel of sets and relations. This theorem turned out to be very powerful since it synthesises many facts about complete distributive lattices, implies various known duality theorems in lattice theory (for example, $\mathsf{Ord}^{\mathrm{op}} \simeq \mathsf{TAL}$ as well as $\mathsf{Set}^{\mathrm{op}} \simeq \mathsf{CABool}$ follow easily), and allows to transfer nice properties and structures from Rel to $\mathsf{CCD}_{\mathrm{sup}}$. Later on, in [Rosebrugh and Wood, 2004] the authors observe that this theorem is not really about lattices but rather a special case of a much more general result about "a mere monad $\mathbb{D}$ on a mere category C where idempotents split". More precisely, they show that the idempotent split completion of the Kleisli category of $\mathbb{D}$ is equivalent to the category of split Eilenberg-Moore algebras for $\mathbb{D}$ (see Section 8). The equivalence above appears now for both the power-set monad on Set and the down-set monad on Ord, and further interesting results can be obtained by considering submonads of the down-set monad on Ord. More important to us, this result paves the road towards similar results for topological, metric and approach spaces. In fact, we argue here that many applications of [Rosebrugh and Wood, 2004] can be found in topology since many interesting classes of spaces can be described as algebras for monads: compact Hausdorff spaces are the algebras for the ultrafilter monad on Set, continuous lattices are the algebras for the filter monad on Set, Ord and Top, stably compact spaces are the algebras for the prime filter monad on Ord and Top, to name a few. Furthermore, in [Clementino and Hofmann, 2009b] we showed already how many of these monads can be described in the language of *modules* which leads us to metric and other variants of filter monads. The principal aim of this paper is to give a systematic study of these monads and their associated duality theory in the spirit of the above-mentioned work of R. Rosebrugh and R.J. Wood.

This work was developed in the context of $(\mathbb{T}, \mathsf{V})$-categories where $\mathbb{T}$ and V are part of a strict topological theory as described in [Hofmann, 2007]. However, we feel that the large amount of special notations needed in the general case makes the actual results less accessible, therefore we decided to present them here in the more familiar context of topological, metric and approach spaces. We stress that most of our results can be derived for strict topological theories in general, just a few are indeed only valid for metric or approach spaces.

The paper is organised as follows. In Section 1 we recall the convergence-relational approach to topological and approach spaces which is the context where "spaces look like categories". Section 2 presents basic facts about ordered sets in the language of modules and adjunction, and Section 3 recalls Lawvere's view on metric spaces as enriched categories (see [Lawvere, 1973]). In Section 4 we define the notion of dual space. Here our approach is slightly different then in previous work [Clementino and Hofmann, 2009a]. In Section 5 we recall the main results on cocomplete spaces of [Hofmann, 2011] and [Clementino and Hofmann, 2009b] and derive further results about cocomplete approach spaces. We show in particular that cocomplete approach spaces are determined by their underlying metric and that they define a Cartesian closed category. In Section 6 we introduce completely distributive spaces and develop their duality theory which resembles closely the situation for Ord. In Section 7 we show that the category of completely distributive topological spaces is equivalent to the category of frames. In Section 8 we recall the idea of relative cocompleteness and apply the techniques of [Rosebrugh and Wood, 1994, 2004] to those monads which correspond to a choice of colimit weights. Finally, in Section 9 we discuss examples of such monads.

Some warnings:

a. The underlying order of a topological space X we define as
$$x \le y \quad \text{whenever } \dot{x} \to y,$$
which is the *dual* of the specialisation order. We do so because we wish to think of the underlying order as the "point shadow" of the convergence relation.

b. In the sequel we consider the Sierpiński space $2 = \{0, 1\}$ with $\{1\}$ closed. This is compatible with the point above since the underlying order gives $0 \le 1$, but note that $\varphi : X \to 2$ is the characteristic map of a closed subset.

1. Topological and approach spaces as categories

First we recall how a topological space can be viewed as a category. The fundamental idea is to think of the points of X as objects and of the convergence $\mathfrak{x} \to x$ of an ultrafilter $\mathfrak{x}$ on X to a point x in X as a morphism in X, so that the convergence relation

$$UX \times X \to 2$$

becomes the "hom-functor" of X. An abstract relation between ultrafilters and points is the convergence relation of a (unique) topology on X if and only if (see [Barr, 1970])

$$(1) \qquad e_X(x) \to x \qquad \text{and} \qquad (\mathfrak{X} \to \mathfrak{x} \ \&\ \mathfrak{x} \to x) \Rightarrow m_X(\mathfrak{X}) \to x,$$

for all $x \in X$, $\mathfrak{x} \in UX$ and $\mathfrak{X} \in UUX$, where $e_X(x) = \dot{x}$ the principal ultrafilter generated by $x \in X$ and

$$m_X(\mathfrak{X}) = \{A \subseteq X \mid A^\# \in \mathfrak{X}\} \qquad (A^\# = \{\mathfrak{x} \in UX \mid A \in \mathfrak{x}\}).$$

The first arrow of (1) one might see as an identity on x, and the second condition of (1) one might interpret as the existence of a "composite" of "composable pairs of arrows". Furthermore, a function $f : X \to Y$ between topological spaces is continuous if and only if $\mathfrak{x} \to x$ in X implies $f(\mathfrak{x}) \to f(x)$ in Y, that is, f associates to each object in X an object in Y and to each arrow in X an arrow in Y between the corresponding (ultrafilter of) objects in Y. As usual, Top denotes the category of topological spaces and continuous maps.

Note that the second condition of (1) talks about the convergence of an ultrafilter of ultrafilters $\mathfrak{X}$ to an ultrafilter $\mathfrak{x}$, which comes from applying the ultrafilter functor U to the *relation* $a : UX \rightarrow\!\!\!\!\!\mapsto X$. In general, for a relation $r : X \rightarrow\!\!\!\!\!\mapsto Y$ from X to Y and ultrafilters $\mathfrak{x} \in UX$ and $\mathfrak{y} \in UY$ one puts

$$\mathfrak{x}\,(Ur)\,\mathfrak{y} \quad \text{whenever} \quad \forall A \in \mathfrak{x}, B \in \mathfrak{y}\, \exists x \in A, y \in B \,.\, x\,r\,y,$$

and obtains this way an extension of the Set-functor U to a functor $U : \mathsf{Rel} \to \mathsf{Rel}$ which, moreover, satisfies $U(r^\circ) = (Ur)^\circ$ (where $r^\circ : Y \rightarrow\!\!\!\!\!\mapsto X$ is defined as $y\,r^\circ\,x$ whenever $x\,r\,y$) and $Ur \subseteq Us$ whenever $r \subseteq s$. Furthermore, the multiplication m is still a natural transformation $m : UU \to U$, but $e : 1 \to U$ satisfies only $e_Y \cdot r \subseteq Ur \cdot e_X$ for any relation $r : X \rightarrow\!\!\!\!\!\mapsto Y$.

To describe approach spaces (introduces in [Lowen, 1989]), it is only necessary to trade relation for ***numerical relation***: $r : X \rightarrow\!\!\!\!\!\mapsto Y$ stands now for $r : X \times Y \to [0, \infty]$. We sketch here very briefly this construction which can be found in [Clementino and Hofmann, 2003], and for questions concerning approach spaces in general we refer to [Lowen, 1997]. Given also $s : Y \rightarrow\!\!\!\!\!\mapsto Z$, one can calculate the composite $s \cdot r : X \rightarrow\!\!\!\!\!\mapsto Z$ by the formula

$$(2) \qquad s \cdot r(x, z) = \inf_{y \in Y}(r(x, y) + s(y, z)).$$

Each relation becomes a numerical relation by interpreting true as 0 and false as ∞, and with this interpretation the identity function is also the identity numerical relation. Taking into account the opposite of the pointwise order on the set of all numerical relations from X to Y, one obtains the ordered category NRel of sets and numerical relations. The "turning around" of the natural order of $[0,\infty]$ has its roots in the translation of "$\mathsf{false} \leq \mathsf{true}$" in $\mathsf{2}$ to "$\infty \geqslant 0$" in $[0,\infty]$. Due to this switch, "$\exists$" becomes "inf" in (2), but also note that "&" is replaced by "+". The implication $x \Rightarrow - : \mathsf{2} \to \mathsf{2}$ is right adjoint to $x \,\&\, - : \mathsf{2} \to \mathsf{2}$ for $x \in \mathsf{2}$; similarly, for $x \in [0,\infty]$, the map "addition with x" $x + - : [0,\infty] \to [0,\infty]$ has a right adjoint, namely $\hom(x,-) : [0,\infty] \to [0,\infty]$, $y \mapsto y \ominus x := \max\{y - x, 0\}$.

As above, the ultrafilter functor $U : \mathsf{Set} \to \mathsf{Set}$ extends to $U : \mathsf{NRel} \to \mathsf{NRel}$ (with the properties mentioned in the topological case) via

$$Ur(\mathfrak{x},\mathfrak{y}) = \sup_{A\in\mathfrak{x},B\in\mathfrak{y}} \inf_{x\in A,y\in B} r(x,y),$$

for a numerical relation $r : X \times Y \to [0,\infty]$. We remark that a different but equivalent formula defining the extension of U to NRel was used in [Clementino and Hofmann, 2003], the one above is taken from [Clementino and Tholen, 2003].

1.1. Remark. Thinking of a relation $r : X \rightarrow\!\!\!\!\!\!\mapsto Y$ as a subset $R \subseteq X \times Y$, it is not hard to see that

$$\mathfrak{x}\,(Ur)\,\mathfrak{y} \iff \exists \mathfrak{w} \in U(X \times Y)\,.\,U\pi_1(\mathfrak{w}) = \mathfrak{x} \,\&\, U\pi_2(\mathfrak{w}) = \mathfrak{y} \,\&\, \mathfrak{w} \in UR$$

for all $\mathfrak{x} \in UX$ and $\mathfrak{y} \in UY$. Similarly, for a numerical relation $r : X \rightarrow\!\!\!\!\!\!\mapsto Y$ one has

$$Ur(\mathfrak{x},\mathfrak{y}) = \inf\{\xi \cdot Ur(\mathfrak{w}) \;\Big|\; \mathfrak{w} \in U(X \times Y), T\pi_1(\mathfrak{w}) = \mathfrak{x}, T\pi_2(\mathfrak{w}) = \mathfrak{y}\},$$

where $\xi : U[0,\infty] \to [0,\infty]$, $\mathfrak{u} \mapsto \sup_{A\in\mathfrak{u}} \inf A$ is the convergence of the Euclidean topology on $[0,\infty]$. The notation here is a bit ambiguous since Ur appears on both sides, but on the ride hand side it stands for the functions $Ur : U(X \times Y) \to U[0,\infty]$. We use the occasion to mention that the $\mathbb{U}$-algebra structure $\xi : U[0,\infty] \to [0,\infty]$ makes $[0,\infty]$ a monoid in the category of compact Hausdorff spaces and continuous maps in two different ways since both $+ : [0,\infty] \times [0,\infty] \to [0,\infty]$ and $\max : [0,\infty] \times [0,\infty] \to [0,\infty]$ are continuous. It is useful to observe that continuity of $+$ and $\max$ mean precisely that the diagrams

$$\begin{tikzcd} U([0,\infty]\times[0,\infty]) \arrow[r, "U(+)"] \arrow[d, "\langle \xi\cdot U\pi_1, \xi\cdot U\pi_2\rangle"'] & U[0,\infty] \arrow[d, "\xi"] \\ {[0,\infty]\times[0,\infty]} \arrow[r, "+"'] & {[0,\infty]} \end{tikzcd} \qquad \begin{tikzcd} U([0,\infty]\times[0,\infty]) \arrow[r, "U(\max)"] \arrow[d, "\langle \xi\cdot U\pi_1, \xi\cdot U\pi_2\rangle"'] & U[0,\infty] \arrow[d, "\xi"] \\ {[0,\infty]\times[0,\infty]} \arrow[r, "\max"'] & {[0,\infty]} \end{tikzcd}$$

commute. Also note that ξ is compatible with the map $\hom : [0,\infty] \times [0,\infty] \to [0,\infty]$ which sends (x,y) to $\hom(x,y) = y \ominus x$ in the sense that $\xi \cdot U(\hom) \geqslant \hom \cdot \langle \xi \cdot U\pi_1, \xi \cdot U\pi_2\rangle$

(see [Hofmann, 2007]).

$$\begin{tikzcd}
U([0,\infty]\times[0,\infty]) \arrow[r, "U(\mathrm{hom})"] \arrow[d, "\langle \xi\cdot U\pi_1, \xi\cdot U\pi_2\rangle"'] \arrow[dr, phantom, "\leqslant"] & U[0,\infty] \arrow[d, "\xi"] \\
{[0,\infty]\times[0,\infty]} \arrow[r, "\mathrm{hom}"'] & {[0,\infty]}
\end{tikzcd}$$

With this notation, an ***approach space*** can be described as a pair (X, a) consisting of a set X and a numerical relation $a : UX \relbar\joinrel\mapstochar\joinrel\rightarrow X$ satisfying

(3) $$0 \geqslant a(\dot{x}, x) \qquad \text{and} \qquad Ua(\mathfrak{X}, \mathfrak{x}) + a(\mathfrak{x}, x) \geqslant a(m_X(\mathfrak{X}), x),$$

and a mapping $f : X \to Y$ between approach spaces $X = (X, a)$ and $Y = (Y, b)$ is a ***contraction*** whenever $a(\mathfrak{x}, x) \geqslant b(Uf(\mathfrak{x}), f(x))$ for all $\mathfrak{x} \in UX$ and $x \in X$. Approach spaces and contraction maps are the main ingredients of the category App.

There is a canonical forgetful functor $\mathsf{App} \to \mathsf{Top}$ sending an approach space (X, a) to the topological space with the same underlying set X and with the convergence relation

$$\mathfrak{x} \to x \text{ whenever } 0 \geqslant a(\mathfrak{x}, x).$$

This functor has a left adjoint $\mathsf{Top} \to \mathsf{App}$ which one obtains by interpreting the convergence relation of a topological space as a numerical relation.

1.2. Remark. The left adjoint functor $\mathsf{Top} \to \mathsf{App}$ has a further left adjoint which can be obtained by first sending an approach space (X, a) to the pseudotopological space X with convergence

$$\mathfrak{x} \to x \text{ whenever } a(\mathfrak{x}, x) < \infty,$$

and then taking its topological reflection. Recall from [Herrlich *et al.*, 1991] that a pseudotopology on a set X is a convergence relation between ultrafilters and points which is only required to satisfy $\dot{x} \to x$, for all $x \in X$.

The pointfree calculus of (numerical) relations allows for a simultaneous treatment of topological and approach spaces emphasising their common nature. For instance, both axioms (1) and (3) read as

(4)
$$\begin{tikzcd}
X \arrow[r, "e_X"] \arrow[dr, "1_X"'] \arrow[dr, phantom, shift left=3, "\sqsubseteq"] & UX \arrow[d, "a", "\shortmid"{marking}] \\
& X
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
UUX \arrow[r, "m_X"] \arrow[d, "Ua"', "\shortmid"{marking}] \arrow[dr, phantom, "\sqsubseteq"] & UX \arrow[d, "a", "\shortmid"{marking}] \\
UX \arrow[r, "a"', "\shortmid"{marking}] & X
\end{tikzcd}$$

$$1_X \sqsubseteq a \cdot e_X \qquad\qquad a \cdot Ua \sqsubseteq a \cdot m_X$$

where $\sqsubseteq$ stands either for $\subseteq$ or $\geqslant$. Since $f : X \to Y$ is continuous respectively contractive if and only if

$$\begin{tikzcd}
UX \arrow[r, "Uf"] \arrow[d, "a"', "\shortmid"{marking}] \arrow[dr, phantom, "\sqsubseteq"] & UY \arrow[d, "b", "\shortmid"{marking}] \\
X \arrow[r, "f"'] & Y,
\end{tikzcd}$$

we can think of Top and App as categories of lax Eilenberg–Moore algebras. Using the fact that $m_X \dashv m_X^\circ$ and $e_X \dashv e_X^\circ$ in the ordered category Rel (and hence in NRel), one can express the axioms (4) as

$$(5) \qquad e_X^\circ \sqsubseteq a \qquad \text{and} \qquad a \cdot Ua \cdot m_X^\circ \sqsubseteq a.$$

In this context it is useful to think of a (numerical) relation $a : UX \rightarrow\!\!\!\!\!\!\mapsto X$ as an endomorphism $a : X \rightharpoondown X$ and, more generally, of $r : UX \rightarrow\!\!\!\!\!\!\mapsto Y$ as an arrow $r : X \rightharpoondown Y$, called $\mathbb{U}$-***relation*** in the sequel. Given also $s : Y \rightharpoondown Z$, one can compose s and r using (a variant of) ***Kleisli composition***:

$$s \circ r := s \cdot Ur \cdot m_X^\circ.$$

The (numerical) relation $e_X^\circ : UX \rightarrow\!\!\!\!\!\!\mapsto X$ behaves almost as an identity arrow $X \rightharpoondown X$ since

$$r \circ e_X^\circ = r \qquad \text{and} \qquad e_Y^\circ \circ r \sqsupseteq r.$$

We can now restate the second condition of (5) as $a \circ a \sqsubseteq a$, or even as $a \circ a = a$ thanks to the first condition.

1.3. Remark. One calls a $\mathbb{U}$-relation $r : X \rightharpoondown Y$ ***unitary*** if $e_Y^\circ \circ r = r$. These relations are not completely unfamiliar to topologists: a reflexive (numerical) relation $a : UX \rightarrow\!\!\!\!\!\!\mapsto X$ is a pretopology (preapproach structure) precisely if $a : X \rightharpoondown X$ is unitary (see [Hofmann, 2006]).

By restricting a convergence relation $a : UX \rightarrow\!\!\!\!\!\!\mapsto X$ to principal ultrafilters one obtains

- an order relation $a_0 := a \cdot e_X : X \rightarrow\!\!\!\!\!\!\mapsto X$ where $x \le y$ whenever $\dot{x} \to y$ (we write $\le$ for a_0 and $\to$ for a) if one starts with a topological space,
- or a metric $a_0 = a \cdot e_X : X \rightarrow\!\!\!\!\!\!\mapsto X$ where $a_0(x, y) = a(\dot{x}, y)$ if one starts with an approach spaces.

Note that for us an order relation does not need to be anti-symmetric. Hence, an ordered set $X = (X, \le)$ consists of a set X and a relation $\le : X \times X \to \mathsf{2}$ satisfying

$$x \le x \qquad \text{and} \qquad (x \le y \,\&\, y \le x) \Rightarrow x \le z.$$

Similarly, a metric d on set X is only required to satisfy

$$0 \geqslant d(x, x) \qquad \text{and} \qquad d(x, y) + d(y, z) \geqslant d(x, z),$$

a "classical" metric is then a ***separated*** ($d(x, y) = 0 = d(y, x)$ implies $x = y$), ***symmetric*** ($d(x, y) = d(y, x)$) and ***finitary*** ($d(x, y) < \infty$) metric. The construction $a \mapsto a \cdot e_X$ results in forgetful functors $\mathsf{Top} \to \mathsf{Ord}$ and $\mathsf{App} \to \mathsf{Met}$ respectively, both have a left adjoint defined by $(X, a_0) \mapsto (X, e_X^\circ \cdot U(a_0))$. Furthermore, one has a forgetful functor

$\mathsf{Met} \to \mathsf{Ord}$ which can be seen as the "point shadow" of $\mathsf{App} \to \mathsf{Top}$: for a metric space (X, d), define

$$x \leq y \text{ whenever } 0 \geqslant d(x, y).$$

As in the "ultrafilter case", $\mathsf{Met} \to \mathsf{Ord}$ has a left adjoint $\mathsf{Ord} \to \mathsf{Met}$ via interpreting an order relation as a numerical relation.

1.4. Remark. The left adjoint $\mathsf{Ord} \to \mathsf{Met}$ has a further left adjoint which sends the metric d on X to the order relation

$$x \leq y \text{ whenever } d(x, y) < \infty$$

on X.

Putting everything together, we have the following commuting diagram of right adjoint forgetful functors:

$$\begin{tikzcd} \mathsf{App} \arrow[r] \arrow[d] & \mathsf{Met} \arrow[d] \\ \mathsf{Top} \arrow[r] & \mathsf{Ord}. \end{tikzcd}$$

The pointwise order makes Ord an ordered category, and these forgetful functors reflect this property into Top, Met and App. Concretely, for morphisms $f, g : X \to Y$

$$\begin{aligned} &\text{in } \mathsf{Top}: && f \leq g \text{ whenever } e_X(f(x)) \to g(x) \\ &\text{in } \mathsf{Met}: && f \leq g \text{ whenever } 0 \geqslant d(f(x), g(x)) \\ &\text{in } \mathsf{App}: && f \leq g \text{ whenever } 0 \geqslant d(e_X(f(x)), g(x)) \end{aligned}$$

for all $x \in X$. We stress that it is in general very useful to realise the ordered nature of ones category since it allows to speak about adjunction, a notion which will be very helpful in our study of injectivity in Top and App.

2. Some facts about complete ordered sets

Our transportation of order-theoretic concepts into the realm of spaces relies on their respective formulation in point-free style using the notions of *module* (also called order-ideal or distributor) and *adjunction*. In this section we give a quick overview, mainly to establish notation; and refer to [Wood, 2004] for a nice presentation of "ordered sets via adjunction".

We recall that an ordered set is ***complete*** if each up-closed subset (up-set for short) has an infimum, dually, it is ***cocomplete*** if each down-set has a supremum. By definition, X is complete if and only if X^{op} is cocomplete. Moreover, X is complete if and only if X is cocomplete.

A subset $A \subseteq X$ of an ordered set X is down-closed if and only if its characteristic map is monotone of type $X^{\mathrm{op}} \to \mathsf{2}$; likewise, A is up-closed if and only if its characteristic map is monotone of type $X \to 2$. Both concepts can be brought under one roof by introducing the notion of ***module*** $\varphi : X \multimapdotboth Y$, which is defined as a relation $\varphi : X \relbar\joinrel\mapstochar\joinrel\rightarrow Y$ compatible with the order relations on X and Y in the sense that $\varphi : X^{\mathrm{op}} \times Y \to 2$ is monotone. One quickly verifies that a relation $\varphi : X \relbar\joinrel\mapstochar\joinrel\rightarrow Y$ is a module if and only if

$$\forall x, x' \in X\, \forall y, y' \in Y \,.\, ((x \le x' \,\&\, x' \,\varphi\, y' \,\&\, y' \le y) \;\Rightarrow\; x \,\varphi\, y),$$

and the pointfree version of this formula reads as $(\le_Y \cdot \varphi \cdot \le_X) \subseteq \varphi$. Since order relations are reflexive one actually has equality, moreover, this condition can be split in two parts so that $\varphi : X \relbar\joinrel\mapstochar\joinrel\rightarrow Y$ is a module if and only if

$$\varphi \cdot \le_X = \varphi \qquad\qquad \text{and} \qquad\qquad \le_Y \cdot \varphi = \varphi.$$

Summing up, a module can be seen either as

a. a relation $\varphi : X \relbar\joinrel\mapstochar\joinrel\rightarrow Y$ satisfying the two equations above, or

b. a monotone map $\varphi : X^{\mathrm{op}} \times Y \to 2$, or

c. a monotone map $\ulcorner \varphi \urcorner : Y \to 2^{X^{\mathrm{op}}}$.

Note that the equivalence between (b) and (c) relies on the fact that Ord is Cartesian closed. In general, for ordered sets X and Y, the exponential Y^X is given by the set of all monotone functions of type $X \to Y$ with the pointwise order: $h \le h'$ whenever $\forall x \in X \,.\, h(x) \le h'(x)$.

The order relation $\le$ on X is an example of a module $\le : X \multimapdotboth X$ since the transitivity axiom gives $\le \cdot \le = \le$. By definition, $\le : X \multimapdotboth X$ is the identity arrow on X in the ordered category Mod of ordered sets and modules between them, where the compositional and order structure is inherited from Rel. Two further important examples of modules are induced by a monotone map $f : X \to Y$:

$$f_* : X \multimapdotboth Y,\; x\, f_*\, y :\iff f(x) \le y \qquad \text{and} \qquad f^* : Y \multimapdotboth X,\; y\, f^*\, x :\iff y \le f(x),$$

and one has $f_* = b \cdot f$ and $f^* = f^\circ \cdot b$. One easily verifies the inequalities $\le_X \subseteq f^* \cdot f_*$ and $f_* \cdot f^* \subseteq \le_Y$ for a monotone map $f : X \to Y$, hence $f_* \dashv f^*$ in Mod. If we think of $x \in X$ as $x : 1 \to X$, then x^* is the down-set $\downarrow x$ generated by x, and x_* is the up-set $\uparrow x$ induced by x. It is also worth noting that these constructions define functors

$$(-)_* : \mathsf{Ord} \to \mathsf{Mod} \qquad\qquad \text{and} \qquad\qquad (-)^* : \mathsf{Ord}^{\mathrm{op}} \to \mathsf{Mod},$$

in particular, the order relation $\le$ in X is both $(1_X)_*$ and 1_X^*. Furthermore, $f \le g$ if and only if $f^* \le g^*$ if and only if $g_* \le f_*$, hence $(-)_*$ is order reversing and $(-)^*$ is order preserving. By this observation, $f \dashv g$ in Ord if and only if $g^* \dashv f^*$ in Mod, which in turn is equivalent to $f_* = g^*$. In pointwise notation, this reads as the familiar formula

$$\forall\, x \in X, y \in Y \,.\, f(x) \le y \iff x \le g(y).$$

Coming back to up-sets and down-sets, we identify a down-set with a module of type $X \mathrel{-\!\!\circ\!\!\to} 1$, and an up-set with a module of type $1 \mathrel{-\!\!\circ\!\!\to} X$. Hence, the ordered set of all down-sets of X can be identified with both the exponential $2^{X^{\mathrm{op}}}$ in Ord and the ordered hom-set $\mathsf{Mod}(X,1)$; and we write PX to denote this object. With the latter interpretation, the mate $\ulcorner\varphi\urcorner : Y \to PX$ of a module $\varphi : X \mathrel{-\!\!\circ\!\!\to} Y$ sends $y \in Y$ to $y^* \cdot \varphi$.

2.1. Remark. The composite $\psi \cdot \varphi$ of a down-set $\psi : X \mathrel{-\!\!\circ\!\!\to} 1$ with an up-set $\varphi : 1 \mathrel{-\!\!\circ\!\!\to} X$ yields a module of type $1 \mathrel{-\!\!\circ\!\!\to} 1$ which is either true or false; it is true precisely if φ and ψ have a common element. On the other hand, $\varphi \cdot \psi : X \mathrel{-\!\!\circ\!\!\to} X$ relates x and y if and only if x belongs to ψ and y belongs to φ; therefore $\varphi \cdot \psi \subseteq \leq$ if and only if each element of ψ is less or equal then each element of φ. From this we conclude that $\varphi \dashv \psi$ in Mod if and only if $\psi = x^*$ and $\varphi = x_*$ for some $x \in X$. Using the Axiom of Choice, we deduce that each adjunction $\varphi \dashv \psi$ in Mod with $\varphi : X \mathrel{-\!\!\circ\!\!\to} Y$ and $\psi : Y \mathrel{-\!\!\circ\!\!\to} X$ is of the form $f_* \dashv f^*$ for some $f : X \to Y$ in Ord. In fact, this statement is equivalent to the Axiom of Choice as shown in [Borceux and Dejean, 1986].

The mate of the identity module $\leq : X \mathrel{-\!\!\circ\!\!\to} X$ is the ***Yoneda embedding*** $\boldsymbol{y}_X : X \to PX$ sending $x \in X$ to its down closure $\downarrow\! x = x^*$, which is indeed fully faithful thanks to the well-known Yoneda Lemma which states

$$\downarrow\! x \subseteq \varphi \iff x \in \varphi.$$

This is a rather trivial statement in the context of ordered sets; however, the reformulation of this result is the key in the translation process from Ord to Top and App. Cocompleteness of an ordered set X gives a map $\mathrm{Sup}_X : PX \to X$ which, when writing down the definition of "Supremum", turns out to be left adjoint to $\boldsymbol{y}_X$. In fact, X is cocomplete if and only if $\boldsymbol{y}_X$ has a left adjoint. With the help of the Yoneda Lemma one easily shows that any *monotone* map $L : PX \to X$ with $L \cdot \boldsymbol{y}_X = 1_X$ is actually left adjoint to $\boldsymbol{y}_X$ (see also 2.3). Clearly, the ordered set PX of down-sets is cocomplete where the supremum of a down-set of down-sets $\Psi \in PPX$ is given by union $\bigcup \Psi$, or, in the language of modules, by $\Psi \cdot (\boldsymbol{y}_X)_* : X \mathrel{-\!\!\circ\!\!\to} 1$.

More generally, an arbitrary union of modules $X \mathrel{-\!\!\circ\!\!\to} Y$ is again a module which tells us that each hom-set in Mod is actually a (co)complete ordered set, moreover, relational composition preserves suprema. Hence, for $\varphi : X \mathrel{-\!\!\circ\!\!\to} Y$, both "composition with φ"-maps $- \cdot \varphi$ and $\varphi \cdot -$ have a right adjoint. Unwinding the definition, a right adjoint to $- \cdot \varphi$ must give, for each $\psi : X \mathrel{-\!\!\circ\!\!\to} Z$, the largest module of type $Y \mathrel{-\!\!\circ\!\!\to} Z$ whose composite with φ is contained in ψ,

$$\begin{tikzcd} X \arrow[r, "\psi", "\circ"{marking}] \arrow[d, "\varphi"', "\circ"{marking}] \arrow[dr, phantom, "\subseteq"] & Z \\ Y \arrow[ur, dotted, "\circ"{marking}] & \end{tikzcd}$$

and a right adjoint to $\varphi \cdot -$ must provide, for each $\psi : Z \mathrel{-\!\!\circ\!\!\to} Y$, the largest module of type

$Z \mathrel{-\!\!\circ\!\!\to} X$ whose composite with φ is contained in ψ.

$$\begin{tikzcd} Y & Z \arrow[l, "\psi"', "\circ" marking] \arrow[dl, dotted, "\circ" marking] \arrow[dl, phantom, "\supseteq", near start, shift right=3] \\ X \arrow[u, "\varphi", "\circ" marking] \end{tikzcd}$$

We denote the right adjoint of $- \cdot \varphi$ as $- \mathrel{\bullet\!\!-} \varphi$, and call $\psi \mathrel{\bullet\!\!-} \varphi$ the ***extension*** of ψ along φ. Similarly, $\varphi \mathrel{-\!\!\bullet} -$ denotes the right adjoint of $\varphi \cdot -$, and $\varphi \mathrel{-\!\!\bullet} \psi$ is called the ***lifting*** of ψ along φ. All what was just said about Mod could have been said earlier about Rel, indeed the operations $\mathrel{\bullet\!\!-}$ and $\mathrel{-\!\!\bullet}$ are just restrictions to modules of these operations on Rel. It is worthwhile noting that, for instance, the extension $\psi \mathrel{\bullet\!\!-} \varphi$ of ψ along φ is given by

$$y\,(\psi \mathrel{\bullet\!\!-} \varphi)\,z \iff \forall x \in X\,.\,(x\,\varphi\,y \Rightarrow x\,\psi\,z) \iff \ulcorner\varphi\urcorner(y) \leq \ulcorner\psi\urcorner(z). \tag{6}$$

2.2. Remark. A supremum of a down-set $\psi : X \mathrel{-\!\!\circ\!\!\to} 1$ is by definition a smallest upper bound. Now, as we observed in 2.1, an up-set $\varphi : 1 \mathrel{-\!\!\circ\!\!\to} X$ consists only of upper bounds of ψ if and only if $\varphi \cdot \psi \subseteq \leq$, and φ is the up-set of all upper bounds precisely if $\varphi = (\leq \mathrel{\bullet\!\!-} \psi)$. Furthermore, $x \in X$ is a smallest upper bound of ψ if and only if $x_* = (\leq \mathrel{\bullet\!\!-} \psi)$. We recall that $\leq = (1_X)_*$, hence an ordered set X is cocomplete if, for each down-set $\psi : X \mathrel{-\!\!\circ\!\!\to} 1$, the extension $(1_X)_* \mathrel{\bullet\!\!-} \psi$ of $(1_X)_*$ along ψ is equal to x_* for some $x \in X$. It is useful to observe here that a cocomplete ordered set X admits all colimits of the following type: for each monotone map $h : A \to X$ and each module $\psi : A \mathrel{-\!\!\circ\!\!\to} B$, there exists a monotone map $f : B \to X$ with $f_* = (h_* \mathrel{\bullet\!\!-} \psi)$. A diagram of the form

$$\begin{tikzcd} A \arrow[r, "h"] \arrow[d, "\psi"', "\circ" marking] & X \\ B \end{tikzcd}$$

is called ***weighted*** (by ψ), and such a monotone map f with $f_* = (h_* \mathrel{\bullet\!\!-} \psi)$ is a ***weighted colimit*** of this diagram. Furthermore, any sup-preserving map preserves also all colimits.

Every monotone map $f : X \to Y$ induces a string of adjunctions between the "down-set-sets": one has the inverse image function $PY \to PX$, $B \mapsto f^{-1}(B)$ which has a left adjoint $Pf : PX \to PY$, $A \mapsto \downarrow f(A)$ and a right adjoint $PX \to PY$, $A \mapsto \{y \in A \mid f^{-1}(\downarrow y) \subseteq A\}$. The "module point of view" allows for an elegant description of these maps using relational composition: the inverse image function is given by $\psi \mapsto \psi \cdot f_*$, its left adjoint by $\varphi \mapsto \varphi \cdot f^*$ and its right adjoint by $\varphi \mapsto \varphi \mathrel{\bullet\!\!-} f_*$.

$$\begin{tikzcd}[column sep=huge] PX \arrow[r, bend left, "(-\cdot f^*)"] \arrow[r, bend right, "(- \mathrel{\bullet\!\!-} f_*)"'] \arrow[r, phantom, shift left=3, "\perp"] \arrow[r, phantom, shift right=3, "\perp"] & PY \arrow[l, "(-\cdot f_*)" description] \end{tikzcd}$$

Note that $f_* \dashv f^*$ in Ord gives $- \cdot f^* \dashv - \cdot f_*$ in Mod. It is interesting to observe that $- \bullet\!\!- (\boldsymbol{y}_X)_*$ is just the Yoneda embedding $\boldsymbol{y}_{PX}$ of PX (use (6)), and therefore $\mathrm{Sup}_{PX} = - \cdot (\boldsymbol{y}_X)_*$.

More generally, for each module $\varphi : X \mathrel{-\!\!\!\circ\!\!\!\rightarrow} Y$ one has an adjunction $- \cdot \varphi \dashv - \bullet\!\!- \varphi$ in Ord. Since Mod is an ordered category, both $- \cdot \varphi : PY \to PX$ and $- \bullet\!\!- \varphi : PX \to PY$ are by definition monotone maps, however, later on we wish to deduce that these maps are continuous respectively contractive which does *not* follow from $\mathbb{U}$-Mod (the ultra-counterpart of Mod) being ordered. Therefore we note here that $- \cdot \varphi$ is the mate of the module $(\boldsymbol{y}_Y)_* \cdot \varphi : X \mathrel{-\!\!\!\circ\!\!\!\rightarrow} PY$, and $- \bullet\!\!- \varphi$ is the mate of $(\ulcorner \varphi \urcorner)_* : Y \mathrel{-\!\!\!\circ\!\!\!\rightarrow} PX$.

The Yoneda embedding $\boldsymbol{y}_X : X \to PX$ has an important universal property: for any monotone map $f : X \to Y$ with cocomplete codomain Y, there exists a unique sup-preserving (=left adjoint) extension $g : PX \to Y$, i.e. $g \cdot \boldsymbol{y}_X \simeq f$. Here g takes a down-set ψ to a supremum of its image in Y. In the language of modules: ψ maps to the supremum of $\psi \cdot f^*$, that is, g can be taken as the composite $\sup_Y \cdot (- \cdot f^*)$. The right adjoint of g is even easier to describe: it is simply the mate $\ulcorner f_* \urcorner : Y \to PX$ of $f_* : X \mathrel{-\!\!\!\circ\!\!\!\rightarrow} Y$. As a consequence, the (non-full) subcategory Sup of Ord consisting of all sup-lattices (=cocomplete anti-symmetric ordered sets) and sup-preserving maps is reflective in Ord, a left adjoint to the inclusion functor is given by the down-set functor $P : \mathsf{Ord} \to \mathsf{Sup}$ which sends X to PX and $f : X \to Y$ to the map $- \cdot f^* : PX \to PY$ ("direct image"). In fact, Sup is monadic over Ord, and the induced monad is given by the down-set functor $P : \mathsf{Ord} \to \mathsf{Ord}$ with units the Yoneda embeddings $\boldsymbol{y}_X : X \to PX$ and multiplications $\boldsymbol{m}_X : PPX \to PX$, $\Psi \mapsto \Psi \cdot (\boldsymbol{y}_X)_*$ ("union"). Its restriction to discrete ordered sets gives the usual power-set monad on Set which has the category Sup as Eilenberg-Moore category too.

2.3. Remark. The down-set monad $\mathbb{P}$ on Ord has a very particular property: $P\boldsymbol{y}_X \leq \boldsymbol{y}_{PX}$ for all ordered sets X. This seemingly harmless property turns out to be very powerful, it implies for instance that $h : PX \to X$ in Ord is the structure morphism of a $\mathbb{P}$-algebra if and only if $h \cdot \boldsymbol{y}_X = 1_X$, moreover, such a map h is necessarily left adjoint to $\boldsymbol{y}_X$. These kinds of monads where introduced independently by A. Kock (in his thesis, but see [Kock, 1995]) and [Zöberlein, 1976], hence one refers to them as of ***Kock-Zöberlein type***. From their results one can extract the following

2.4. Theorem. *Let $\mathbb{T} = (T, e, m)$ be a monad on a ordered category X where T is a 2-functor. Furthermore, assume that* $\hom(Y, TX)$ *is separated, for all objects X, Y in X. Then the following assertions are equivalent.*

i. $Te_X \leq e_{TX}$ *for all* $X \in \mathsf{X}$.

ii. *For all $X \in \mathsf{X}$, a X-morphism $h : TX \to X$ is the structure morphism of a $\mathbb{T}$-algebra if and only if $h \cdot e_X = 1_X$ (and then $h \dashv e_X$).*

iii. $m_X \dashv e_{TX}$ *for all* $X \in \mathsf{X}$.

iv. $Te_X \dashv m_X$ *for all* $X \in \mathsf{X}$.

It is also well-known that the category $\mathsf{Ord}_{\mathrm{sep}}$ of separated ordered sets and monotone maps is dually equivalent to the category TAL of totally algebraic lattices (defined below) and sup- and inf-preserving maps. We refer to [Rosebrugh and Wood, 1994] for a nice presentation of this particular result, and to [Porst and Tholen, 1991] for a nice presentation of duality theory in general. This duality can be obtained by first constructing an adjunction

$$D \dashv S,\ D : \mathsf{Ord} \to \mathsf{CCD}^{\mathrm{op}},\ S : \mathsf{CCD}^{\mathrm{op}} \to \mathsf{Ord}$$

between Ord and the dual of the category CCD of ***(constructively) completely distributive lattices*** and sup- and inf-preserving maps. We recall from [Fawcett and Wood, 1990] that a complete lattice X is (ccd) if $\mathrm{Sup}_X : PX \to X$ has a left adjoint $t_X : X \to PX$. Note that t_X corresponds to a module of type $X \multimap X$, and this relation is precisely the totally-below relation $\lll$ studied first in [Raney, 1952]. Clearly, any lattice of the form PX is (ccd) since one has the string of adjunctions

$$y_{PX} = - \multimap (y_X)_* \vdash - \cdot (y_X)_* \vdash - \cdot (y_X)^* = Py_X\,.$$

The functor $D : \mathsf{Ord} \to \mathsf{CCD}^{\mathrm{op}}$ sends an ordered set X to $DX := PX = 2^{X^{\mathrm{op}}}$ and a monotone map $f : X \to Y$ to $Df := (- \cdot f_*) : DY \to DX$ (inverse image function). For $L \in \mathsf{CCD}$ with $y_L \vdash \mathrm{Sup}_L \vdash t_L$, one defines $SL := A$ where A is the equaliser

$$A \xrightarrow{\ i\ } L \underset{y_L}{\overset{t_L}{\rightrightarrows}} PL.$$

Hence, A can be taken as $\{x \in L \mid x \lll x\}$, that is, A consists precisely of the ***totally compact*** elements of L. Given also $M \in \mathsf{CCD}$ with corresponding equaliser $SM := B \to M$ and a sup- and inf-preserving map $f : L \to M$, then its left adjoint $g : M \to L$ restricts to $g_0 : B \to A$. With $Sf := g_0$ one obtains a functor $S : \mathsf{CCD}^{\mathrm{op}} \to \mathsf{Ord}$. Note that we need here anti-symmetry of (ccd)-lattices, otherwise S is only a pseudo-functor. By the Yoneda Lemma, $y_X : X \to PX$ is fully faithful and its image is precisely the equaliser of Py_X and y_{PX}. Hence,

$$X \xrightarrow{\ y_X\ } PX \underset{y_{PX}}{\overset{Py_X}{\rightrightarrows}} PPX$$

is an equaliser diagram for each anti-symmetric ordered set X. From that we get a natural equivalence $\eta : 1 \to SD$ which is a natural isomorphism if we restrict η to anti-symmetric ordered sets. For $L \in \mathsf{CCD}$, one defines $\varepsilon_L : L \to DS(L)$ as the composite (of right adjoints) $L \xrightarrow{y_L} PL \xrightarrow{-\cdot i_*} PA$, where $i : A \hookrightarrow L$ is the inclusion map. Clearly, ε_L preserves infima, and it is not difficult to verify that ε_L preserves also suprema. Therefore $\varepsilon_L : L \to DS(L)$ lives in CCD and is indeed the L-component of a natural transformation $\varepsilon : 1 \to DS$. The necessary equations are now easily verified, therefore one obtains the desired dual adjunction. We will now determine the fixed subcategories. There is nothing left to do on the Ord-side, we observed already that $\mathrm{Fix}(\eta) = \mathsf{Ord}_{\mathrm{sep}}$. Therefore we concentrate now on $L \in \mathsf{CCD}$. The left adjoint $c : PA \to L$ of $\varepsilon_L : L \to PA$ (where

$A = SL$) sends $\psi \in PA$ to $\mathrm{Sup}_L(\psi \cdot i^*)$ (where $i : A \hookrightarrow L$ is the inclusion map). In fact, one always has $\varepsilon_L \cdot c = 1$, hence ε_L is an equivalence if $c \cdot \varepsilon_L \geq 1$, that is, every $x \in L$ is a supremum of the totally compact elements below x. A (ccd)-lattice with this property is called ***totally algebraic***, and we obtain $\mathsf{Ord}_{\mathrm{sep}} \simeq \mathsf{TAL}^{\mathrm{op}}$ where TAL denotes the full subcategory of CCD defined by the totally algebraic lattices.

2.5. Remark. Firstly, instead of $X \mapsto \mathsf{2}^{X^{\mathrm{op}}}$ one can also work with $X \mapsto \mathsf{2}^X$, and construct the dual adjunction above as

$$\begin{tikzcd} \mathsf{Ord} \arrow[r, shift left, "\hom(-{,}\mathsf{2})"] \arrow[r, phantom, "\perp"] & \mathsf{CCD}^{\mathrm{op}}. \arrow[l, shift left, "\hom(-{,}\mathsf{2})"] \end{tikzcd}$$

In fact, one construction can be obtain from the other by composing it with the equivalence $(-)^{\mathrm{op}} : \mathsf{Ord} \to \mathsf{Ord}$.

2.6. Remark. Secondly, as explained in [Rosebrugh and Wood, 1994], the duality $\mathsf{Ord}_{\mathrm{sep}} \simeq \mathsf{TAL}^{\mathrm{op}}$ is the restriction of a "bigger" duality involving the category $\mathsf{CCD}_{\mathrm{sup}}$ of (ccd)-lattices and sup-preserving maps on one side and the idempotent split completion $\mathrm{kar}(\mathsf{Rel})$ of Rel on the other side. This result is then further generalised in [Rosebrugh and Wood, 2004]. We come back to this in Section 8.

3. A short visit to metric spaces

The discussion of the previous section can be easily brought to metric spaces by considering numerical relations, which amounts to substituting $\mathsf{2}$ by $[0,\infty]$, & by $+$, true by 0, $x \Rightarrow y$ sometimes by $x \geqslant y$ and sometimes by $y \ominus x$ (truncated minus)[1], $\exists$ by inf, $\forall$ by sup, and so on. Most notably, we will usually not consider the Cartesian structure (=max-metric) on $X \times Y$ but rather the $+$-metric, and denote the resulting space as $X \otimes Y$. This comes with the advantage that, although Met is not Cartesian closed, it is ***monoidal closed*** in the sense that $X \otimes -$ has a right adjoint $-^X$. Here Y^X can be taken as the set of all contraction maps of type $X \to Y$ together with the sup-metric $d(h,k) = \sup_{x \in X} b(h(x), h'(x))$. We are especially interested in $PX := [0,\infty]^{X^{\mathrm{op}}}$, where the distance on $[0,\infty]$ is given by $\delta(x,y) = y \ominus x$, and consequently on PX by $[\varphi,\psi] = \sup_{x \in X}(\psi(x) \ominus \varphi(x))$. One should compare this with the order case where the truth value of $[\varphi \subseteq \psi]$ is given by $\forall x \in X \,.\, \varphi(x) \Rightarrow \psi(x)$. A ***module*** $\varphi : X \,{\circ\!\!\!\rightarrow}\, Y$ between metric spaces $X = (X,a)$ and $Y = (Y,b)$ can be seen as either

a. a numerical relation $\varphi : X \,{-\!\!\!\!\!\!\!\rightarrow}\, Y$ satisfying $\varphi \cdot a = \varphi$ and $b \cdot \varphi = \varphi$, or

b. a contraction map $\varphi : X^{\mathrm{op}} \otimes Y \to [0,\infty]$, or

c. a contraction map $\ulcorner \varphi \urcorner : Y \to PX$.

[1]Since $\Rightarrow$ sometimes denotes the right adjoint to & ($x\&- \dashv x \Rightarrow -$), and sometimes is used to express the inclusion $r \subseteq r'$ of relations pointwise.

As before,

- each contraction map $f : X \to Y$ induces modules $f_* : X \relbar\!\!\!\!\circ\!\!\!\to Y$, $f_*(x,y) = b(f(x),y)$ and $f^* : Y \relbar\!\!\!\!\circ\!\!\!\to X$, $f^*(y,x) = b(y,f(x))$ with $f_* \dashv f^*$,
- the metric a of $X = (X,a)$ is the identity module $X \relbar\!\!\!\!\circ\!\!\!\to X$ on X,
- which induces the Yoneda embedding $\mathpzc{y}_X : X \to PX$ sending x to x^*,
- the Yoneda Lemma states now that $[\mathpzc{y}_X(x), \psi] = \psi(x)$,
- a metric space is cocomplete whenever $\mathpzc{y}_X$ has a left adjoint $\mathrm{Sup}_X : PX \to X$,
- the cocomplete metric spaces are precisely the injective ones (where a metric space X is injective whenever, for every fully faithful $i : A \to B$ in Met and every contraction map $f : A \to X$, there is a contraction map $g : B \to X$ so that $g \cdot i \simeq f$),
- the subcategory $\mathsf{Cocts}_{\mathrm{sep}}$ of cocomplete and separated metric spaces and suprema preserving contraction maps is reflective (in fact, monadic) in Met, and the Yoneda embedding $\mathpzc{y}_X : X \to PX$ serves as a reflection map.

An immediate question is now how the important notion of *Cauchy-completeness* fits into this framework. The answer can be found in F.W. Lawvere's 1973 paper [Lawvere, 1973] where he made the amazing discovery that equivalence classes of Cauchy sequences correspond precisely to right adjoint modules $\psi : X \relbar\!\!\!\!\circ\!\!\!\to 1$, and a Cauchy sequence converges to x if and only if x is a supremum of the corresponding module. Consequently, X is Cauchy complete if and only if the restriction $\mathpzc{y}_X : X \to \tilde{X}$ of the Yoneda embedding to the subspace $\tilde{X}$ of PX defined by all right adjoint modules has a left adjoint in Met. Since $\mathpzc{y}_X : X \to \tilde{X}$ is dense (in the usual metric sense), this simply means that $\mathpzc{y}_X : X \to \tilde{X}$ is surjective. Furthermore, $\mathpzc{y}_X : X \to \tilde{X}$ is a Cauchy completion for any space X. It is also worth noting that $\tilde{X} \hookrightarrow PX$ is the equaliser of

$$PX \underset{\mathpzc{y}_{PX}}{\overset{P\mathpzc{y}_X}{\rightrightarrows}} PPX \quad \text{(see also Lemma 6.5).}$$

As for ordered sets, one can built a dual adjunction between Met and CDMet, which restricts to a dual equivalence between the full subcategories of Cauchy complete metric spaces and algebraic metric spaces. The reader has certainly no difficulties in writing down the definitions of completely distributive metric space and consequently of the category CDMet as well as of algebraic metric space.

3.1. Remark. Since Met is not Cartesian closed one might wonder what the exponentiable objects are. They are characterised in [Clementino and Hofmann, 2006] (see also [Clementino *et al.*, 2009]) as those spaces $X = (X,a)$ where, for all $x, y \in X$, $u+v = a(x,y)$ and $\varepsilon > 0$, there exists some $z \in X$ with $a(x,z) \leq u + \varepsilon$ and $a(z,y) \leq v + \varepsilon$. One easily sees that a cocomplete (=injective) metric space satisfies this property, just consider (with $w = a(x, y)$)

$$\begin{tikzcd} \{0 \xrightarrow{w} 2\} \arrow[r, tail] \arrow[dr, "f"'] & \{0 \xrightarrow{u} 1 \xrightarrow{v} 2\} \arrow[d, dotted, "g"] \\ & X \end{tikzcd}$$

where $f(0) = x$, $f(2) = y$ and $g(1)$ gives the desired $z \in X$. Furthermore, with Y is also Y^X cocomplete (=injective), to see this just pass from

$$\begin{tikzcd} A \arrow[r, tail] \arrow[dr] & B \\ & Y^X \end{tikzcd} \qquad \text{to} \qquad \begin{tikzcd} X \times A \arrow[r, tail] \arrow[dr] & X \times B \\ & Y. \end{tikzcd}$$

Since the product of cocomplete spaces is also cocomplete, we conclude that the *full* subcategory of Met defined by all cocomplete spaces is Cartesian closed. This observation contradicts Theorem 2.2 of [Wagner, 1994] stating that the largest Cartesian closed full subcategory of Met is the category of ultrametric spaces; however, the proof given there seems to me incorrect. In fact, in the proof of this theorem the author assumes not only that $X \times -$ has a right adjoint but also that this right adjoint coincides with the right adjoint of $X \otimes -$.

4. The dual space

In the remaining sections we will go further and lift notions and results (such as dual category, module, presheaf-construction and the Yoneda Lemma) from the theory of enriched categories to topological and approach spaces. The first obstacle waits right at the beginning as the fundamental notion of down-set $\psi : X^{\mathrm{op}} \to \mathsf{2}$ involves the dual ordered set, a concept which has no obvious counterpart in Top and App. Clearly, one cannot directly dualise the convergence relation $\mathfrak{x} \to x$ of a topological space to "$x \to \mathfrak{x}$", it is necessary to move into a more symmetric environment. Our experience shows so far that a good candidate for such an environment is L. Nachbin's notion of ordered compact Hausdorff space (see [Nachbin, 1950]) as well as its metric counterpart. In fact, in these space we can dualise the order respectively metric, and then return to topological respectively approach spaces through an adjunction.

4.1. DEFINITION. *An* **ordered compact Hausdorff space** *is a triple* $(X, \leq, \alpha)$ *where* $(X, \leq)$ *is an ordered set and* α *is (the convergence relation of) a compact Hausdorff topology on* X *so that* $\{(x, y) \mid x \leq y\}$ *is closed in* $X \times X$.

In the sequel we write $\mathsf{OrdCompHaus}$ for the category of ordered compact Hausdorff spaces and maps preserving both the order and the topology. We emphasise again that we do not assume the order relation to be anti-symmetric. It is shown in [Flagg, 1997] that the full subcategory $\mathsf{OrdCompHaus}_{\mathrm{sep}}$ of $\mathsf{OrdCompHaus}$ defined by the objects with anti-symmetric order is the category of Eilenberg-Moore algebras for the prime filter monad (of up-sets) $\mathbb{B}$ on Ord, and the "non-separated" version of this result can be found in [Tholen, 2009] with the prime filter monad substituted by the ultrafilter monad. In fact, based on its extension to Rel, the ultrafilter monad $\mathbb{U} = (U, e, m)$ on Set extends to a monad on Ord where $U : \mathsf{Ord} \to \mathsf{Ord}$ sends $(X, \leq)$ to $(UX, U\leq)$, and with this definition e_X and m_X are monotone maps. Then, by Remark 1.1, $\{(x, y) \mid x \leq y\}$ is closed in $X \times X$ if and only if $\alpha : U(X, \leq) \to (X, \leq)$ is monotone. Therefore the category $\mathsf{OrdCompHaus}$ of ordered compact Hausdorff spaces and continuous monotone maps is precisely the Eilenberg-Moore category $\mathsf{Ord}^{\mathbb{U}}$. For each ordered set X there is a canonical map $\rho_X : UX \twoheadrightarrow BX$, $\mathfrak{x} \mapsto \{\uparrow A \mid A \in \mathfrak{x}\}$ which turns out to be the X-component of a monad morphism $\rho : \mathbb{U} \to \mathbb{B}$. It is shown in [Flagg, 1997, Lemma 5] that ρ_X is even surjective, and one easily verifies that $\rho_X(\mathfrak{x}) \leq \rho_X(\mathfrak{x}') \iff \mathfrak{x} \leq \mathfrak{x}'$. Hence, $\rho_X : UX \twoheadrightarrow BX$ is the anti-symmetric reflection of UX, and composition with ρ induces the inclusion functor $\mathsf{OrdCompHaus}_{\mathrm{sep}} \to \mathsf{OrdCompHaus}$.

As a byproduct of this discussion we obtain a notion of ***metric compact Hausdorff spaces*** as the Eilenberg-Moore algebras for the extension of $\mathbb{U}$ to Met based on its extension to numerical relations, that is, we define $\mathsf{MetCompHaus} := \mathsf{Met}^{\mathbb{U}}$.

4.2. Remark. The functor U does not restrict to an endofunctor on $\mathsf{Ord}_{\mathrm{sep}}$ respectively $\mathsf{Met}_{\mathrm{sep}}$. For instance, the order relation of $U\mathbb{N}$ is not anti-symmetric, where $\mathbb{N}$ has the natural order. To see this, just take $\mathfrak{x} \in UX$ such that each $A \in \mathfrak{x}$ contains arbitrary large odd numbers, and $\mathfrak{y} \in UX$ such that each $B \in \mathfrak{y}$ contains arbitrary large even numbers. Then $\mathfrak{x} \leq \mathfrak{y}$ and $\mathfrak{y} \leq \mathfrak{x}$, but $\mathfrak{x}$ can be chosen different from $\mathfrak{y}$.

There are canonical forgetful functors

$$K : \mathsf{OrdCompHaus} \to \mathsf{Top} \qquad \text{and} \qquad K : \mathsf{MetCompHaus} \to \mathsf{App},$$

both send (X, a_0, α) to $(X, a_0 \cdot \alpha)$ where a_0 is either an order relation or a metric.

4.3. Examples. The ordered set $\mathsf{2} = \{0, 1\}$ with the discrete (compact Hausdorff) topology lives in $\mathsf{OrdCompHaus}$ and gives us the Sierpiński space $\mathsf{2}$ where $\{1\}$ is closed and $\{0\}$ is open. The metric space $[0, \infty]$ with distance $\delta(x, y) = y \ominus x$ equipped with the usual compact Hausdorff topology where $\mathfrak{x}$ converges to $\xi(\mathfrak{x}) := \sup_{A \in \mathfrak{x}} \inf A$ is a metric compact Hausdorff space which gives the usual approach structure $\lambda(\mathfrak{x}, x) = x \ominus \xi(\mathfrak{x})$ on $[0, \infty]$ (see [Lowen, 1997]).

One easily verifies that K has a left adjoint

$$M : \mathsf{Top} \to \mathsf{OrdCompHaus} \qquad \text{respectively} \qquad M : \mathsf{App} \to \mathsf{MetCompHaus}$$

which sends $X = (X, a)$ to $(UX, Ua \cdot m_X^\circ, m_X)$.

4.4. Examples. For a topological space $X = (X, a)$, the order relation

$$UX \xrightarrow{m_X^\circ} UUX \xrightarrow{Ua} UX$$

is described by

$$\mathfrak{x} \leq \mathfrak{y} \qquad \text{whenever } \overline{A} \in \mathfrak{y} \text{ for every } A \in \mathfrak{x}.$$

Hence, $\mathfrak{x} \leq \mathfrak{y}$ if and only if every closed set in $\mathfrak{x}$ belongs to $\mathfrak{y}$ if and only if every open set in $\mathfrak{y}$ belongs to $\mathfrak{x}$. For an approach space $X = (X, a)$, the metric $UX \xrightarrow{m_X^\circ} UUX \xrightarrow{Ua} UX$ gives

$$\inf\{u \in [0,\infty] \mid \forall A \in \mathfrak{x}\,.\, A^{(u)} \in \mathfrak{y}\}$$

as distance from $\mathfrak{x}$ to $\mathfrak{y}$ (where $A^{(u)} = \{x \in X \mid \delta(A,x) \leqslant u\}$ and $\delta(A,x) = \inf\{a(\mathfrak{x},x) \mid A \in \mathfrak{x}\}$). To see this, with $\hat{a} = Ua \cdot m_X^\circ$, let

$$v := \hat{a}(\mathfrak{x},\mathfrak{x}') = \inf\{Ua(\mathfrak{X},\mathfrak{x}') \mid \mathfrak{X} \in UUX, m_X(\mathfrak{X}) = \mathfrak{x}\}$$

and

$$w := \inf\{u \in [0,\infty] \mid \forall A \in \mathfrak{x}\,.\, A^{(u)} \in \mathfrak{x}'\}.$$

Since

$$v \geqslant \sup_{A\in\mathfrak{x}} \sup_{B\in\mathfrak{x}'} \inf_{\mathfrak{a}\in A^{\#}} \inf_{y\in B} a(\mathfrak{a},y),$$

for every $\varepsilon > 0$, $A \in \mathfrak{x}$, and $B \in \mathfrak{x}'$, there exist $\mathfrak{a} \in A^{\#}$ and $y \in B$ with $a(\mathfrak{a},y) \leqslant v+\varepsilon$; hence, $\delta(A,y) \leqslant v+\varepsilon$. Therefore, $A^{(v+\varepsilon)} \cap B \neq \varnothing$, and we conclude that $A^{(v+\varepsilon)} \in \mathfrak{x}'$ for every $A \in \mathfrak{x}$ and $\varepsilon > 0$. This proves $w \leqslant v$. For the reverse inequality, note that $A^{(w+\varepsilon)} \cap B \neq \varnothing$, for every $\varepsilon > 0$, $A \in \mathfrak{x}$ and $B \in \mathfrak{x}'$; this implies that

$$\sup_{A\in\mathfrak{x}} \sup_{B\in\mathfrak{x}'} \inf_{\mathfrak{a}\in A^{\#}} \inf_{y\in B} a(\mathfrak{a},y) \leqslant w+\varepsilon.$$

Hence, by [Hofmann, 2006, Lemma 1.5] there is some $\mathfrak{X} \in UUX$ with

$$\{A^{\#} \mid A \in \mathfrak{x}\} \qquad \text{and} \qquad Ua(\mathfrak{X},\mathfrak{x}') \leqslant w+\varepsilon,$$

so that $v \leqslant w$.

The functors $(-)^{\mathrm{op}} : \mathsf{Ord} \to \mathsf{Ord}$ and $(-)^{\mathrm{op}} : \mathsf{Met} \to \mathsf{Met}$ lift to functors $(-)^{\mathrm{op}} : \mathsf{OrdCompHaus} \to \mathsf{OrdCompHaus}$ and $(-)^{\mathrm{op}} : \mathsf{MetCompHaus} \to \mathsf{MetCompHaus}$ by putting $(X, a_0, \alpha)^{\mathrm{op}} = (X, a_0^\circ, \alpha)$. Using the adjunction $M \dashv K$, we can now define the dual of a topological space and an approach space.

4.5. Definition. *The functors* $(-)^{\mathrm{op}} : \mathsf{Top} \to \mathsf{Top}$ *and* $(-)^{\mathrm{op}} : \mathsf{App} \to \mathsf{App}$ *are defined as the composites*

$$\begin{tikzcd} \mathsf{Top} \arrow[r, dotted, "(-)^{\mathrm{op}}"] \arrow[d, "M"'] & \mathsf{Top} \\ \mathsf{OrdCompHaus} \arrow[r, "(-)^{\mathrm{op}}"'] & \mathsf{OrdCompHaus} \arrow[u, "K"'] \end{tikzcd} \qquad \textit{and} \qquad \begin{tikzcd} \mathsf{App} \arrow[r, dotted, "(-)^{\mathrm{op}}"] \arrow[d, "M"'] & \mathsf{App} \\ \mathsf{MetCompHaus} \arrow[r, "(-)^{\mathrm{op}}"'] & \mathsf{MetCompHaus}. \arrow[u, "K"'] \end{tikzcd}$$

4.6. Examples. By definition, an ultrafilter $\mathfrak{X} \in UUX$ of ultrafilters converges to $\mathfrak{x} \in UX$ in X^{op} whenever $\mathfrak{x} \leq m_X(\mathfrak{X})$, which is equivalent to $A^{\#} \in \mathfrak{X}$ for each closed set $A \in \mathfrak{x}$. From this one obtains that all sets $A^{\#}$ for $A \subseteq X$ closed form a basis for the topology on X^{op}. In this sense, we dualise X by making the closed subsets of X open. A continuous map $\psi : X^{\mathrm{op}} \to 2$ can be identified with a closed subset $\mathcal{A} \subseteq UX$, where $\mathcal{A} \subseteq UX$ is closed if and only if $\mathcal{A}$ is Zariski closed (i.e. closed for the compact Hausdorff topology m_X on UX) and down-closed (with respect to the order $\leq$ on UX).

It is well-known that both Top and App are not Cartesian closed. However, as we shall see, the opposite topological space X^{op} always is exponentiable; that is, the functor $X^{\mathrm{op}} \times - : \mathsf{Top} \to \mathsf{Top}$ has a right adjoint. Similar to the metric case, for approach spaces we are interested in the $+$-approach structure rather then the max-structure on the product space. More in detail, for approach spaces $X = (X, a)$ and $Y = (Y, b)$, we put $X \otimes Y = (X, \times Y, d)$ where $d(\mathfrak{w}, (x, y)) = a(T\pi_1(\mathfrak{w}), x) + b(T\pi_2(\mathfrak{w}), y)$ (see [Hofmann, 2007]). Then an approach space X is called ***+-exponentiable*** whenever $X \otimes - : \mathsf{App} \to \mathsf{App}$ has a right adjoint. We recall from [Pisani, 1999] and [Hofmann, 2007] that a topological/approach space $X = (X, a)$ is exponentiable/$+$-exponentiable if and only if the diagram

$$\begin{tikzcd} UUX \arrow[r, "m_X"] \arrow[d, "Ua"', "\shortmid" marking] & UX \arrow[d, "a", "\shortmid" marking] \\ UX \arrow[r, "a"', "\shortmid" marking] & X \end{tikzcd}$$

commutes.

4.7. Proposition. *For each ordered compact Hausdorff space X, KX is exponentiable in* Top*. Likewise, for each metric compact Hausdorff space, KX is $+$-exponentiable in* App*.*

Proof. Let $X = (X, a_0, \alpha)$ be in $\mathsf{OrdCompHaus}$ or $\mathsf{MetCompHaus}$. We have to show that $a := a_0 \cdot \alpha$ satisfies $a \cdot Ua \sqsupseteq a \cdot m_X$ (since the other inequality holds anyway), where $\sqsubseteq$ stands either for $\subseteq$ or $\geqslant$. But this follows easily:

$$a \cdot Ua = a_0 \cdot \alpha \cdot U(a_0) \cdot U\alpha \sqsupseteq a_0 \cdot \alpha \cdot U\alpha = a_0 \cdot \alpha \cdot m_X = a \cdot m_X.$$

∎

4.8. Corollary. *For each topological (approach) space X, X^{op} is (+-)exponentiable.*

4.9. Remark. Clearly, both $\mathsf{Ord}^{\mathbb{U}}$ and $\mathsf{Met}^{\mathbb{U}}$ inherit products from Ord and Met respectively. However, more important to us is the monoidal structure on Met defined by the plus-metric, and therefore we are interested in transporting this structure to $\mathsf{Met}^{\mathbb{U}}$. This problem is addressed in general in [Moerdijk, 2002] where the author introduces the notion of a ***Hopf monad*** on a monoidal category C, which captures exactly what is needed to transport the monoidal structure on C to the category of Eilenberg–Moore algebras. For space reasons we must refer to [Moerdijk, 2002] for the definition of Hopf monad, and simply state here that the monad $\mathbb{U} = (U, e, m)$ on Met is an example of a monad with a Hopf structure since

$$\tau_{X,Y} : U(X \otimes Y) \to UX \otimes UY,\ \mathfrak{w} \mapsto (T\pi_1(\mathfrak{w}), T\pi_2(\mathfrak{w})) \qquad\qquad ! : U1 \to 1$$

are contraction maps. This is clear for the second map, and for the first one it follows using Remark 1.1. Consequently, $\mathsf{Met}^{\mathbb{U}}$ inherits the monoidal structure from Met: for $X = (X, a, \alpha)$ and $Y = (Y, b, \beta)$, $X \otimes Y$ becomes equipped with the plus-metric $a \otimes b$ and the product topology $U(X \times Y) \xrightarrow{\tau_{X,Y}} UX \times UY \xrightarrow{\alpha\times\beta} X \times Y$. Recall from Example 4.3 that $[0, \infty]$ lives in $\mathsf{Met}^{\mathbb{U}}$, and it is now clear that $+ : [0, \infty] \otimes [0, \infty] \to [0, \infty]$ is a $\mathbb{U}$-homomorphism. We also remark that $K : \mathsf{Met}^{\mathbb{U}} \to \mathsf{App}$ is a strict monoidal functor.

In [Simmons, 1982; Wyler, 1984] it is shown that $\mathsf{OrdCompHaus}_{\mathrm{sep}}$ is also monadic over Top where the monad is the prime filter (of opens) monad. Similarly, the adjunction $M \dashv K$ induces a monad on Top respectively App, in fact, it lifts the ultrafilter monad $\mathbb{U} = (U, e, m)$ to these categories. One easily verifies that the ultrafilter monad $\mathbb{U}$ on App is a Hopf monad witnessed by the maps $\tau_{X,Y}$ and ! described above.

4.10. Proposition. *The ultrafilter monad $\mathbb{U} = (U, e, m)$ on Top is of Kock-Zöberlein type.*

Proof. First note that an ultrafilter $\mathfrak{X} \in UUX$ converges to $\mathfrak{x} \in UX$ in UX if and only if $m_X(\mathfrak{X}) \le \mathfrak{x}$, and this is equivalent to $A^\# \in \mathfrak{X}$ for all open subsets $A \subseteq X$ with $A \in \mathfrak{x}$. Therefore all sets $A^\#$ where A is open in X form a basis for the topology in UX. We now show $m_X \dashv e_{UX}$ (see Remark 2.3). To see $1_{UUX} \le e_{UX} \cdot m_X$, let $\mathfrak{X} \in UUX$ and $\mathcal{A} \in e_{UX} \cdot m_X(\mathfrak{X})$ be open. Hence $m_X(\mathfrak{X}) \in \mathcal{A}$, and there is some open subset $A \subseteq X$ with $A \in m_X(\mathfrak{X})$ and $A^\# \subseteq \mathcal{A}$. Consequently, $A^\# \in \mathfrak{X}$ and therefore also $\mathcal{A} \in \mathfrak{X}$. Since $m_X \cdot e_{UX} = 1_{UX}$, we conclude $m_X \dashv e_{UX}$. ■

For an approach space $X = (X, a)$ with underlying topological space X_t and $\mathfrak{x}, \mathfrak{y} \in UX$, from Example 4.4 we obtain that $\mathfrak{x} \le \mathfrak{y}$ in $U(X_t)$ implies $0 = \hat{a}(\mathfrak{x}, \mathfrak{y})$ (where $\hat{a} = Ua \cdot m_X$), and therefore:

4.11. Corollary. *The ultrafilter monad $\mathbb{U} = (U, e, m)$ on App is of Kock-Zöberlein type.*

Hence, the algebra structure $l : UX \to X$ is left adjoint to $e_X : X \to UX$ in Top respectively App. Moreover, any left inverse $l : UX \to X$ of $e_X : X \to UX$ (that is, $l \cdot e_X = 1_X$) in $\mathsf{Top}/\mathsf{App}$ is left adjoint to e_X and makes X a pseudoalgebra for $\mathbb{U}$. In particular, a topological/approach T_0-space is an Eilenberg-Moore algebra for $\mathbb{U}$ precisely if $e_X : X \to UX$ admits a left inverse.

4.12. Proposition. $\mathsf{Top}^{\mathbb{U}} \simeq \mathsf{Ord}^{\mathbb{U}}$ *and* $\mathsf{App}^{\mathbb{U}} \simeq \mathsf{Met}^{\mathbb{U}}$.

PROOF. For $X = (X, a_0, \alpha)$ in $\mathsf{OrdCompHaus}$ or $\mathsf{MetCompHaus}$, $\alpha : UX \to X$ turns out to be continuous respectively contractive, hence the functors $K : \mathsf{Ord}^{\mathbb{U}} \to \mathsf{Top}$ respectively $K : \mathsf{Met}^{\mathbb{U}} \to \mathsf{App}$ can be seen a functor $\mathsf{Ord}^{\mathbb{U}} \to \mathsf{Top}^{\mathbb{U}}$ respectively $\mathsf{Met}^{\mathbb{U}} \to \mathsf{App}^{\mathbb{U}}$. On the other hand, for $X = (X, a)$ in $\mathsf{Top}^{\mathbb{U}}$ or $\mathsf{App}^{\mathbb{U}}$, the underlying ordered set (X, a_0) together with the algebra structure l_X lives in $\mathsf{Ord}^{\mathbb{U}}/\mathsf{Met}^{\mathbb{U}}$. Furthermore, $l_X \dashv e_X$ in Top respectively App and consequently in Ord respectively in Met, and one observes that the underlying order/metric of UX is given by $\hat{a} = Ua \cdot m_X^\circ$. From

$$a_0(l_X(\mathfrak{x}), x) = \hat{a}(\mathfrak{x}, e_X(x)) = a(\mathfrak{x}, x)$$

one reaches eventually to the conclusion that $\mathsf{Top}^{\mathbb{U}} \simeq \mathsf{Ord}^{\mathbb{U}}$ and $\mathsf{App}^{\mathbb{U}} \simeq \mathsf{Met}^{\mathbb{U}}$. ∎

5. Cocomplete spaces

With the notion of dual space at our disposal, one can now define $\mathbb{U}$-*modules* between topological spaces and approach spaces and develop their basic properties following closely what was done for ordered sets. For topological spaces $X = (X, a)$ and $Y = (Y, b)$, a $\mathbb{U}$-module $\varphi : X \,{-\!\!\circ\!\!\rightharpoonup}\, Y$ is a $\mathbb{U}$-relation $\varphi : X \,{-\!\!\!\rightharpoonup}\, Y$ so that $X^{\mathrm{op}} \times Y \to 2$ is continuous; and for approach spaces $X = (X, a)$ and $Y = (Y, b)$, a $\mathbb{U}$-module $\varphi : X \,{-\!\!\circ\!\!\rightharpoonup}\, Y$ is a $\mathbb{U}$-relation $\varphi : X \,{-\!\!\!\rightharpoonup}\, Y$ so that $X^{\mathrm{op}} \otimes Y \to [0, \infty]$ is contractive. By Corollary 4.8, $\mathbb{U}$-modules correspond to continuous/contractive maps $\ulcorner \varphi \urcorner : Y \to PX$, where $PX := 2^{X^{\mathrm{op}}}$ in the topological case and $PX := [0, \infty]^{X^{\mathrm{op}}}$ in the approach case. It is not completely trivial that the module-property can be also expressed with the help of Kleisli composition, but it is shown in [Clementino and Hofmann, 2009a] that a $\mathbb{U}$-relation $\varphi : X \,{-\!\!\!\rightharpoonup}\, Y$ is a $\mathbb{U}$-module if and only if $b \circ \varphi = \varphi$ and $\varphi \circ a = \varphi$. This correspondence will be particularly useful when establishing cont(inuity/ractivity) of a map of type $Y \to PX$ as it is occasionally easier to verify these two equalities.

5.1. REMARK. It should be noted that the dual space introduced in this paper is different from what was considered in [Clementino and Hofmann, 2009a; Hofmann and Tholen, 2010; Hofmann, 2011; Clementino and Hofmann, 2009b], the two ingredients of an ordered/metric compact Hausdorff space were kept separately there. Since the presheaf space PX there is defined as a subspace of the exponential with respect to the compact Hausdorff topology only, it is not automatically clear that this gives the same space. The following result tells us that there is no problem.

5.2. LEMMA. *For any (X, a_0, α) in* $\mathsf{OrdCompHaus}$ *or* $\mathsf{MetCompHaus}$ *and any Y in* Top *respectively* App*, the exponential $Y^{(X, a_0 \cdot \alpha)} \to Y^{(X, \alpha)}$ of $(X, \alpha) \to (X, a_0 \cdot \alpha)$ is an embedding.*

PROOF. To prove this, we recall that the function space structure on Y^X (with $Y = (Y, b)$ and $X = (X, a)$) is defined as the largest one making the evaluation map $\mathrm{ev} : Y^X \times X \to Y$ continuous (respectively $\mathrm{ev} : Y^X \otimes X \to Y$ contractive in the approach case). Explicitly, for $\mathfrak{p} \in U(Y^X)$ and $h \in Y^X$, one has

$$\mathfrak{p} \to h \iff \begin{cases} \text{for all } \mathfrak{w} \in U(Y^X \times X),\ x \in X \text{ with } \mathfrak{w} \mapsto \mathfrak{p} \\ \mathfrak{x} \to x \ \Rightarrow\ U\mathrm{ev}(\mathfrak{w}) \to h(x)), \\ (\text{where } \mathfrak{w} \mapsto \mathfrak{x} \in UX) \end{cases}$$

in the topological case and

$$d(\mathfrak{p}, h) = \sup\{b(U\mathrm{ev}(\mathfrak{w}), h(x)) \ominus a(\mathfrak{x}, x) \mid \mathfrak{w} \in U(Y^X \otimes X) \text{ with } \mathfrak{w} \mapsto \mathfrak{p}, x \in X, (\mathfrak{w} \mapsto \mathfrak{x})\}$$

in the approach case. Now, in $Y^{(X,\alpha)}$ one has

$$d_2(\mathfrak{p}, h) = \sup\{b(U\mathrm{ev}(\mathfrak{w}), h(\alpha(\mathfrak{x}))) \mid \mathfrak{w} \in U(Y^X \otimes X) \text{ with } \mathfrak{w} \mapsto \mathfrak{p}, (\mathfrak{w} \mapsto \mathfrak{x})\},$$

and in $Y^{(X,a_0\cdot\alpha)}$

$$d_1(\mathfrak{p}, h) = \sup\{b(U\mathrm{ev}(\mathfrak{w}), h(x)) \ominus a_0(\alpha(\mathfrak{x}), x) \mid \mathfrak{w} \in U(Y^X \otimes X), \mathfrak{w} \mapsto \mathfrak{p}, x \in X, (\mathfrak{w} \mapsto \mathfrak{x})\}.$$

To conclude $d_1(\mathfrak{p}, h) \leqslant d_2(\mathfrak{p}, h)$, we show that

$$b(U\mathrm{ev}(\mathfrak{w}), h(\alpha(\mathfrak{x}))) \geqslant b(U\mathrm{ev}(\mathfrak{w}), h(x)) \ominus a_0(\alpha(\mathfrak{x}), x)$$

for any $x \in X$. In fact, the inequality above is equivalent to

$$b(U\mathrm{ev}(\mathfrak{w}), h(\alpha(\mathfrak{x}))) + a_0(\alpha(\mathfrak{x}), x) \geqslant b(U\mathrm{ev}(\mathfrak{w}), h(x)),$$

which is indeed true since

$$\begin{aligned} b(U\mathrm{ev}(\mathfrak{w}), h(\alpha(\mathfrak{x}))) + a_0(\alpha(\mathfrak{x}), x) &\geqslant b(U\mathrm{ev}(\mathfrak{w}), h(\alpha(\mathfrak{x}))) + b_0(h(\alpha(\mathfrak{x})), h(x)) \\ &\geqslant b(U\mathrm{ev}(\mathfrak{w}), h(x)). \end{aligned}$$

Here b_0 denotes the underlying metric of the approach structure b on Y. For topological spaces one can argue in a similar way. ∎

Consequently, the function space PX is essentially the exponential of a compact Hausdorff space, therefore its topology is the compact-open topology. An approach variant of this topology was introduced in [Lowen and Sioen, 2004].

5.3. Example. In [Hofmann and Tholen, 2010] it is shown that the topological space PX is homeomorphic to the space $F_0(X)$ of all filters (including the improper one) on the lattice $\mathcal{O}X$ of open sets of X, where the topology on $F_0(X)$ has

$$\{\mathfrak{f} \in F_0(X) \mid A \in \mathfrak{f}\} \qquad\qquad (A \subseteq X \text{ open})$$

as basic open sets (see [Escardó, 1997]). Here we can identify an element $\psi \in PX = 2^{X^{\mathrm{op}}}$ with a closed (=Zariski closed and down-closed) subset $\mathcal{A}$ of UX. With this identification, the maps

$$PX \xrightarrow{\Phi} F_0(X),\ \mathcal{A} \mapsto (\bigcap \mathcal{A}) \cap \mathcal{O}X \qquad \text{and} \qquad F_0(X) \xrightarrow{\Pi} PX,\ \mathfrak{f} \mapsto \{\mathfrak{x} \in UX \mid \mathfrak{f} \subseteq \mathfrak{x}\}$$

are indeed continuous and inverse to each other.

The structure a of a space $X = (X, a)$ is a $\mathbb{U}$-module $X \mathrel{\circ\!\!\!\!\longrightarrow} X$ and indeed the identity arrow on X in the ordered category $\mathbb{U}$-Mod of topological/approach spaces and $\mathbb{U}$-modules between them, composition is given by Kleisli-composition and the order structure is inherited from Rel respectively NRel. Each continuous/contractive map $f : X \to Y$ gives rise to $\mathbb{U}$-modules

$$\begin{aligned} f_* : X \mathrel{\circ\!\!\!\!\longrightarrow} Y,\ f_*(\mathfrak{x}, y) &= b(Uf(\mathfrak{x}), y) &\qquad \text{and} \qquad f^* : Y \mathrel{\circ\!\!\!\!\longrightarrow} X,\ f^*(\mathfrak{y}, x) &= b(\mathfrak{y}, f(x)) \\ f_* &= b \cdot Uf & f^* &= f^\circ \cdot b \end{aligned}$$

which form an adjunction $f_* \dashv f^*$ in $\mathbb{U}$-Mod, and these constructions define functors $(-)_* : \mathsf{Top} \to \mathbb{U}$-$\mathsf{Mod}$ and $(-)^* : \mathsf{Top}^{\mathrm{op}} \to \mathbb{U}$-$\mathsf{Mod}$ respectively $(-)_* : \mathsf{App} \to \mathbb{U}$-$\mathsf{Mod}$ and $(-)^* : \mathsf{App}^{\mathrm{op}} \to \mathbb{U}$-$\mathsf{Mod}$. The order on the hom-sets of Top and App are reflections from their respective module categories since

$$f \le h \iff f^* \sqsubseteq h^* \iff h_* \sqsubseteq f_*.$$

From this it follows that $f \dashv g$ in $\mathsf{Top}/\mathsf{App}$ if and only if $g^* \dashv f^*$ in $\mathbb{U}$-Mod if and only if $g^* = f_*$, which in pointwise notation reads as

$$b(Uf(\mathfrak{x}), y) = a(\mathfrak{x}, g(y)),$$

or, in the particular case of topological spaces, as

$$Uf(\mathfrak{x}) \to y \iff \mathfrak{x} \to g(y).$$

The ordered category $\mathbb{U}$-Mod has (co)complete hom-sets, and Kleisli-composition with a $\mathbb{U}$-module $\varphi : X \mathrel{\circ\!\!\!\!\longrightarrow} Y$ from the right preserves suprema. As in the case of ordered sets, a right adjoint to $- \circ \varphi$ gives, for each $\psi : X \mathrel{\circ\!\!\!\!\longrightarrow} Z$, the largest $\mathbb{U}$-module of type $Y \mathrel{\circ\!\!\!\!\longrightarrow} Z$ which composite with φ is less or equal then ψ:

(7)
$$\begin{tikzcd} X \arrow[r, "\psi", "\circ"{marking}] \arrow[d, "\varphi"', "\circ"{marking}] \arrow[dr, phantom, "\sqsubseteq"] & Z \\ Y \arrow[ur, dotted, "\circ"{marking}] & \end{tikzcd}$$

This $\mathbb{U}$-module is called ***extension of*** ψ ***along*** φ, and we write $\psi \mathrel{\circ\!\!-} \varphi$. It can be calculated in Rel respectively NRel as $\psi \mathrel{\bullet\!\!-} (U\varphi \cdot m_X^\circ)$.

5.4. Theorem. $\psi \mathrel{\circ\!\!-} \varphi(\mathfrak{y}, z) = [\![U\ulcorner\varphi\urcorner(\mathfrak{y}), \ulcorner\psi\urcorner(z)]\!]$.

Proof. See [Hofmann, 2011, Theorem 1.10]. ∎

Since the structure a of $X = (X, a)$ is a $\mathbb{U}$-module $X \mathrel{-\!\!\circ\!\!\rightharpoonup} X$, we obtain as its mate the ***Yoneda embedding*** $\mathscr{y}_X = \ulcorner a \urcorner : X \to PX$ which sends x to $x^* = a(-, x)$. Choosing in (7) φ as the identity module and $\psi : X \mathrel{-\!\!\circ\!\!\rightharpoonup} 1$, the theorem above specialises to the Yoneda

5.5. LEMMA. $[\![U\mathscr{y}_X(\mathfrak{x}), \psi]\!] = \psi(\mathfrak{x})$.

As usual, the lemma above tells us that the Yoneda embedding is fully faithful (=initial). For a topological space X, the Yoneda Lemma says that, when identifying $\psi \in PX$ with a filter $\mathfrak{f} \in F_0(X)$,

$$U\mathscr{y}_X(\mathfrak{x}) \to \mathfrak{f} \iff \mathfrak{x} \supseteq \mathfrak{f},$$

which follows also easily from the definition of the topology on $F_0(X)$ (see Example 5.3).

Each module $\varphi : X \mathrel{-\!\!\circ\!\!\rightharpoonup} Y$ induces maps $- \circ \varphi : PY \to PX$ and $- \mathrel{\circ\!\!-} \varphi : PX \to PY$ which are both continuous respectively contractive as $- \circ \varphi$ is the mate of the module $(\mathscr{y}_Y)_* \circ \varphi : X \mathrel{-\!\!\circ\!\!\rightharpoonup} PY$, and $- \mathrel{\circ\!\!-} \varphi$ is the mate of $(\ulcorner \varphi \urcorner)_* : Y \mathrel{-\!\!\circ\!\!\rightharpoonup} PX$, and therefore form an adjunction $- \circ \varphi \dashv - \mathrel{\circ\!\!-} \varphi$ in $\mathsf{Top}/\mathsf{App}$. Hence, for $f : X \to Y$ in $\mathsf{Top}/\mathsf{App}$, one has

$$\begin{tikzcd}
PX \arrow[r, bend left=40, "(-\circ f^*)"] \arrow[r, bend right=40, "(- \circ\!\!- f_*)"'] & PY. \arrow[l, "(-\circ f_*)" description]
\end{tikzcd}$$

In the sequel we write Pf for $- \circ f^*$. Note that $\psi \mathrel{\circ\!\!-} (\mathscr{y}_X)_* = [\![-, \psi]\!] = \psi^*$, hence $- \mathrel{\circ\!\!-} (\mathscr{y}_X)_* = \mathscr{y}_{PX}$.

5.6. DEFINITION. *A topological/approach space is called* **cocomplete** *if the Yoneda embedding* $\mathscr{y}_X : X \to PX$ *has a left adjoint* $\mathrm{Sup}_X : PX \to X$ *in* $\mathsf{Top}/\mathsf{App}$.

If, for a topological space X, we think of PX as the filter space $F_0(X)$, then Sup_X produces for each filter $\mathfrak{f} \in F_0(X)$ a smallest convergence point. In [Hofmann, 2011] it is shown that many properties of cocomplete spaces resemble closely the ones of cocomplete ordered sets:

- cocomplete=injective,
- PX is cocomplete where a supremum $\mathrm{Sup}_X : PPX \to PX$ is given by $- \circ (\mathscr{y}_X)_*$,
- the subcategory $\mathsf{Cocts}_{\mathrm{sep}}$ of $\mathsf{Top}/\mathsf{App}$ consisting of cocomplete T_0-spaces and left adjoint morphisms is reflective, and the Yoneda embedding provides a universal arrow,
- $\mathsf{Cocts}_{\mathrm{sep}}$ is monadic over $\mathsf{Top}/\mathsf{App}$ where the induced monad $\mathbb{P}$ is of Kock-Zöberlein type and has P as functor, the Yoneda embeddings $\mathscr{y}_X : X \to PX$ as units and $\mathscr{m}_X := - \circ (\mathscr{y}_X)_* : PPX \to PX$ as multiplications (providing us with the filter monad in the topological case and with what one might call now ***approach filter monad*** in the approach case),

- $\mathsf{Cocts}_{\mathrm{sep}}$ is also monadic over Set and $\mathsf{Ord}/\mathsf{Met}$.

Recall that a topological/approach space $X = (X, a)$ induces an order/metric $\hat{a} := Ua \cdot m_X^\circ$ on UX, and $\hat{a} : UX \mathrel{\relbar\!\!\mapstochar\!\!\rightarrow} UX$ can be also viewed as a $\mathbb{U}$-relation $\hat{a} : X \mathrel{\relbar\!\!\mapstochar\!\!\rightharpoonup} UX$. This relation is actually a $\mathbb{U}$-module $\hat{a} : X \mathrel{\relbar\!\!\circ\!\!\rightharpoonup} UX$ as one easily verifies:

$$\begin{aligned}
\hat{a} \circ a &= Ua \cdot m_X^\circ \cdot Ua \cdot m_X^\circ = \hat{a} \cdot \hat{a} = \hat{a}, \text{ and}\\
(Ua \cdot m_X^\circ \cdot m_X) \circ \hat{a} &= Ua \cdot m_X^\circ \cdot m_X \cdot UUa \cdot Um_X^\circ \cdot m_X^\circ\\
&= Ua \cdot m_X^\circ \cdot Ua \cdot m_{UX} \cdot m_{UX}^\circ \cdot m_X^\circ = Ua \cdot m_X^\circ \cdot Ua \cdot m_X^\circ = \hat{a} \cdot \hat{a} = \hat{a}.
\end{aligned}$$

Therefore $\hat{a} : X \mathrel{\relbar\!\!\circ\!\!\rightharpoonup} UX$ induces a morphism $\mathcal{Y}_X : UX \to PX$, for each toplogical/approach space X. Moreover, $\mathcal{Y}_X : UX \to PX$ can be seen as a "second" Yoneda embedding:

5.7. Lemma. *For $\mathfrak{X} \in UUX$ and $\psi \in PX$, $[\![U\mathcal{Y}_X(\mathfrak{X}), \psi]\!] = \psi(m_X(\mathfrak{X}))$.*

Proof. In fact, with $\hat{a} = Ua \cdot m_X^\circ$ one has

$$U\hat{a} \cdot m_X^\circ = UUa \cdot Um_X^\circ \cdot m_X^\circ = UUa \cdot m_{UX}^\circ \cdot m_X^\circ = m_X^\circ \cdot Ua \cdot m_X^\circ = m_X^\circ \cdot \hat{a},$$

and therefore

$$[\![U\mathcal{Y}_X(-), \psi]\!] = \psi \mathrel{\circ\!\!-} \hat{a} = \psi \mathrel{\bullet\!\!-} (m_X^\circ \cdot \hat{a}) = (\psi \mathrel{\bullet\!\!-} \hat{a}) \cdot m_X = [\ulcorner\hat{a}\urcorner(m_X(-)), \psi] = \psi(m_X(-)).$$

Here $[-,-]$ denotes the underlying order/metric on PX and we made use of the Yoneda Lemma for ordered sets respectively metric spaces. ∎

5.8. Proposition. *$\mathcal{Y} = (\mathcal{Y}_X)_X$ is a monad morphism $\mathcal{Y} : U \to P$.*

Proof. To check naturality, let $X = (X, a)$, $Y = (Y, b)$ be topological/approach spaces and $f : X \to Y$ be a cont(inous/ractive). Furthermore, let $\hat{a} := Ua \cdot m_Y^\circ$ and $\hat{b} := Ub \cdot m_Y^\circ$, and note that

$$U(f^*) \cdot m_X^\circ = Uf^\circ \cdot Ub \cdot m_X^\circ = Uf^\circ \cdot \hat{b} = (Uf)^*,$$

where $(Uf)^*$ is the module induced by $Uf : UX \to UY$ in Ord respectively Met. With this in mind, the left-lower path in

$$\begin{tikzcd}
UX \arrow[r, "\mathcal{Y}_X"] \arrow[d, "Uf"'] & PX \arrow[d, "Pf"] \\
UY \arrow[r, "\mathcal{Y}_Y"'] & PY
\end{tikzcd}$$

sends $\mathfrak{x}$ to $\hat{b}(-, Uf(\mathfrak{x})) = Uf^*(-, \mathfrak{x})$, and the upper-right path sends $\mathfrak{x}$ to

$$\mathcal{Y}_X(\mathfrak{x}) \circ f^* = \hat{a}(-, \mathfrak{x}) \cdot Uf^* = Uf^*(-, \mathfrak{x}).$$

One easily verifies that the diagram

$$\begin{tikzcd} UX \arrow[rr, "\mathcal{Y}_X"] & & PX \\ & X \arrow[ul, "e_X"] \arrow[ur, "y_X"'] & \end{tikzcd}$$

commutes for each topological/approach space X, and the assertion follows since both monads $\mathbb{U}$, $\mathbb{P}$ are of Kock-Zöberlein type and PX is always separated. ■

We conclude that every $\mathbb{P}$-algebra X is also a $\mathbb{U}$-algebra and therefore also a compact Hausdorff space with convergence

(8)
$$\begin{tikzcd} UX \arrow[r, "\mathcal{Y}_X"] \arrow[rr, bend left, "l_X"] & PX \arrow[r, "\mathrm{Sup}_X"] & X; \end{tikzcd}$$

moreover, $l_X : UX \to X$ is characterised as being left adjoint to $e_X : X \to UX$ in Top respectively App. For a cocomplete topological T_0-space X, this topology is known as the ***Lawson topology*** (see [Gierz *et al.*, 2003]) and, for $\mathfrak{x} \in UX$,

$$l_X(\mathfrak{x}) = \bigvee_{A\in\mathfrak{x}} \bigwedge_{x\in A} x = \bigwedge_{A\in\mathfrak{x}} \bigvee_{x\in A} x.$$

If X is not necessarily a T_0-space, the formula above still describes the left adjoint to $e_X : X \to UX$ in Top. From this description of l_X one concludes that this convergence is already encoded in the underlying order, therefore the convergence a of the topology of X can be recovered from the underlying order $a_0 = a \cdot e_X$ alone since $l_X \dashv e_X$ gives

$$a(\mathfrak{x}, x) = a_0(l_X(\mathfrak{x}), x),$$

for all $\mathfrak{x} \in UX$ and $x \in X$ (see also Proposition 4.12). In fact, injective topological T_0-spaces are known to be precisely the continuous lattices (see [Scott, 1972]). It also follows that, for injective space X and Y, a monotone map $f : X \to Y$ (between the underlying ordered sets) is continuous provided that it preserves co-directed infima[2].

5.9. Example. Since PX is cocomplete it is also a metric compact Hausdorff space where the convergence $UPX \to PX$ sends $\mathfrak{p} \in UPX$ to $\mathcal{Y}_{PX}(\mathfrak{p}) \circ (y_X)_*$ in PX. Recall from Lemma 5.5 that $(y_X)_* : X \multimap PX$ is given by the evaluation relation $\mathrm{ev} : UX \to\!\!\!\!\to PX$, $\mathrm{ev}(\mathfrak{x}, \psi) = \psi(\mathfrak{x})$. Therefore, for any $\mathfrak{x} \in UX$, one has

$$\begin{aligned}(\mathcal{Y}_{PX}(\mathfrak{p}) \circ (y_X)_*)(\mathfrak{x}) &= U(\llbracket -,- \rrbracket) \cdot m_{PX}^\circ \cdot U\, y_X(\mathfrak{x}, \mathfrak{p}) \\ &= U(\llbracket -,- \rrbracket \cdot U\, y_X) \cdot m_X^\circ(\mathfrak{x}, \mathfrak{p}) = U\mathrm{ev} \cdot m_X^\circ(\mathfrak{x}, \mathfrak{p}).\end{aligned}$$

In the remainder of this section we have a closer look at cocomplete (=injective) approach spaces. Motivated by the situation for topological spaces, we define:

[2]Recall that we consider the dual of the specialisation order. We should also mention that continuity is even equivalent to preservation of these infima.

5.10. Definition. *A metric space is called* **continuous** *if it underlies an injective approach space.*

For an injective approach space $X = (X, a)$, we consider l_X as in (8), hence $l_X \cdot e_X \simeq 1_X$ and $l_X \dashv e_X$ in App. By Examples 4.4, the identity map on UX is continuous of type $U(X_t) \to (UX)_t$ where $(-)_t$ refers to the underlying topological space, therefore $l_X : U(X_t) \to X_t$ is continuous and $l_X \cdot e_X \simeq 1_X$ in Top, hence also $l_X \dashv e_X$ in Top. In conclusion, the approach structure of an injective approach space can be recovered from its underlying metric since l_X is determined by the underlying order of the underlying metric; and a contraction map between continuous metric spaces is a contraction map between the corresponding approach spaces if it preserves co-directed infima (i.e. if it is continuous with respect to the Scott-topologies of the underlying lattices).

5.11. Proposition. *For a metric space $X = (X, d)$, there exists at most one injective approach space $X = (X, a)$ so that $d(x, y) = a(\dot{x}, y)$, for all $x, y \in X$.*

The full subcategory of App consisting of all injective approach spaces we denote as $\mathsf{ContMet}$. By the corollary above, $\mathsf{ContMet}$ can be also viewed as a (non-full) subcategory of Met.

5.12. Example. We consider the approach space $[0, \infty]$ with $\lambda(\mathfrak{x}, x) = x \ominus \xi(\mathfrak{x})$ (see 4.3). In the underlying topology,

$$\mathfrak{x} \to x \iff 0 \geqslant x \ominus \xi(\mathfrak{x}) \iff \xi(\mathfrak{x}) \geqslant x.$$

In particular, any interval $[0, u]$ is closed. Take now the filter base $\mathfrak{g} := \{(1, 1+\varepsilon) \mid 0 < \varepsilon\}$ and let $\mathfrak{y} \in U[0, \infty]$ be with $\mathfrak{g} \subseteq \mathfrak{y}$. Then $\dot{1} \not\leq \mathfrak{y}$ (since $[0, 1] \notin \mathfrak{y}$) but $\delta(\dot{1}, \mathfrak{y}) = 0$ (since every $B \in \mathfrak{y}$ contains elements arbitrary close to 1 from the right).

5.13. Remark. The metric space $[0, \infty]$ is continuous since it underlies the injective approach space $[0, \infty]$. Certainly, every continuous metric space is also a continuous lattice via its underlying order; however, it should be noted a continuous lattice (via its free metric) is in general not a continuous metric space. For instance, the Sierpiński space $\mathsf{2}$ is not injective in App. To see this, just consider the embedding $\{0, \infty\} \hookrightarrow [0, \infty]$ and $f : \{0, \infty\} \to \mathsf{2}$ with $f(0) = \mathsf{true}$ and $f(\infty) = \mathsf{false}$, and observe that there is no contraction map $g : [0, \infty] \to \mathsf{2}$ extending f since there exists $\mathfrak{x} \in U[0, \infty)$ with $\lambda(\mathfrak{x}, \infty) = 0$.

It is well-known that the full subcategory of Top defined by all injective spaces is Cartesian closed (see [Scott, 1972]). We will now show that $\mathsf{ContMet}$ is a Cartesian closed category as well. To this end, recall that the approach space $[0, \infty]$ is actually a monoid in the monoidal category App since addition $+$ is a contraction map $+ : [0, \infty] \otimes [0, \infty] \to [0, \infty]$. Hence it induces a monad $\mathbb{M} = (M, 0, +)$ on App where $M = - \otimes [0, \infty]$. For each approach space X,

$$\boldsymbol{t}_X : X \otimes [0, \infty] \to PX, (u, x) \mapsto a(-, x) + u$$

is a contraction map since it is the mate of the composite

$$X^{\mathrm{op}} \otimes X \otimes [0,\infty] \xrightarrow{a\otimes 1} [0,\infty] \otimes [0,\infty] \xrightarrow{+} [0,\infty]$$

of contraction maps. Thinking of $u \in [0,\infty]$ as a $\mathbb{U}$-module $u : 1 \multimap 1$, then $\boldsymbol{t}_X(x,u)$ is the $\mathbb{U}$-module $u \circ x^*$. One easily confirms that the family $\boldsymbol{t} = (\boldsymbol{t}_X)_X$ is a monad morphism $\mathbb{M} \to \mathbb{P}$. Therefore each injective approach space admits an action

$$+ := \mathrm{Sup}_X \cdot \boldsymbol{t}_X : X \otimes [0,\infty] \to X,$$

which satisfies

$$a_0(x+u, y) = a_0(\mathrm{Sup}_X(u \circ x^*), y) = [u \circ x^*, y^*] = a_0(x,y) \ominus u.$$

Note that the universal property above determines the action $+$ uniquely, that is, an approach space X admits up to equivalence at most one action $+ : X \otimes [0,\infty] \to X$ in App with $a_0(x+u,y) = a_0(x,y) \ominus u$.

For a numerical relation $\varphi : X \to\!\!\!\!\!\mapsto Y$ and $u \in [0,\infty]$, we write $\varphi \mathbin{\boxplus} u$ for the relation defined by $\varphi \mathbin{\boxplus} u(x,y) := \varphi(x,y) + u$. Note that $U(\varphi \mathbin{\boxplus} u) = U\varphi \mathbin{\boxplus} u$, and, given also $\psi : Y \to\!\!\!\!\!\mapsto Z$ and $v \in [0,\infty]$, $(\psi \mathbin{\boxplus} v) \cdot (\varphi \mathbin{\boxplus} u) = (\psi \cdot \varphi) \mathbin{\boxplus} (v+u)$.

5.14. Theorem. *Let X be an approach space which admits a left adjoint $l : UX \to X$ of $e_X : X \to UX$ and an action $+ : X \otimes [0,\infty] \to X$ in* App *with $a_0(x+u,y) = a_0(x,y) \ominus u$. Then X is exponentiable in* App.

Proof. Firstly, fixing $u \in [0,\infty]$, one obtains $t_u : X \to X$, $x \mapsto x+u$ in App. Moreover, from

$$a_0(x,y) \geqslant a_0(x+u, y+u) = a_0(x, y+u) \ominus u$$

it follows that $a_0(x,y) + u \geqslant a_0(x, y+u)$, for all $x, y \in X$; and hence also

$$a(\mathfrak{x}, y) + u = a_0(l(\mathfrak{x}), y) + u \geqslant a_0(l(\mathfrak{x}), y+u) = a(\mathfrak{x}, y+u)$$

for all $\mathfrak{x} \in TX$ and $y \in X$. With notation introduced above, this reads as $a \mathbin{\boxplus} u \geqslant t_u^\circ \cdot a$, which allows us to conclude

$$(Ua) \mathbin{\boxplus} u = U(a \mathbin{\boxplus} u) \geqslant Ut_u^\circ \cdot Ua,$$

that is, $Ua(\mathfrak{X}, \mathfrak{x}) + u \geqslant Ua(\mathfrak{X}, Ut_u(\mathfrak{x}))$, for all $\mathfrak{X} \in TTX$ and $\mathfrak{x} \in TX$. Furthermore, $l \cdot Ut_u \leq t_u \cdot l$ since t_u is a contraction map, and therefore

$$a(Ut_u(\mathfrak{x}), x) = a_0(l \cdot Ut_u(\mathfrak{x}), x) \leqslant a_0(l(\mathfrak{x}) + u, x) = a(\mathfrak{x}, x) \ominus u,$$

for all $\mathfrak{x} \in TX$ and $x \in X$.

Secondly, recall from [Hofmann, 2006] that an approach space $X = (X,a)$ is exponentiable if, for all $\mathfrak{X} \in UUX$ and $x \in X$ with $a(m_X(\mathfrak{X}), x) < \infty$, all $v, u \in [0,\infty)$ with $u + v = a(m_X(\mathfrak{X}), x)$ and all $\varepsilon > 0$, there exists an ultrafilter $\mathfrak{x} \in UX$ such that

$$Ua(\mathfrak{X}, \mathfrak{x}) \leqslant u + \varepsilon \qquad \text{and} \qquad a(\mathfrak{x}, x) \leqslant v + \varepsilon.$$

Assume now that $X = (X, a)$ is injective in App, and let $\mathfrak{X} \in UUX$, $x \in X$ with $w := a(m_X(\mathfrak{X}), x) < \infty$ and $u, v \in [0, \infty]$ with $u+v = w$. Put $\mathfrak{y} := Ul(\mathfrak{X})$ and $\mathfrak{x} := Ut_u(\mathfrak{y})$. Then

$$\begin{aligned} Ua(\mathfrak{X}, \mathfrak{x}) &\leqslant Ua(\mathfrak{X}, \mathfrak{y}) + u = u, \text{ and} \\ a(\mathfrak{x}, x) &\leqslant a(\mathfrak{y}, x) \ominus u \\ &= a_0(l \cdot Ul(\mathfrak{X}), x) \ominus u \\ &= a_0(l \cdot m_X(\mathfrak{X}), x) \ominus u \\ &= w \ominus u = v, \end{aligned}$$

and the assertion follows. ∎

5.15. Corollary. *Each injective approach space is exponentiable in* App.

With the same argument as in Remark 3.1 one can show that with Y and X also Y^X and $Y \times X$ are injective approach spaces, hence

5.16. Theorem. $\mathsf{ContMet}$ *is Cartesian closed.*

6. A seemingly unnatural dual adjunction

At the end of Section 2 we briefly discussed the dual adjunction between Ord and CCD. In this section we show that this dual adjunction has a natural analogue when we replace Ord with Top or App. It is interesting to observe that the considerations in this section only apply to $X \mapsto \mathsf{2}^{X^{\mathrm{op}}}$, the construction $X \mapsto \mathsf{2}^X$ (see Remark 2.5) is a completely different story and is studied in general in [Hofmann and Stubbe, 2011]. Note that $(-)^{\mathrm{op}} : \mathsf{Top} \to \mathsf{Top}$ is no longer an equivalence, and also that $\mathsf{2}^{X^{\mathrm{op}}}$ is a (very particular) topological space but $\mathsf{2}^X$ in general not since Top is not Cartesian closed. Of course, $X \mapsto \mathsf{2}^X$ leads to the well-known dual adjunction between Top and Frm, so lets look now at $X \mapsto \mathsf{2}^{X^{\mathrm{op}}}$.

6.1. Definition. *A cocomplete topological/approach space X is called* **completely distributive (cd)** *if* $\mathsf{Sup}_X : PX \to X$ *has a left adjoint in* $\mathsf{Top}/\mathsf{App}$.

We remark immediately that every space of type PX is (cd), witnessed by the string of adjunctions

$$\mathcal{Y}_{PX} = - \multimap (\mathcal{Y}_X)_* \vdash - \circ (\mathcal{Y}_X)_* \vdash - \circ (\mathcal{Y}_X)^* = P\mathcal{Y}_X \, .$$

We let CDTop (CDApp) denote the category of completely distributive topological (approach) T_0-spaces and left-and-right adjoint continuous (contractive) maps. The presheaf construction defines functors

$$D : \mathsf{Top}^{\mathrm{op}} \to \mathsf{CDTop} \qquad \text{respectively} \qquad D : \mathsf{App}^{\mathrm{op}} \to \mathsf{CDApp}$$

sending $f : X \to Y$ to $- \circ f_* : PY \to PX$, that is, $DX = PX$ and $Pf \dashv Df$. To define functors in the opposite direction, we note that a completely distributive space L comes together with $\mathscr{y}_L : L \to PL$ and $t_L : L \to PL$ where $t_L \dashv \mathrm{Sup}_L$. As in the Ord-case, we consider now the equaliser

$$A \xrightarrow{i} L \underset{\mathscr{y}_L}{\overset{t_L}{\rightrightarrows}} PL. \tag{9}$$

in $\mathsf{Top}/\mathsf{App}$. Let also M be a completely distributive space with corresponding equaliser $j : B \hookrightarrow M$ and $f : L \to M$ in $\mathsf{CDTop}/\mathsf{CDApp}$, hence f preserves suprema and has a left adjoint $g : M \to L$. Therefore the diagrams

$$\begin{tikzcd} M \arrow[r, "\mathscr{y}_M"] \arrow[d, "g"'] & PM \arrow[d, "Pg"] \\ L \arrow[r, "\mathscr{y}_L"'] & PL \end{tikzcd} \qquad \text{and} \qquad \begin{tikzcd} PL \arrow[r, "\mathrm{Sup}_L"] \arrow[d, "Pf"'] & L \arrow[d, "f"] \\ PM \arrow[r, "\mathrm{Sup}_M"'] & M \end{tikzcd}$$

commute (up to equivalence), and from the latter it follows that the diagram of the corresponding left adjoints

$$\begin{tikzcd} M \arrow[r, "t_M"] \arrow[d, "g"'] & PM \arrow[d, "Pg"] \\ L \arrow[r, "t_L"'] & PL \end{tikzcd}$$

commutes (up to equivalence, but PL is separated, so it really commutes). We conclude that $g : M \to L$ restricts to a continuous/contractive map $g_0 : B \to A$. Summing up:

6.2. Proposition. *With the notation used above, there are functors*

$$S : \mathsf{CDTop} \to \mathsf{Top}^{\mathrm{op}} \qquad \textit{respectively} \qquad S : \mathsf{CDApp} \to \mathsf{App}^{\mathrm{op}}$$

where $SL = A$ and $Sf = g_0$.

We will now show that S and D form a dual adjunction. To construct a natural transformation $\eta : 1 \to SD$, we start by observing that $P\mathscr{y}_X \cdot \mathscr{y}_X = \mathscr{y}_{PX} \cdot \mathscr{y}_X$ for any X in $\mathsf{Top}/\mathsf{App}$; however, $\mathscr{y}_X$ is in general not the equaliser of $P\mathscr{y}_X$ and $\mathscr{y}_{PX}$. Nevertheless, the universal property of the equaliser produces a continuous/contractive map $\eta_X : X \to SD(X)$ which is just the corestriction of the Yoneda embedding, and $\eta = (\eta_X)_X$ is indeed a natural transformation. Let now L in $\mathsf{CDTop}/\mathsf{CDApp}$ with equaliser diagram (9), we put

$$\begin{tikzcd} L \arrow[r, "\mathscr{y}_L"'] \arrow[rr, bend left, "\varepsilon_L"] & PL \arrow[r, "-\circ i_*"'] & PA = DS(L). \end{tikzcd}$$

Then ε_L is a right adjoint since both $\mathscr{y}_L$ and $-\circ i_*$ are. To see that ε_L is also left adjoint, we show that

$$\begin{tikzcd} PL \arrow[r, "P\varepsilon_L"] \arrow[d, "\mathrm{Sup}_L"'] & PPA \arrow[d, "\sup_{PA} = -\circ(\mathscr{y}_A)_*"] \\ L \arrow[r, "\varepsilon_L"'] & PA \end{tikzcd}$$

commutes. Let $\psi \in PL$ and $\mathfrak{a} \in UA$. Then (with $L = (L, a)$)

$$\begin{aligned}
\varepsilon_L \cdot \mathrm{Sup}_L(\psi)(\mathfrak{a}) &= a(Ui(\mathfrak{a}), \mathrm{Sup}_L(\psi)) \\
&= [\![U(t_L \cdot i)(\mathfrak{a}), \psi]\!] && (t_L \dashv \mathrm{Sup}_L) \\
&= [\![U\mathcal{y}_L(Ui(\mathfrak{a})), \psi]\!] && (t_L \cdot i = \mathcal{y}_L \cdot i) \\
&= \psi(Ui(\mathfrak{a})) = \psi \circ i_*(\mathfrak{a}) && (\text{Yoneda Lemma})
\end{aligned}$$

and

$$\begin{aligned}
\mathrm{Sup}_{PA} \cdot P\varepsilon_L(\psi) &= \psi \circ \varepsilon_L^* \circ (\mathcal{y}_A)_* \\
&= \psi \circ \mathcal{y}_L^* \circ (- \circ i_*)^* \circ (\mathcal{y}_A)_* && (\varepsilon_L = (- \circ i_*) \cdot \mathcal{y}_L) \\
&= \psi \circ \mathcal{y}_L^* \circ (Pi)_* \circ (\mathcal{y}_A)_* && (Pi \dashv (- \circ i_*),\ \text{hence } (Pi)_* = (- \circ i_*)^*) \\
&= \psi \circ \mathcal{y}_L^* \circ (\mathcal{y}_L)_* \circ i_* = \psi \circ i_*.
\end{aligned}$$

Next we show that $\varepsilon = (\varepsilon_L)_L$ is a natural transformation $\varepsilon : 1 \to DS$. To this end, let $f : L \to M$ in $\mathsf{CDTop}/\mathsf{CDApp}$ with left adjoint $g : M \to L$. We have to convince ourself that

$$\begin{tikzcd}
L \arrow[r, "\varepsilon_L"] \arrow[d, "f"'] & PA \arrow[d, "-\circ(g_0)_*"] \\
M \arrow[r, "\varepsilon_M"'] & PB
\end{tikzcd}$$

commutes (we use here the notation introduced above), which we do by pasting the commutative diagrams

$$\begin{tikzcd}
L \arrow[r, "y_L"] \arrow[d, "f"'] & PL \arrow[d, "Pf"] \\
M \arrow[r, "y_M"'] & PM
\end{tikzcd} \qquad \text{and} \qquad \begin{tikzcd}
PL \arrow[r, "-\circ i_*"] \arrow[d, "-\circ g_*"'] & PA \arrow[d, "-\circ(g_0)_*"] \\
PM \arrow[r, "-\circ j_*"'] & PB
\end{tikzcd}$$

together. This is indeed possible since from $Pg \dashv Pf$ and $Pg \dashv (- \circ g_*)$ it follows that $Pf = - \circ g_*$. Finally, the composites

$$\begin{array}{ccccc}
SL & \xrightarrow{\eta_{SL}} & SDS(L) & \xrightarrow{S(\varepsilon_L)} & SL \\
x & \longmapsto & x^* & \longmapsto & \mathrm{Sup}_L(x^*) \ = x
\end{array}$$

and

$$\begin{array}{ccccc}
DX & \xrightarrow{\varepsilon_{DX}} & DSD(X) & \xrightarrow{D(\eta_X)} & DX \\
\psi & \longmapsto & \psi^* \circ i_* & \longmapsto & \psi^* \circ i_* \circ (\eta_X)_* \ = \psi^* \circ (\mathcal{y}_X)_* = \psi
\end{array}$$

are both equal to the identity, where $i : SDX \hookrightarrow DX$ denotes the inclusion map.

6.3. Theorem. *$(D, S, \eta, \varepsilon)$ defines a dual adjunction*

$$\mathsf{Top}^{\mathrm{op}} \leftrightarrows \mathsf{CDTop} \textit{ respectively } \mathsf{App}^{\mathrm{op}} \leftrightarrows \mathsf{CDApp}.$$

6.4. Remark. The dual adjunction above does not seem to be induced by a dualising object. Certainly, $S \simeq \hom(-,2)$ respectively $S \simeq \hom(-,[0,\infty])$, but there is no space X with $D \simeq \hom(-,X)$. This indicates that the "obvious" forgetful functor $\mathsf{CDTop} \to \mathsf{Set}$ respectively $\mathsf{CDApp} \to \mathsf{App}$ is a "bad" choice, in fact, we will later on (Remark 7.17) see that there is a better candidate.

As for any dual adjunction, one obtains a dual equivalence between the fixed full subcategories

$$\mathrm{Fix}(\eta) := \{X \mid \eta_X \text{ is an isomorphism}\} \quad \text{and} \quad \mathrm{Fix}(\varepsilon) := \{L \mid \varepsilon_L \text{ is an isomorphism}\}$$

which we determine now.

6.5. Lemma. *For each topological/approach space X and $\psi \in PX$,*

$$P\boldsymbol{y}_X(\psi) = \boldsymbol{y}_{PX}(\psi) \iff \psi \text{ is right adjoint.}$$

Proof. Our proof uses the fact obtained in [Hofmann and Tholen, 2010] that

$$\tilde{X} := \{\psi \in PX \mid \psi \text{ is right adjoint}\}$$

is the Lawvere closure of $\boldsymbol{y}_X(X)$ in PX. Clearly, the equaliser of $\boldsymbol{y}_{PX}$ and $P\boldsymbol{y}_X$ is Lawvere closed and contains $\boldsymbol{y}_X(X)$, and the implication "$\Leftarrow$" follows. To see "$\Rightarrow$", note that from $P\boldsymbol{y}_X(\psi) = \boldsymbol{y}_{PX}(\psi)$ it follows that $\psi^* = \psi \circ \boldsymbol{y}_X^*$, hence $\psi \circ \boldsymbol{y}_X^*(\dot{\psi})$ is true respectively 0. Since $Ue_Y \cdot e_Y = m_Y^\circ \cdot e_Y$[3] for any Y,

$$\psi \circ \boldsymbol{y}_X^*(\dot{\psi}) = \psi \cdot U\,\boldsymbol{y}_X^*(e_{UPX} \cdot e_{PX}(\psi)) = \bigvee_{\mathfrak{x} \in UX} \psi(x) \otimes U[\![-,-]\!](e_{UPX} \cdot e_{PX}(\psi), T\,\boldsymbol{y}_X(\mathfrak{x}))$$

where $[\![-,-]\!]$ denotes the structure on PX, $\otimes$ is either & or $+$, and $\bigvee$ is either $\exists$ or inf. The result follows now from Proposition 4.16 (3.16 in the arXiv-version) of [Hofmann and Tholen, 2010]. ∎

Hence, X belongs to $\mathrm{Fix}(\eta)$ precisely if each right adjoint module ψ is representable as $\psi = x^*$ for a unique $x \in X$. But this is precisely the definition of a ***Lawvere complete*** (also called Cauchy complete) separated space as introduced in [Clementino and Hofmann, 2009a]. In both the topological and the approach case, Lawvere completeness together with separateness means soberness (see [Clementino and Hofmann, 2009a, Subsections 6.3 and 6.4]), so that $\mathrm{Fix}(\eta)$ is precisely the category $\mathsf{Sob}/\mathsf{ASob}$ of sober topological/approach spaces and continuous/contraction maps (see [Banaschewski *et al.*, 2006] and [Van Olmen, 2005] for the notion of sober approach space).

[3]The same holds for any monad where $T1 = 1$.

6.6. Example. For a topological space X, a $\mathbb{U}$-module $\varphi : 1 \,{\circ\!\!\rightharpoonup}\, X$ corresponds to a closed subset $A \subseteq X$, and $\psi : X \,{\circ\!\!\rightharpoonup}\, 1$ to a closed subset $\mathcal{A} \subseteq UX$. With this identification, $\varphi \dashv \psi$ means that (see [Clementino and Hofmann, 2009a])

- $\mathcal{A} = \{\mathfrak{x} \in UX \mid \forall x \in A \,.\, \mathfrak{x} \to x\}$,
- there exists an ultrafilter $\mathfrak{x}_0 \in \mathcal{A}$ with $A \in \mathfrak{x}_0$.

Hence, for any $\mathfrak{x} \in \mathcal{A}$ and any $B \in \mathfrak{x}$, $A \subseteq \overline{B}$ and therefore $B \in \mathfrak{x}_0$. We conclude that $\mathfrak{x} \leq \mathfrak{x}_0$, hence $\mathcal{A} = \downarrow\!\mathfrak{x}_0$.

For L in $\mathsf{CDTop}/\mathsf{CDApp}$, $\varepsilon_L : L \to PA$ has a left adjoint $c : PA \to L$ which sends $\psi \in PA$ to $\mathrm{Sup}_L(\psi \circ i^*)$. Since ε_L preserves suprema and $\varepsilon \cdot i = y_A$, we see that even $\varepsilon_L \cdot c = 1$ since

$$\begin{aligned}\varepsilon_L \cdot c(\psi) = \varepsilon_L(\mathrm{Sup}_L(\psi \circ i^*)) = \mathrm{Sup}_{PA}(P\varepsilon_L(\psi \circ i^*)) &= \mathrm{Sup}_{PA}(\psi \circ i^* \circ \varepsilon_L^*)\\ &= \mathrm{Sup}_{PA}(\psi \circ y_A^*) = m_A \cdot P\, y_A(\psi) = \psi.\end{aligned}$$

6.7. Definition. *We call a completely distributive topological/approach space L* **totally algebraic** *if $c \cdot \varepsilon_L \simeq 1$, that is, if*

$$\mathrm{Sup}_L(x^* \circ i_* \circ i^*) \simeq x$$

for each $x \in X$.

6.8. Example. By definition, a topological space X is totally algebraic if each element $x \in X$ is a supremum of the distributor $x^* \circ i_* \circ i^* : X \,{\circ\!\!\rightharpoonup}\, 1$. Intuitively, $x^* \circ i_* \circ i^*$ is the down-set of all totally algebraic elements below x, and in fact, $\mathfrak{x} \in UX$ belongs to $x^* \circ i_* \circ i^*$ if and only if there is some $\mathfrak{a} \in UA$ with $\mathfrak{x} \leq \mathfrak{a}$ and $\mathfrak{a} \to x$.

Clearly, $\mathrm{Fix}(\varepsilon)$ is the full subcategory of $\mathsf{CDTop}/\mathsf{CDApp}$ consisting of all totally algebraic T_0-spaces; we denote this category as TATop respectively as TAApp. In conclusion,

6.9. Theorem. $\mathsf{Sob}^{\mathrm{op}} \simeq \mathsf{TATop}$ *and* $\mathsf{ASob}^{\mathrm{op}} \simeq \mathsf{TAApp}$.

In Section 8 we will develop a more general theory which contains the result above as a special case (see Theorem 8.20).

6.10. Proposition. *The inclusion functors* $\mathsf{Sob} \hookrightarrow \mathsf{Top}$*,* $\mathsf{ASob} \hookrightarrow \mathsf{App}$*,* $\mathsf{TATop} \hookrightarrow \mathsf{CDTop}$ *and* $\mathsf{TAApp} \hookrightarrow \mathsf{CDApp}$ *are right adjoint.*

Proof. It is well-known (see, for instance,Theorem 2.0 of [Lambek and Rattray, 1979]) that these fixed subcategories are reflective if and only if η_{SL} respectively ε_{DX} are isomorphisms, that is, SL is sober respectively DX is totally algebraic. Now, any completely distributive space is cocomplete, hence Lawvere complete (=sober), and SL is L-closed (see [Hofmann and Tholen, 2010]) in L since it is the equaliser of y_L and t_L. Therefore SL is sober. Certainly, $DX = PX$ is totally algebraic for each sober space X. For an arbitrary space X, the induced $\mathbb{U}$-module i_* of the sobrification $i : X \to \tilde{X}$ satisfies $i^* \circ i_* = 1$ and $i_* \circ i^* = 1$, therefore $PX \simeq P\tilde{X}$ is totally algebraic as well. ∎

7. Frames vs. complete distributivity

In the previous section we have studied the dual adjunctions

$$\mathsf{Top}^{\mathrm{op}} \leftrightarrows \mathsf{CDTop} \qquad\text{and}\qquad \mathsf{App}^{\mathrm{op}} \leftrightarrows \mathsf{CDApp}$$

which are quite different from the "traditional" ones with frames (see [Isbell, 1972]) respectively approach frames (see [Banaschewski *et al.*, 2006] and [Van Olmen, 2005]). In fact, for a topological space X, the two constructions $X \mapsto \mathsf{Top}(X, 2)$ and $X \mapsto \mathsf{Top}(X^{\mathrm{op}}, 2)$ produce quite different objects: the former one an ordered set satisfying certain completeness properties, and the latter one a very particular topological space. This stands in sharp contrast to the situation for an ordered set X where both $X \mapsto \mathsf{Ord}(X, 2)$ and $X \mapsto \mathsf{Ord}(X^{\mathrm{op}}, 2)$ lead to complete lattices. One reason for this discrepancy is the fact that X is in general not exponentiable but X^{op} is always exponentiable. Another reason is the asymmetry in the definition of convergence: whereby the domain and codomain of $x \le y$ are points of X, the domain of $\mathfrak{x} \to x$ is an ultrafilter but the codomain is a point. Nevertheless, all these adjunctions restrict to dual equivalences involving sober spaces; therefore one might ask about the relationship between frames and completely distributive spaces. In this section we will consider only the topological case and show that CDTop is equivalent to Frm. Unfortunately, so far I do not know if a similar result is true for approach spaces.

Recall from Example 5.3 that PX is homeomorphic to the filter space $F\mathcal{O}X$, where $\mathcal{O}X$ denotes as usual the frame of open subsets of a topological space X. Therefore we can hope that there is a commutative diagram

$$\begin{tikzcd} & \mathsf{Top}^{\mathrm{op}} \arrow[dl, "\mathcal{O}"'] \arrow[dr, "D"] & \\ \mathsf{Frm} \arrow[rr, "F"'] & & \mathsf{CDTop} \end{tikzcd}$$

of functors, where FL denotes the usual filter space of a frame. More general, for a meet semi-lattice L one puts

$$FL := \{\mathfrak{f} \subseteq L \mid \mathfrak{f} \text{ is a (possibly improper) filter}\}$$

which is a topological space with

$$x^{\#} = \{\mathfrak{f} \in FL \mid x \in \mathfrak{f}\} \qquad (x \in L)$$

as basic open set. Note that $1^{\#} = FL$ and $(x \wedge y)^{\#} = x^{\#} \cap y^{\#}$. Furthermore, the underlying order on FL is given by

$$\mathfrak{f} \le \mathfrak{g} \iff \dot{\mathfrak{f}} \to \mathfrak{g} \iff \forall x \in \mathfrak{g}\,.\,\mathfrak{f} \in x^{\#} \iff \mathfrak{g} \subseteq \mathfrak{f},$$

which also tells us that FL is separated (=T_0). For a meet semi-lattice homomorphism $f : L \to M$, the mapping

$$Ff : FL \to FM,\ \mathfrak{f} \mapsto \Uparrow\{f(x) \mid x \in \mathfrak{f}\}$$

is continuous since

$$Ff^{-1}(y^{\#}) = \{\mathfrak{f} \in FL \mid \exists x \in \mathfrak{f} \,.\, f(x) \leq y\} = \bigcup_{x:f(x)\leq y} x^{\#},$$

and so is

$$f_! : FM \to FL,\ \mathfrak{g} \mapsto f^{-1}(\mathfrak{g}).$$

since

$$f_!^{-1}(x^{\#}) = \{\mathfrak{g} \mid f_!(\mathfrak{g}) \in x^{\#}\} = \{\mathfrak{g} \mid f(x) \in \mathfrak{g}\} = f(x)^{\#}. \tag{10}$$

Furthermore, one easily verifies that $f_! \dashv Ff$ in Top. Given also $g : L \to M$ with $f \leq g$ and $\mathfrak{f} \in FL$, then

$$\{g(x) \mid x \in \mathfrak{f}\} \subseteq \Uparrow\{f(x) \mid x \in \mathfrak{f}\} = Ff(\mathfrak{f})$$

and therefore $Ff(\mathfrak{f}) \leq Fg(\mathfrak{f})$. We write $\mathsf{Top}_{\mathrm{inf}}$ for the 2-category of T_0-spaces and right adjoint continuous maps with the pointwise order on hom-sets, and SLat denotes the 2-category of meet semi-lattices and meet semi-lattice homomorphisms with the pointwise order on hom-sets. Therefore, we have proven the following

7.1. Proposition. *$F : \mathsf{SLat} \to \mathsf{Top}_{\mathrm{inf}}$ is a 2-functor.*

We will now show that F restricts to an equivalence functor between the full subcategories of SLat and $\mathsf{Top}_{\mathrm{inf}}$ defined by all frames and all completely distributive T_0-spaces respectively.

Given a meet semi-lattice L, one has the mapping

$$\alpha_L : L \to \mathcal{O}(FL),\ x \mapsto x^{\#}$$

which is an order-embedding since $x^{\#} \subseteq y^{\#} \iff \uparrow x \in y^{\#} \iff x \leq y$. Furthermore, α_L preserves all existing infima in L. To see this, observe first that

$$\mathrm{int}(\mathcal{A}) = \{\mathfrak{f} \in FL \mid \exists x \in \mathfrak{f} \,.\, x^{\#} \subseteq \mathcal{A}\}$$

Let now $(x_i)_{i\in I}$ be a family of elements of L with infimum $x \in L$. Then

$$\begin{aligned}\bigwedge_{i\in I} x_i^{\#} = \mathrm{int}(\bigcap_{i\in I} x_i^{\#}) &= \{\mathfrak{f} \in FL \mid \exists z \in \mathfrak{f}\,\forall i \in I \,.\, z^{\#} \subseteq x_i^{\#}\}\\ &= \{\mathfrak{f} \in FL \mid \exists z \in \mathfrak{f}\,\forall i \in I \,.\, z \leq x_i\} = \{\mathfrak{f} \in FL \mid x \in \mathfrak{f}\} = x^{\#}.\end{aligned}$$

If L is complete, then $\alpha_L : L \to \mathcal{O}(FL)$ has a left adjoint $\beta_L : \mathcal{O}(FL) \to L$ which is necessarily given by

$$\beta_L(\mathcal{A}) = \bigwedge\{x \in L \mid \mathcal{A} \subseteq x^{\#}\}.$$

7.2. LEMMA. *Assume that L is complete. For any open subset $\mathcal{A} \subseteq FL$,*

$$\bigwedge\{x \in L \mid \mathcal{A} \subseteq x^{\#}\} = \bigvee\{y \in L \mid y^{\#} \subseteq \mathcal{A}\}.$$

PROOF. We only need to show "$\leq$". We put $z = \bigvee\{y \in L \mid y^{\#} \subseteq \mathcal{A}\}$ and show $\mathcal{A} \subseteq z^{\#}$. To this end, let $\mathfrak{f} \in \mathcal{A}$. Since $\mathcal{A}$ is open, there is some $u \in \mathfrak{f}$ with $u^{\#} \subseteq \mathcal{A}$. Hence $u \leq z$ and therefore $\mathfrak{f} \in z^{\#}$. ■

7.3. PROPOSITION. *For every frame L, $\beta_L : \mathcal{O}(FL) \to L$ is a frame homomorphism.*

PROOF. Clearly, $\beta_L(FL) = \top$. Let now $\mathcal{A}, \mathcal{B} \in \mathcal{O}(FL)$. Then

$$\begin{aligned}\beta_L(\mathcal{A}) \wedge \beta_L(\mathcal{B}) &= \bigvee\{y \in L \mid y^{\#} \subseteq \mathcal{A}\} \wedge \bigvee\{z \in L \mid z^{\#} \subseteq \mathcal{B}\} \\ &= \bigvee\{y \wedge z \mid y^{\#} \subseteq \mathcal{A}, z^{\#} \subseteq \mathcal{B}\} = \bigvee\{x \in L \mid x^{\#} \subseteq \mathcal{A} \cap \mathcal{B}\} = \beta_L(\mathcal{A} \cap \mathcal{B}).\end{aligned}$$

■

Hence, for any frame L, one has

$$\begin{tikzcd} FL \arrow[r, bend left, "F\alpha_L"] \arrow[r, bend right, "(\beta_L)_!"'] & F\mathcal{O}F(L) \arrow[l, "F\beta_L" description] \end{tikzcd} .$$

Since $P(FL) \simeq F\mathcal{O}F(L)$ and

$$F\alpha_L(\mathfrak{f}) = \langle\{x^{\#} \mid x \in \mathfrak{f}\}\rangle = y_{FL}(\mathfrak{f}),$$

we conclude that FL is a completely distributive T_0-space.

7.4. PROPOSITION. $F : \mathsf{SLat} \to \mathsf{Top}_{\mathrm{inf}}$ *restricts to a 2-functor* $F : \mathsf{Frm}_{\wedge} \to \mathsf{CDTop}_{\mathrm{inf}}$ *where* $\mathsf{Frm}_{\wedge}$ *denotes the full subcategory of* SLat *defined by those meet-semilattices which are frames, and* $\mathsf{CDTop}_{\mathrm{inf}}$ *denotes the 2-category of completely distributive T_0-spaces and right adjoint continuous maps.*

To show that $F : \mathsf{Frm}_{\wedge} \to \mathsf{CDTop}_{\mathrm{inf}}$ is an equivalence of categories, we will now describe its inverse $\mathrm{Pt} : \mathsf{CDTop}_{\mathrm{inf}} \to \mathsf{Frm}_{\wedge}$. To motivate our construction, note that this functor should send a completely distributive space Y of the form $Y \simeq PX$ for $X \in \mathsf{Top}$ to the frame $\mathcal{O}X \simeq \mathsf{Top}(X, 2)^{\mathrm{op}}$ of opens of X. By the universal property of the Yoneda embedding,

$$\mathrm{Map}(\mathsf{Top})(PX, 2) \to \mathsf{Top}(X, 2),\ g \mapsto g \cdot y_X$$

is an isomorphism in Ord; where we write $\mathrm{Map}(-)$ for the subcategory defined by all left adjoint morphisms. Its inverse sends $\varphi : X \to 2$ to the left adjoint

$$\varphi_L := \mathrm{Sup}_2 \cdot P\varphi : PX \to 2. \tag{11}$$

Therefore we consider, for any topological space X,

$$\Lambda(X) := \{\varphi : X \to \mathsf{2} \mid \varphi \text{ is continuous and left adjoint}\}$$

which becomes an ordered set with the pointwise order. In the sequel we will write $\mathcal{C}(X)$ for the coframe of all continuous maps of type $X \to \mathsf{2}$. Note that $\varphi : X \to \mathsf{2}$ is left adjoint in Top if and only if it is continuous and left adjoint in Ord (with respect to the underlying orders). The first hint that we are on the right track is

7.5. LEMMA. *For each frame L, the map $\rho_L : L \to \Lambda(FL)^{\mathrm{op}}$ sending $x \in L$ to*

$$\varphi_x : FL \to \mathsf{2},\, \mathfrak{f} \mapsto \begin{cases} 1 & x \notin \mathfrak{f} \\ 0 & x \in \mathfrak{f} \end{cases}$$

is an isomorphism in Ord.

PROOF. First note that φ_x is the characteristic map of the complement of $x^{\#}$, hence it is continuous. Furthermore, φ_x preserves suprema (=intersection), hence it is left adjoint. From

$$x \le y \iff \forall \mathfrak{f} \in FL\,.\,(x \in \mathfrak{f} \Rightarrow y \in \mathfrak{f}) \iff \varphi_y \le \varphi_x$$

we deduce that $L \to \Lambda(FL)^{\mathrm{op}}$ is an order-embedding. Let now $\varphi : FL \to \mathsf{2}$ be continuous and left adjoint. Put $\mathcal{B} = \varphi^{-1}(0)$ and $\mathfrak{f} = \bigvee \mathcal{B}$. Since φ preserves suprema, $\varphi(\mathfrak{f}) = 0$ and therefore $\mathfrak{f} \in \mathcal{B}$. Since $\mathcal{B}$ is open, there is some $x \in \mathfrak{f}$ with $x^{\#} \subseteq \mathcal{B}$. Hence $\uparrow x \le \mathfrak{f}$, that is, $\mathfrak{f} \subseteq \uparrow x$, and therefore $\mathfrak{f} = \uparrow x$. We conclude that $\varphi = \varphi_x$. ∎

7.6. PROPOSITION. *Let X be a completely distributive space with $t_X \dashv \mathrm{Sup}_X \dashv y_X$. Then the inclusion map $i : \Lambda(X) \to \mathcal{C}(X)$ has a right adjoint $r : \mathcal{C}(X) \to \Lambda(X)$ given by $r(\varphi) = \varphi_L \cdot t_X$ (see* (11)*). Moreover, r preserves finite suprema.*

PROOF. First note that $r(\varphi)$ is left adjoint since it is a composite of left adjoint. Furthermore, $i \cdot r \le 1$ since $\varphi = \varphi_L \cdot y_X \ge \varphi_L \cdot t_X$ for any $\varphi \in \mathcal{C}(X)$, and $r \cdot i = 1$ since $\varphi = \varphi \cdot \mathrm{Sup}_X \cdot t_X = \mathrm{Sup}_{\mathsf{2}} \cdot P\varphi \cdot t_X = \varphi_L \cdot t_X$ for each left adjoint $\varphi : X \to \mathsf{2}$. Finally, $r : \mathcal{C}(X) \to \Lambda(X)$ is the corestriction of

$$\mathcal{C}(X) \xrightarrow{\simeq} \Lambda(PX) \xrightarrow{\text{left adjoint}} \mathcal{C}(PX) \xrightarrow{\text{coframe homom. induced by } t_X} \mathcal{C}(X),$$

therefore r preserves finite suprema. ∎

7.7. COROLLARY. *For each completely distributive space X, $\Lambda(X)$ is a coframe.*

For any left adjoint $g : Y \to X$ in Top, composition with g defines a monotone map

$$\Lambda(g) : \Lambda(X) \to \Lambda(Y),\, \varphi \mapsto \varphi \cdot g.$$

Furthermore, since

$$\begin{tikzcd} \Lambda(X) \arrow[r, "\Lambda(g)"] \arrow[d] & \Lambda(Y) \arrow[d] \\ \mathcal{C}(X) \arrow[r, "\mathcal{C}(g)"'] & \mathcal{C}(Y) \end{tikzcd}$$

commutes, $\Lambda(g)$ preserves finite suprema. For X in $\mathsf{CDTop}_{\inf}$ we put $\mathrm{Pt}(X) := \Lambda(X)^{\mathrm{op}}$, and for $f : X \to Y$ in $\mathsf{CDTop}_{\inf}$ with left adjoint $g : Y \to X$ we define $\mathrm{Pt}(f) = \Lambda(g)^{\mathrm{op}}$. Then

7.8. Proposition. $\mathrm{Pt} : \mathsf{CDTop}_{\inf} \to \mathsf{Frm}_{\wedge}$ *is a 2-functor.*

Furthermore, we revise Lemma 7.5:

7.9. Lemma. ρ_L *is the L-component of a natural isomorphism* $\rho : 1_{\mathsf{Frm}_{\wedge}} \to \mathrm{Pt}\, F$.

Proof. Use (10) to conclude naturality. ∎

For a space X in $\mathsf{CDTop}_{\inf}$, we put

$$\sigma_X : X \to F\,\mathrm{Pt}(X),\; x \mapsto \{\varphi \in \Lambda(X) \mid \varphi(x) = 0\}.$$

7.10. Lemma. σ_X *is surjective.*

Proof. Let $\mathfrak{j} \subseteq \Lambda(X)$ be an ideal. For any $\varphi \in \mathfrak{j}$, put $A_\varphi := \{x \in X \mid \varphi(x) = 0\}$ and $x_\varphi := \bigvee A_\varphi$. Since $x_\psi \le x_\varphi$ for $\varphi \le \psi \in \mathfrak{j}$, the association $\varphi \mapsto x_\varphi$ defines a codirected diagram $D : \mathfrak{j}^{\mathrm{op}} \to X$. Let $x = \bigwedge_{\varphi \in \mathfrak{j}} x_\varphi$. By continuity, $\varphi(x) = 0$ for every $\varphi \in \mathfrak{j}$. Let now $\varphi_0 \in \Lambda(X)$ with $\varphi_0 \notin \mathfrak{j}$. For any $\varphi \in \mathfrak{j}$, $\varphi_0 \not\le \varphi$ and therefore there is some $x \in A_\varphi$ with $\varphi_0(x) = 1$, hence $\varphi_0(x_\varphi) = 1$. Consequently, $\varphi_0(x) = 1$. ∎

By definition, any space $X = FL$ for some frame L has a basis for the closed sets formed by the complements of the opens $x^{\#}$ $(x \in L)$. The characteristic map of such a basic closed set is left adjoint (see Lemma 7.5), hence any $\varphi \in \mathcal{C}(X)$ is the infimum of elements of $\Lambda(X)$. Via the adjunction $t_X \dashv \mathrm{Sup}_X$ one can transport this property to any completely distributive space X as follows. For any $\varphi \in \mathcal{C}(X)$, $\varphi \cdot \mathrm{Sup}_X \in \mathcal{C}(PX)$, hence $\varphi \cdot \mathrm{Sup}_X \simeq \bigwedge_i \varphi_i$ in $\mathcal{C}(PX)$ with all $\varphi_i : PX \to 2$ left adjoint, and therefore $\varphi \simeq \varphi \cdot \mathrm{Sup}_X \cdot t_X \simeq (\bigwedge_i \varphi_i) \cdot t_X \simeq \bigwedge_i (\varphi_i \cdot t_X)$.

7.11. Lemma. *For each completely distributive space X and $x, y \in X$ with $x \not\simeq y$, $\sigma_X(x) \neq \sigma(y)$.*

Proof. If, for instance, $y \notin \mathrm{cl}\{x\}$, then there exists some "left adjoint closed subset" $B \subseteq X$ with $y \notin B$ and $x \in B$. ∎

7.12. Proposition. *For any $X \in \mathsf{CDTop}_{\mathrm{inf}}$, $\sigma_X : X \to F\,\mathrm{Pt}(X)$ is an isomorphism.*

Proof. We know already that $\sigma_X : X \to F\,\mathrm{Pt}(X)$ is bijective. To see continuity, notice that
$$\sigma_X^{-1}(\varphi^{\#}) = \{x \in X \mid \varphi(x) = 0\}$$
for any $\varphi \in \Lambda(X)$. Let now $B \subseteq X$ be closed with left adjoint characteristic map $\varphi : X \to 2$. Then
$$\sigma_X(B) = \{\sigma_X(x) \mid x \in B\} = F\,\mathrm{Pt}(X) \setminus (\varphi^{\#}).$$
Clearly, $\varphi \notin \sigma_X(x)$ for any $x \in B$. Let now $\mathfrak{j} \subseteq \Lambda(X)$ be an ideal with $\varphi \notin \mathfrak{j}$. One has $\mathfrak{j} = \sigma_X(x)$ for some $x \in X$ and, since $\varphi \notin \sigma_X(x)$, $x \in B$. ∎

7.13. Lemma. $\sigma = (\sigma_X)_X$ *is a natural isomorphism* $\sigma : 1_{\mathsf{CDTop}_{\mathrm{inf}}} \to F\,\mathrm{Pt}$.

Proof. We have to show the naturality condition. To this end, let $f : X \to Y$ in $\mathsf{CDTop}_{\mathrm{inf}}$ with left adjoint $g : Y \to X$. We identify $\Lambda(X)$ with the set of all "left adjoint closed subsets" of X, and $\sigma_X(x) = \{A \in \Lambda(X) \mid x \notin A\}$. Then
$$\Downarrow\!\{g^{-1}(A) \mid x \notin A\} = \{B \in \Lambda(Y) \mid x \notin f^{-1}(B)\} = \{B \in \Lambda(Y) \mid f(x) \notin B\}.$$
∎

7.14. Theorem. $F : \mathsf{Frm}_\wedge \to \mathsf{CDTop}_{\mathrm{inf}}$ *and* $\mathrm{Pt} : \mathsf{CDTop}_{\mathrm{inf}} \to \mathsf{Frm}_\wedge$ *define an equivalence of categories.*

7.15. Corollary. *A topological space is equivalent to the filter space of some frame if and only if it is completely distributive.*

Throughout we have emphasised that both F and Pt are 2-functors, hence the subcategories of $\mathsf{Frm}_\wedge$ and $\mathsf{CDTop}_{\mathrm{inf}}$ defined by the left adjoint morphisms are equivalent as well. Therefore

7.16. Theorem. Frm *is equivalent to* CDTop.

7.17. Remark. The results of this section tell us that CDTop is actually a very nice category: it is monadic over Set. However, we have to take here the "right" forgetful functor $\mathsf{CDTop} \to \mathsf{Set}$ (see also Remark 6.4); namely the one which sends $X \in \mathsf{CDTop}$ to the set of all right adjoint continuous maps of type $2 \to X$. Any such map necessarily sends 1 to the top element of X, hence it is completely determined by the image of 0. But note that, unlike in ordered sets, not every $x \in X$ defines a right adjoint via $0 \mapsto x$. Therefore our result really extends the well-known fact that the canonical forgetful functor $\mathsf{CCD} \to \mathsf{Set}$ is monadic. I do not know yet if the corresponding functor $\mathsf{CDApp} \to \mathsf{Set}$, $X \mapsto \mathrm{Map}(\mathsf{App})(X, [0,\infty])$ is monadic.

8. Spaces with weighted colimits of a certain class

So far we have studied spaces which admit *all* suprema; however, it is often desirable to limit the discussion to certain chosen ones. This is, for instance, the case in domain theory where one typically considers directed cocomplete ordered sets, and the "directed version" of complete distributivity is called continuity. The main point for us here is that many results are valid for both cases, one just has to write JX (the ordered set of all directed down-sets) instead of PX everywhere.

Therefore it seems reasonable to start with a specification of certain $\mathbb{U}$-modules, and to study spaces which admit all suprema of $\mathbb{U}$-modules belonging to this specified class. This is indeed a well-known procedure in the context of enriched category theory, we refer to [Kelly, 1982; Albert and Kelly, 1988; Kelly and Schmitt, 2005; Kelly and Lack, 2000]. A similar investigation of relative cocompleteness for $(\mathbb{T},\mathsf{V})$-categories (hence for topological and approach spaces) was done in [Clementino and Hofmann, 2009b]. There seems to be no equal treatment of relative distributivity (or continuity) in the literature, but some initial steps are done in [Hofmann and Waszkiewicz, 2011]. We also wish to point the reader to [Stubbe, 2007] where an extensive study of complete distributivity in the context of quantaloid enriched categories can be found.

Following [Kelly and Schmitt, 2005], one might want to start with a collection $\Phi[X]$ of $\mathbb{U}$-modules of type $X \,{-\!\!\circ\!\!\rightharpoonup}\, 1$, for each space X, where $\Phi[X]$ contains all representable modules $x^* : X \,{-\!\!\circ\!\!\rightharpoonup}\, 1$ ($x \in X$). Then a Φ-weighted diagram in a space X is given by a morphism $d : D \to X$ in $\mathsf{Top}/\mathsf{App}$ and a $\mathbb{U}$-module $\psi : D \,{-\!\!\circ\!\!\rightharpoonup}\, 1$ in $\Phi[D]$. A colimit of such a diagram is an element $x \in X$ which represents $d_* \multimapdotinv \psi$, that is, $x_* = d_* \multimapdotinv \psi$. One calls x a ψ-weighted colimit of d and writes $x \simeq \operatorname{colim}(d,\varphi)$. Furthermore, a continuous/contractive map $f : X \to Y$ preserves the ψ-weighted colimit of d if $f(\operatorname{colim}(\psi,d)) \simeq \operatorname{colim}(\psi, f\cdot d)$. If the family $\Phi[X]$ is functorial in the sense that, for all $f : X \to Y$ in $\mathsf{Top}/\mathsf{App}$ and all $\psi \in \Phi[X]$, $\psi \circ f^* \in \Phi[Y]$, then it is enough to consider weighted diagrams where d is the identity $1_X : X \to X$ since the diagrams $(d : D \to X, \psi : D \,{-\!\!\circ\!\!\rightharpoonup}\, 1)$ and $(1_X : X \to X, \psi \circ d^* : X \,{-\!\!\circ\!\!\rightharpoonup}\, 1)$ share the same colimit. Finally, it is often convenient to assume that the family $\Phi[X]$ is ***saturated***, meaning that $i : \Phi[X] \hookrightarrow PX$ is closed under Φ-weighted colimits, for each space X. As we will see below, saturated implies functorial.

One would then call a space X Φ-cocomplete if X admits all colimits weighted by some $\psi : X \,{-\!\!\circ\!\!\rightharpoonup}\, 1$ in $\Phi[X]$. However, the situation for spaces is a bit more complicated than the one for enriched categories which is already visible in the global case where $\Phi[X] = PX$ contains all $\mathbb{U}$-modules of type $X \,{-\!\!\circ\!\!\rightharpoonup}\, 1$. If X is cocomplete, then $\operatorname{Sup}_X : PX \to X$ calculates for each weighted diagram $1_X : X \to X$, $\psi : X \,{-\!\!\circ\!\!\rightharpoonup}\, 1$ in X a colimit $\operatorname{Sup}_X(\psi)$, however, the existence of all weighted colimits does not guarantee cocompleteness of X. In fact, [Hofmann and Waszkiewicz, 2011] presents an example of a topological space X which admits all suprema of $\mathbb{U}$-modules of type $X \,{-\!\!\circ\!\!\rightharpoonup}\, 1$ but X is not cocomplete. The problem here is that the induced map $PX \to X, \psi \mapsto x$ does not need to be continuous, and therefore is in general only a right adjoint to $\boldsymbol{y}_X : X \to PX$ in Ord.

The situation changes if we allow $\mathbb{U}$-modules $\psi : D \,{-\!\!\circ\!\!\rightharpoonup}\, A$ in the definition of weighted colimits, where A might be different from the one-point space 1. A ***colimit*** of a weighted diagram (weighted colimit for short) is now a continuous/contractive map $g : A \to X$ which represents $d_* \multimapinv \psi$, that is, $g_* = d_* \multimapinv \psi$. With this modification it is indeed true that X is cocomplete if and only if X admits all weighted colimits (see [Hofmann, 2011]). In other words, X admits "continuously" suprema of all $\mathbb{U}$-modules $\psi : X \relbar\joinrel\mapstochar\joinrel\rightharpoonup 1$ if and only if X admits weighted colimits of all $\mathbb{U}$-modules $\psi : X \relbar\joinrel\mapstochar\joinrel\rightharpoonup A$.

The following example of a weighted colimit is essentially taken from [Stubbe, 2010] and will be used later.

8.1. Example. [Composition as a colimit] Let $\varphi : X \relbar\joinrel\mapstochar\joinrel\rightharpoonup Y$ and $\psi : Y \relbar\joinrel\mapstochar\joinrel\rightharpoonup Z$ be $\mathbb{U}$-modules, and consider the diagram

$$\begin{tikzcd} Y \arrow[r, "\ulcorner\varphi\urcorner"] \arrow[d, "\psi"', "\circ" marking] & PX. \\ Z & \end{tikzcd}$$

Then $\mathrm{colim}(\ulcorner\varphi\urcorner, \psi) = \ulcorner\psi \circ \varphi\urcorner$.

Therefore what we need is not just a choice of $\mathbb{U}$-modules of type $X \relbar\joinrel\mapstochar\joinrel\rightharpoonup 1$ but rather a class $\mathcal{K}$ of $\mathbb{U}$-modules $\varphi : X \relbar\joinrel\mapstochar\joinrel\rightharpoonup Y$. One possibility is to extend the given family $\Phi[X]$ to such a class by defining, for $\varphi : X \relbar\joinrel\mapstochar\joinrel\rightharpoonup Y$ in $\mathbb{U}\text{-}\mathsf{Mod}$,

$$\varphi : X \relbar\joinrel\mapstochar\joinrel\rightharpoonup Y \text{ in } \mathcal{K} \quad \text{whenever} \quad \forall y \in Y \,.\, y^* \circ \varphi \in \Phi[X].$$

Note that, for any $f : Z \to Y$ in $\mathsf{Top}/\mathsf{App}$, the $\mathbb{U}$-module f^* belongs to $\mathcal{K}$, and $f^* \circ \varphi$ is in $\mathcal{K}$ whenever $\varphi : X \relbar\joinrel\mapstochar\joinrel\rightharpoonup Y$ is in $\mathcal{K}$. In [Stubbe, 2010] it is shown (in the context of quantaloid-enriched categories, but the argument is based on Example 8.1 and therefore adapts easily to our case) that the family $\Phi[X]$ is saturated if and only if the correspdonding class $\mathcal{K}$ is actually a subcategory of $\mathbb{U}\text{-}\mathsf{Mod}$. In [Clementino and Hofmann, 2009b] we went the other way around and started with a class $\mathcal{K}$ of $\mathbb{U}$-modules containing all $\mathbb{U}$-modules of the form f^*, closed under certain compositions (see below), and such that

$$(\forall y \in Y \,.\, y^* \circ \varphi \in \mathcal{K}) \iff \varphi \in \mathcal{K} \tag{12}$$

for all $\varphi : X \relbar\joinrel\mapstochar\joinrel\rightharpoonup Y \in \mathbb{U}\text{-}\mathsf{Mod}$. Note that (12) guarantees already that $\mathcal{K}$ is closed under compositions of the form $f^* \circ \varphi$. Combining [Stubbe, 2010] with [Clementino and Hofmann, 2009b] gives

8.2. Theorem. *Assume that a family $\Phi[X]$ of $\mathbb{U}$-modules of type $X \relbar\joinrel\mapstochar\joinrel\rightharpoonup 1$ (X in Top or App) is given, and define $\mathcal{K}$ as above. Then the following assertions are equivalent.*

i. *The family $\Phi[X]$ is saturated.*

ii. *$\mathcal{K}$ is a subcategory of $\mathbb{U}\text{-}\mathsf{Mod}$.*

iii. *For all $\psi : X \relbar\joinrel\mapstochar\joinrel\rightharpoonup 1$ in $\Phi[X]$ and all continuous/contractive maps $f : X \to Y$ and $g : Y \to X$ where $g_* \in \mathcal{K}$,*

$$\psi \circ f^* \in \Phi[Y] \qquad \textit{and} \qquad \psi \circ g_* \in \Phi[Y].$$

PROOF. By definition, $\varphi : X \,{-\!\!\circ\!\!\rightharpoondown}\, Y$ belongs to $\mathcal{K}$ if and only if $\ulcorner\varphi\urcorner : Y \to PX$ factors through $\Phi[X] \hookrightarrow PX$. Assume (i) and let $\varphi : X \,{-\!\!\circ\!\!\rightharpoondown}\, Y$ and $\psi : Y \,{-\!\!\circ\!\!\rightharpoondown}\, Z$ be in $\mathcal{K}$. Then $\ulcorner z^* \circ \psi \circ \varphi\urcorner : 1 \to PX$ factors through $\Phi[X] \hookrightarrow PX$, for each $z \in Z$, hence $\psi \circ \varphi$ belongs to $\mathcal{K}$. The implication (ii)$\Rightarrow$(iii) is clear, so assume now (iii). Since $\mathcal{K}$ is closed under compositions of the form $\varphi \circ f^*$, it is enough to show that $i : \Phi[X] \hookrightarrow PX$ is closed under suprema of $\mathbb{U}$-modules of type $\Phi[X] \,{-\!\!\circ\!\!\rightharpoondown}\, 1$ in $\mathcal{K}$. Let $\psi : \Phi[X] \,{-\!\!\circ\!\!\rightharpoondown}\, 1$ be in $\mathcal{K}$. Then the colimit of i and ψ in PX is given by $\psi \circ i^* \circ (\boldsymbol{y}_X)_* \in \Phi[X]$. ∎

8.3. DEFINITION. *A subcategory $\mathcal{K}$ of $\mathbb{U}$-Mod is called* **saturated** *whenever $\mathcal{K}$ satisfies* (12) *and contains f^*, for every $f : X \to Y$ in Top respectively in App.*

8.4. PROPOSITION. *Every saturated family $\Phi[X]$ (X in Top or App) of $\mathbb{U}$-modules of type $X \,{-\!\!\circ\!\!\rightharpoondown}\, 1$ defines via* (12) *a saturated subcategory $\mathcal{K} = (\mathbb{U}, \Phi)$-$\mathsf{Mod}$ of $\mathbb{U}$-Mod; and every saturated subcategory $\mathcal{K}$ of $\mathbb{U}$-Mod defines a saturated family $\Phi[X] := \mathcal{K}(X, 1)$ (X in Top or App) of $\mathbb{U}$-modules of type $X \,{-\!\!\circ\!\!\rightharpoondown}\, 1$. Moreover, these two constructions are inverse to each other.*

Due to the considerations above, throughout this section we assume that a saturated subcategory $(\mathbb{U}, \Phi)$-Mod of $\mathbb{U}$-Mod is given. Following the nomenclature of [Clementino and Hofmann, 2009b], a continuous/contractive map $f : X \to Y$ is called ***Φ-dense*** if $f_* \in (\mathbb{U}, \Phi)$-$\mathsf{Mod}$, and a topological/approach space X is called ***Φ-injective*** if it is injective w.r.t. Φ-dense embeddings. Furthermore, we define

$$\Phi X = \{\psi \in PX \mid \psi \in (\mathbb{U}, \Phi)\text{-}\mathsf{Mod}\}$$

as a subspace of PX. One verifies easily that the Yoneda embedding $\boldsymbol{y} : X \to PX$ corestricts to a Φ-dense mapping $\boldsymbol{y}_X^\Phi : X \to \Phi X$. For each $\mathbb{U}$-module $\varphi : X \,{-\!\!\circ\!\!\rightharpoondown}\, Y$, $\varphi \in (\mathbb{U}, \Phi)$-$\mathsf{Mod}$ if and only if its mate $\ulcorner\varphi\urcorner : Y \to PX$ factors through the embedding $\Phi X \hookrightarrow PX$.

For a $\mathbb{U}$-module $\varphi : X \,{-\!\!\circ\!\!\rightharpoondown}\, Y$ in $(\mathbb{U}, \Phi)$-Mod, $- \circ \varphi : PX \to PY$ sends $\psi \in \Phi X$ to $\psi \circ \varphi \in \Phi X$ and therefore restricts to $- \circ \varphi : \Phi X \to \Phi Y$. In particular, $Pf : PX \to PY$ restricts to $\Phi f : \Phi X \to \Phi Y$ since $f^* \in (\mathbb{U}, \Phi)$-$\mathsf{Mod}$. The right adjoint $- \circ f_*$ of Pf restricts to a right adjoint of Φf if f is Φ-dense. In fact, it is shown in [Clementino and Hofmann, 2009b] that f is Φ-dense if and only if Φf has a right adjoint.

8.5. DEFINITION. *A topological/approach space X is called* **Φ-cocomplete** *whenever X has all weighted colimits where the weight $\psi : D \,{-\!\!\circ\!\!\rightharpoondown}\, A$ belongs to $(\mathbb{U}, \Phi)$-Mod. A continuous/contractive map $f : X \to Y$ is called* **Φ-cocontinuous** *if it preserves all Φ-weighted colimits which exist in X.*

The following four results can be found in [Clementino and Hofmann, 2009b].

8.6. THEOREM. *The following assertions are equivalent, for a topological/approach space X.*

i. X is Φ-cocomplete.

ii. $\boldsymbol{y}^{\Phi}_X : X \to \Phi X$ *has a left adjoint* $\mathrm{Sup}^{\Phi}_X : \Phi X \to X$.

iii. X *is* Φ*-injective.*

8.7. Proposition. *Let* $f : X \to Y$ *be a continuous/contractive map between* Φ*-cocomplete spaces.*

a. f *is* Φ*-cocontinuous if and only if* $f \cdot \mathrm{Sup}^{\Phi}_X \simeq \mathrm{Sup}^{\Phi}_Y \cdot \Phi f$.

b. f *is* Φ*-cocontinuous and* Φ*-dense if and only if* f *is left adjoint.*

8.8. Corollary. *For each space* X*,* ΦX *is* Φ*-cocomplete where* $\mathrm{Sup}^{\Phi}_{\Phi X} = - \circ (\boldsymbol{y}^{\Phi}_X)_*$*. Furthermore, the inclusion map* $\Phi X \hookrightarrow PX$ *is* Φ*-cocontinuous.*

8.9. Theorem. *The subcategory* $\Phi\text{-}\mathsf{Cocts}_{\mathrm{sep}}$ *of* $\mathsf{Top}/\mathsf{App}$ *consisting of* Φ*-cocomplete* T_0*-spaces and* Φ*-cocontinuous morphisms is reflective with the Yoneda embedding as universal arrow. Furthermore, the inclusion functor* $\Phi\text{-}\mathsf{Cocts}_{\mathrm{sep}} \to \mathsf{Top}/\mathsf{App}$ *is even monadic. The induced monad* $\mathbb{\Phi} = (\Phi, \boldsymbol{y}^{\Phi}, \boldsymbol{m}^{\Phi})$ *is of Kock-Zöberlein type and has* Φ *as functor, the Yoneda embeddings* $\boldsymbol{y}^{\Phi}_X : X \to \Phi X$ *as units and* $\boldsymbol{m}^{\Phi}_X := - \circ (\boldsymbol{y}^{\Phi}_X)_* : \Phi\Phi X \to \Phi X$ *as multiplications.*

8.10. Theorem. *Let* $(\mathbb{U}, \Phi)\text{-}\mathsf{Mod}$ *be a saturated subcategory of* $\mathbb{U}\text{-}\mathsf{Mod}$*. Then* $(\mathbb{U}, \Phi)\text{-}\mathsf{Mod}$ *is dually 2-equivalent to the Kleisli category of* $\mathbb{\Phi} = (\Phi, \boldsymbol{y}^{\Phi}, \boldsymbol{m}^{\Phi})$ *on* $\mathsf{Top}/\mathsf{App}$*.*

Proof. We have seen already that $\mathbb{U}$-modules $X \multimapdot Y$ in $(\mathbb{U}, \Phi)\text{-}\mathsf{Mod}$ are in bijection with continuous/contractive maps of type $Y \to \Phi X$, where the identity distributor $a : X \multimapdot X$ corresponds to the Yoneda embedding $\boldsymbol{y}^{\Phi}_X : X \to \Phi X$. Let $\varphi : X \multimapdot Y$ and $\psi : Y \multimapdot Z$ be in $(\mathbb{U}, \Phi)\text{-}\mathsf{Mod}$. By Example 8.1,

$$\ulcorner \psi \circ \varphi \urcorner = \mathrm{colim}(\ulcorner \varphi \urcorner, \psi) = \mathrm{Sup}^{\Phi}_{\Phi X} \cdot \Phi \ulcorner \varphi \urcorner \cdot \ulcorner \psi \urcorner = \boldsymbol{m}^{\Phi}_X \cdot \Phi \ulcorner \varphi \urcorner \cdot \ulcorner \psi \urcorner.$$

∎

The notion of complete distributivity generalises in an obvious way to this relative case, and was studied in this context under the name "continuity" in [Hofmann and Waszkiewicz, 2011].

8.11. Definition. *Let* $(\mathbb{U}, \Phi)\text{-}\mathsf{Mod}$ *be a saturated subcategory of* $\mathbb{U}\text{-}\mathsf{Mod}$*. A* Φ*-cocomplete topological/approach space* X *is called* **Φ-distributive** *whenever* $\mathrm{Sup}^{\Phi}_X : \Phi X \to X$ *has a left adjoint* $t : X \to \Phi X$*.*

One naturally expects that the proofs of Section 6 can be adapted to this case leading to a duality theorem for "Φ-algebraic spaces". It is the aim of the remainder of this section to show that this is indeed the case.

More generally, R. Rosebrugh and R.J. Wood showed in [Rosebrugh and Wood, 1994] that the category $\mathsf{CCD}_{\mathrm{sup}}$ of constructively completely distributive lattices and suprema preserving maps is equivalent to the idempotent split completion kar(Rel) of the category Rel of sets and relations, as well as to the idempotent split completion kar(Mod) of the category Mod of ordered sets and modules. Later on, in [Rosebrugh and Wood, 2004] they observed that this theorem is "not really about lattices" but rather a special case of a much more general result about "a mere monad $\mathbb{D}$ on a mere category C".

8.12. THEOREM. *Let $\mathbb{D}$ be a monad on a category C where idempotents split. Then*

$$\mathrm{kar}(\mathsf{C}_{\mathbb{D}}) \simeq \mathrm{Spl}(\mathsf{C}^{\mathbb{D}}).$$

Here $\mathsf{C}_{\mathbb{D}}$ denotes the Kleisli and $\mathsf{C}^{\mathbb{D}}$ the Eilenberg–Moore category of $\mathbb{D}$, $\mathrm{kar}(\mathsf{C}_{\mathbb{D}})$ the idempotent split completion of $\mathsf{C}_{\mathbb{D}}$ and $\mathrm{Spl}(\mathsf{C}^{\mathbb{D}})$ the full subcategory of $\mathsf{C}^{\mathbb{D}}$ defined by those algebras (X,α) which admit a splitting $t : X \to DX$, $\alpha \cdot t = 1_X$ in $\mathsf{C}^{\mathbb{D}}$.

PROOF. See [Rosebrugh and Wood, 2004]. ∎

Note that, as shown in [Rosebrugh and Wood, 2004], if $\mathbb{D}$ is of Kock-Zöberlein type, then a $\mathbb{D}$-algebra (X,α) admits at most one splitting which is necessarily left adjoint to α.

Since it is important for the remainder of this section, below we explain this result more in detail. Recall that an idempotent morphism $e : X \to X$ in a category A splits if $e = s \cdot r$ for $r : X \to Y$ and $s : Y \to X$ in A with $r \cdot s = 1_Y$. One says that idempotents split in A if every idempotent is of this form. Most "everyday" categories have this property since s can be taken as the equaliser of e and 1_X and necessarily r as the induced morphism, or r as the coequaliser of e and 1_X and s as the induced morphism; supposing here that these (co)limits exist. The arguably most prominent example of a (highly) non-complete category is Rel, and for instance the idempotent relation $<: \mathbb{R} \to\!\!\!\!\!\!\mapsto \mathbb{R}$ does not split in Rel. In any case, the idempotent split completion $\mathrm{kar}(\mathsf{A})$ of A has as objects pairs (X,e) where e is idempotent, and a morphism $f : (X,e) \to (X',e')$ in $\mathrm{kar}(\mathsf{A})$ is an A-morphism $f : X \to X'$ such that $e' \cdot f = f = f \cdot e$. The category A is fully embedded into $\mathrm{kar}(\mathsf{A})$ via $X \mapsto (X,1_X)$, all idempotents split in $\mathrm{kar}(\mathsf{A})$ and it is indeed the free idempotent split completion of A. To explain the latter, let $F : \mathsf{A} \to \mathsf{B}$ be a functor where idempotents split in B. One can construct now the (essentially unique) extension $\tilde{F} : \mathrm{kar}(\mathsf{A}) \to \mathsf{B}$ as follows. For any object (X,e) in $\mathrm{kar}(\mathsf{A})$, define $\tilde{F}(X,e)$ as the idempotent split $FX \xrightarrow{r} \tilde{F}(X,e) \xrightarrow{s} FX$ of the idempotent Fe in B; and for a morphism $f : (X,e) \to (X',e')$ in $\mathrm{kar}(\mathsf{A})$ put $\tilde{F}f = r' \cdot Ff \cdot s$ where r' and s' split Fe'.

A category where idempotents split is sometimes also called Cauchy complete, due to the fact that in the language of modules both properties (for categories and metric spaces respectively) are instances of the same definition. Therefore many properties we know about Cauchy completion of metric spaces are shared by $\mathrm{kar}(\mathsf{A})$, for instance:

8.13. LEMMA. *Let A be a full subcategory of B and assume that idempotents split in B. Let $\overline{\mathsf{A}}$ be the full subcategory of B defined by the retracts of the objects in A. Then idempotents split in $\overline{\mathsf{A}}$ and $\mathsf{A} \to \overline{\mathsf{A}}$ is the free idempotent split completion of A.*

PROOF. Every idempotent in $\overline{\mathsf{A}}$ splits in B and the splitting belongs to $\overline{\mathsf{A}}$. By definition, every B in $\overline{\mathsf{A}}$ splits some idempotent $e : A \to A$ in A. If B splits also $e' : A' \to A'$

in A, so that $A \xrightarrow{r} B \xrightarrow{s} A$ and $A' \xrightarrow{r'} B \xrightarrow{s} A'$ with $e = sr$, $rs = 1_B$ and $e' = s'r'$, $r's' = 1_B$, then $s'r : (A, e) \to (A', e')$ and $sr' : (A', e') \to (A, e)$ are inverse to each other in $\mathrm{kar}(\mathsf{A})$. Choosing for every B in $\overline{\mathsf{A}}$ such an idempotent $e : A \to A$ in A defines the object part of a functor $G : \overline{\mathsf{A}} \to \mathrm{kar}(\mathsf{A})$, which sends a morphism $f : B \to B'$ in $\overline{\mathsf{A}}$ to $s'fr : (A, e) \to (A', e')$ in $\mathrm{kar}(\mathsf{A})$. With $F : \mathrm{kar}(\mathsf{A}) \to \overline{\mathsf{A}}$ denoting a functor induced by the universal property, one verifies that both GF and FG are naturally isomorphic to the identity. ∎

Clearly, every algebra (X, α) which admits a splitting $t : X \to DX$ is a retract of the free algebra DX. Vice versa, if (X, α) is a retract of a free algebra, then (X, α) is projective with respect to those morphisms in $\mathsf{C}^{\mathbb{D}}$ which are split epimorphisms in C, hence $\alpha : DX \to X$ admit a splitting $t : X \to DX$. Consequently, $\mathrm{Spl}(\mathsf{C}^{\mathbb{D}})$ is the free idempotent split completion of full subcategory of $\mathsf{C}^{\mathbb{D}}$ defined by the free algebras which is known to be equivalent to $\mathsf{C}_{\mathbb{D}}$, and Theorem 8.12 follows.

Our principal object of interest here is the monad $\Phi = (\Phi, \boldsymbol{y}^{\Phi}, \boldsymbol{m}^{\Phi})$ on $\mathsf{Top}/\mathsf{App}$, for a given saturated subcategory $(\mathbb{U}, \Phi)$-Mod of $\mathbb{U}$-Mod. We know already (see Theorem 8.9) that the category of Eilenberg–Moore algebras of Φ has Φ-cocomplete T_0-spaces as objects, and Φ-cocontinuous maps as morphisms. The objects of $\mathrm{Spl}(\mathsf{Top}^{\Phi})$ respectively $\mathrm{Spl}(\mathsf{App}^{\Phi})$ are precisely the Φ-distributive T_0-spaces, and we denote the category of Φ-distributive T_0-spaces and Φ-cocontinuous maps as Φ-$\mathsf{DTop}_{\mathrm{cocts}}/\Phi$-$\mathsf{DApp}_{\mathrm{cocts}}$.

Combining Theorem 8.12 with Theorem 8.10 yields

8.14. Theorem. $\mathrm{kar}((\mathbb{U}, \Phi)\text{-}\mathsf{Mod})^{\mathrm{op}} \simeq \Phi\text{-}\mathsf{DTop}_{\mathrm{cocts}}/\Phi\text{-}\mathsf{DApp}_{\mathrm{cocts}}$.

Below we give a description of the corresponding equivalence functors

$$\mathrm{kar}((\mathbb{U}, \Phi)\text{-}\mathsf{Mod})^{\mathrm{op}} \xrightarrow{S} \Phi\text{-}\mathsf{DTop}_{\mathrm{cocts}}/\Phi\text{-}\mathsf{DApp}_{\mathrm{cocts}}$$

and

$$\Phi\text{-}\mathsf{DTop}_{\mathrm{cocts}}/\Phi\text{-}\mathsf{DApp}_{\mathrm{cocts}} \xrightarrow{I} \mathrm{kar}((\mathbb{U}, \Phi)\text{-}\mathsf{Mod})^{\mathrm{op}}.$$

Of course, the equivalence of Theorem 8.14 is induced by the equivalence

$$\varphi : X \,-\!\!\circ\!\!\rightharpoonup X' \mapsto (- \circ \varphi) : \Phi X' \to \Phi X$$

between $(\mathbb{U}, \Phi)$-$\mathsf{Mod}^{\mathrm{op}}$ and the full subcategory of Φ-$\mathsf{Cocts}_{\mathrm{sep}}$ defined by the free algebras. Accordingly, the functors S and I can be constructed as follows. For (X, θ) in $\mathrm{kar}((\mathbb{U}, \Phi)\text{-}\mathsf{Mod})$, let $\Phi X \xrightarrow{r} S(X, \theta) \xrightarrow{s} \Phi X$ be a splitting of the idempotent $- \circ \theta : \Phi X \to \Phi X$; to have something concrete,

$$S(X, \theta) = \{\psi \in \Phi \mid \psi \circ \theta = \psi\},$$

$r : \Phi X \to S(X, \theta)$, $\psi \mapsto \psi \circ \theta$ and $s : S(X, \theta) \to \Phi X$ is the inclusion functor. If $\varphi : (X, \theta) \to (X', \theta')$, then $S\varphi : S(X, \theta) \to S(X', \theta')$ sends $\psi \in S(X, \theta)$ to $\psi \circ \theta$. Let now

X be a Φ-distributive T_0-space with $\mathsf{y}_X^\Phi \vdash \mathrm{Sup}_X^\Phi \vdash t$. Then $t : X \to \Phi X$ corresponds to a module $\theta : X \mathrel{-\!\!\circ\!\!\rightharpoonup} X$ in $(\mathbb{U},\Phi)$-Mod which is necessarily idempotent. Furthermore, $\Phi X \xrightarrow{\mathrm{Sup}_X^\Phi} X \xrightarrow{t} \Phi X$ splits the idempotent $-\circ\theta : \Phi X \to \Phi X$, and therefore $I(X)$ can be taken as (X,θ). Accordingly, for $f : X \to X'$ one calculates now $I(f) = \theta' \circ f^* \circ \theta$, in the sequel we denote $\theta' \circ f^* \circ \theta$ also as $f^\#$. Note that both functors S and I are actually 2-functors.

For a Φ-distributive T_0-space X, the natural isomorphism $X \simeq SI(X)$ stems from the fact that both X and $S(X,\theta)$ split the idempotent $-\circ\theta : \Phi X \to \Phi X$. Hence,

$$X \to S(X,\theta),\ x \mapsto x^* \circ \theta \qquad \text{and} \qquad S(X,\theta) \to X,\ \psi \mapsto \mathrm{Sup}_X^\Phi(\psi)$$

are inverse to each other. Certainly, also $(X,\theta) \simeq IS(X,\theta)$ for every object (X,θ) in $\mathrm{kar}((\mathbb{U},\Phi)\text{-}\mathsf{Mod})$, but to describe the natural isomorphism $(X,\theta) \mathrel{-\!\!\circ\!\!\rightharpoonup} IS(X,\theta)$ we need some notation.

For (X,θ) in $\mathrm{kar}((\mathbb{U},\Phi)\text{-}\mathsf{Mod})$ we define $\widehat{\theta} = (-\circ\theta)\cdot \ulcorner\theta\urcorner : X \to S(X,\theta)$, which is indeed just the corestriction of $\ulcorner\theta\urcorner : X \to \Phi X$ to $S(X,\theta)$. Furthermore, we put $\widehat{\theta}_+ = \widehat{\theta}_* \circ \theta$ and $\widehat{\theta}^+ = \theta \circ \widehat{\theta}^*$. Note that $\widehat{\theta}^+ \circ \widehat{\theta}_+ = \theta$ since $\widehat{\theta}^* \circ \widehat{\theta}_* = \ulcorner\theta\urcorner^* \circ \ulcorner\theta\urcorner_* = [\![U\ulcorner\theta\urcorner(-), \ulcorner\theta\urcorner(-)]\!] = \theta \multimapinv \theta$ by Theorem 5.4, idempotency of θ gives $\theta \le \theta \multimapinv \theta$, and therefore $\theta = \theta \circ \theta \circ \theta \le \theta \circ (\theta \multimapinv \theta) \circ \theta \le \theta \circ \theta = \theta$. One easily verifies that the suprema in $S(X,\theta)$ are given by

$$\mathrm{Sup}_{S(X,\theta)}^\Phi : \Phi S(X,\theta) \to S(X,\theta),\ \Psi \mapsto \Psi \circ \widehat{\theta}_+,$$

and the left adjoint of $\mathrm{Sup}_{S(X,\theta)}^\Phi$ by

$$t : S(X,\theta) \to \Phi S(X,\theta),\ \psi \mapsto \psi \circ \widehat{\theta}^+.$$

Therefore $t \cdot \mathrm{Sup}_{S(X,\theta)}^\Phi$ sends ψ to $\psi \circ \widehat{\theta}_+ \circ \widehat{\theta}^+$, hence $t = \ulcorner\omega\urcorner$ for $\omega = \widehat{\theta}_+ \circ \widehat{\theta}^+$. Since $S(X,\theta)$ splits both $-\circ\theta$ and $-\circ\omega$,

$$\begin{tikzcd}
\Phi X \arrow[rr, "-\circ\theta"] \arrow[dr] & & \Phi X \\
& S(X,\theta) \arrow[ur] \arrow[dr, "t"] & \\
\Phi S(X,\theta) \arrow[ur, "\mathrm{Sup}_{S(X,\theta)}^\Phi"] \arrow[rr, "-\circ\omega"'] & & \Phi S(X,\theta)
\end{tikzcd}$$

(X,θ) and $(S(X,\theta),\omega)$ are naturally isomorphic in $\mathrm{kar}((\mathbb{U},\Phi)\text{-}\mathsf{Mod})$ via

$$\widehat{\theta}_+ : (X,\theta) \mathrel{-\!\!\circ\!\!\rightharpoonup} (S(X,\theta),\omega) \qquad \text{and} \qquad \widehat{\theta}^+ : (S(X,\theta),\omega) \mathrel{-\!\!\circ\!\!\rightharpoonup} (X,\theta).$$

Finally, we note that the diagrams

$$\begin{tikzcd}
(X,\theta) \arrow[r, "\varphi", "\circ"{marking}] \arrow[d, "\widehat{\theta}_+"', "\circ"{marking}] & (X',\theta') \arrow[d, "\widehat{\theta'}_+", "\circ"{marking}] \\
(S(X,\theta),\omega) \arrow[r, "(-\circ\varphi)^\#"', "\circ"{marking}] & (S(X',\theta'),\omega')
\end{tikzcd}
\qquad
\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "\widehat{\theta}"'] & X' \arrow[d, "\widehat{\theta'}"] \\
S(X,\theta) \arrow[r, "-\circ f^\#"'] & S(X',\theta')
\end{tikzcd}$$

commute, for $\varphi : (X,\theta) \to (X',\theta')$ in $\mathrm{kar}((\mathbb{U},\Phi)\text{-}\mathsf{Mod})$ and $f : X \to Y$ in $\Phi\text{-}\mathsf{DTop}_{\mathrm{cocts}}$ respectively $\Phi\text{-}\mathsf{DApp}_{\mathrm{cocts}}$.

8.15. Definition. *We call a Φ-distributive T_0-space X* **Φ-algebraic** *if X is isomorphic to a space of form ΦY.*

Moving to the other side of the equivalence, X is Φ-algebraic if and only if (X,θ) is isomorphic to some $(Y,(1_Y)_*)$ in $\mathrm{kar}((\mathbb{U},\Phi)\text{-}\mathsf{Mod})$. Let X be Φ-algebraic, and assume that $\alpha : (Y,(1_Y)_*) \,{-\!\!\circ\!\!\rightharpoonup}\, (X,\theta)$ and $\beta : (X,\theta) \,{-\!\!\circ\!\!\rightharpoonup}\, (Y,(1_Y)_*)$ are inverse to each other in $\mathrm{kar}((\mathbb{U},\Phi)\text{-}\mathsf{Mod})$. As above one verifies that $\alpha : Y \,{-\!\!\circ\!\!\rightharpoonup}\, X$ is left adjoint to $\beta : X \,{-\!\!\circ\!\!\rightharpoonup}\, Y$ in $(\mathbb{U},\Phi)\text{-}\mathsf{Mod}$, and, since X is Φ-cocomplete, $\alpha = f_*$ and $\beta = f^*$ for some $f : Y \to X$. Furthermore, f equalises $\boldsymbol{y}^\Phi_X, \ulcorner\theta\urcorner : X \to \Phi X$ since $f^* \circ \theta = f^*$. We write $i : A \to X$ for the equaliser of $\boldsymbol{y}^\Phi_X, \ulcorner\theta\urcorner : X \to \Phi X$, and $h : Y \to A$ for the map induced by f. Then $f^* = h^* \circ i^*$, hence $h^* = f^* \circ i_*$ and

$$\begin{aligned} i^* \circ f_* \circ h^* &= i^* \circ f_* \circ f^* \circ i_* = i^* \circ \theta \circ i_* = i^* \circ i_* = (1_A)_*,\\ f_* \circ h^* \circ i^* &= f_* \circ f^* = \theta \le (1_X)_*. \end{aligned}$$

Therefore $f_* \circ h^* \dashv i^*$ in $\mathbb{U}\text{-}\mathsf{Mod}$, which implies $i_* = f_* \circ h^* \in (\mathbb{U},\Phi)\text{-}\mathsf{Mod}$. Clearly, $i^* \circ i_* = (1_A)_*$, but also $i_* \circ i^* = \theta$ since

$$\theta = f_* \circ f^* = i_* \circ h_* \circ h^* \circ i^* \le i_* \circ i^* = i_+ \circ i^+ \le \theta.$$

8.16. Proposition. *Let X a Φ-distributive T_0-space, and $i : A \to X$ be the equaliser of $\boldsymbol{y}^\Phi_X, \ulcorner\theta\urcorner : X \to \Phi X$. Then X is Φ-algebraic if and only if i is Φ-dense and $i_* \circ i^* = \theta$.*

The full subcategory of $\Phi\text{-}\mathsf{DTop}$ respectively $\Phi\text{-}\mathsf{DApp}$ determined by the Φ-algebraic spaces we denotes as $\Phi\text{-}\mathsf{ATop}$ and $\Phi\text{-}\mathsf{AApp}$ respectively.

8.17. Theorem. $(\mathbb{U},\Phi)\text{-}\mathsf{Mod}$ *is dually equivalent to* $\Phi\text{-}\mathsf{ATop}_{\mathrm{cocts}}/\Phi\text{-}\mathsf{AApp}_{\mathrm{cocts}}$.

The functor $S : (\mathbb{U},\Phi)\text{-}\mathsf{Mod}^{\mathrm{op}} \to \Phi\text{-}\mathsf{ATop}/\Phi\text{-}\mathsf{AApp}$ is just the restriction of the functor $S : \mathrm{kar}((\mathbb{U},\Phi)\text{-}\mathsf{Mod})^{\mathrm{op}} \to \Phi\text{-}\mathsf{DTop}/\Phi\text{-}\mathsf{DApp}$, its inverse $C : \Phi\text{-}\mathsf{ATop}/\Phi\text{-}\mathsf{AApp} \to (\mathbb{U},\Phi)\text{-}\mathsf{Mod}^{\mathrm{op}}$ substitutes (X,θ) by the isomorphic $(A,(1_A)_*)$ where $i : A \to X$ denotes the equaliser of $\boldsymbol{y}^\Phi_X, \ulcorner\theta\urcorner : X \to \Phi X$, and accordingly sends $f : X \to X'$ to the restriction of f^* to A and A', that is, to $i^* \circ f^* \circ i'_*$. One easily verifies (see Lemma 6.5):

8.18. Lemma. *For X in* $(\mathbb{U},\Phi)\text{-}\mathsf{Mod}$*, the equaliser of $\Phi(\boldsymbol{y}^\Phi_X), \boldsymbol{y}^\Phi_{\Phi X} : \Phi X \to \Phi\Phi X$ is given by*

$$\tilde{X}_\Phi := \{\psi \in \Phi X \mid \psi : X \,{-\!\!\circ\!\!\rightharpoonup}\, 1 \text{ is right adjoint in } \mathbb{U}\text{-}\mathsf{Mod}\} \hookrightarrow \Phi X$$

We write $\eta^\Phi_X : X \to CS(X)$ for the restriction of the Yoneda embedding $\boldsymbol{y}^\Phi_X$ to $\tilde{X}_\Phi$, then the isomorphism $X \,{-\!\!\circ\!\!\rightharpoonup}\, CS(X)$ is given by $(\eta^\Phi_X)_*$. For a Φ-algebraic space X, the isomorphism $SC(X) \to X$ is the restriction of Sup^Φ_X to ΦA.

Since both S and C are indeed 2-functors, we obtain immediately that the category $\mathrm{Map}((\mathbb{U},\Phi)\text{-}\mathsf{Mod})$ of left adjoint modules in $(\mathbb{U},\Phi)\text{-}\mathsf{Mod}$ is dually equivalent to the category $\Phi\text{-}\mathsf{ATop}/\Phi\text{-}\mathsf{AApp}$ of Φ-algebraic (topological/approach) spaces and right adjoint Φ-cocontinuous maps between them.

8.19. Definition. *We call a T_0-space X* **Φ-sober** *if each left adjoint $\varphi : Y \multimap X$ in* $(\mathbb{U},\Phi)$-Mod *is of the form $\varphi = f_*$ for some (unique) $f : Y \to X$.*

Note that each space of the form ΦX is Φ-sober. More important, also $\tilde{X}_\Phi$ is Φ-sober which can be seen as follows. For any $\Psi : \tilde{X}_\Phi \multimap 1$ in $(\mathbb{U},\Phi)$-Mod which is right adjoint in $\mathbb{U}$-Mod put $\psi = \Psi \circ (\eta_X^\Phi)_*$, then $\psi \in \tilde{X}_\Phi$ and $\Psi = \psi \circ (\eta_X^\Phi)^* = \psi^* \circ (\eta_X^\Phi)_* \circ (\eta_X^\Phi)^* = \psi^*$. We write Φ-Sob for the category of Φ-sober spaces and Φ-dense maps, the considerations above imply that $(-)_* : \Phi\text{-}\mathsf{Sob} \to \mathrm{Map}((\mathbb{U},\Phi)\text{-}\mathsf{Mod})$ is an equivalence of categories. We conclude:

8.20. Theorem. Φ-Sob *is dually equivalent to* Φ-ATop/Φ-AApp.

It is high time to present examples.

9. Examples of choices of weighting

In this section we describe some possible choices of $(\mathbb{U},\Phi)$-Mod and derive properties of spaces and maps arising from these choices (Φ-cocomplete spaces and Φ-dense maps, for instance), and in some of these cases we spell out the meaning of the duality theorems of the previous sections. We have to admit right at the beginning that, unfortunately, we do not have yet intrinsic topological descriptions of Φ-distributivity or Φ-algebraicity in general. Nevertheless, we hope to be able to convince the reader that these spaces have nice properties and that it is therefore desirably to have such descriptions.

In the topological case, we know that $\mathbb{P}$ is isomorphic to the filter monad on Top. Consequently, the monad Φ corresponding to a choice $(\mathbb{U},\Phi)$-Mod of $\mathbb{U}$-modules is isomorphic to a submonad of the filter monad, which puts us in the context of [Escardó and Flagg, 1999] where many semantic domains are identified as the algebras for certain submonads of the filter monad. In [Clementino and Hofmann, 2009b] we showed already how the defining properties of these submonads translate into the language of modules. It was also observed there that many of these examples can be described in a uniform manner as follows: take $(\mathbb{U},\Phi)$-Mod as the category all those modules $\varphi : X \multimap Y$ where "φ-colimits commute with certain limits" (see [Kelly and Schmitt, 2005]), that is, where the monotone/contractive map

$$\varphi \circ - : \mathbb{U}\text{-}\mathsf{Mod}(1,X) \to \mathbb{U}\text{-}\mathsf{Mod}(1,Y)$$

preserves chosen limits.

9.1. The absolute case. Certainly we can choose no limits at all, and then $(\mathbb{U},\Phi)$-Mod is the category $\mathbb{U}$-Mod of all $\mathbb{U}$-modules. The results of the previous section restate Theorem 6.9 and, more general, tell us that the category $\mathsf{CDTop}_{\mathrm{cocts}}$ respectively $\mathsf{CDApp}_{\mathrm{cocts}}$ of completely distributive T_0-spaces and left adjoints in $\mathsf{Top}/\mathsf{App}$ is dually equivalent to the idempotent split completion $\mathrm{kar}(\mathbb{U}\text{-}\mathsf{Mod})$ of $\mathbb{U}$-Mod, and that the category $\mathsf{TATop}_{\mathrm{cocts}}$ respectively $\mathsf{TAApp}_{\mathrm{cocts}}$ of totally algebraic T_0-spaces and left adjoint continuous/contractive maps is dually equivalent to $\mathbb{U}$-Mod.

9.2. The "inhabited" case. Our next example is $(\mathbb{U},\Phi)$-Mod being the category of all $\mathbb{U}$-modules $\varphi : X \to Y$ where $\varphi \circ -$ preserves the top element, we call such an $\mathbb{U}$-module ***inhabited***. Explicitly, $\varphi : X \,{-\!\!\!\circ\!\!\rightharpoonup}\, Y$ is inhabited if and only if

$$\forall y \in Y\, \exists \mathfrak{x} \in UX\,.\, \mathfrak{x}\varphi y \qquad\qquad \text{resp.} \qquad\qquad 0 \geqslant \sup_{y\in Y} \inf_{\mathfrak{x}\in UX} \varphi(\mathfrak{x}, y).$$

A continuous map f between topological spaces is Φ-dense if and only if f is dense in the usual topological sense, and a topological space X is Φ-cocomplete if and only if X is densely injective, that is, a Scott domain (see [Gierz *et al.*, 2003]). Correspondingly, we call a contraction map $f : X \to Y$ between approach spaces $X = (X, a)$ and $Y = (Y, b)$ ***dense*** if f is Φ-dense, that is, if

$$0 \geqslant \inf_{\mathfrak{x}\in UX} b(Uf(\mathfrak{x}), y)$$

for all $y \in Y$. Every right adjoint $\mathbb{U}$-module is inhabited, hence a topological/approach space is Φ-sober precisely if it is sober. The results of the previous section tells us now that the category $\mathsf{Sob}_{\text{dense}}$ respectively $\mathsf{ASob}_{\text{dense}}$ of sober spaces and dense maps is dually equivalent to the category of "inhabited algebraic" spaces and right adjoint continuous/contractive maps which preserve inhabited suprema.

9.3. The prime case. One can go further and consider $(\mathbb{U},\Phi)$-Mod being the category of all $\mathbb{U}$-modules $\varphi : X \,{-\!\!\!\circ\!\!\rightharpoonup}\, Y$ where $\varphi \circ -$ preserves finite or countable suprema, or even all weighted limits. The latter case is not very interesting since for this choice a $\mathbb{U}$-module φ belongs to $(\mathbb{U},\Phi)$-Mod if and only if φ is right adjoint. Colimits weighted by right adjoints are absolute, that is, every continuous/contractive map preserves them. Moreover, a T_0-space X is Φ-cocomplete if and only if X is Φ-distributive if and only if X is Φ-algebraic if and only if X is sober. Consequently, Theorem 8.20 just tells us that the category of sober spaces and left adjoints is dually equivalent to the category of sober spaces and right adjoints.

The first case, on the other hand, seems to be more promising. First of all, we find it interesting that this definition, applied to metric spaces, yields forward Cauchy completeness as shown in [Vickers, 2005]: for a metric space X, the modules $\psi : X \,{-\!\!\circ\!\!\!\rightarrow}\, 1$ where $\psi \cdot -$ preserves finite infima correspond precisely to forward Cauchy filters, and x is a supremum of ψ if and only if x is a limit point of the corresponding filter.

Turning now to the topological case, the induced monad Φ on Top is isomorphic to the prime filter (of opens) monad which we encountered already in Section 4. Recall from Section 4 that Top^{Φ} is equivalent to the category $\mathsf{OrdCompHaus}_{\text{sep}}$ of anti-symmetric ordered compact Hausdorff space and monotone continuous maps. These spaces are also known under the designation ***stably compact*** (see [Gierz *et al.*, 2003]) as they are precisely those spaces which are sober, locally compact, and have the property that their compact down-sets[4] are closed under finite intersections. As usual, it is enough to require stability under empty and binary intersections, and stability under empty intersection

[4]Recall that our underlying order is dual to the specialisation order.

translates to compactness of X. Note that a sober space is locally compact if and only if it is core-compact if and only if it is exponentiable. A stably compact space is called ***spectral*** (or ***coherent***) if the compact opens form a basis for the topology of X. One easily verifies that each space of the form ΦX is spectral, and with an argument similar to the one used before Lemma 7.11 one shows that every Φ-distributive space is spectral.

A continuous map $f : (X, a) \to (Y, b)$ between topological spaces is Φ-dense if it is dense in a very strong sense: for each $y \in Y$, there must exist a largest ultrafilter $\mathfrak{x} \in UX$ with $Uf(\mathfrak{x}) \to y$. For lack of a better name we call these maps ultra-dense. The general results of [Clementino and Hofmann, 2009b] tell us that a topological T_0-space is stably compact if and only if it is injective with respect to ultra-dense embeddings. Furthermore, by Theorem 8.20, the category $\mathsf{Sob}_{\text{ultra-dense}}$ of sober spaces and ultra-dense maps is dually equivalent to the category of Φ-algebraic spaces (which are very special spectral spaces) and right adjoint continuous maps which preserve smallest convergence points of ultrafilters.

Every Φ-cocomplete approach T_0-space X is sober since $(\mathbb{U}, \Phi)$-Mod contains all right adjoint modules. Furthermore, for every ultrafilter $\mathfrak{x} \in UX$,

$$\mathcal{Y}_X(\mathfrak{x}) \circ \varphi = \xi \cdot U\varphi(\mathfrak{x}),$$

for all $\varphi : 1 \multimap X$, and therefore $\mathcal{Y}_X(\mathfrak{x}) \circ - : \mathbb{U}\text{-}\mathsf{Mod}(1, X) \to [0, \infty]$ preserves finite suprema (which are infima in the natural order of $[0, \infty]$). Therefore, by Proposition 4.7, X is also $+$-exponentiable. Unfortunately, we do not know yet a characterisation of Φ-cocomplete approach spaces.

9.4. The ultrafilter case. One obtains a closely related example using the monad morphism $\mathcal{Y} : U \to P$ (see Proposition 5.8): for a space X, let $\Phi[X]$ be the image of $\mathcal{Y}_X$. Of course, for topological spaces one gets the prime filter monad discussed above, but the situation is different for approach spaces. We observed already that $\mathcal{Y}_X(\mathfrak{x}_0) \circ - : \mathbb{U}\text{-}\mathsf{Mod}(1, X) \to [0, \infty]$ preserves finite suprema. Furthermore, using Remark 1.1 one shows that

$$\mathcal{Y}_X(\mathfrak{x}_0) \circ (\hom(u, \varphi)) \geqslant \hom(u, \mathcal{Y}_X(\mathfrak{x}_0) \circ \varphi)$$

for every $\varphi : 1 \multimap X$ and $u \in [0, \infty]$. Since for every contraction map $\mathbb{U}\text{-}\mathsf{Mod}(1, X) \to [0, \infty]$ one has the reverse inequality, we conclude that $\mathcal{Y}_X(\mathfrak{x}_0) \circ -$ preserves even the operation $\hom(u, -)$ on $\mathbb{U}\text{-}\mathsf{Mod}(1, X)$. This begs the question if every module $\psi : X \multimap 1$ where $\psi \circ -$ preserves all finite suprema and "homing" with all $u \in [0, \infty]$ is of the form $\psi = \mathcal{Y}_X(\mathfrak{x})$ for some $\mathfrak{x} \in UX$. If this is the case it follows that the corresponding class $(\mathbb{U}, \Phi)$-Mod of $\mathbb{U}$-modules is a subcategory of $\mathbb{U}$-Mod (see Theorem 8.2); however, since we do not know this yet we present a different argument.

Recall that the functor $M_0 : \mathsf{App} \to \mathsf{Met}$ sends $X = (X, a)$ to $M_0(X) = (UX, \tilde{a})$ where $\tilde{a} = Ua \cdot m_X^\circ$. More general, for an arbitrary $\mathbb{U}$-relation $\varphi : X \relbar\joinrel\rightharpoonup Y$ we define $\tilde{\varphi} = U\varphi \cdot m_X^\circ : UX \relbar\joinrel\rightharpoonup UY$. Given also $\psi : Y \relbar\joinrel\rightharpoonup Z$, then

$$\widetilde{\psi \circ \varphi} = U\psi \cdot UU\varphi \cdot Um_X^\circ \cdot m_X^\circ = U\psi \cdot UU\varphi \cdot m_{UX}^\circ \cdot m_X^\circ = U\psi \cdot m_X^\circ \cdot U\varphi \cdot m_X^\circ = \tilde{\psi} \cdot \tilde{\varphi}.$$

Consequently, if $\varphi : X \longrightarrow\!\!\!\!\circ\; Y$ is $\mathbb{U}$-module, then $\tilde{\varphi} : M_0(X) \longrightarrow\!\!\!\!\circ\; M_0(Y)$ is a module between metric spaces. We also remark that φ can be seen as a module $\varphi : M_0(X) \longrightarrow\!\!\!\!\circ\; Y_0$. By definition, $\varphi : X \longrightarrow\!\!\!\!\circ\; Y$ belongs to $(\mathbb{U},\Phi)$-Mod if there is a function $f : Y \to UX$ such that

$$\varphi = \tilde{a}(-, f(-)) = f^\circ \cdot \tilde{a} = f^*.$$

Note that $f : M_0(X) \to Y_0$ is necessarily contractive since $f^* = \varphi$ is a module between metric spaces. Let now $\varphi : (X,a) \longrightarrow\!\!\!\!\circ\; (Y,b)$ and $\psi : (Y,b) \to (Z,c)$ be in $(\mathbb{U},\Phi)$-Mod with $\psi = g^*$ and $\varphi = f^*$. Then

$$\begin{aligned} &\psi \circ \varphi = g^\circ \cdot \tilde{b} \cdot U\varphi \cdot m_X^\circ = g^\circ \cdot \tilde{b} \cdot \tilde{\varphi} = g^\circ \cdot \widetilde{b \circ \varphi} = g^\circ \cdot \tilde{\varphi} = g^\circ \cdot U\varphi \cdot m_X^\circ \\ &= g^\circ \cdot Uf^\circ \cdot UUa \cdot Um_X^\circ \cdot m_X^\circ = g^\circ \cdot Uf^\circ \cdot m_X^\circ \cdot Ua \cdot m_X^\circ = (m_X \cdot Uf \cdot g) \cdot \tilde{a} = (m_X \cdot Uf \cdot g)^*. \end{aligned}$$

The following lemma contains the approach counterpart to Example 6.6.

9.5. Lemma. *$\Phi[X]$ contains all right adjoint $\mathbb{U}$-modules $\psi : X \longrightarrow\!\!\!\!\circ\; 1$.*

Proof. We make use of the description of maps $\varphi : X \to [0,\infty]$ as variable sets $(A_v)_{v\in[0,\infty]}$ where $A_v = \{x \in X \mid \varphi(x) \leqslant v\}$. Let now $\psi : X \longrightarrow\!\!\!\!\circ\; 1$ with left adjoint $\varphi : 1 \longrightarrow\!\!\!\!\circ\; X$. By [Banaschewski *et al.*, 2006, Proposition 5.7], there is some ultrafilter $\mathfrak{x}_0 \in UX$ with $\varphi = a(\mathfrak{x}_0, -)$ and $A_v \in \mathfrak{x}_0$, for all $v > 0$. Furthermore, by [Clementino and Hofmann, 2009a, Subsection 6.4], the variable set $(\mathcal{A}_v)_{v\in[0,\infty]}$ corresponding to ψ is given by

$$\mathcal{A}_v = \{\mathfrak{x} \in UX \mid \forall u \in [0,\infty]\, \forall x \in X\,.\,(a(\mathfrak{x}_0, x) \leqslant u \;\Rightarrow\; a(\mathfrak{x}, x) \leqslant u + v)\}.$$

We show that $\mathfrak{x} \in \mathcal{A}_v \iff \mathcal{Y}(\mathfrak{x}_0)(\mathfrak{x}) \leqslant v$, for all $v \in [0,\infty]$. Assume first $\mathcal{Y}(\mathfrak{x}_0)(\mathfrak{x}) = \hat{a}(\mathfrak{x}, \mathfrak{x}_0) \leqslant v$, with $\hat{a} = Ua \cdot m_X^\circ$. If $a(\mathfrak{x}_0, x) \leqslant u$, then, since a is transitive, $a(\mathfrak{x}, x) \leqslant \hat{a}(\mathfrak{x}, \mathfrak{x}_0) + a(\mathfrak{x}_0, x) \leqslant v + u$; and therefore $\mathfrak{x} \in \mathcal{A}_v$. Assume now $\mathfrak{x} \in \mathcal{A}_v$ and let $B \in \mathfrak{x}$. Let $u > 0$. Then, by hypothesis, $A_u \subseteq A^{(v+u)}$, hence $A^{(v+u)} \in \mathfrak{x}_0$. Consequently (see Example 4.4), $\mathcal{Y}(\mathfrak{x}_0)(\mathfrak{x}) = \hat{a}(\mathfrak{x}, \mathfrak{x}_0) \leqslant v$. ∎

By definition, the corresponding monad Φ appears in the (epi,mono)-factorisation $\mathbb{U} \twoheadrightarrow \Phi \rightarrowtail \mathbb{P}$ of the monad morphism $\mathcal{Y} : \mathbb{U} \to \mathbb{P}$, and the monad morphism $\mathbb{U} \twoheadrightarrow \Phi$ induces full embeddings $\mathsf{App}^\Phi \to \mathsf{MetCompHaus}$. By the "second Yoneda Lemma" (Lemma 5.7), $\mathcal{Y}_X : UX \to PX$ is fully faithful. Therefore $UX \twoheadrightarrow \Phi X$ is a quotient map, in fact, $UX \twoheadrightarrow \Phi X$ gives the T_0-reflection of UX. Consequently, every separated metric compact Hausdorff space X is also a Φ-algebra since the universal property of $UX \twoheadrightarrow \Phi X$ provides us with an inverse $\mathrm{Sup}_X^\Phi : \Phi X \to X$ of $y_X^\Phi : X \to \Phi X$. We conclude that App^Φ is equivalent to the category of separated metric compact Hausdorff spaces.

Given an approach space $X = (X,a)$ which is a Φ-algebra, then X is $+$-exponentiable by Proposition 4.7. Furthermore, the structure map $\alpha : UX \to X$ picks, for each ultrafilter $\mathfrak{x}$, a supremum of the $\mathbb{U}$-module $\mathcal{Y}_X(\mathfrak{x}) : X \longrightarrow\!\!\!\!\circ\; 1$, that is, a point $\alpha(\mathfrak{x}) \in X$ such that, for each $x \in X$, $a(\mathfrak{x}, x) = a_0(\alpha(\mathfrak{x}), x)$. Conversely, assume now that an approach space $X = (X,a)$ admits all suprema of $\mathbb{U}$-module $\mathcal{Y}_X(\mathfrak{x}) : X \longrightarrow\!\!\!\!\circ\; 1$ where $\mathfrak{x} \in UX$. Let $l : UX \to X$ be any map which chooses a supremum of $\mathcal{Y}_X(\mathfrak{x})$, for each $\mathfrak{x} \in UX$. Then

$l : UX \to X$ is a morphism in Met but in general not in App. However, if X is in addition $+$-exponentiable, then l is indeed a morphism in App. To see this, recall from [Hofmann, 2007] that $+$-exponentiability of X is equivalent to commutativity of

$$\begin{tikzcd} UUX \arrow[r, "Ua", "\shortmid"{marking}] \arrow[d, "m_X"'] & UX \arrow[d, "a", "\shortmid"{marking}] \\ UX \arrow[r, "a"', "\shortmid"{marking}] & X \end{tikzcd}$$

in NRel. Then, with $a = a_0 \cdot l$, one obtains

$$\begin{aligned} l \cdot Ua \cdot m_X^\circ \cdot m_X \le a_0 \cdot l \cdot Ua \cdot m_X^\circ \cdot m_X &= a \cdot m_X = a \cdot Ua \\ &= a \cdot Ua_0 \cdot Ul = a \cdot Ua \cdot Ue_X \cdot Ul \le a \cdot Ua \cdot m_X^\circ \cdot Ul \le a \cdot Ul. \end{aligned}$$

We conclude:

9.6. Proposition. *An approach space X is Φ-cocomplete if and only if X is $+$-exponentiable and, for each ultrafilter $\mathfrak{x} \in UX$, there exists a point $x_0 \in X$ such that $a(\mathfrak{x}, x) = a_0(x_0, x)$, for all $x \in X$.*

A contraction map $f : (X, a) \to (Y, b)$ is Φ-dense if and only if, for each $y \in Y$, there is some $\mathfrak{x}_y \in UX$ with $b(Uf(\mathfrak{x}), y) = \hat{a}(\mathfrak{x}, \mathfrak{x}_y)$, for all $\mathfrak{x} \in UX$ (where $\hat{a} = Ua \cdot m_X^\circ$).

9.7. The ultra-and-tensor case. Given X in App, we define $\Phi[X]$ as the set of all $\mathbb{U}$-modules $\psi : X \relbar\!\!\circ\!\!\rightharpoonup 1$ of the form $\psi = \mathcal{Y}_X(\mathfrak{x}) \pitchfork u$ for some $\mathfrak{x} \in UX$ and $u \in [0, \infty]$ (see Section 5, before Theorem 5.14). Hence, a $\mathbb{U}$-module $\varphi : X \relbar\!\!\circ\!\!\rightharpoonup Y$ between approach spaces $X = (X, a)$ and $Y = (Y, b)$ belongs to $(\mathbb{U}, \Phi)$-Mod precisely if there exist functions $h : Y \to UX$ and $\alpha : Y \to [0, \infty]$ with

$$\varphi(\mathfrak{x}, y) = \tilde{a}(\mathfrak{x}, h(y)) + \alpha(y),$$

for all $\mathfrak{x} \in UX$ and $y \in Y$. As above, we use Theorem 8.2 to show that $(\mathbb{U}, \Phi)$-Mod is closed under compositions in $\mathbb{U}$-Mod. Let $X = (X, a)$ and $Y = (Y, b)$ be approach spaces and assume that $\psi : X \relbar\!\!\circ\!\!\rightharpoonup 1$ belongs to $\Phi[X]$ with corresponding $\mathfrak{x}_0 \in UX$ and $u \in [0, \infty]$. For $f : X \to Y$ in App and $\mathfrak{y} \in UY$ one has

$$\psi \circ f^*(\mathfrak{y}) = \psi \cdot (Uf^\circ \cdot \tilde{b})(\mathfrak{y}) = \inf_{\mathfrak{x} \in UX} \tilde{b}(\mathfrak{y}, Uf(\mathfrak{x})) + \tilde{a}(\mathfrak{x}, \mathfrak{x}_0) + u = \tilde{c}(\mathfrak{z}, Uf(\mathfrak{x}_0)) + u.$$

Therefore $\psi \circ f^*$ belongs to $(\mathbb{U}, \Phi)$-Mod. Let now $g : Y \to X$ be in App such that $g_* : Y \relbar\!\!\circ\!\!\rightharpoonup X$ is in $(\mathbb{U}, \Phi)$-Mod, witnessed by $k : X \to UY$ and $\beta : X \to [0, \infty]$. Hence, for all $\mathfrak{y} \in UY$ and $x \in X$,

$$a \cdot Ug(\mathfrak{y}, x) = g_*(\mathfrak{y}, x) = \tilde{b}(\mathfrak{y}, k(x)) + \beta(x).$$

To see that $\psi \circ g_*$ belongs to $\Phi[Y]$, observe first that, for a numerical relation $r : X \relbar\joinrel\mapstochar\joinrel\rightarrow Y$, a function $\gamma : Y \to [0, \infty]$ and for $s(x, y) = r(x, y) + \gamma(y)$, one has

$$Us(\mathfrak{x}, \mathfrak{y}) = Ur(\mathfrak{x}, \mathfrak{y}) + \xi \cdot U\gamma(\mathfrak{x})$$

for all $\mathfrak{x} \in UX$ and $\mathfrak{y} \in UY$, where $\xi : U[0,\infty] \to [0,\infty]$ is defined as $\xi(\mathfrak{u}) = \sup_{A\in\mathfrak{u}} \inf A$. From this one concludes

$$\begin{aligned}
\tilde{a}(Ug(\mathfrak{y}),\mathfrak{x}) &= Ua \cdot m_X^\circ \cdot Ug(\mathfrak{y},\mathfrak{x})\\
&= U(a\cdot Ug)\cdot m_Y^\circ(\mathfrak{y},\mathfrak{x})\\
&= \inf_{\mathfrak{Y},\, m_Y(\mathfrak{Y})=\mathfrak{y}} U(a\cdot Ug)(\mathfrak{Y},\mathfrak{x})\\
&= \inf_{\mathfrak{Y},\, m_Y(\mathfrak{Y})=\mathfrak{y}} U\tilde{b}(\mathfrak{Y},Uk(\mathfrak{x})) + \xi\cdot U\beta(\mathfrak{x})\\
&= U\tilde{b}\cdot m_Y^\circ(\mathfrak{y},Uk(\mathfrak{x})) + \xi\cdot U\beta(\mathfrak{x})\\
&= \tilde{b}(\mathfrak{y}, m_Y\cdot Uk(\mathfrak{x})) + \xi\cdot U\beta(\mathfrak{x}),
\end{aligned}$$

and finally obtains

$$\psi\circ g_*(\mathfrak{y}) = \psi(Ug(\mathfrak{y})) = \tilde{a}(Ug(\mathfrak{y}),\mathfrak{x}_0) + u = \tilde{b}(\mathfrak{y}, m_Y\cdot Uk(\mathfrak{x}_0)) + \xi\cdot U\beta(\mathfrak{x}_0) + u,$$

for all $\mathfrak{y} \in UY$.

By the preceding example, $\Phi[X]$ contains all right adjoint $\mathbb{U}$-modules $\psi : X \multimapdotinv 1$, hence every Φ-cocomplete approach T_0-space is sober. Furthermore, both contraction maps

$$t_X : X\otimes[0,\infty] \to PX,\ (u,x)\mapsto a(-,x)+u \quad\text{and}\quad \mathcal{Y}_X : UX \to PX,\ \mathfrak{x}\mapsto \tilde{a}(-,\mathfrak{x})$$

factor through $\Phi X \hookrightarrow PX$, hence, by Theorem 5.14:

9.8. Proposition. *Every Φ-cocomplete approach space is exponentiable.*

9.9. Monads over Set. So far we have exploited the fact that the category Φ-Cocts is monadic over Top respectively App. However, under further conditions on $(\mathbb{U},\Phi)$-Mod, Φ-Cocts is also monadic over Set, and therefore Theorem 8.12 applies to the induced monad on Set. To finish this paper we briefly discuss this case.

Recall from [Clementino and Hofmann, 2009b] that Φ-Cocts is monadic over Set provided that, in addition to the condition imposed in Section 8, $(\mathbb{U},\Phi)$-Mod satisfies the following condition which we assume from now on: for each surjective continuous/contractive map f, f_* belongs to $(\mathbb{U},\Phi)$-Mod. Hence, under these conditions, Φ-$\mathsf{Cocts} \simeq \mathsf{Set}^{\Phi_0}$ where Φ_0 is the restriction of the monad Φ on Top respectively App to Set. A morphism from X to Y in the Kleisli category Set_{Φ_0} is a map $X\to\Phi Y$ (necessarily continuous respectively contractive) where we consider $X=(X,e_X^\circ)$ and $Y=(Y,e_Y^\circ)$ with the discrete structure, and it corresponds to a $\mathbb{U}$-module $X \multimapdotinv Y$ in $(\mathbb{U},\Phi)$-Mod. We write Φ-URel for the ordered category of all unitary $\mathbb{U}$-relations $\varphi : X \relbar\joinrel\rightharpoonup Y$ where $\varphi : (X,e_X^\circ) \multimapdotinv (Y,e_Y^\circ)$ belongs to $(\mathbb{U},\Phi)$-Mod, the composition is Kleisli composition and the order on hom-sets is the pointwise one. Then the morphisms $\varphi : X \relbar\joinrel\rightharpoonup Y$ of Φ-URel correspond precisely to the morphisms $\ulcorner\varphi\urcorner : Y\to\Phi X$ in Set_{Φ_0}, and with the help of Example 8.1 one concludes that the compositional structures match. In conclusion, Φ-$\mathsf{URel} \simeq \mathsf{Set}_{\Phi_0}$, even as ordered categories. By definition, Φ-URel embeds fully into

$(\mathbb{U},\Phi)$-Mod by considering a set as a discrete space. For a topological/approach space $X=(X,a)$, the convergence relation $a: X \relbar\joinrel\rightharpoonup X$ is unitary and idempotent. Furthermore, $a=i^*\circ i_*$ where $i:(X,e_X^\circ)\to(X,a)$, $x\mapsto x$, hence $a:(X,e_X^\circ)\relbar\joinrel\rightharpoonup(X,e_X^\circ)$ belongs to $(\mathbb{U},\Phi)$-Mod. From this one obtains a full embedding $(\mathbb{U},\Phi)$-$\mathsf{Mod}\to\mathrm{kar}(\Phi$-$\mathsf{URel})$, and therefore $\mathrm{kar}((\mathbb{U},\Phi)$-$\mathsf{Mod})\simeq\mathrm{kar}(\Phi$-$\mathsf{URel})$. From Theorem 8.14 we infer now that

$$\mathrm{kar}(\Phi\text{-}\mathsf{URel})^{\mathrm{op}}\simeq\Phi\text{-}\mathsf{DTop}_{\mathrm{cocts}}/\Phi\text{-}\mathsf{DApp}_{\mathrm{cocts}}.$$

For the choice of all $\mathbb{U}$-modules on topological spaces, the result above tells us that $\mathsf{CDTop}_{\mathrm{cocts}}$ is dually equivalent to $\mathrm{kar}(\mathsf{URel})$, where URel denotes the ordered category of sets and unitary $\mathbb{U}$-relations. Hence, by Theorem 7.16:

9.10. Theorem. *The category* Frm *is dually equivalent to category* $\mathrm{Map}(\mathrm{kar}(\mathsf{URel}))$ *defined by the left adjoint morphisms in* $\mathrm{kar}(\mathsf{URel})$.

By the theorem above, frames correspond to "spaces" (X,a) where $a: UX \relbar\joinrel\mapsto X$ is an idempotent convergence relation but not necessarily reflexive (that is, the principal ultrafilter $\dot{x}$ does not need to converge to x). Equivalently, one can describe these spaces as pair (X,N) where $N: X\to FX$ is a neighbourhood system which satisfies all axiom of a topology except for that a point does not need to belong to all of its neighbourhoods.

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