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THE GLEASON COVER OF A REALIZABILITY TOPOS

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ABSTRACT. Recently Benno van den Berg [1] introduced a new class of realizability toposes which he christened Herbrand toposes. These toposes have strikingly different properties from ordinary realizability toposes, notably the (related) properties that the 'constant object' functor from the topos of sets preserves finite coproducts, and that De Morgan's law is satisfied. In this paper we show that these properties are no accident: for any Schönfinkel algebra Λ, the Herbrand realizability topos over Λ may be obtained as the Gleason cover (in the sense of [8]) of the ordinary realizability topos over Λ. As a corollary, we obtain the functoriality of the Herbrand realizability construction on the category of Schönfinkel algebras and computationally dense applicative morphisms.

1. Introduction

Realizability toposes were first introduced by Martin Hyland in the late 1970s [7, 6] and many particular examples have now been extensively studied (see for example [16]). However, whilst the 2-category of Grothendieck toposes, geometric morphisms and natural transformations between them has long been well understood, there has been relatively little progress until recently on understanding the structure of geometric morphisms between realizability toposes. Indeed, such results as have been established have tended to be negative in character (e.g., if $\mathcal{E}$ is a realizability topos and $\mathcal{F}$ is a Grothendieck topos then there are no geometric morphisms $\mathcal{E} \to \mathcal{F}$, and (up to isomorphism) only one morphism $\mathcal{F} \to \mathcal{E}$, namely that which factors through **Set**), and have tended to raise doubts about whether the well-developed techniques for studying geometric morphisms actually have any utility in the world of realizability. The present paper is perhaps the first to show that a nontrivial geometric construction developed for Grothendieck toposes (specifically, the Gleason cover, introduced — at about the same time as realizability toposes — by the present author [8]) does have a significant rôle to play in this world.

As in [11], we use the term *Schönfinkel algebra* for what most people call a partial combinatory algebra: that is, a set Λ equipped with a partial binary operation (denoted by juxtaposition) and constants K and S such that $\mathsf{K}\lambda\mu = \lambda$ for all λ and μ, and $\mathsf{S}\lambda\mu\nu = \lambda\nu(\mu\nu)$ whenever $\lambda\nu(\mu\nu)$ is defined. For simplicity, we shall assume that all Schönfinkel algebras we consider are *proper* in the sense that $\mathsf{S}\lambda\mu$ is always defined; but we shall not require them to be *strict* in the sense that $\mathsf{S}\lambda\mu\nu$ is defined only when $\lambda\nu(\mu\nu)$ is defined.

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We shall make free use of the combinators $\mathsf{I}, \mathsf{B}, \mathsf{E}$ which respectively correspond to the closed λ-terms $\lambda x.x$, $\lambda xyz.x(yz)$ and $\lambda xy.yx$; of the pairing combinator D corresponding to $\lambda xyz.zyx$ and the unpairing combinators $\mathsf{P}_1 = \mathsf{E}(\mathsf{KI})$ and $\mathsf{P}_2 = \mathsf{EK}$ (note that $\mathsf{P}_1(\mathsf{D}\lambda\mu) = \lambda$ and $\mathsf{P}_2(\mathsf{D}\lambda\mu) = \mu$ for all λ and μ); and of the Church numerals C_n corresponding to $\lambda xy.x(x(\cdots(xy)\cdots))$ where x appears n times in the body of the term.

We recall briefly the construction of a topos from a tripos, which is described in greater detail in [7] and in [16]. By a *tripos* on **Set**, we mean a **Set**-indexed preorder $\mathbb{T} = (A \mapsto T^A)$, such that (a) each preorder T^A is a Heyting prealgebra (i.e., its poset reflection is a Heyting algebra), and the transition functors $u^*\colon T^A \to T^B$ induced by morphisms $u\colon B \to A$ in **Set** preserve this structure; (b) $\mathbb{T}$ is complete and cocomplete (i.e., the functors u^* have left and right adjoints $\exists_u$ and $\forall_u$ satisfying the Beck–Chevalley conditions); and (c) $\mathbb{T}$ has an *exemplary element*, i.e. an element $\sigma \in T^\Sigma$ for some set Σ, such that every element of T^A (for any A) is isomorphic to $u^*(\sigma)$ for some (not necessarily unique) $u\colon A \to \Sigma$. In all the examples we consider, the elements of T^A will be actual functions from A to a fixed set T, the functors u^* will be induced by composition (so that the exemplary element may be taken to be 1_T), the Heyting prealgebra structure will be induced by suitable operations on T, and the quantifiers will be defined 'pointwise' in the sense that $\exists_u f(a)$ and $\forall_u f(a)$ depend only on the set $\{f(b) \mid u(b) = a\}$ (so that the Beck–Chevalley conditions are automatic). (In fact there is no loss of generality in assuming that these conditions on $\mathbb{T}$ are always satisfied, see [7].)

Such a structure gives rise to an allegory (in the sense of [3]) whose objects are sets, and whose morphisms $A \looparrowright B$ are isomorphism classes of elements of $T^{A\times B}$, the composition of $[\phi\colon A \looparrowright B]$ and $[\psi\colon B \looparrowright C]$ being given by $[\psi \circ \phi]$, where

$$(\psi \circ \phi)(a, c) = \exists_b(\phi(a, b) \wedge \psi(b, c)) \ .$$

The topos $\mathbf{Set}\langle\mathbb{T}\rangle$ is obtained from this allegory by splitting all symmetric idempotents and then cutting down to the subcategory of maps: more specifically,

1.1. Definition.

(i) *An object of* $\mathbf{Set}\langle\mathbb{T}\rangle$ *is a pair* (A, δ) *where* A *is a set and* $\delta\colon A \times A \to T$ *is 'symmetric and transitive in the logic of* $\mathbb{T}$*', i.e. the inequalities* $\delta(a, a') \le \delta(a', a)$ *and* $(\delta(a, a') \wedge \delta(a', a'')) \le \delta(a, a'')$ *hold in* $T^{A\times A}$ *and* $T^{A\times A\times A}$ *respectively.*

(ii) *Morphisms* $(A, \delta) \to (B, \epsilon)$ *are named by functions* $F\colon A \times B \to T$ *which are* extensional*,* strict*,* single-valued *and* total *in the sense that the inequalities*

$$\begin{gathered}(\delta(a', a) \wedge F(a, b) \wedge \epsilon(b, b')) \le F(a', b') \ ,\\ F(a, b) \le (\delta(a, a) \wedge \epsilon(b, b)) \ ,\\ (F(a, b) \wedge F(a, b')) \le \epsilon(b, b') \text{ and}\\ \delta(a, a) \le \exists_b F(a, b)\end{gathered}$$

hold; and two such functions name the same morphism iff they are isomorphic as elements of $T^{A\times B}$*.*

We omit the detailed verification that $\mathbf{Set}\langle\mathbb{T}\rangle$ is indeed a topos (the reader may find it in [7] or [16]), but we note for future reference that its subobject classifier Ω is the object $(T,\Leftrightarrow)$ where $\Leftrightarrow$ denotes the Heyting bi-implication.

1.2. EXAMPLES.

(*a*) The tripos corresponding to ordinary realizability, over a Schönfinkel algebra Λ, is obtained by taking T to be the power-set $P\Lambda$, with Heyting operations given by

$$\begin{array}{rcl}(p\Rightarrow q) & = & \{\lambda\in\Lambda \mid \text{for all } \mu\in p,\ \lambda\mu \text{ is defined and } \lambda\mu\in q\}\ ,\\ (p\wedge q) & = & \{\mathsf{D}\lambda\mu \mid \lambda\in p \text{ and } \mu\in q\} \text{ and}\\ (p\vee q) & = & \{\mathsf{DK}\lambda \mid \lambda\in p\}\cup\{\mathsf{D}(\mathsf{KI})\mu \mid \mu\in q\}\ .\end{array}$$

The preorder on $P\Lambda^A$ is the 'uniform' ordering given by $f\leq g$ iff $\bigcap\{(f(a)\Rightarrow g(a)) \mid a\in A\}$ is inhabited; and the quantifiers are given by $\exists_u f(a)=\bigcup\{f(b)\mid u(b)=a\}$ and $\forall_u f(a)=\bigcap\{(\Lambda\Rightarrow f(b))\mid u(b)=a\}$. We denote this tripos by $\mathbb{P}\Lambda$.

(*b*) Given a frame (i.e., a complete Heyting algebra) L, we write $\mathbb{L}$ for the tripos in which L^A is the set of all maps $A\to L$, with pointwise ordering. The topos $\mathbf{Set}\langle\mathbb{L}\rangle$ is simply the topos of L-valued sets (equivalently, of sheaves on L for the canonical coverage; cf. [10], C1.3.11). In particular, it is a Grothendieck topos.

By a *geometric morphism* $f\colon\mathbb{T}\to\mathbb{S}$ of triposes, we mean an adjoint pair of indexed functors $(f^*\colon\mathbb{S}\to\mathbb{T}\dashv f_*\colon\mathbb{T}\to\mathbb{S})$, of which the left adjoint f^* additionally preserves finite meets. Given such an adjunction, it is easy to see that the mapping $((A,\delta)\mapsto(A,f^*\delta))$ defines the object-map of a functor (also denoted f^*) from $\mathbf{Set}\langle\mathbb{S}\rangle$ to $\mathbf{Set}\langle\mathbb{T}\rangle$, whose effect on morphisms is similarly given by $([F]\mapsto[f^*F])$; and this functor preserves finite limits. In general, f_* does not induce a functor in the same simple-minded way, since it need not commute with existential quantification; thus, although $(A,f_*\delta)$ is an object of $\mathbf{Set}\langle\mathbb{S}\rangle$ whenever (A,δ) is an object of $\mathbf{Set}\langle\mathbb{T}\rangle$, and f_*F is extensional, strict and single-valued whenever F represents a morphism of $\mathbf{Set}\langle\mathbb{T}\rangle$, it may fail to be total since the definition of totality involves an existential quantifier. Fortunately, however, there is a 'weak completion' process for objects of $\mathbf{Set}\langle\mathbb{T}\rangle$, such that every object is isomorphic to its weak completion, and f_* *does* preserve totality of functions representing morphisms whose codomain is weakly complete. Specifically, we say (B,ϵ) is *weakly complete* if, for any (A,δ) and any $F\colon A\times B\to T$ representing a morphism $(A,\delta)\to(B,\epsilon)$, there exists a function $f\colon A\to B$ in $\mathbf{Set}$ for which $\exists_b F(a,b)\cong F(a,f(a))$ in T^A. (The word 'weakly' is a reminder that we do not, and cannot reasonably, demand that the function f should be unique.) The following result is well known, but does not seem to be in the literature.

1.3. LEMMA. *The object* Ω *is weakly complete.*

PROOF. Recall that $\Omega=(T,\Leftrightarrow)$. If $F\colon A\times T\to T$ represents a morphism $(A,\delta)\to\Omega$, then from the fact that it is extensional and single-valued we deduce

$$F(a,t)\leq(F(a,\top)\Leftrightarrow(t\Leftrightarrow\top))\ ;$$

but $(t \Leftrightarrow \top)$ is isomorphic to t, and hence we have $F(a,t) \leq F(a, F(a,\top))$ by a further application of extensionality. So we may take $f(a)$ to be $F(a,\top)$. ∎

We define $f_* \colon \mathbf{Set}\langle\mathbb{T}\rangle \to \mathbf{Set}\langle\mathbb{S}\rangle$ by setting $f_*(A,\delta) = (\widetilde{A}, f_*\widetilde{\delta})$ where $(\widetilde{A},\widetilde{\delta})$ is a weak completion of (A,δ); it is then straightforward to verify that this yields a right adjoint for f^* at the topos level. In other words, *every geometric morphism of triposes* $\mathbb{T} \to \mathbb{S}$ *induces a geometric morphism of toposes* $\mathbf{Set}\langle\mathbb{T}\rangle \to \mathbf{Set}\langle\mathbb{S}\rangle$. We do not assert that every geometric morphism between toposes of the form $\mathbf{Set}\langle\mathbb{T}\rangle$ arises in this way — although that is the case if both triposes are induced by frames as in 1.2(b), and it was recently shown [11] that the same is true if they are both induced by Schönfinkel algebras as in 1.2(a).

For future reference, we note also that if $f \colon \mathbb{T} \to \mathbb{S}$ is a reflection (i.e., the counit of $(f^* \dashv f_*)$ is an isomorphism), then the counit of the induced geometric morphism of toposes is also an isomorphism, i.e. the latter is an inclusion. If the morphism of triposes is a coreflection, we cannot conclude the same condition for the induced morphism of toposes (unless f_* happens to commute with existential quantification, so that its extension to $\mathbf{Set}\langle\mathbb{T}\rangle$ may be defined in the 'simple-minded' way); but we can say that $f^* \colon \mathbf{Set}\langle\mathbb{S}\rangle \to \mathbf{Set}\langle\mathbb{T}\rangle$ is faithful, since it reflects isomorphisms between the functions representing morphisms of $\mathbf{Set}\langle\mathbb{S}\rangle$ — in other words, the induced geometric morphism of toposes is a surjection.

Given a tripos $\mathbb{T}$, we write L_T (or simply L) for the Heyting algebra obtained from T^1 by identifying isomorphic elements, and $q \colon T \to L_T$ for the quotient map. If $e \colon L_T \to T$ denotes $\exists_q(1_T)$, we say that $\mathbb{T}$ *has standard existential quantification* (briefly, $\mathbb{T}$ is $\exists$-*standard*) if $qe = 1_L$. (If existential quantification is pointwise, so that it is in effect induced by a join map $\bigvee \colon PT \to T$, this is equivalent to saying that the join of each isomorphism class is a member of the class.) It is shown in [7] that if this condition holds, then L is a frame, and the pair (q,e) induces a geometric morphism of triposes $\mathbb{L} \to \mathbb{T}$. So we obtain a geometric morphism (in fact an inclusion) $\mathbf{Set}\langle\mathbb{L}\rangle \to \mathbf{Set}\langle\mathbb{T}\rangle$. If in addition $\mathbb{T}$ is two-valued (that is, L_T has only two isomorphism classes $[\top]$ and $[\bot]$), this becomes a geometric inclusion $\mathbf{Set} \to \mathbf{Set}\langle\mathbb{T}\rangle$; since it is clearly dense (i.e., its direct image preserves the initial object) and $\mathbf{Set}$ is Boolean, it identifies $\mathbf{Set}$ with the subtopos $\mathbf{sh}_{\neg\neg}(\mathbf{Set}\langle\mathbb{T}\rangle)$. Given a set A, we write ∇A for A considered as a $\neg\neg$-sheaf in $\mathbf{Set}\langle\mathbb{T}\rangle$; it may be identified with (A,Δ) where $\Delta(a,a) = e([\top])$ for all $a \in A$, and $\Delta(a,a') = e([\bot])$ if $a \neq a'$. (The reader should be warned that this functor does not always coincide with the functor $\mathbf{Set} \to \mathbf{Set}\langle\mathbb{T}\rangle$ which is denoted ∇ in [16]: it does so for an ordinary realizability topos, but not for a Herbrand realizability topos.) All the triposes that we consider in this paper are $\exists$-standard, and the majority of them (including $\mathbb{P}\Lambda$) are two-valued.

Ordinary realizability toposes are decidedly non-classical. It has been known for some time that an ordinary realizability topos does not admit any geometric morphism to a Boolean topos, but the following stronger result has not been published before. Recall that a geometric morphism $f \colon \mathcal{F} \to \mathcal{E}$ is said to be *skeletal* [9] if f_* maps $\neg\neg$-sheaves to $\neg\neg$-sheaves; of course, any morphism with Boolean codomain is skeletal, and it was shown in [11] that any geometric morphism between ordinary realizability toposes is skeletal. More generally, if $\mathbb{S}$ and $\mathbb{T}$ are two-valued $\exists$-standard triposes, then any geometric morphism $\mathbf{Set}\langle\mathbb{T}\rangle \to \mathbf{Set}\langle\mathbb{S}\rangle$ which is induced by a geometric morphism of triposes is skeletal, since the triangle

$$\begin{tikzcd} \mathbf{2} \arrow[r] \arrow[dr] & \mathbb{T} \arrow[d] \\ & \mathbb{S} \end{tikzcd}$$

clearly commutes.

1.4. Lemma. *Let Λ be a Schönfinkel algebra. Then there is no skeletal geometric morphism $f\colon \mathbf{Set}\langle\mathbb{P}\Lambda\rangle \to \mathcal{E}$ where $\mathcal{E}$ satisfies De Morgan's law.*

Proof. Suppose given such a morphism f; then since $f_*(0)$ is a $\neg\neg$-sheaf and hence a complemented subterminal object of $\mathcal{E}$, we see that the image of f is dense in a clopen subtopos of $\mathcal{E}$, and hence also satisfies De Morgan's law. And the surjective part of the image factorization of f is still skeletal ([9], 3.5(i)); thus we may reduce to the case when f itself is surjective. For any set A, $f_*(\nabla A)$ is a $\neg\neg$-sheaf and hence decidable in $\mathcal{E}$ ([10], D4.6.2(v)), so that $f^*f_*(\nabla A)$ is decidable in $\mathbf{Set}\langle\mathbb{P}\Lambda\rangle$ and hence a modest assembly ([16], p. 153). Now $\mathcal{E}\,(1, f_*(\nabla A))$ injects into $\mathbf{Set}\langle\mathbb{P}\Lambda\rangle\,(1, f^*f_*(\nabla A))$, and thus has cardinality bounded by that of Λ, since distinct morphisms into a modest assembly must be tracked by distinct elements of Λ. But we also have $\mathcal{E}\,(1, f_*(\nabla A)) \cong \mathbf{Set}\langle\mathbb{P}\Lambda\rangle\,(1, \nabla A) \cong A$; taking A to be the power-set of Λ, we obtain a contradiction. ■

2. Herbrand Realizability

Herbrand realizability, introduced by Benno van den Berg in [1], may be viewed as a further modification of modified realizability (for which see [14]); it resembles the latter in that propositions come equipped with two sets p and a of 'potential' and 'actual' realizers, but differs in that the actual realizers are not individual members of p but are (in effect) finite subsets of p. To this end, we need to suppose that our Schönfinkel algebra is equipped with a notion of coding, not just for pairs, but for arbitrary finite sequences; of course, the latter may easily be constructed from the D-combinator and the Church numerals, but in particular cases there may be simpler ways of achieving it — for example, if Λ has underlying set $\mathbb{N}$, we may code a finite sequence $(n_1, \dots, n_k)$ by the number $2^{n_1+1}3^{n_2+1}\cdots p_k^{n_k+1}$, where p_k denotes the kth prime. We write $\langle\lambda_1, \dots, \lambda_k\rangle$ for the element coding the sequence $(\lambda_1, \dots, \lambda_k)$ (and $\langle\rangle$ for the element coding the empty sequence), and if λ codes a sequence we shall write $|\lambda|$ for the length of this sequence and λ_i for its ith term (it being understood that λ_i may be undefined if $i > |\lambda|$). We shall also write $\lambda * \mu$ for the element coding the concatenated sequence $(\lambda_1, \dots, \lambda_{|\lambda|}, \mu_1, \dots, \mu_{|\mu|})$. We assume that the coding function $(\lambda_1, \dots, \lambda_k) \mapsto \langle\lambda_1, \dots, \lambda_k\rangle$, the projection functions $(-)_i$, the function $\lambda \mapsto \mathsf{C}_{|\lambda|}$, and the function $(\lambda, \mu) \mapsto \lambda * \mu$ are all recursive — that is, they are 'tracked' by suitable elements of Λ.

Given a set $p \subseteq \Lambda$, we shall write $!p$ for the set of codes for finite sequences (of arbitrary length) whose members all belong to p. Note that $\langle\rangle \in !p$ for any p; we shall also write $!^+p$ for the set of codes for nonempty sequences of members of p. We preorder $!p$ by setting $\lambda \preceq \mu$ if every term in the sequence coded by λ also occurs in that coded by μ (but not necessarily in the same order). (The partial order reflection of $!p$ is of course isomorphic to the free join-semilattice Kp, i.e. the set of finite subsets of p.) A realizer λ for $(p \Rightarrow q)$ easily yields a realizer $!\lambda$ for $(!p \Rightarrow !q)$, given by $(!\lambda)\mu = \langle \lambda\mu_1, \ldots, \lambda\mu_{|\mu|}\rangle$. In fact the mapping $p \mapsto !p$ defines an indexed monad on the indexed preorder $\mathbb{P}\Lambda$; we shall not make explicit use of this structure, but we reserve the letters ι and υ for elements of Λ coding the recursive functions $\Lambda \to !\Lambda$ and $!!\Lambda \to !\Lambda$ which yield the unit and multiplication of this monad, namely $\lambda \mapsto \langle\lambda\rangle$ and

$$\langle \mu_1, \mu_2, \ldots, \mu_n\rangle \mapsto \mu_1 * \mu_2 * \cdots * \mu_n \ .$$

Note also that ι and $(-)_1$ are uniform realizers for the implications $(p \Rightarrow !^+p)$ and $(!^+p \Rightarrow p)$ respectively.

We shall use the 'exponential isomorphism' linking $!(-)$ to the join and meet operations of $\mathbb{P}\Lambda$:

2.1. Lemma. *The implications* $!(p \vee q) \Rightarrow (!p \wedge !q)$ *and* $(!p \wedge !q) \Rightarrow !(p \vee q)$ *are uniformly realizable.*

Proof. Given an element λ of $!(p \vee q)$, we map it to $\mathsf{D}\mu\nu$, where μ codes the sequence of those elements $\mathsf{P}_2\lambda_i$ ($i \leq |\lambda|$) for which $\mathsf{P}_1\lambda_i = \mathsf{K}$, and ν similarly codes the sequence of those $\mathsf{P}_2\lambda_i$ for which $\mathsf{P}_1\lambda_i = \mathsf{KI}$. In the opposite direction, given an element $\mathsf{D}\mu\nu$ of $(!p \wedge !q)$, we map it to $(!(\mathsf{DK})\mu) * (!(\mathsf{D}(\mathsf{KI}))\nu)$. It is easy to see that both these mappings are recursive — that is, they may be 'tracked' by elements of Λ. ∎

The mappings constructed above are not literally inverse to each other, but they are inverse modulo the equivalence relation induced by the preorder $\preceq$; so in what follows — where we are primarily interested in (upwards-closed, and hence) equivalence-closed subsets of sets of the form $!p$ — we shall regard them as entitling us to identify $!(p \vee q)$ with $!p \wedge !q$.

We now define $H\Lambda$ to be the set of pairs (p, a) where $p \subseteq \Lambda$ and a is an upwards-closed subset of $!p$. (The idea is that, if we have specified enough potential realizers to give an actual realization of some proposition, we cannot destroy it by adding more potential realizers.) For any set A, we preorder the set $H\Lambda^A$ by setting $f \leq g$ if the implications $(!f_1(a) \Rightarrow !g_1(a))$ and $(f_2(a) \Rightarrow g_2(a))$ are simultaneously realized, uniformly in a (where we write $f_1(a)$ and $f_2(a)$ for the first and second components of $f(a)$).

We define meet and join operations on $H\Lambda$ by setting

$$\begin{array}{rcl} ((p,a) \wedge (q,b)) & = & ((p \vee q), (a \wedge b)) \quad \text{and} \\ ((p,a) \vee (q,b)) & = & ((p \vee q), (a \wedge !q) \cup (!p \wedge b)) \end{array}$$

(note the use we have made of the exponential isomorphism); and the implication is given by

$$((p,a) \Rightarrow (q,b)) = ((!p \Rightarrow !q), c) \ ,$$

where c the set of codes for sequences which contain at least one term in $(a \Rightarrow b)$.

2.2. PROPOSITION. *The above definitions make $H\Lambda^A$ into a Heyting prealgebra for any set A, and the indexed preorder $\mathbb{H}\Lambda = (A \mapsto H\Lambda^A)$ into a tripos.*

We omit the proof, whose details may be found in [1]. However, we note that existential quantification in $\mathbb{H}\Lambda$ is given by taking $\exists_u f(a)$ to have as its first component $\bigcup\{!f_1(b) \mid u(b) = a\}$, and as its second component the set of codes for sequences which contain an element of $f_2(b)$ for some b with $u(b) = a$. It follows that $\mathbb{H}\Lambda$ is $\exists$-standard as well as two-valued (the two isomorphism classes in $H\Lambda^1$ consist of those pairs (p, a) with a inhabited, and those with $a = \emptyset$), so as usual we have a geometric inclusion $\mathbf{Set} \to \mathbf{Set}\langle\mathbb{H}\Lambda\rangle$, representing the former as the subcategory of $\neg\neg$-sheaves in the latter. However, the most striking difference between Herbrand and ordinary (or modified) realizability lies in the fact that the direct image functor ∇ preserves finite coproducts; hence, by [10], D4.6.2(xiii), $\mathbf{Set}\langle\mathbb{H}\Lambda\rangle$ satisfies De Morgan's law. (Once again, we refer to [1] for the proof.)

3. A Tale of Three Triposes

In [14], Jaap van Oosten showed that, at least when Λ is the Kleene algebra, the modified realizability topos over Λ may be identified with a closed subtopos of an ordinary realizability topos over the Sierpiński topos $[\mathbf{2}, \mathbf{Set}]$, whose open complement is the ordinary realizability topos $\mathbf{Set}\langle\mathbb{P}\Lambda\rangle$. Fortunately, thanks to the theorem of A.M. Pitts on iterated tripos extensions (see [16], 2.7.1), we do not have to concern ourselves here with internal Schönfinkel algebras in $[\mathbf{2}, \mathbf{Set}]$, since the ordinary realizability topos which we need can also be presented as $\mathbf{Set}\langle\mathbb{P}_1\Lambda\rangle$ for a suitable tripos $\mathbb{P}_1\Lambda$ on $\mathbf{Set}$. Specifically, $P_1\Lambda$ is the set $\{(p, q) \in P\Lambda \times P\Lambda \mid q \subseteq p\}$, with Heyting operations defined by

$$\begin{array}{rcl} (p,q) \wedge (p',q') &=& (p \wedge p', q \wedge q')\ , \\ (p,q) \vee (p',q') &=& (p \vee p', q \vee q') \quad \text{and} \\ (p,q) \Rightarrow (p',q') &=& ((p \Rightarrow p'), (p \Rightarrow p') \cap (q \Rightarrow q')) \end{array}$$

and preordering on $P_1\Lambda^A$ defined by $f \leq g$ iff $\bigcap\{(f(a) \Rightarrow g(a))_2 \mid a \in A\}$ is inhabited. The verification that this yields a tripos (with quantifiers, like join and meet, defined 'componentwise') is straightforward. We note in passing that $\mathbb{P}_1\Lambda$, though $\exists$-standard, is not two-valued: in addition to $[\top]$ and $[\bot]$, there is a third isomorphism class in $P_1\Lambda^1$, consisting of those pairs $(p, \emptyset)$ with $p \neq \emptyset$.

Between $\mathbb{P}\Lambda$ and $\mathbb{P}_1\Lambda$, we have a string of five indexed adjoint functors $(f_1 \dashv f_2 \dashv f_3 \dashv f_4 \dashv f_5)$, induced respectively by composition with $(p \mapsto (p, \emptyset))$, $((p, q) \mapsto p)$, $(p \mapsto (p, p))$, $((p, q) \mapsto q)$ and $(p \mapsto (\Lambda, (\Lambda \Rightarrow p)))$. (The adjunctions are all trivial to verify except for the last: if λ realizes $(q \Rightarrow r)$, then $\mathsf{B}(\mathsf{B}\lambda)\mathsf{K}$ realizes $((p, q) \Rightarrow (\Lambda, (\Lambda \Rightarrow r)))$, and if μ realizes $((p, q) \Rightarrow (\Lambda, (\Lambda \Rightarrow r)))$, then $\mathsf{B}(\mathsf{EK})\mu$ realizes $(q \Rightarrow r)$.) Although f_1 does not preserve the top element, it does preserve binary meets, and hence composition with it defines a full embedding $\mathbf{Set}\langle\mathbb{P}\Lambda\rangle \to \mathbf{Set}\langle\mathbb{P}_1\Lambda\rangle$, whose image consists of those objects (A, δ) such that the second component of $\delta(a,a')$ is empty for all a,a'. But these are exactly the objects admitting a morphism to the unique nontrivial subterminal object U of $\mathbf{Set}\langle\mathbb{P}_1\Lambda\rangle$ (which may be taken to be $(\{*\},\delta)$ where $\delta(*,*)=(p,\emptyset)$ for some inhabited p); so we may identify $\mathbf{Set}\langle\mathbb{P}\Lambda\rangle$ with $\mathbf{Set}\langle\mathbb{P}_1\Lambda\rangle/U$, in such a way that the functor induced by f_1 is identified with Σ_U — and hence the functors induced by f_2 and f_3 are identified with U^* and Π_U respectively, i.e. they form an open geometric inclusion $u\colon \mathbf{Set}\langle\mathbb{P}\Lambda\rangle \to \mathbf{Set}\langle\mathbb{P}_1\Lambda\rangle$.

The open inclusion u has a left adjoint $g\colon \mathbf{Set}\langle\mathbb{P}_1\Lambda\rangle \to \mathbf{Set}\langle\mathbb{P}\Lambda\rangle$ induced by f_3 and f_4, and this in turn has a left adjoint $v\colon \mathbf{Set}\langle\mathbb{P}\Lambda\rangle \to \mathbf{Set}\langle\mathbb{P}_1\Lambda\rangle$ induced by f_4 and f_5. Since v is an inclusion (equivalently, g is connected), g is a local geometric morphism in the sense of [12].

3.1. LEMMA. *The inclusions u and v define disjoint subtoposes of* $\mathbf{Set}\langle\mathbb{P}_1\Lambda\rangle$.

PROOF. The local operators $\mathbf{j}$ and $\mathbf{k}$ on $\mathbb{P}_1\Lambda$ corresponding to u and v are respectively given by $(p,q)\mapsto(p,p)$ and $(p,q)\mapsto(\Lambda,(\Lambda\Rightarrow q))$; we have to show that the join ($\mathbf{m}$, say) of these two operators is isomorphic to $(p,q)\mapsto(\Lambda,\Lambda)$. But this is easy, since the idempotency of $\mathbf{m}$ yields $\mathbf{jk}\le\mathbf{m}$, and the composite $\mathbf{jk}$ is exactly $(p,q)\mapsto(\Lambda,\Lambda)$. ■

An alternative proof of 3.1 could be given by observing that $v_*(0)$ is the subterminal object U; recall that, for any geometric morphism f, $f_*(0)$ is the open complement of the closure of the image of f.

As we remarked above, in [14] Jaap van Oosten identified the closed complement of the open subtopos u as the *modified realizability topos* $\mathbf{Set}\langle\mathbb{M}\Lambda\rangle$ over Λ, at least in the case when Λ is the (first) Kleene algebra. Moreover, by 3.1 the non-open inclusion v factors through this closed subtopos; hence the composite $\mathbf{Set}\langle\mathbb{M}\Lambda\rangle \to \mathbf{Set}\langle\mathbb{P}_1\Lambda\rangle \to \mathbf{Set}\langle\mathbb{P}\Lambda\rangle$ is still local.

More recently, van Oosten [private communication] observed:

3.2. LEMMA. *There is a geometric inclusion $w\colon \mathbf{Set}\langle\mathbb{H}\Lambda\rangle \to \mathbf{Set}\langle\mathbb{P}_1\Lambda\rangle$. Moreover, this subtopos is disjoint from u.*

PROOF. w is induced by a geometric morphism of triposes whose direct image is $((p,a)\mapsto(!p,a))$ and whose inverse image is $((p,q)\mapsto(p,s(p,q)))$ where $s(p,q)\subseteq{!p}$ is the set of codes for sequences containing at least one term in q. It is easy to see that both these maps are order-preserving. If λ realizes $((p,s(p,q))\Rightarrow(r,a))$, then $\mathsf{B}\lambda\iota$ realizes $((p,q)\Rightarrow(!r,a))$; and conversely if μ realizes $((p,q)\Rightarrow(!r,a))$ then $\mathsf{B}\upsilon(!\mu)$ realizes $((p,s(p,q))\Rightarrow(r,a))$. A realizer for the fact that w^* preserves binary meets is given by a code for the function sending $\mathsf{D}\lambda\mu$ (where λ and μ code sequences of lengths m and n respectively) to the code for the sequence of length mn whose $(rn+s)$th term is $\mathsf{D}\lambda_{r+1}\mu_s$. And ι realizes $(1\le w^*w_*)$, so the adjunction is a reflection.

For the second assertion, we again consider the composite $\mathbf{jl}$, where $\mathbf{l}$ is the local operator $((p,q)\mapsto(!p,!^+p))$ corresponding to w. We have $\mathbf{jl}(p,q)=(!p,!p)$ for all (p,q); and since $\langle\rangle\in{!p}$ for all p this is isomorphic to the constant function with value (Λ,Λ). ■

The composite $h = gw$ is not a local morphism: the left adjoint v of g cannot factor through w by 1.4, since such a factorization would be (a dense inclusion, and hence) skeletal. But it is surjective, since it corresponds to the geometric morphism of triposes given by $((p, a) \mapsto a)$ and $(p \mapsto (p, !^+p))$, and as we observed earlier the inequality $h_*h^*(p) \leq p$ is realized by $(-)_1$.

3.3. Lemma. $h\colon \mathbf{Set}\langle\mathbb{H}\Lambda\rangle \to \mathbf{Set}\langle\mathbb{P}\Lambda\rangle$ *is a closed map.*

Proof. By the tripos version of [10], C3.2.1, it suffices to verify the 'co-Frobenius' condition

$$(h_*((p, a) \vee h^*(q)) \Leftrightarrow (h_*(p, a) \vee q)) \ .$$

But the two sides reduce to $(a \wedge !q) \cup (!p \wedge !^+q)$ and to $(a \vee q)$ respectively; so a realizer for the left-to-right implication is given by a code for the function sending $\mathsf{D}\lambda\mu$ to $\mathsf{DK}\lambda$ if $\mu = \langle\rangle$ and to $\mathsf{D}(\mathsf{KI})\mu_1$ otherwise, and the right-to-left implication is realized by a code for the function sending $\mathsf{D}\lambda\mu$ to $\mathsf{D}\mu\langle\rangle$ if $\lambda = \mathsf{K}$, and to $\mathsf{D}\langle\rangle(\iota\mu)$ if $\lambda = \mathsf{KI}$. ■

4. Herbrand Realizability as a Gleason Cover

The Gleason cover construction takes its name from a paper by A.M. Gleason [4], in which he showed how to construct a 'best possible' covering of an arbitrary compact Hausdorff space by an extremally disconnected one. This construction was subsequently extended to larger classes of spaces by various authors, and in [8] the present author showed that it may be extended to arbitrary toposes. Given a topos $\mathcal{E}$, its Gleason cover $\gamma\mathcal{E}$ is defined to be the topos of $\mathcal{E}$-valued sheaves on the internal frame $\mathrm{Idl}(\Omega_{\neg\neg})$ of ideals of the complete Boolean algebra $\Omega_{\neg\neg}$ in $\mathcal{E}$. $\gamma\mathcal{E}$ always satisfies De Morgan's law, and it comes equipped with a geometric morphism $f\colon \gamma\mathcal{E} \to \mathcal{E}$ which is a proper separated map and a minimal localic surjection; moreover, $\gamma\mathcal{E}$ is characterized up to equivalence in $\mathsf{Top}/\mathcal{E}$ by these properties (see [10], D4.6.8). The morphism f is also skeletal (indeed, it restricts to an equivalence between categories of $\neg\neg$-sheaves), and the construction $\mathcal{E} \mapsto \gamma\mathcal{E}$ may be viewed as a right adjoint to the inclusion, in the 2-category of toposes and skeletal geometric morphisms, of the full sub-2-category of toposes satisfying De Morgan's law ([10], D4.6.12).

In our present context, the domain of $h\colon \mathbf{Set}\langle\mathbb{H}\Lambda\rangle \to \mathbf{Set}\langle\mathbb{P}\Lambda\rangle$ satisfies De Morgan's law; h is skeletal (either by direct computation, or by the argument of Lemma 2.1 in [11]) and localic (as are all morphisms induced by geometric morphisms of triposes), and it is minimal surjective since the only proper closed subtopos of its domain is the degenerate topos. So, in order to identify it with the Gleason cover of $\mathbf{Set}\langle\mathbb{P}\Lambda\rangle$, it would suffice to show that it is proper and separated. As we saw in 3.3, it is a closed map, which is one part of propriety; the other is preservation of filtered $\mathbf{Set}\langle\mathbb{P}\Lambda\rangle$-indexed colimits by its direct image. Although filtered colimits in realizability toposes were studied by van Oosten in [15], we have chosen not to follow this route, but instead to argue directly to show that the internal frame in $\mathbf{Set}\langle\mathbb{P}\Lambda\rangle$ which corresponds to h is isomorphic to the frame of ideals of $\Omega_{\neg\neg} = \nabla 2$.

Since Ω is weakly complete in $\mathbf{Set}\langle\mathbb{H}\Lambda\rangle$ by 1.3, we may compute $h_*(\Omega)$ in the 'simple-minded' way: that is, its underlying set is simply $H\Lambda$, with $P\Lambda$-valued equality predicate obtained by applying h_* to the $H\Lambda$-valued predicate $\Leftrightarrow$ (that is, by forgetting the first coordinate of the latter). We have to show that this object is isomorphic to $\mathrm{Idl}(\nabla 2)$ in $\mathbf{Set}\langle\mathbb{P}\Lambda\rangle$.

Of course, an ideal $I \rightarrowtail \nabla 2$ is completely determined by a pair of truth-values (= subsets of Λ) $p = [\![0 \in I]\!]$ and $q = [\![1 \in I]\!]$; but the assertion that these truth-values define an ideal involves realizers for the assertions that I is downwards closed and closed under finite joins — i.e., we must be given specified realizers $\alpha \in [\![0 \in I]\!] = p$,

$$\beta \in \bigcap\{([\![i \in I]\!] \Rightarrow [\![j \in I]\!]) \mid i \geq j\} = (p \Rightarrow p) \cap (q \Rightarrow p) \cap (q \Rightarrow q)\ , \text{ and}$$

$$\begin{aligned}\gamma \in \bigcap\{(([\![i \in I]\!] \wedge [\![j \in I]\!]) \Rightarrow [\![i \vee j \in I]\!]) \mid i, j \in 2\}\\ = ((p \wedge p) \Rightarrow p) \cap (p \wedge q) \Rightarrow q) \cap ((q \wedge p) \Rightarrow q) \cap ((q \wedge q) \Rightarrow q)\ .\end{aligned}$$

We note that if (p, a) is any element of $H\Lambda$ then the pair $(!p, a)$ carries this structure, with $\alpha = \langle\rangle$, $\beta = \mathsf{I}$ and γ taken to be a code for the mapping $(\mathsf{D}\lambda\mu \mapsto \lambda * \mu)$. In showing that any ideal is isomorphic to one of this form, we proceed in two steps:

4.1. Lemma.

(i) *Given any ideal (p, q) (with realizers α, β, γ), we may uniformly construct an isomorphic ideal (p', q') for which $q' \subseteq p'$ and $\beta' = \mathsf{I}$.*

(ii) *Given an ideal (p, q) with $q \subseteq p$ (and realizers $\alpha, \mathsf{I}, \gamma$), we may uniformly construct an isomorphic ideal $(!p', a)$ where a is an upwards-closed subset of $!p'$.*

Proof. (i) We define $p' = p$ and $q' = p \cap q$; clearly this is an ideal, with the same realizers α and γ as (p, q) and with $\beta' = \mathsf{I}$. Moreover, β realizes the inequality $(p, q) \leq (p', q')$, and I realizes the converse.

(ii) Again, we take $p' = p$, and we take a to be the set of codes for sequences including at least one term in q. The inequality $(p, q) \leq (!p, a)$ is realized by ι; to realize the converse, we take a code for the function $f : !p \to p$ recursively defined by

$$\begin{aligned} f(\lambda) &= \alpha && \text{if } \lambda = \langle\rangle \\ &= \gamma(\mathsf{D}(f(\langle\lambda_1, \ldots, \lambda_{|\lambda|-1}\rangle))\lambda_{|\lambda|}) && \text{if } |\lambda| > 0\ . \end{aligned}$$

It is clear that if λ codes a sequence including a member of q then $f(\lambda) \in q$. ∎

4.2. Theorem. *For any Schönfinkel algebra Λ, the internal frames $h_*(\Omega)$ and $\mathrm{Idl}(\nabla 2)$ are isomorphic in $\mathbf{Set}\langle\mathbb{P}\Lambda\rangle$. Hence $\mathbf{Set}\langle\mathbb{H}\Lambda\rangle$ is equivalent to the Gleason cover of $\mathbf{Set}\langle\mathbb{P}\Lambda\rangle$.*

PROOF. We may take $\mathrm{Idl}(\nabla 2)$ to be the set of pairs (p,q) of subsets of Λ for which there exist realizers α, β, γ as above, with $[\![(p,q) \in \mathrm{Idl}(\nabla 2)]\!]$ taken to be the set of coded triples $\langle \alpha, \beta, \gamma \rangle$ which satisfy the conditions for p and q, and equality predicate δ given by

$$\begin{aligned}&\delta((p,q),(p',q'))\\ &\qquad = [\![(p,q) \in \mathrm{Idl}(\nabla 2)]\!] \wedge [\![(p',q') \in \mathrm{Idl}(\nabla 2)]\!] \wedge ((p,q) \leq (p',q')) \wedge ((p',q') \leq (p,q))\ ,\end{aligned}$$

where $((p,q) \leq (p',q'))$ denotes $(p \Rightarrow p') \cap (q \Rightarrow q')$. As already mentioned, $h_*(\Omega)$ is the set of elements $(p,a) \in H\Lambda$, with equality predicate $\epsilon((p,a),(p',a')) = (((p,a) \leq (p',a')) \wedge ((p',a') \leq (p,a)))$, where $((p,a) \leq (p',a'))$ is the set of codes for sequences of elements of $(!p \Rightarrow !p')$ which contain at least one term in $(a \Rightarrow a')$. Note that, given such a code λ, we may uniformly construct an element of $((!p \Rightarrow !p') \cap (a \Rightarrow a'))$, coding the function which applies each of the λ_i for $i \leq |\lambda|$ to an element of $!p$ and then concatenates the results.

It is now straightforward to verify that, if we define $F\colon \mathrm{Idl}(\nabla 2) \times H\Lambda \to P\Lambda$ by

$$F((p,q),(r,a)) = \delta((p,q),(!r,a))\ ,$$

then F represents a bijection $f\colon \mathrm{Idl}(\nabla 2) \to h_*(\Omega)$ in $\mathbf{Set}\langle \mathbb{P}\Lambda \rangle$. Strictness is easy, as is extensionality in the first variable (p,q); extensionality in the second variable follows from the last sentence of the previous paragraph; single-valuedness in either direction is again easy; totality is the left-to-right direction follows from 4.1, and totality from right to left follows from the fact which we noted earlier that $[\![(!r,a) \in \mathrm{Idl}(\nabla 2)]\!]$ has a uniform realizer.

Moreover, f is an isomorphism of ordered sets (and hence of frames), since the order relation on each of the two objects is induced in the obvious way by the relations on their elements which we denoted $\leq$, and F is clearly compatible (in either direction) with these relations. But we have $\mathbf{Set}\langle \mathbb{H}\Lambda \rangle \simeq \mathbf{Sh}(h_*(\Omega))$ since h is localic, so

$$\mathbf{Set}\langle \mathbb{H}\Lambda \rangle \simeq \mathbf{Sh}(\mathrm{Idl}(\nabla 2)) \simeq \gamma(\mathbf{Set}\langle \mathbb{P}\Lambda \rangle)\ .$$

∎

5. Functoriality of Herbrand Realizability

The functoriality of the construction $\Lambda \mapsto \mathbf{Set}\langle \mathbb{P}\Lambda \rangle$ has been extensively studied, first by John Longley [13] and more recently by Hofstra and van Oosten [5] and by the present author [11]. The conclusion of these researches can be summarized as follows:

5.1. THEOREM. *The assignment $\Lambda \mapsto \mathbf{Set}\langle \mathbb{P}\Lambda \rangle$ is a full embedding of 2-categories $\mathsf{Sch\ddot{o}n}_{\mathrm{qs}}^{\mathrm{op}} \to \mathsf{Top}$, where $\mathsf{Sch\ddot{o}n}$ denotes the 2-category of Schönfinkel algebras and applicative morphisms, $\mathsf{Sch\ddot{o}n}_{\mathrm{qs}}$ is its subcategory whose 1-arrows are quasi-surjective morphisms, and Top is the 2-category of toposes and geometric morphisms.* ∎

We recall that Longley defined an *applicative morphism* $\theta \colon \Lambda \nrightarrow \mathrm{M}$ of Schönfinkel algebras to be an entire relation (i.e. one relating each element of Λ to at least one element of M) which has a *witness* $\tau \in \mathrm{M}$ such that, if $\theta(\lambda, \mu)$ and $\theta(\lambda', \mu')$ hold and $\lambda\lambda'$ is defined in Λ, then $\tau\mu\mu'$ is defined in M and $\theta(\lambda\lambda', \tau\mu\mu')$ holds. Such a morphism is said to be *quasi-surjective* if there exists a function $r \colon \mathrm{M} \to \Lambda$ and an element $\rho \in \mathrm{M}$ such that for all $\mu, \mu' \in \mathrm{M}$, if $\theta(r(\mu), \mu')$ holds, then $\rho\mu' = \mu$. In [11], this was shown to be equivalent to the notion of computational density introduced by Hofstra and van Oosten in [5], which is in turn equivalent to the assertion that the indexed functor $\mathbb{P}\Lambda \to \mathbb{P}\mathrm{M}$ induced by composition with θ has a right adjoint (and is thus the inverse image of a geometric morphism of triposes).

We have already observed that the 2-full embedding of 5.1 takes values in the sub-2-category $\mathsf{Top}_{\mathrm{sk}}$ of toposes and skeletal geometric morphisms. But the Gleason cover construction is 2-functorial on this 2-category; in fact, as shown in [10], D4.6.12, it is right adjoint to the inclusion $\mathsf{DMTop}_{\mathrm{sk}} \to \mathsf{Top}_{\mathrm{sk}}$, where $\mathsf{DMTop}_{\mathrm{sk}}$ is the full sub-2-category of toposes satisfying De Morgan's law. Thus we may immediately conclude:

5.2. Corollary. *The assignment* $\Lambda \mapsto \mathbf{Set}\langle \mathbb{H}\Lambda \rangle$ *is a* 2*-functor* $\mathsf{Schön}_{\mathrm{qs}}^{\mathrm{op}} \to \mathsf{Top}$. ∎

No doubt it would be possible to give a direct proof of Corollary 5.2, by showing that a quasi-surjective morphism of Schönfinkel algebras induces a geometric morphism between the corresponding Herbrand triposes. We leave this as an exercise for the reader! (Note, however, that we do not make any fullness claim for the 2-functor of 5.2: we do not even know whether every geometric morphism $\mathbf{Set}\langle \mathbb{H}\Lambda \rangle \to \mathbf{Set}\langle \mathbb{H}\mathrm{M} \rangle$ is induced by a geometric morphism of triposes $\mathbb{H}\Lambda \to \mathbb{H}\mathrm{M}$.)

We conclude with an open problem. In [2], Olivia Caramello showed that every topos $\mathcal{E}$ contains a largest dense subtopos which satisfies De Morgan's law; if $\mathcal{E}$ is two-valued, so that its only proper closed subtopos is degenerate, then the word 'dense' may be omitted from this characterization. We have seen that, at least when Λ is the Kleene algebra, $\mathbf{Set}\langle \mathbb{H}\Lambda \rangle$ is a dense subtopos of the modified realizability topos $\mathbf{Set}\langle \mathbb{M}\Lambda \rangle$; indeed, the latter is the closure of the former as a subtopos of $\mathbf{Set}\langle \mathbb{P}_1\Lambda \rangle$. It would therefore be of interest to know whether the Herbrand realizability topos could also be obtained by applying Caramello's construction to the modified realizability topos. It seems highly likely that this is true; for if $\mathcal{E}$ is any subtopos of $\mathbf{Set}\langle \mathbb{M}\Lambda \rangle$ satisfying De Morgan's law, then the composite $\mathcal{E} \to \mathbf{Set}\langle \mathbb{M}\Lambda \rangle \to \mathbf{Set}\langle \mathbb{P}\Lambda \rangle$ is skeletal, and hence factors uniquely through the Gleason cover $\mathbf{Set}\langle \mathbb{H}\Lambda \rangle$ by [10], D4.6.12. However, this argument does not suffice to show that the triangle

$$\begin{tikzcd} \mathcal{E} \arrow[r] \arrow[dr] & \mathbf{Set}\langle \mathbb{H}\Lambda \rangle \arrow[d] \\ & \mathbf{Set}\langle \mathbb{M}\Lambda \rangle \end{tikzcd}$$

commutes.

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