Theory and Applications of Categories, Vol. 28, No. 6, 2013, pp. 166–205.

TANNAKA DUALITY AND CONVOLUTION FOR DUOIDAL CATEGORIES

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ABSTRACT. Given a horizontal monoid M in a duoidal category $\mathscr{F}$, we examine the relationship between bimonoid structures on M and monoidal structures on the category $\mathscr{F}^{*M}$ of right M-modules which lift the vertical monoidal structure of $\mathscr{F}$. We obtain our result using a variant of the so-called Tannaka adjunction; that is, an adjunction inducing the equivalence which expresses Tannaka duality. The approach taken utilizes hom-enriched categories rather than categories on which a monoidal category acts ("actegories"). The requirement of enrichment in $\mathscr{F}$ itself demands the existence of some internal homs, leading to the consideration of convolution for duoidal categories. Proving that certain hom-functors are monoidal, and so take monoids to monoids, unifies classical convolution in algebra and Day convolution for categories. Hopf bimonoids are defined leading to a lifting of closed structures on $\mathscr{F}$ to $\mathscr{F}^{*M}$. We introduce the concept of warping monoidal structures and this permits the construction of new duoidal categories.

1. Introduction

This paper initiates the development of a general theory of duoidal categories. In addition to providing the requisite definition of a duoidal $\mathscr{V}$-category, various "classical" concepts are reinterpreted and new notions put forth, including: produoidal $\mathscr{V}$-categories, convolution structures and duoidal cocompletion, enrichment in a duoidal $\mathscr{V}$-category, Tannaka duality, lifting closed structures to a category of representations (Hopf opmonoidal monads), and discovering new duoidal categories by "warping" a given monoidal structure.

Some background references for this work are as follows. Standard categorical notions can be obtained from Mac Lane [18] and basic enriched category theory from Kelly [15]. For treatments of Tannaka duality, see Joyal-Street [13], Street [27], and the more recent McCurdy [19]. Promonoidal categories and monoidal convolution are due to Brian Day; see [7].

Duoidal categories, some examples, and applications, have appeared in the Aguiar-Mahajan book [1] (under the name "2-monoidal categories"), in the recently published work of Batanin-Markl [2] and in a series of lectures by the second author [26]. Taken to-

The first author was supported by an Australian Postgraduate Award. The second author gratefully acknowledges the support of an Australian Research Council Discovery Grant DP1094883.

Received by the editors 2013-02-14 and, in revised form, 2013-03-04.

Transmitted by Walter Tholen. Published on 2013-04-01.

2010 Mathematics Subject Classification: 18D35; 18D10; 20J06.

Key words and phrases: duoidal; duoid; bimonoid; Tannaka duality; monoidal category; closed category; Hopf monoid.

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gether with this paper, the vast potential of duoidal category theory is only now becoming apparent.

An encapsulated definition is that a duoidal $\mathscr{V}$-category $\mathscr{F}$ is a pseudomonoid in the 2-category Mon($\mathscr{V}$-Cat) of monoidal $\mathscr{V}$-categories, monoidal $\mathscr{V}$-functors and monoidal $\mathscr{V}$-natural transformations. Since Mon($\mathscr{V}$-Cat) is equivalently the category of pseudomonoids in $\mathscr{V}$-Cat we are motivated to call a pseudomonoid in a monoidal bicategory a *monoidale* (i.e. a monoidal object). Thus a duoidal $\mathscr{V}$-category is an object of $\mathscr{V}$-Cat equipped with two monoidal structures, one called horizontal and the other called vertical, such that one is monoidal with respect to the other. We distinguish the structure we are referring to by subscripts h and v respectively; in particular, $\mathscr{F}_h$ is the $\mathscr{V}$-category $\mathscr{F}$ equipped with the horizontal monoidal structure alone. We introducing the term *duoidale* for objects equipped with a duoidal structure in other monoidal bicategories besides $\mathscr{V}$-Cat; in particular, in $\mathscr{M} = \mathscr{V}$-Mod. By exhibiting and using a canonical monoidal structure on the $\mathscr{V}$-Cat-valued hom of any left unit closed monoidal bicategory $\mathscr{M}$ (see Section 2), where $\mathscr{V} = \mathscr{M}(\mathcal{I}, \mathcal{I})$, we see that a duoidale in $\mathscr{M} = \mathscr{V}$-Mod is precisely the notion of promonoidal category lifted to the duoidal setting; that is, a produoidal $\mathscr{V}$-category.

A study of duoidal cocompletion (in light of the produoidal $\mathscr{V}$-category material) leads to Section 5 where we consider enrichment in a duoidal $\mathscr{V}$-category base. We observe that if $\mathscr{F}$ is a duoidal $\mathscr{V}$-category then the vertical monoidal structure $\circ$ lifts to give a monoidal structure on $\mathscr{F}_h$-Cat. If $\mathscr{F}$ is then a horizontally left closed duoidal $\mathscr{V}$-category then $\mathscr{F}$ is in fact a monoidale $(\mathscr{F}_h, \hat{\circ}, \ulcorner \mathbf{1} \urcorner)$ in $\mathscr{F}_h$-Cat with multiplication $\hat{\circ} : \mathscr{F}_h \circ \mathscr{F}_h \longrightarrow \mathscr{F}_h$ defined using the evaluation of homs. That is, $\mathscr{F}_h$ is an $\mathscr{F}_h$-category.

Section 6 revisits Tannaka duality as an equivalence obtained (as in [27]) from an adjunction that has been called the Tannaka adjunction (see for example [19]). Our purpose here is to adapt this adjunction to duoidal $\mathscr{V}$-categories. We write $\mathscr{F}_h$-Cat $\downarrow^{\mathrm{ps}} \mathscr{F}_h$ for the 2-category like the slice $\mathscr{F}_h$-Cat $\downarrow \mathscr{F}_h$ except that the 1-cells are those triangles that commute up to a given isomorphism. Post composition with the monoidale multiplication $\hat{\circ}$ yields a tensor product $\overline{\circ}$ on $\mathscr{F}_h$-Cat $\downarrow^{\mathrm{ps}} \mathscr{F}_h$ and we write $\mathscr{F}$-Cat $\downarrow^{\mathrm{ps}} \mathscr{F}$ for this monoidal 2-category. Let $\mathscr{F}^{*M}$ be the $\mathscr{F}_h$-category of Eilenberg-Moore algebras for the monad $-*M$. There is a monoidal functor mod : $(\mathrm{Mon}\,\mathscr{F})^{\mathrm{op}} \longrightarrow \mathscr{F}$-Cat $\downarrow^{\mathrm{ps}} \mathscr{F}$ defined by taking a monoid M to the object $U_M : \mathscr{F}^{*M} \longrightarrow \mathscr{F}_h$. Here Mon $\mathscr{F}$ is only being considered as a monoidal category, not a 2-category. Representable objects of $\mathscr{F}$-Cat $\downarrow^{\mathrm{ps}} \mathscr{F}$ are closed under the monoidal structure $\overline{\circ}$ which motivates restricting to $\mathscr{F}$-Cat $\downarrow^{\mathrm{ps}}_{\mathrm{rep}} \mathscr{F}$. Since representable functors are "tractable" and the functor end : $\mathscr{F}$-Cat $\downarrow^{\mathrm{ps}}_{\mathrm{rep}} \mathscr{F} \longrightarrow$ Mon $\mathscr{F}$ is strong monoidal we have the biadjunction

$$(\mathrm{Bimon}\ \mathscr{F}_h)^{\mathrm{op}} \underset{\mathrm{mod}}{\overset{\mathrm{end}}{\underset{\longrightarrow}{\overset{\longleftarrow}{\bot}}}} \mathrm{Mon}_{\mathrm{ps}}(\mathscr{F}\text{-Cat} \downarrow^{\mathrm{ps}}_{\mathrm{rep}} \mathscr{F})$$

giving the correspondence between bimonoid structures on M and isomorphism classes of monoidal structures on $\mathscr{F}^{*M}$ such that the underlying functor is strong monoidal into

the vertical structure on $\mathscr{F}$. The non-duoidal version of this result is attributed to Bodo Pareigis (see [22], [20] and [21]).

The notion of a Hopf opmonoidal monad is found in the paper of Bruguières-Lack-Virelizier [4]. We adapt their work to the duoidal setting in order to lift closed structures on the monoidale (monoidal $\mathscr{F}_h$-category) $(\mathscr{F}, \hat{\circ}, \ulcorner \mathbf{1} \urcorner)$ to the $\mathscr{F}_h$-category of right modules $\mathscr{F}^{*M}$ for a bimonoid M. In particular, Proposition 7.7 says that the monoidal $\mathscr{F}_h$-category $(\mathscr{F}, \hat{\circ}, \ulcorner \mathbf{1} \urcorner)$ is closed if and only if $\mathscr{F}_v$ is a closed monoidal $\mathscr{V}$-category and there exists $\mathscr{V}$-natural isomorphisms $X \circ (W * Y) \cong W * (X \circ Y) \cong (W * X) \circ Y$. In light of $\mathscr{F}$ being a duoidal $\mathscr{V}$-category, Proposition 7.9 gives a refinement of this result which taken together with Proposition 7.7 yields two isomorphisms

$$X * (J \circ Y) \cong X \circ Y \cong Y * (X \circ J)$$

and

$$Y \circ (W * \mathbf{1}) \cong W * Y \cong (W * \mathbf{1}) \circ Y \,.$$

This result implies that in order to know $\circ$ we only need to know $*$ and $J \circ -$ or $- \circ J$. Similarly to know $*$ we need only know $\circ$ and $\mathbf{1} * -$ or $- * \mathbf{1}$. This extreme form of interpolation motivates the material of Section 8.

We would like a way to generate new duoidal categories. One possible method presented here is the notion of a *warped monoidal structure*. In its simplest presentation, a warping for a monoidal category $\mathscr{A} = (\mathscr{A}, \otimes)$ is a perturbation of $\mathscr{A}$'s tensor product by a "suitable" endo-functor $T : \mathscr{A} \longrightarrow \mathscr{A}$ such that the new tensor product is defined by

$$A \,\square\, B = TA \otimes B \,.$$

We lift this definition to the level of a monoidale A in a monoidal bicategory $\mathscr{M}$. Proposition 8.4 observes that a warping for a monoidale determines another monoidale structure on A. If $\mathscr{F}$ is a duoidal $\mathscr{V}$-category satisfying the right-hand side of the second isomorphism above then a vertical warping of $\mathscr{F}$ by $T = - * \mathbf{1}$ recovers $\mathscr{F}_h$. This is precisely a warping of the monoidale $\mathscr{F}_v$ in $\mathscr{M} = \mathscr{V}$-Cat. The last example given generates a duoidal category by warping the monoidal structure of any lax braided monoidal category viewed as a duoidal category with $* = \circ = \otimes$ and $\gamma = 1 \otimes c \otimes 1$.

We are grateful to the referees for helping us to clarify parts of our exposition. In particular, there was a question referring to our Proposition 6.2 and Proposition 7.7 which involve Yoneda's Lemma and naturality for categories enriched in a base which is not symmetric. This was quite a fair point since most publications on enriched categories, including Kelly's book [15], require the base to be symmetric closed monoidal. We believe the first person to point out in print the possibility of having "covariant and contravariant $\mathscr{V}$-functors into $\mathscr{V}$ itself", without $\mathscr{V}$ closed or symmetric, was Linton [17]. This possibility seems strange at first since we apparently need $\mathscr{V}$ closed to be a $\mathscr{V}$-category and symmetric to define opposites.

However, a natural context for this phenomenon is that of $\mathscr{V}$-modules (= profunctors = distributors). Our reference for this is [25] but we could also give [3] (both of these

references have the base a bicategory and so include the case of a non-symmetric monoidal category when the bicategory has one object).

It is quite clear how to define a $\mathscr{V}$-module $M : \mathscr{A} \to \mathscr{X}$ with only a monoidal structure on $\mathscr{V}$: we need objects $M(X, A)$ of $\mathscr{V}$, a left action

$$\mathscr{A}(A, B) \otimes M(X, A) \to M(X, B)$$

and a right action

$$M(X, A) \otimes \mathscr{X}(Y, X) \to M(Y, A)$$

satisfying the five "bimodule" conditions. A module morphism $\theta : M \Rightarrow N$ consists of morphisms $\theta_{X,A} : M(X, A) \Rightarrow N(X, A)$ preserving the actions. If $\mathscr{V}$ is symmetric closed monoidal, a module $M : \mathscr{A} \to \mathscr{X}$ is a $\mathscr{V}$-functor

$$M : \mathscr{X}^{\mathrm{op}} \otimes \mathscr{A} \to \mathscr{V} \ ,$$

and a module morphism $\theta : M \Rightarrow N$ is a $\mathcal{V}$-natural transformation. So, for $\mathscr{V}$ merely monoidal, we still have modules without having the $\mathscr{V}$-categories $\mathscr{A}^{\mathrm{op}}$, $\mathscr{A} \otimes \mathscr{B}$ or $\mathscr{V}$.

At the beginning of Section 7, we do open the discussion of *right* modules. It was clear from the referee's question that we needed to reinforce this particularly in relation to Proposition 7.7. The point is that, in the proof, $[-, A]$ is a module $M : \mathcal{J} \to \mathcal{F}_h$. Here $\mathcal{F}_h$ is left closed and so is itself an $\mathcal{F}_h$-category with hom $\mathcal{F}_h(X, Y) = [X, Y]$. The right action of this module $M = [-, A]$ is the composition operation

$$[X, A] \otimes [Y, X] \to [Y, A]$$

as defined soon after Proposition 5.1.

2. The monoidality of hom

Let $(\mathscr{V}, \otimes)$ be a symmetric closed complete and cocomplete monoidal category. Recall from [15] that a $\mathscr{V}$-natural transformation θ between $\mathscr{V}$-functors $T, S : \mathscr{A} \longrightarrow \mathscr{X}$ consists of a *$\mathscr{V}$-natural family*

$$\theta_A : TA \longrightarrow SA,\ A \in \mathscr{A} \ ,$$

such that the diagram

$$\begin{array}{ccc} \mathscr{A}(A, B) & \xrightarrow{\ T\ } & \mathscr{X}(TA, TB) \\ {\scriptstyle S}\downarrow & & \downarrow{\scriptstyle \mathscr{X}(1,\theta_B)} \\ \mathscr{X}(SA, SB) & \xrightarrow[\mathscr{X}(\theta_A,1)]{} & \mathscr{X}(TA, SB) \end{array}$$

commutes in the base category $\mathscr{V}$.

If $(\mathscr{C}, \boxtimes)$ is a monoidal $\mathscr{V}$-category with tensor product $\boxtimes$ then the associativity isomorphisms $a_{A,B,C} : (A \boxtimes B) \boxtimes C \longrightarrow A \boxtimes (B \boxtimes C)$ are necessarily a $\mathscr{V}$-natural family, which amounts to the commutativity of the diagram

$$\begin{array}{ccc}
(\mathscr{C}(A,A') \otimes \mathscr{C}(B,B')) \otimes \mathscr{C}(C,C') & \xrightarrow{\boxtimes(\boxtimes\otimes 1)} & \mathscr{C}((A \boxtimes B) \boxtimes C, (A' \boxtimes B') \boxtimes C') \\
\cong \downarrow & & \\
\mathscr{C}(A,A') \otimes (\mathscr{C}(B,B') \otimes \mathscr{C}(C,C')) & Nat_a & \downarrow \mathscr{C}(1, a_{A',B',C'}) \\
\boxtimes(1\otimes\boxtimes) \downarrow & & \\
\mathscr{C}(A \boxtimes (B \boxtimes C), A' \boxtimes (B' \boxtimes C')) & \xrightarrow[\mathscr{C}(a_{A,B,C},1)]{} & \mathscr{C}((A \boxtimes B) \boxtimes C, A' \boxtimes (B' \boxtimes C'))
\end{array}$$

Similarly the $\mathscr{V}$-naturality of the unit isomorphisms

$$\ell_A : I \boxtimes A \longrightarrow A \qquad \text{and} \qquad r_A : A \boxtimes I \longrightarrow A$$

amounts to the commutativity of

$$\begin{array}{ccc}
\mathscr{C}(A,A') & \xrightarrow{I\boxtimes -} & \mathscr{C}(I \boxtimes A, I \boxtimes A') \\
\| & Nat_\ell & \downarrow \mathscr{C}(1, \ell_{A'}) \\
\mathscr{C}(A,A') & \xrightarrow[\mathscr{C}(\ell_A,1)]{} & \mathscr{C}(I \boxtimes A, A')
\end{array}
\qquad\qquad
\begin{array}{ccc}
\mathscr{C}(A,A') & \xrightarrow{-\boxtimes I} & \mathscr{C}(A \boxtimes I, A' \boxtimes I) \\
\| & Nat_r & \downarrow \mathscr{C}(1, r_{A'}) \\
\mathscr{C}(A,A') & \xrightarrow[\mathscr{C}(r_A,1)]{} & \mathscr{C}(A \boxtimes I, A')
\end{array}$$

2.1. Proposition. *If $(\mathscr{C}, \boxtimes)$ is a monoidal $\mathscr{V}$-category then the $\mathscr{V}$-functor*

$$\mathscr{C}(-,-) : \mathscr{C}^{\mathrm{op}} \otimes \mathscr{C} \longrightarrow \mathscr{V}$$

is equipped with a canonical monoidal structure.

Proof. For $\mathscr{C}(-,-)$ to be monoidal we require the morphisms

$$\boxtimes : \mathscr{C}(W,X) \otimes \mathscr{C}(Y,Z) \longrightarrow \mathscr{C}(W \boxtimes Y, X \boxtimes Z)$$

and

$$j_I : I \longrightarrow \mathscr{C}(I,I)$$

to satisfy the axioms

$$\begin{array}{ccc}
(\mathscr{C}(U,V) \otimes \mathscr{C}(W,X)) \otimes \mathscr{C}(Y,Z) & \xrightarrow{\boxtimes\otimes 1} & \mathscr{C}(U \boxtimes W, V \boxtimes X) \otimes \mathscr{C}(Y,Z) \\
\cong \downarrow & & \downarrow \boxtimes \\
\mathscr{C}(U,V) \otimes (\mathscr{C}(W,X) \otimes \mathscr{C}(Y,Z)) & & \mathscr{C}((U \boxtimes W) \boxtimes Y, (V \boxtimes X) \boxtimes Z) \\
1\otimes\boxtimes \downarrow & & \downarrow \mathscr{C}(a^{-1}_{U,W,Y}, a_{V,X,Z}) \\
\mathscr{C}(U,V) \otimes \mathscr{C}(W \boxtimes Y, X \boxtimes Z) & \xrightarrow[\boxtimes]{} & \mathscr{C}(U \boxtimes (W \boxtimes Y), V \boxtimes (X \boxtimes Z))
\end{array}$$

and

$$\begin{array}{ccc} \mathscr{C}(I,I)\otimes\mathscr{C}(Y,Z) & \xrightarrow{\boxtimes} & \mathscr{C}(I\boxtimes Y, I\boxtimes Z) \\ {\scriptstyle j_I\otimes 1}\uparrow & & \downarrow{\scriptstyle \mathscr{C}(\ell_Y^{-1},\ell_Z)} \\ I\otimes\mathscr{C}(Y,Z) & \xrightarrow[\ell]{} & \mathscr{C}(Y,Z) \end{array} \qquad \begin{array}{ccc} \mathscr{C}(W,X)\otimes I & \xrightarrow{r} & \mathscr{C}(W,X) \\ {\scriptstyle 1\otimes j_I}\downarrow & & \uparrow{\scriptstyle \mathscr{C}(r_W^{-1},r_X)} \\ \mathscr{C}(W,X)\otimes\mathscr{C}(I,I) & \xrightarrow[\boxtimes]{} & \mathscr{C}(W\boxtimes I, X\boxtimes I) \end{array}$$

These diagrams are simply reorganizations of the diagrams Nat_a, Nat_ℓ, and Nat_r above. ∎

2.2. Corollary. *If C is a comonoid and A is a monoid in the monoidal $\mathscr{V}$-category $\mathscr{C}$ then $\mathscr{C}(C,A)$ is canonically a monoid in $\mathscr{V}$.*

Proof. We observe that monoidal $\mathscr{V}$-functors take monoids to monoids and (C,A) is a monoid in $\mathscr{C}^{\mathrm{op}}\otimes\mathscr{C}$. ∎

2.3. Proposition. *If $\mathscr{C}$ is a braided monoidal $\mathscr{V}$-category then*

$$\mathscr{C}(-,-):\mathscr{C}^{\mathrm{op}}\otimes\mathscr{C}\longrightarrow\mathscr{V}$$

is a braided monoidal $\mathscr{V}$-functor.

Proof. Let $c_{X,Y}: X\boxtimes Y\longrightarrow Y\boxtimes X$ denote the braiding on $\mathscr{C}$. The requirement of $\mathscr{V}$-naturality for this family of isomorphisms amounts precisely to the commutativity of

$$\begin{array}{ccc} \mathscr{C}(W,X)\otimes\mathscr{C}(Y,Z) & \xrightarrow{\boxtimes} & \mathscr{C}(W\boxtimes Y, X\boxtimes Z) \\ {\scriptstyle \cong}\downarrow & & \downarrow{\scriptstyle \mathscr{C}(c^{-1},c)} \\ \mathscr{C}(Y,Z)\otimes\mathscr{C}(W,X) & \xrightarrow[\boxtimes]{} & \mathscr{C}(Y\boxtimes W, Z\boxtimes X) \end{array}$$

which is exactly the braiding condition for the monoidal functor $\mathscr{C}(-,-)$ of Proposition 2.1. ∎

We now give a spiritual successor to the above by moving to the level of monoidal bicategories.

2.4. Proposition. *If $\mathscr{M}$ is a monoidal bicategory then the pseudofunctor*

$$\mathscr{M}(-,-):\mathscr{M}^{\mathrm{op}}\times\mathscr{M}\longrightarrow\mathrm{Cat}$$

is equipped with a canonical monoidal structure.

Proof. We avail ourselves of the coherence theorem of [11] by assuming that $\mathscr{M}$ is a Gray monoid (see [8]). The definition of a monoidal pseudofunctor (called a "weak monoidal homomorphism") between Gray monoids is defined on pages 102 and 104 of [8]. Admittedly Cat is not a Gray monoid, but the adjustment to compensate for this is not too challenging.

In the notation of [8], the pseudonatural transformation χ is defined at objects to be the functor

$$\otimes:\mathscr{M}(A,A')\times\mathscr{M}(B,B')\longrightarrow\mathscr{M}(A\otimes B, A'\otimes B')$$

and at the morphisms to be the isomorphism

$$\begin{array}{ccc} \mathscr{M}(A,A')\times\mathscr{M}(B,B') & \xrightarrow{\otimes} & \mathscr{M}(A\otimes B, A'\otimes B') \\ {\scriptstyle \mathscr{M}(f,f')\times\mathscr{M}(g,g')}\downarrow & \cong & \downarrow{\scriptstyle \mathscr{M}(f\otimes g, f'\otimes g')} \\ \mathscr{M}(C,C')\times\mathscr{M}(D,D') & \xrightarrow[\otimes]{} & \mathscr{M}(C\otimes D, C'\otimes D') \end{array}$$

whose component

$$(f'uf)\otimes(g'vg)\cong(f'\otimes g')(u\otimes v)(f\otimes g)$$

at $(u,v)\in\mathscr{M}(A,A')\times\mathscr{M}(B,B')$ is the canonical isomorphism associated with the pseudofunctor $\otimes:\mathscr{M}\times\mathscr{M}\longrightarrow\mathscr{M}$ (see the top of page 102 of [8]). For ι, we have the functor $1\longrightarrow\mathscr{M}(I,I)$ which picks out 1_I. For ω, we have the natural isomorphism

$$\begin{array}{ccc} \mathscr{M}(A,A')\times\mathscr{M}(B,B')\times\mathscr{M}(C,C') & \xrightarrow{\otimes\times 1} & \mathscr{M}(A\otimes B,A'\otimes B')\times\mathscr{M}(C,C') \\ {\scriptstyle 1\times\otimes}\downarrow & \omega\Downarrow & \downarrow{\scriptstyle \otimes} \\ \mathscr{M}(A,A')\times\mathscr{M}(B\otimes C,B'\otimes C') & \xrightarrow[\otimes]{} & \mathscr{M}(A\otimes B\otimes C,A'\otimes B'\otimes C') \end{array}$$

whose component at (u,v,w) is the canonical isomorphism

$$(u\otimes v)\otimes w\cong u\otimes(v\otimes w)$$

associated with $\otimes:\mathscr{M}\times\mathscr{M}\longrightarrow\mathscr{M}$. For ξ and κ, we have the natural isomorphisms

$$\begin{array}{ccccc} & & \mathscr{M}(A,A')\times\mathscr{M}(I,I) & & \\ & {\scriptstyle 1\times\ulcorner 1_I\urcorner}\nearrow & \cong\Downarrow & \searrow{\scriptstyle \otimes} & \\ \mathscr{M}(A,A') & & \xrightarrow[1]{} & & \mathscr{M}(A,A') \end{array}$$

and

$$\begin{array}{ccccc} & & \mathscr{M}(I,I)\times\mathscr{M}(A,A') & & \\ & {\scriptstyle \ulcorner 1_I\urcorner\times 1}\nearrow & \cong\Downarrow & \searrow{\scriptstyle \otimes} & \\ \mathscr{M}(A,A') & & \xrightarrow[1]{} & & \mathscr{M}(A,A') \end{array}$$

with canonical components

$$u\otimes 1_I\cong u\qquad\text{and}\qquad 1_I\otimes u\cong u\,.$$

The two required axioms are then a consequence of the coherence conditions for pseudofunctors in the case of $\otimes:\mathscr{M}\times\mathscr{M}\longrightarrow\mathscr{M}$. ∎

2.5. Corollary. *([8]; page 110, Proposition 4) If A is a pseudomonoid and C is a pseudocomonoid in a monoidal bicategory $\mathscr{M}$ then the category $\mathscr{M}(C,A)$ is equipped with a canonical monoidal structure.*

2.6. Proposition. *If $\mathscr{M}$ is a braided monoidal bicategory then*

$$\mathscr{M}(-,-) : \mathscr{M}^{\mathrm{op}} \times \mathscr{M} \longrightarrow Cat$$

is a braided monoidal pseudofunctor.

Proof. The required data of page 122, Definition 14 in [8] is provided by the invertible modification

$$\begin{array}{ccc} \mathscr{M}(A,A') \times \mathscr{M}(B,B') & \xrightarrow{\cong} & \mathscr{M}(B,B') \times \mathscr{M}(A,A') \\ {\scriptstyle\otimes}\downarrow & \Longrightarrow & \downarrow{\scriptstyle\otimes} \\ \mathscr{M}(A\otimes B, A'\otimes B') & \xrightarrow[\mathscr{M}(\rho^{-1},\rho)]{} & \mathscr{M}(B\otimes A, B'\otimes A') \end{array}$$

whose component at (u,v) is

$$\begin{array}{ccccc} B\otimes A & \xrightarrow{\rho} & A\otimes B & \xrightarrow{u\otimes v} & A'\otimes B' \\ & \overset{\cong}{\searrow^{1}} & \downarrow{\scriptstyle\rho} & \cong \rho_{u,v} & \downarrow{\scriptstyle\rho} \\ & & B\otimes A & \xrightarrow[v\otimes u]{} & B'\otimes A' \end{array}$$

■

What we really want is a presentation of these results lifted to the level of enriched monoidal bicategories.

Suppose $\mathscr{M}$ is a monoidal bicategory. Put $\mathscr{V} = \mathscr{M}(I,I)$, regarding it as a monoidal category under composition $\circ$. There is another "multiplication" on $\mathscr{V}$ defined by the composite

$$\mathscr{M}(I,I) \times \mathscr{M}(I,I) \xrightarrow{\otimes} \mathscr{M}(I\otimes I, I\otimes I) \quad\cong\quad \mathscr{M}(I,I)$$

with the same unit 1_I as $\circ$. By Proposition 5.3 of [14], a braiding is obtained on $\mathscr{V}$.

Furthermore, each hom category $\mathscr{M}(X,Y)$ has an action

$$\mathscr{M}(I,I) \times \mathscr{M}(X,Y) \xrightarrow{\otimes} \mathscr{M}(I\otimes X, I\otimes Y) \quad\simeq\quad \mathscr{M}(X,Y)$$

by $\mathscr{V}$ which we abusively write as

$$(v,m) \longmapsto v\otimes m\ .$$

We call $\mathscr{M}$ *left unit closed* when each functor

$$-\otimes m : \mathscr{V} \longrightarrow \mathscr{M}(X,Y)$$

has a right adjoint

$$[m, -] : \mathscr{M}(X, Y) \longrightarrow \mathscr{V} .$$

That is, we have a natural isomorphism

$$\mathscr{M}(X, Y)(v \otimes m, n) \cong \mathscr{V}(v, [m, n]) .$$

In particular, this implies $\mathscr{V}$ is a left closed monoidal category and that each hom category $\mathscr{M}(X, Y)$ is $\mathscr{V}$-enriched with $\mathscr{V}$-valued hom defined by $[m, n]$. Furthermore, since $\mathscr{V}$ is braided, the 2-category $\mathscr{V}$-Cat of $\mathscr{V}$-categories, $\mathscr{V}$-functors and $\mathscr{V}$-natural transformations is monoidal; see Remark 5.2 of [14].

2.7. Proposition. *If the monoidal bicategory $\mathscr{M}$ is left unit closed then the monoidal pseudofunctor of Proposition 2.4 lifts to a monoidal pseudofunctor*

$$\mathscr{M}(-, -) : \mathscr{M}^{\mathrm{op}} \times \mathscr{M} \longrightarrow \mathscr{V}\text{-Cat}$$

where $\mathscr{V} = \mathscr{M}(I, I)$ as above.

Proof. We use the fact that, for tensored $\mathscr{V}$-categories $\mathscr{A}$ and $\mathscr{B}$, enrichment of a functor $F : \mathscr{A} \longrightarrow \mathscr{B}$ to a $\mathscr{V}$-functor can be expressed in terms of a lax action morphism structure

$$\overline{\chi}_{V,A} : V \otimes FA \longrightarrow F(V \otimes A)$$

for $V \in \mathscr{V}$, $A \in \mathscr{A}$. Given such $\mathscr{V}$-functors $F, G : \mathscr{A} \longrightarrow \mathscr{B}$, a family of morphisms

$$\theta_A : FA \longrightarrow GA$$

is $\mathscr{V}$-natural if and only if the diagrams

$$\begin{array}{ccc} V \otimes FA & \xrightarrow{\overline{\chi}_{V,A}} & F(V \otimes A) \\ {\scriptstyle 1\otimes\theta_A}\downarrow & & \downarrow{\scriptstyle \theta_{V\otimes A}} \\ V \otimes GA & \xrightarrow[\overline{\chi}_{V,A}]{} & G(V \otimes A) \end{array}$$

commute. Therefore, to see that the functors

$$\mathscr{M}(f, g) : \mathscr{M}(X, Y) \longrightarrow \mathscr{M}(X', Y') ,$$

for $f : X' \longrightarrow X$ and $g : Y \longrightarrow Y'$, are $\mathscr{V}$-enriched, we require 2-cells

$$v \otimes (g \circ m \circ f) \longrightarrow g \circ (v \otimes m) \circ f$$

which constitute a lax action morphism. As in the proof of Proposition 2.4, we assume that $\mathscr{M}$ is a Gray monoid where we can take these 2-cells to be the canonical isomorphisms.

It is then immediate that the 2-cells $\sigma : f \Longrightarrow f'$ and $\tau : g \Longrightarrow g'$ induce $\mathscr{V}$-natural transformations $\mathscr{M}(\sigma, \tau) : \mathscr{M}(f, g) \Longrightarrow \mathscr{M}(f', g')$.

For the monoidal structure on $\mathscr{M}(-,-)$, we need to see that the effect of the tensor of $\mathscr{M}$ on homs defines a $\mathscr{V}$-functor

$$\otimes : \mathscr{M}(A, A') \otimes \mathscr{M}(B, B') \longrightarrow \mathscr{M}(A \otimes B, A' \otimes B') \,.$$

Again we make use of the coherent isomorphisms; in this case they are

$$v \otimes (m \otimes n) \cong (v \circ m) \otimes (v \circ n)$$

for $v : I \longrightarrow I$, $m : A \longrightarrow A'$, $n : B \longrightarrow B'$. It is clear that ι can be regarded as a $\mathscr{V}$-functor $\iota : \mathcal{I} \longrightarrow \mathscr{M}(I, I)$. The $\mathscr{V}$-naturality of all the 2-cells involved in the monoidal structure on $\mathscr{M}(-,-)$ now follows automatically from the naturality of the Gray monoid constraints. ∎

2.8. Proposition. *In the situation of Proposition 2.7, if $\mathscr{M}$ is also symmetric then so is $\mathscr{M}(-,-)$.*

Proof. If $\mathscr{M}$ is symmetric, so too is $\mathscr{V} = \mathscr{M}(I, I)$. Consequently, $\mathscr{V}$-Cat is also symmetric. Referring to the proof of Proposition 2.6, we see that the techniques of the proof of Proposition 2.7 apply. ∎

2.9. Example. Let $\mathscr{V}$ be any braided monoidal category which is closed complete and cocomplete. Put $\mathscr{M} = \mathscr{V}$-Mod, the bicategory of $\mathscr{V}$-categories, $\mathscr{V}$-modules (i.e. $\mathscr{V}$-distributors or equivalently $\mathscr{V}$-profunctors), and $\mathscr{V}$-module morphisms. This $\mathscr{M}$ is a well-known example of a monoidal bicategory (see [8]). We can easily identify $\mathscr{V}$ with $\mathscr{V}$-Mod$(\mathcal{I}, \mathcal{I})$ and the action on $\mathscr{M}(\mathscr{A}, \mathscr{X})$ with the functor

$$\mathscr{V} \times \mathscr{V}\text{-Mod}(\mathscr{A}, \mathscr{X}) \longrightarrow \mathscr{V}\text{-Mod}(\mathscr{A}, \mathscr{X})$$

given by the mapping

$$(V, M) \longmapsto V \otimes M \,.$$

This mapping is defined by $(V \otimes M)(X, A) = V \otimes M(X, A)$ equipped with a left module action

$$\mathscr{A}(A, B) \otimes V \otimes M(X, A) \xrightarrow[\cong]{c \otimes 1} V \otimes \mathscr{A}(A, B) \otimes M(X, A) \xrightarrow{1 \otimes act_\ell} V \otimes M(X, B)$$

and a right module action

$$V \otimes M(X, A) \otimes \mathscr{X}(Y, X) \xrightarrow{1 \otimes_r} V \otimes M(Y, A) \,,$$

where c is the braiding of $\mathscr{V}$. We have ignored associativity isomorphisms. To see that $\mathscr{M} = \mathscr{V}$-Mod is left unit closed we easily identify $[M, N] \in \mathscr{V}$ for $M, N \in \mathscr{V}$-Mod$(\mathscr{A}, \mathscr{X})$ with the usual $\mathscr{V}$-valued hom for the $\mathscr{V}$-category $[\mathscr{X}^{\mathrm{op}} \otimes \mathscr{A}, \mathscr{V}]$; namely,

$$[M, N] = \int_{X,A} [M(X, A), N(X, A)] \,,$$

the "object of $\mathscr{V}$-natural transformations". Therefore, in this case, Proposition 2.7 is about the pseudofunctor

$$\mathscr{V}\text{-Mod}^{\mathrm{op}} \times \mathscr{V}\text{-Mod} \longrightarrow \mathscr{V}\text{-Cat}\,,$$

given by the mapping

$$(\mathscr{A}, \mathscr{X}) \longmapsto [\mathscr{X}^{\mathrm{op}} \otimes \mathscr{A}, \mathscr{V}]\,,$$

asserting monoidality. When $\mathscr{V}$ is symmetric, Proposition 2.8 assures us the pseudofunctor is also symmetric.

2.10. Remark. There is presumably a more general setting encompassing the results of this section. For a monoidal bicategory $\mathscr{K}$, it is possible to define a notion of *$\mathscr{K}$-bicategory* $\mathscr{M}$ by which we mean that the homs $\mathscr{M}(X,Y)$ are objects of $\mathscr{K}$. For Proposition 2.1 we would take $\mathscr{K}$ to be $\mathscr{V}$ as a locally discrete bicategory and $\mathscr{M}$ to be $\mathscr{C}$. For Proposition 2.4, $\mathscr{K}$ would be Cat. For Proposition 2.7, $\mathscr{K}$ would be $\mathscr{V}$-Cat. Then, as in these cases, we would require $\mathscr{K}$ to be braided in order to define the *tensor product of $\mathscr{K}$-bicategories* and so *monoidal $\mathscr{K}$-bicategories*. With all this properly defined, we expect

$$\mathscr{M}(-,-) : \mathscr{M}^{\mathrm{op}} \otimes \mathscr{M} \longrightarrow \mathscr{K}$$

to be a monoidal $\mathscr{K}$-pseudofunctor.

3. Duoidal $\mathscr{V}$-categories

Throughout $\mathscr{V}$ is a symmetric monoidal closed, complete and cocomplete category. The following definition agrees with that of Batanin and Markl in [2] and, under the name 2-monoidal category, Aguiar and Mahajan in [1].

3.1. Definition. A *duoidal structure* on a $\mathscr{V}$-category $\mathscr{F}$ consists of two $\mathscr{V}$-monoidal structures

$$* : \mathscr{F} \otimes \mathscr{F} \longrightarrow \mathscr{F}, \quad \ulcorner J \urcorner : \mathbf{1} \longrightarrow \mathscr{F}, \tag{3.1}$$

$$\circ : \mathscr{F} \otimes \mathscr{F} \longrightarrow \mathscr{F}, \quad \ulcorner 1 \urcorner : \mathbf{1} \longrightarrow \mathscr{F}, \tag{3.2}$$

such that either of the following equivalent conditions holds:

(i) the $\mathscr{V}$-functors $\circ$ and $\ulcorner 1 \urcorner$ of (3.2) and their coherence isomorphisms are monoidal with respect to the monoidal $\mathscr{V}$-category $\mathscr{F}_h$ of (3.1).

(ii) the $\mathscr{V}$-functors $*$ and $\ulcorner J \urcorner$ of (3.1) and their coherence isomorphisms are opmonoidal with respect to the monoidal $\mathscr{V}$-category $\mathscr{F}_v$ of (3.2).

We call the monoidal $\mathscr{V}$-category $\mathscr{F}_h$ of (3.1) *horizontal* and the monoidal $\mathscr{V}$-category $\mathscr{F}_v$ of (3.2) *vertical*; this terminology comes from an example of derivation schemes due to [2] (also see [26]).

The extra elements of structure involved in (i) and (ii) are a $\mathscr{V}$-natural middle-of-four interchange transformation

$$\gamma : (A \circ B) * (C \circ D) \longrightarrow (A * C) \circ (B * D) \,,$$

and maps

$$\mathbf{1} * \mathbf{1} \xrightarrow{\mu} \mathbf{1} \xleftarrow{\tau} J \xrightarrow{\delta} J \circ J$$

such that the diagrams

$$\begin{array}{ccc} ((A \circ B) * (C \circ D)) * (E \circ F) & \xrightarrow{\cong} & (A \circ B) * ((C \circ D) * (E \circ F)) \\ \downarrow{\scriptstyle \gamma * 1} & & \downarrow{\scriptstyle 1 * \gamma} \\ ((A * C) \circ (B * D)) * (E \circ F) & & (A \circ B) * ((C * E) \circ (D * F)) \\ \downarrow{\scriptstyle \gamma} & & \downarrow{\scriptstyle \gamma} \\ ((A * C) * E) \circ ((B * D) * F) & \xrightarrow[\cong]{} & (A * (C * E)) \circ (B * (D * F)) \end{array} \tag{3.3}$$

$$\begin{array}{ccc} ((A \circ B) \circ C) * ((D \circ E) \circ F) & \xrightarrow{\cong} & (A \circ (B \circ C)) * (D \circ (E \circ F)) \\ \downarrow{\scriptstyle \gamma} & & \downarrow{\scriptstyle \gamma} \\ ((A \circ B) * (D \circ E)) \circ (C * F) & & (A * D) \circ ((B \circ C) * (E \circ F)) \\ \downarrow{\scriptstyle \gamma \circ 1} & & \downarrow{\scriptstyle 1 \circ \gamma} \\ ((A * D) \circ (B * E)) \circ (C * F) & \xrightarrow[\cong]{} & (A * D) \circ ((B * E) \circ (C * F)) \end{array} \tag{3.4}$$

and

$$\begin{array}{ccc} J * (A \circ B) & \xrightarrow{\delta * 1} & (J \circ J) * (A \circ B) \\ \uparrow{\scriptstyle \cong} & & \downarrow{\scriptstyle \gamma} \\ A \circ B & \xrightarrow[\cong]{} & (J * A) \circ (J * B) \end{array} \qquad \begin{array}{ccc} (A \circ B) * J & \xrightarrow{1 * \delta} & (A \circ B) * (J \circ J) \\ \uparrow{\scriptstyle \cong} & & \downarrow{\scriptstyle \gamma} \\ A \circ B & \xrightarrow[\cong]{} & (A * J) \circ (B * J) \end{array} \tag{3.5}$$

$$\begin{array}{ccc} \mathbf{1} \circ (A * B) & \xleftarrow{\mu \circ 1} & (\mathbf{1} * \mathbf{1}) \circ (A * B) \\ \uparrow{\scriptstyle \cong} & & \uparrow{\scriptstyle \gamma} \\ A * B & \xrightarrow[\cong]{} & (\mathbf{1} \circ A) * (\mathbf{1} \circ B) \end{array} \qquad \begin{array}{ccc} (A * B) \circ \mathbf{1} & \xleftarrow{1 \circ \mu} & (A * B) \circ (\mathbf{1} * \mathbf{1}) \\ \uparrow{\scriptstyle \cong} & & \uparrow{\scriptstyle \gamma} \\ A * B & \xrightarrow[\cong]{} & (A \circ \mathbf{1}) * (B \circ \mathbf{1}) \end{array} \tag{3.6}$$

commute, together with the requirement that $(\mathbf{1}, \mu, \tau)$ is a monoid in $\mathscr{F}_h$ and (J, δ, τ) is a comonoid in $\mathscr{F}_v$.

3.2. Example. A braided monoidal category $\mathscr{C}$ with braid isomorphism $c : A \otimes B \cong B \otimes A$ is an example of a duoidal category with $\otimes = * = \circ$ and γ, determined by $1_A \otimes c \otimes 1_D$ and re-bracketing, invertible.

3.3. Example. Let $\mathscr{C}$ be a monoidal $\mathscr{V}$-category. An important example is the $\mathscr{V}$-category $\mathscr{F} = [\mathscr{C}^{\mathrm{op}} \otimes \mathscr{C}, \mathscr{V}]$ of $\mathscr{V}$-modules $\mathscr{C} \longrightarrow \mathscr{C}$ and $\mathscr{V}$-module homomorphisms. We see that $\mathscr{F}$ becomes a duoidal $\mathscr{V}$-category with $*$ the convolution tensor product for $\mathscr{C}^{\mathrm{op}} \otimes \mathscr{C}$ and $\circ$ the tensor product "over $\mathscr{C}$". This example can be found in the last section of [26].

3.4. Definition. A *duoidal functor* $F : \mathscr{F} \longrightarrow \mathscr{F}'$ is a functor F that is equipped with monoidal structures $\mathscr{F}_h \longrightarrow \mathscr{F}'_h$ and $\mathscr{F}_v \longrightarrow \mathscr{F}'_v$ which are compatible with the duoidal data γ, μ, δ, and τ.

3.5. Definition. A *bimonoidal functor* $F : \mathscr{F} \longrightarrow \mathscr{F}'$ is a functor F that is equipped with a monoidal structure $\mathscr{F}_h \longrightarrow \mathscr{F}'_h$ and an opmonoidal structure $\mathscr{F}_v \longrightarrow \mathscr{F}'_v$ both of which are compatible with the duoidal data γ, μ, δ, and τ.

3.6. Definition. A *bimonoid* A in a duoidal category $\mathscr{F}$ is a bimonoidal functor $\ulcorner A \urcorner : \mathbf{1} \longrightarrow \mathscr{F}$. That is, it is an object A equipped with the structure of a monoid for $*$ and a comonoid for $\circ$, compatible via the axioms

$$\begin{array}{ccccc} A * A & \xrightarrow{\mu} & A & \xrightarrow{\delta} & A \circ A \\ {\scriptstyle \delta * \delta}\downarrow & & = & & \uparrow{\scriptstyle \mu \circ \mu} \\ (A \circ A) * (A \circ A) & & \xrightarrow[\gamma]{} & & (A * A) \circ (A * A) \end{array} \tag{3.7}$$

$$\begin{array}{ccc} A * A & \xrightarrow{\mu} & A \\ {\scriptstyle \epsilon * \epsilon}\downarrow & = & \downarrow{\scriptstyle \epsilon} \\ \mathbf{1} * \mathbf{1} & \xrightarrow[\mu]{} & \mathbf{1} \end{array} \qquad \begin{array}{ccc} J \circ J & \xleftarrow{\delta} & J \\ {\scriptstyle \eta \circ \eta}\downarrow & = & \downarrow{\scriptstyle \eta} \\ A \circ A & \xleftarrow[\delta]{} & A \end{array} \tag{3.8}$$

$$\begin{array}{ccc} J & & \\ \downarrow{\scriptstyle \tau} & \searrow^{\eta} & \\ & = & A \\ & \swarrow_{\epsilon} & \\ \mathbf{1} & & \end{array} \tag{3.9}$$

These are a lifting of the usual axioms for a bimonoid in a braided monoidal category.

4. Duoidales and produoidal $\mathscr{V}$-categories

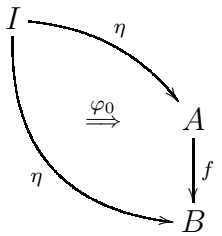
Recall the two following definitions and immediately following example from [8] where $\mathscr{M}$ is a monoidal bicategory.

4.1. Definition. A *pseudomonoid* A in $\mathscr{M}$ is an object A of $\mathscr{M}$ together with multiplication and unit morphisms $\mu : A \otimes A \longrightarrow A$, $\eta : I \longrightarrow A$, and invertible 2-cells $a : \mu(\mu \otimes 1) \Longrightarrow \mu(1 \otimes \mu)$, $\ell : \mu(\eta \otimes 1) \Longrightarrow 1$, and $r : \mu(1 \otimes \eta) \Longrightarrow 1$ satisfying the coherence conditions given in [8].

4.2. Definition. A *(lax-)morphism* f between pseudomonoids A and B in $\mathscr{M}$ is a morphism $f : A \longrightarrow B$ equipped with

$$\begin{array}{ccc} A \otimes A & \xrightarrow{\mu} & A \\ {\scriptstyle f\otimes f}\downarrow & \overset{\varphi}{\Longrightarrow} & \downarrow{\scriptstyle f} \\ B \otimes B & \xrightarrow[\mu]{} & B \end{array}$$

and



subject to three axioms.

4.3. Example. If $\mathscr{M}$ is the cartesian closed 2-category of categories, functors, and natural transformations then a monoidal category is precisely a pseudomonoid in $\mathscr{M}$.

This example motivates calling a pseudomonoid in a monoidal bicategory $\mathscr{M}$ a *monoidale* (short for a monoidal object of $\mathscr{M}$). A morphism $f : M \to N$ of monoidales is then a morphism of pseudomonoids (i.e. a monoidal morphism between monoidal objects). We write $\mathrm{Mon}(\mathscr{M})$ for the 2-category of monoidales in $\mathscr{M}$, monoidal morphisms, and monoidal 2-cells. If $\mathscr{M}$ is symmetric monoidal then so is $\mathrm{Mon}(\mathscr{M})$.

4.4. Definition. A *duoidale* F in the symmetric monoidal bicategory $\mathscr{M}$ is an object F together with two monoidale structures

$$* : F \otimes F \longrightarrow F, \quad J : I \longrightarrow F \tag{4.1}$$

$$\circ : F \otimes F \longrightarrow F, \quad \mathbf{1} : I \longrightarrow F \tag{4.2}$$

such that $\circ$ and $\mathbf{1}$ are monoidal morphisms with respect to $*$ and J.

4.5. Remark. If $\mathscr{M} = \mathscr{V}$-Cat then a duoidale in $\mathscr{M}$ is precisely a duoidal $\mathscr{V}$-category.

Let $\mathscr{M} = \mathscr{V}$-Mod be the symmetric monoidal bicategory of $\mathscr{V}$-categories, $\mathscr{V}$-modules, and $\mathscr{V}$-module morphisms. By Proposition 2.8, there is a symmetric monoidal pseudofunctor

$$\mathscr{M}(\mathscr{I}, -) : \mathscr{M} \longrightarrow \mathscr{V}\text{-Cat}$$

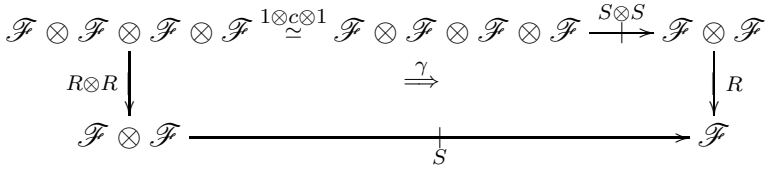
defined by taking a $\mathscr{V}$-category $\mathscr{A}$ to the $\mathscr{V}$-category $[\mathscr{A}^{\mathrm{op}}, \mathscr{V}]$ of $\mathscr{V}$-functors and $\mathscr{V}$-natural transformations.

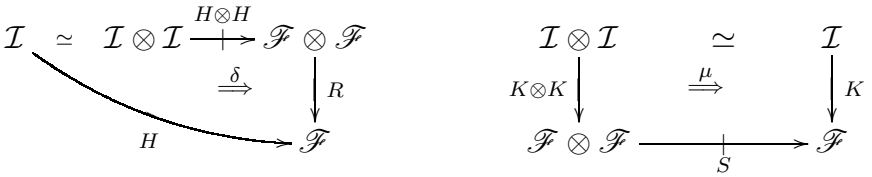
4.6. Definition. A *produoidal $\mathscr{V}$-category* is a duoidale in $\mathscr{V}$-Mod.

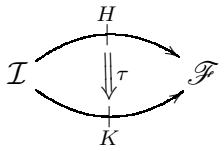
If $\mathscr{F}$ is a produoidal $\mathscr{V}$-category then there are $\mathscr{V}$-modules

$$S : \mathscr{F} \otimes \mathscr{F} \longrightarrow\!\!\!\!\!\rightarrow \mathscr{F}, \qquad H : \mathcal{I} \longrightarrow\!\!\!\!\!\rightarrow \mathscr{F},$$
$$R : \mathscr{F} \otimes \mathscr{F} \longrightarrow\!\!\!\!\!\rightarrow \mathscr{F}, \qquad K : \mathcal{I} \longrightarrow\!\!\!\!\!\rightarrow \mathscr{F},$$

where R and K are monoidal with respect to S so that there are 2-cells γ, δ, and τ:







compatible with the two pseudomonoid structures. By composition of $\mathscr{V}$-modules these 2-cells have component morphisms

$$\int^{X,Y} R(X; A, B) \otimes R(Y; C, D) \otimes S(E; X, Y)$$
$$\downarrow \gamma$$
$$\int^{U,V} S(U; A, C) \otimes S(V; B, D) \otimes R(E; U, V)$$

$$H(A) \xrightarrow{\ \delta\ } \int^{X,Y} H(X) \otimes H(Y) \otimes R(A; X, Y)$$

$$\int^{X,Y} K(X) \otimes K(Y) \otimes S(A; X, Y) \xrightarrow{\ \mu\ } K(A)$$

$$H(A) \xrightarrow{\ \tau\ } K(A)$$

in $\mathscr{V}$.

Given any duoidal $\mathscr{V}$-category $\mathscr{F}$ we obtain a produoidal $\mathscr{V}$-category structure on $\mathscr{F}$ by setting

$$S(A; B, C) = \mathscr{F}(A, B * C)$$

and

$$R(A; B, C) = \mathscr{F}(A, B \circ C)$$

that is, we pre-compose the $\mathscr{V}$-valued hom of $\mathscr{F}$ with (3.1) and (3.2) of Definition 3.1.

4.7. Proposition. *If $\mathscr{F}$ is a produoidal $\mathscr{V}$-category then $\mathscr{M}(\mathscr{I},\mathscr{F}) = [\mathscr{F}^{\mathrm{op}},\mathscr{V}]$ is a duoidal $\mathscr{V}$-category.*

Proof. By Proposition 2.8 and by using the obvious symmetric monoidal structure on $\mathscr{I}$, we see that $\mathscr{M}(\mathscr{I},-)$ is a symmetric monoidal pseudofunctor and so takes duoidales to duoidales. The result follows from this. However, we spell out some details. Consider the $\mathscr{V}$-category of $\mathscr{V}$-functors and $\mathscr{V}$-natural transformations $[\mathscr{F}^{\mathrm{op}},\mathscr{V}]$. The two monoidale structures on $\mathscr{F}$ translate to two monoidal structures on $[\mathscr{F}^{\mathrm{op}},\mathscr{V}]$ by Day-convolution

$$(M * N)(A) = \int^{X,Y} S(A;X,Y)\otimes M(X)\otimes N(Y) \tag{4.3}$$

$$(M \circ N)(B) = \int^{U,V} R(B;U,V)\otimes M(U)\otimes N(V) \tag{4.4}$$

such that the duoidale 2-cell structure morphisms lift to give a duoidal $\mathscr{V}$-category. More specifically the maps (γ,δ,μ,τ) lift to $[\mathscr{F}^{\mathrm{op}},\mathscr{V}]$ and satisfy the axioms (3.3), (3.4), (3.5) and (3.6) in Definition 3.1. Demonstrating the lifting and commutativity of the requisite axioms uses iterated applications of the $\mathscr{V}$-enriched Yoneda lemma and Fubini's interchange theorem as in [15]. ∎

Our final theorem for this section permits us to apply the theory of categories enriched in a duoidal $\mathscr{V}$-category $\mathscr{F}$ even if the monoidal structures on $\mathscr{F}$ are not closed.

4.8. Theorem. *Let $\mathscr{F}$ be a small duoidal $\mathscr{V}$-category. The Yoneda embedding $y : \mathscr{F} \longrightarrow [\mathscr{F}^{\mathrm{op}},\mathscr{V}]$ gives $[\mathscr{F}^{\mathrm{op}},\mathscr{V}]$ as the duoidal cocompletion of $\mathscr{F}$ with both monoidal structures closed. The universal property is that, for any cocomplete duoidal category $\mathscr{X}$ with both monoidal structures closed, there is an equivalence between the $\mathscr{V}$-category of duoidal $\mathscr{V}$-functors $\mathscr{F} \longrightarrow \mathscr{X}$ and the $\mathscr{V}$-category of colimit-preserving duoidal $\mathscr{V}$-functors $[\mathscr{F}^{\mathrm{op}},\mathscr{V}] \longrightarrow \mathscr{X}$.*

Proof. This theorem is essentially an extension of some results of Im and Kelly in [12] which themselves are largely extensions of results in [6] and [15]. In particular, if $\mathscr{A}$ is a monoidal $\mathscr{V}$-category then $\hat{\mathscr{A}} = [\mathscr{A}^{\mathrm{op}},\mathscr{V}]$ is the free monoidal closed completion with the convolution monoidal structure. If $\mathscr{F}$ is a duoidal $\mathscr{V}$-category then, by Proposition 4.1 of [12], the monoidal structures $*$ and $\circ$ on $\mathscr{F}$ give two monoidal biclosed structures on $\hat{\mathscr{F}} = [\mathscr{F}^{\mathrm{op}},\mathscr{V}]$ with the corresponding Yoneda embeddings strong monoidal functors. As per [12] the monoidal products are given by Day convolution

$$P \mathbin{\hat{*}} Q = \int^{A,B} P(A)\otimes Q(B)\otimes \mathscr{F}(-,A*B) \tag{4.5}$$

$$P \mathbin{\hat{\circ}} Q = \int^{A,B} P(A)\otimes Q(B)\otimes \mathscr{F}(-,A\circ B) \tag{4.6}$$

as the left Kan-extension of $y\otimes y$ along the composites $y*$ and $y\circ$ respectively. Write $\hat{J}$ and $\hat{\mathbf{1}}$ for the tensor units $y(J) = \mathscr{F}(-,J)$ and $y(\mathbf{1}) = \mathscr{F}(-,\mathbf{1})$ respectively. The duoidal data (γ,μ,δ,τ) lifts directly to give duoidal data $(\hat{\gamma},\hat{\mu},\hat{\delta},\hat{\tau})$ for $\mathscr{F}$. ∎

5. Enrichment in a duoidal $\mathscr{V}$-category base

Let $\mathscr{F}$ be a duoidal $\mathscr{V}$-category. There is a 2-category $\mathscr{F}_h$-Cat of $\mathscr{F}_h$-categories, $\mathscr{F}_h$-functors, and $\mathscr{F}_h$-natural transformations in the usual Eilenberg-Kelly sense; see [15]. We write $\mathcal{J}$ for the one-object $\mathscr{F}_h$-category whose hom is the horizontal unit J in $\mathscr{F}$.

Let $\mathscr{A}$ and $\mathscr{B}$ be $\mathscr{F}_h$-categories and define $\mathscr{A} \circ \mathscr{B}$ to be the $\mathscr{F}_h$-category with objects pairs (A, B) and hom-objects $(\mathscr{A} \circ \mathscr{B})((A, B), (A', B')) = \mathscr{A}(A, A') \circ \mathscr{B}(B, B')$ in $\mathscr{F}_h$. Composition is defined using the middle-of-four map γ as follows

$$\begin{array}{c}
(\mathscr{A} \circ \mathscr{B})((A', B'), (A'', B'')) * (\mathscr{A} \circ \mathscr{B})((A, B), (A', B')) \\
\downarrow{\scriptstyle \cong} \\
(\mathscr{A}(A', A'') \circ \mathscr{B}(B', B'')) * (\mathscr{A}(A, A') \circ \mathscr{B}(B, B')) \\
\downarrow{\scriptstyle \gamma} \\
(\mathscr{A}(A', A'') * \mathscr{A}(A, A')) \circ (\mathscr{B}(B', B'') * \mathscr{B}(B, B')) \\
\downarrow{\scriptstyle comp \circ comp} \\
\mathscr{A}(A, A'') \circ \mathscr{B}(B, B'') \\
\downarrow{\scriptstyle \cong} \\
(\mathscr{A} \circ \mathscr{B})((A, B), (A'', B''))\,.
\end{array}$$

Identities are given by the composition

$$J \xrightarrow{\delta} J \circ J \xrightarrow{\hat{id}_A \circ \hat{id}_B} \mathscr{A}(A, A) \circ \mathscr{B}(B, B)\,.$$

The monoidal unit is the $\mathscr{F}_h$-category $\overline{\mathbf{1}}$ consisting of a single object $\bullet$ and hom-object $\overline{\mathbf{1}}(\bullet, \bullet) = \mathbf{1}$.

Checking the required coherence conditions proves the following result of [2].

5.1. Proposition. *The $\circ$ monoidal structure on $\mathscr{F}_h$ lifts to a monoidal structure on the 2-category $\mathscr{F}_h$-Cat.*

We write $\mathscr{F}$-Cat for the monoidal 2-category $\mathscr{F}_h$-Cat with $\circ$ as the tensor product.

Let $\mathscr{F}$ be a duoidal $\mathscr{V}$-category such that the horizontal monoidal structure $*$ is left-closed. That is, we have

$$\mathscr{F}(X * Y, Z) \cong \mathscr{F}(X, [Y, Z])$$

with the "evaluation" counit $ev : [Y, Z] * Y \longrightarrow Z$.

This gives $\mathscr{F}_h$ as an $\mathscr{F}_h$-category in the usual way by defining the composition operation $[Y, Z] * [X, Y] \longrightarrow [X, Z]$ as corresponding to

$$([Y, Z] * [X, Y]) * X \quad \cong \quad [Y, Z] * ([X, Y] * X) \xrightarrow{1 * ev} [Y, Z] * Y \xrightarrow{ev} Z$$

and identities $\hat{id}_X : J \longrightarrow [X, X]$ as corresponding to $\ell : J * X \longrightarrow X$.

The duoidal structure of $\mathscr{F}$ provides a way of defining $[X, X'] \circ [Y, Y'] \longrightarrow [X \circ Y, X' \circ Y']$ using the middle-of-four interchange map:

$$\begin{array}{ccc} ([X, X'] \circ [Y, Y']) * (X \circ Y) & \longrightarrow & X' \circ Y' \\ \downarrow{\scriptstyle \gamma} & \nearrow_{ev \circ ev} & \\ ([X, X'] * X) \circ ([Y, Y'] * Y) & & \end{array} \tag{5.1}$$

The above shows that $\mathscr{F}$ is a monoidale (pseudo-monoid) in the category of $\mathscr{F}_h$-categories with multiplication given by the $\mathscr{F}_h$-functor $\hat{\circ} : \mathscr{F}_h \circ \mathscr{F}_h \longrightarrow \mathscr{F}_h$ as defined.

Let $\mathrm{Mon}(\mathscr{F}_h)$ be the category of (horizontal) monoids $(M, \mu : M * M \longrightarrow M, \eta : J \longrightarrow M)$ in $\mathscr{F}_h$. Let M and N be objects of $\mathrm{Mon}(\mathscr{F}_h)$ and define the monoid multiplication map of $M \circ N$ to be the composition

$$(M \circ N) * (M \circ N) \xrightarrow{\gamma} (M * M) \circ (N * N) \xrightarrow{\mu \circ \mu} M \circ N$$

and the unit to be

$$J \xrightarrow{\delta} J \circ J \xrightarrow{\eta \circ \eta} M \circ N \ .$$

This tensor product of monoids is the restriction to one-object $\mathscr{F}_h$-categories of the tensor of $\mathscr{F}$-Cat. So we have the following result which was also observed in [1].

5.2. Proposition. *The monoidal structure $\circ$ on $\mathscr{F}$ lifts to a monoidal structure on the category Mon($\mathscr{F}_h$).*

We write $\mathrm{Mon}\ \mathscr{F}$ for the monoidal category $\mathrm{Mon}(\mathscr{F}_h)$ with $\circ$.

5.3. Remark. A monoid in $(\mathrm{Mon}\ \mathscr{F})^{\mathrm{op}}$ is precisely a bimonoid in $\mathscr{F}$.

6. The Tannaka adjunction revisited

Let $\mathscr{F}$ be a horizontally left closed duoidal $\mathscr{V}$-category. Each object M of $\mathscr{F}$ determines an $\mathscr{F}_h$-functor

$$- * M : \mathscr{F}_h \longrightarrow \mathscr{F}_h$$

defined on objects by $A \mapsto A * M$ and on homs by taking

$$- * M : [A, B] \longrightarrow [A * M, B * M] \tag{6.1}$$

to correspond to

$$[A, B] * (A * M) \quad \cong \quad ([A, B] * A) * M \xrightarrow{ev * 1} B * M \ .$$

If M is a monoid in $\mathscr{F}_h$ then $- * M$ becomes a monad in $\mathscr{F}_h$-Cat in the usual way.

We write $\mathscr{F}^{*M}$ for the Eilenberg-Moore $\mathscr{F}_h$-category of algebras for the $\mathscr{F}_h$-monad $-*M$; see [16] and [23]. It is the $\mathscr{F}_h$-category of right M-modules in $\mathscr{F}$. If $\mathscr{F}$ has equalizers then $\mathscr{F}^{*M}$ is assured to exist; the $\mathscr{F}_h$-valued hom is the equalizer of the pair

$$\begin{array}{ccc} [A,B] & \xrightarrow{\quad[\alpha,1]\quad} & [A*M,B] \\ & {\scriptstyle -*M}\searrow \quad \nearrow{\scriptstyle [1,\beta]} & \\ & [A*M,B*M] & \end{array} \tag{6.2}$$

where $\alpha : A*M \longrightarrow A$ and $\beta : B*M \longrightarrow B$ are the actions of A and B as objects of $\mathscr{F}^{*M}$.

Let $U_M : \mathscr{F}^{*M} \longrightarrow \mathscr{F}_h$ denote the underlying $\mathscr{F}_h$-functor which forgets the action and whose effect on homs is the equalizer of (6.2). There is an $\mathscr{F}_h$-natural transformation

$$\chi : U_M * M \longrightarrow U_M \tag{6.3}$$

which is the universal action of the monad $-*M$; its component at A in $\mathscr{F}^{*M}$ is precisely the action $\alpha : A*M \longrightarrow A$ of A.

An aspect of the strong enriched Yoneda Lemma is the $\mathscr{F}_h$-natural isomorphism

$$\mathscr{F}^{*M}(M,B) \cong U_M B. \tag{6.4}$$

In this special case, the result comes from the equalizer

$$B \xrightarrow{\hat{\beta}} [M,B] \underset{[1,\beta](-*M)}{\overset{[\mu,1]}{\rightrightarrows}} [M*M,B].$$

In other words, the $\mathscr{F}_h$-functor U_M is representable with M as the representing object.

Each $\mathscr{F}_h$-functor $U : \mathscr{A} \longrightarrow \mathscr{F}_h$ defines a functor

$$U * - : \mathscr{F} \longrightarrow \mathscr{F}_h\text{-Cat}(\mathscr{A}, \mathscr{F}_h) \tag{6.5}$$

taking $X \in \mathscr{F}$ to the composite $\mathscr{F}_h$-functor

$$\mathscr{A} \xrightarrow{\quad U \quad} \mathscr{F}_h \xrightarrow{\quad -*X \quad} \mathscr{F}_h$$

and $f : X \longrightarrow Y$ to the $\mathscr{F}_h$-natural transformation $U * f$ with components

$$1 * f : UA * X \longrightarrow UA * Y.$$

We shall call $U : \mathscr{A} \longrightarrow \mathscr{F}_h$ *tractable* when the functor $U * -$ has a right adjoint denoted

$$\{U, -\} : \mathscr{F}_h\text{-Cat}(\mathscr{A}, \mathscr{F}_h) \longrightarrow \mathscr{F}. \tag{6.6}$$

This means that morphisms $t : X \longrightarrow \{U, V\}$ are in natural bijection with $\mathscr{F}_h$-natural transformations $\theta : U * X \longrightarrow V$.

Let us examine what $\mathscr{F}_h$-naturality of $\theta : U * X \longrightarrow V$ means. By definition it means commutativity of

$$\begin{array}{ccc} & [VA,VB] & \\ {\scriptstyle V_{A,B}}\nearrow & & \searrow{\scriptstyle [\theta_A,1]} \\ \mathscr{A}(A,B) & & [UA*X,VB] \\ {\scriptstyle U_{A,B}}\downarrow & & \uparrow{\scriptstyle [1,\theta_B]} \\ [UA,UB] & \xrightarrow[-*X]{} & [UA*X,UB*X]\,. \end{array} \tag{6.7}$$

This is equivalent to the module-morphism condition

$$\begin{array}{ccccc} & & \mathscr{A}(A,B)*(UA*X) & \xrightarrow{1*\theta_A} & \mathscr{A}(A,B)*VA \\ & \overset{\cong}{\nearrow} & & & \downarrow{\scriptstyle \overline{V}_{A,B}} \\ (\mathscr{A}(A,B)*UA)*X & & & & \\ & \searrow{\scriptstyle \overline{U}_{A,B}*X} & & & \\ & & UB*X & \xrightarrow[\theta_B]{} & VB. \end{array} \tag{6.8}$$

under left closedness of $\mathscr{F}_h$. Notice that tractability of an object Z of $\mathscr{F}$, regarded as an $\mathscr{F}_h$-functor $\ulcorner Z \urcorner : \mathcal{J} \longrightarrow \mathscr{F}_h$, is equivalent to the existence of a horizontal right hom $\{Z,-\}$:

$$\mathscr{F}(X,\{Z,Y\}) \cong \mathscr{F}(Z*X,Y). \tag{6.9}$$

Assuming all of the objects UA and $\mathscr{A}(A,B)$ in $\mathscr{F}$ are tractable, we can rewrite (6.8) in the equivalent form

$$\begin{array}{ccccc} & & \{UA,VA\} & \xrightarrow{\{1,\hat{V}_{AB}\}} & \{UA,\{\mathscr{A}(A,B),VB\}\} \\ & {\scriptstyle \hat{\theta}_A}\nearrow & & & \downarrow{\scriptstyle \cong} \\ X & & & & \\ & \searrow{\scriptstyle \hat{\theta}_B} & & & \\ & & \{UB,VB\} & \xrightarrow{\{\overline{U}_{AB},1\}} & \{\mathscr{A}(A,B)*UA,VB\}. \end{array} \tag{6.10}$$

6.1. Proposition. *If $\mathscr{F}$ is a complete, horizontally left and right closed, duoidal $\mathscr{V}$-category and $\mathscr{A}$ is a small $\mathscr{F}_h$-category then every $\mathscr{F}_h$-functor $U : \mathscr{A} \longrightarrow \mathscr{F}_h$ is tractable.*

However, some U can still be tractable even when $\mathscr{A}$ is not small.

6.2. Proposition. *(Yoneda Lemma) If $U : \mathscr{A} \longrightarrow \mathscr{F}_h$ is an $\mathscr{F}_h$-functor represented by an object K of $\mathscr{A}$ then U is tractable and*

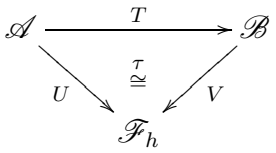
$$\{U,V\} \cong VK.$$

PROOF. By the "weak Yoneda Lemma" (see [15]) we have

$$\mathscr{F}_h\text{-Cat}(U * X, V) \cong \mathscr{F}_h\text{-Cat}(U, [X, V]) \cong \mathscr{F}(J, [X, VK]) \cong \mathscr{F}(X, VK).$$

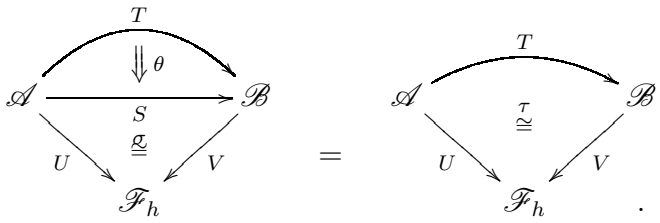
■

Consider the 2-category $\mathscr{F}_h$-Cat $\downarrow^{\text{ps}}$ $\mathscr{F}_h$ defined as follows. The objects are $\mathscr{F}_h$-functors $U : \mathscr{A} \longrightarrow \mathscr{F}_h$. The morphisms $(T, \tau) : U \longrightarrow V$ are triangles



(6.11)

in $\mathscr{F}_h$-Cat. The 2-cells $\theta : (T, \tau) \Longrightarrow (S, \sigma)$ are $\mathscr{F}_h$-natural transformations $\theta : T \Longrightarrow S$ such that

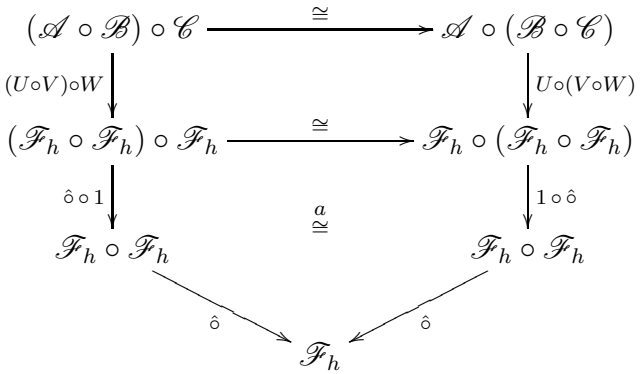


(6.12)

We define a *vertical tensor product* $\underline{\circ}$ on the 2-category $\mathscr{F}_h$-Cat $\downarrow^{\text{ps}}$ $\mathscr{F}_h$ making it a monoidal 2-category, which we denote by $\mathscr{F}$-Cat $\downarrow^{\text{ps}}$ $\mathscr{F}$. For $\mathscr{F}_h$-functors $U : \mathscr{A} \longrightarrow \mathscr{F}_h$ and $V : \mathscr{B} \longrightarrow \mathscr{F}_h$, define $U \underline{\circ} V : \mathscr{A} \circ \mathscr{B} \longrightarrow \mathscr{F}_h$ to be the composite

$$\mathscr{A} \circ \mathscr{B} \xrightarrow{U \circ V} \mathscr{F}_h \circ \mathscr{F}_h \xrightarrow{\hat{\circ}} \mathscr{F}_h \,. \tag{6.13}$$

The unit object is $\ulcorner \mathbf{1} \urcorner : \overline{\mathbf{1}} \longrightarrow \mathscr{F}_h$. The associativity constraints are explained by the diagram



(6.14)

where a is the associativity constraint for the vertical structure on $\mathscr{F}$. The unit constraints are similar.

6.3. Remark. We would like to emphasise that, although there are conceivable 2-cells for Mon $\mathscr{F}$ as a sub-2-category of $\mathscr{F}_h$-Cat (see [23]), we are only regarding Mon $\mathscr{F}$ as a monoidal category, not a monoidal 2-category.

Next we specify a monoidal functor

$$\mathrm{mod} : (\mathrm{Mon}\ \mathscr{F})^{\mathrm{op}} \longrightarrow \mathscr{F}\text{-Cat} \downarrow^{\mathrm{ps}} \mathscr{F}_h\ . \tag{6.15}$$

For each monoid M in $\mathscr{F}_h$, we put

$$\mathrm{mod}\ M = (U_M : \mathscr{F}^{*M} \longrightarrow \mathscr{F}_h).$$

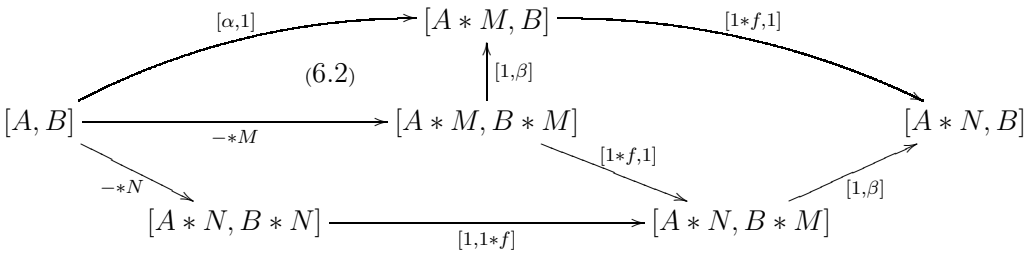
For a monoid morphism $f : N \longrightarrow M$, we define

$$\begin{array}{ccc} \mathscr{F}^{*M} & \xrightarrow{\mathrm{mod}\ f} & \mathscr{F}^{*N} \\ & {\scriptstyle U_M} \searrow \quad = \quad \swarrow {\scriptstyle U_N} & \\ & \mathscr{F}_h & \end{array} \tag{6.16}$$

by

$$(\mathrm{mod}\ f)(A * M \xrightarrow{\alpha} A) = (A * N \xrightarrow{1*f} A * M \xrightarrow{\alpha} A).$$

To see that mod f is an $\mathscr{F}_h$-functor, we recall the equalizer of (6.2) and point to the following diagram in which the empty regions commute.



Alternatively, we could use the universal property of mod N as the universal action of the monad $- * N$ on $\mathscr{F}$.

For the monoidal structure on mod, we define an $\mathscr{F}_h$-functor $\Phi_{M,N}$ making the square

$$\begin{array}{ccc} \mathscr{F}^{*M} \circ \mathscr{F}^{*N} & \xrightarrow{\Phi_{M,N}} & \mathscr{F}^{*(M \circ N)} \\ {\scriptstyle U_M \circ U_N} \downarrow & & \downarrow {\scriptstyle U_{M \circ N}} \\ \mathscr{F}_h \circ \mathscr{F}_h & \xrightarrow[\hat{\circ}]{} & \mathscr{F}_h \end{array} \tag{6.17}$$

commute; put

$$\Phi_{M,N}(A * M \xrightarrow{\alpha} A, B * N \xrightarrow{\beta} B) =$$

$$((A \circ B) * (M \circ N) \xrightarrow{\gamma} (A * M) \circ (B * N) \xrightarrow{\alpha\circ\beta} A \circ B)$$

and use the universal property of $\mathrm{mod}(M \circ N)$ to define $\Phi_{M,N}$ on homs.

For tractable $U : \mathscr{A} \longrightarrow \mathscr{F}_h$, we have an evaluation $\mathscr{F}_h$-natural transformation

$$ev : U * \{U, V\} \longrightarrow V,$$

corresponding under the adjunction (6.6), to the identity of $\{U, V\}$. We have a "composition morphism"

$$\mu : \{U, V\} * \{V, W\} \longrightarrow \{U, W\}$$

corresponding to the composite

$$U * \{U, V\} * \{V, W\} \xrightarrow{ev*1} V * \{V, W\} \xrightarrow{ev} W.$$

In particular,

$$\mu : \{U, U\} * \{U, U\} \longrightarrow \{U, U\}$$

together with

$$\eta : J \longrightarrow \{U, U\},$$

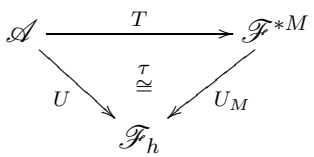
corresponding to $U * J \cong U$, gives $\{U, U\}$ the structure of a monoid, denoted end U, in $\mathscr{F}_h$.

6.4. Proposition. *For each tractable $\mathscr{F}_h$-functor $U : \mathscr{A} \longrightarrow \mathscr{F}_h$, there is an equivalence of categories*

$$(\mathrm{Mon}\,\mathscr{F}_h)(M, \mathrm{end}\,U) \simeq (\mathscr{F}_h\text{-Cat} \downarrow^{\mathrm{ps}} \mathscr{F}_h)(U, \mathrm{mod}\,M)$$

pseudonatural in monoids M in $\mathscr{F}_h$.

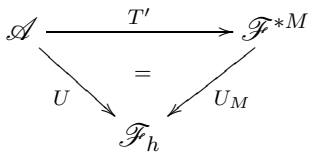
Proof. Morphisms $t : M \longrightarrow \mathrm{end}\,U$ in $\mathscr{F}$ are in natural bijection (using (6.6)) with $\mathscr{F}_h$-natural transformations $\theta : U * M \longrightarrow U$. It is easy to see that t is a monoid morphism if and only if θ is an action of the monad $- * M$ on $U : \mathscr{A} \longrightarrow \mathscr{F}_h$. By the universal property of the Eilenberg-Moore construction [23], such actions are in natural bijection with liftings of U to $\mathscr{F}_h$-functors $\mathscr{A} \longrightarrow \mathscr{F}^{*M}$. This describes a bijection between $(\mathrm{Mon}\ \mathscr{F}_h)(M, \mathrm{end}\ U)$ and the full subcategory of $(\mathscr{F}_h\text{-Cat} \downarrow^{\mathrm{ps}} \mathscr{F}_h)(U, \mathrm{mod}\ M)$ consisting of the morphisms



for which τ is an identity. It remains to show that every general such morphism (T, τ) is isomorphic to one for which τ is an identity. However, each (T, τ) determines an action

$$U * M \overset{\tau}{\cong} U_M T * M = (U_M * M)T \xrightarrow{\chi T} U_M T \overset{\tau^{-1}}{\cong} U$$

of the monad $- * M$ on U. By the universal property, we induce a morphism



and an invertible 2-cell $(T, \tau) \cong (T', 1)$ in $\mathscr{F}_h$-Cat $\downarrow^{\mathrm{ps}} \mathscr{F}_h$. ∎

In other words, we have a biadjunction

$$(\mathrm{Mon}\ \mathscr{F}_h)^{\mathrm{op}} \underset{\mathrm{mod}}{\overset{\mathrm{end}}{\underset{\longrightarrow}{\overset{\longleftarrow}{\bot}}}} \mathscr{F}_h\text{-Cat} \downarrow^{\mathrm{ps}}_{\mathrm{tract}} \mathscr{F}_h \tag{6.18}$$

where the 2-category on the right has objects restricted to the tractable U. As a consequence, notice that end takes each 2-cell to an identity (since all 2-cells in Mod $\mathscr{F}_h$ are identities). Notice too from the notation that we are ignoring the monoidal structure in (6.18). This is because tractable U are not generally closed under the monoidal structure of $\mathscr{F}$-Cat $\downarrow^{\mathrm{ps}} \mathscr{F}$.

6.5. Proposition. *Representable objects of $\mathscr{F}$-Cat $\downarrow^{\mathrm{ps}} \mathscr{F}$ are closed under the monoidal structure.*

Proof.

$$\mathscr{A} \circ \mathscr{B} \xrightarrow{\mathscr{A}(A,-)\circ\mathscr{B}(B,-)} \mathscr{F}_h \circ \mathscr{F}_h \xrightarrow{\hat{o}} \mathscr{F}_h \ . \qquad \cong \qquad (\mathscr{A}\circ\mathscr{B})((A,B),-)$$

and

$$\ulcorner \mathbf{1} \urcorner = \overline{\mathbf{1}}(\bullet, -) : \overline{\mathbf{1}} \longrightarrow \mathscr{F}_h \ .$$

∎

Let $\mathscr{F}$-Cat $\downarrow^{\mathrm{ps}}_{\mathrm{rep}} \mathscr{F}$ denote the monoidal full sub-2-category of $\mathscr{F}$-Cat $\downarrow^{\mathrm{ps}} \mathscr{F}$ consisting of the representable objects. The biadjunction (6.18) restricts to a biadjunction

$$(\mathrm{Mon}\ \mathscr{F}_h)^{\mathrm{op}} \underset{\mathrm{mod}}{\overset{\mathrm{end}}{\underset{\longrightarrow}{\overset{\longleftarrow}{\bot}}}} \mathscr{F}_h\text{-Cat} \downarrow^{\mathrm{ps}}_{\mathrm{rep}} \mathscr{F}_h \tag{6.19}$$

and we have already pointed out that mod is monoidal; see (6.17). In fact, we shall soon see that this is a monoidal biadjunction.

First note that, if $U : \mathscr{A} \longrightarrow \mathscr{F}_h$ is represented by K then by Proposition 6.2 we have a monoidal isomorphism

$$\mathrm{end}\ U = \{U, U\} \overset{6.2}{\cong} UK \cong \mathscr{A}(K, K). \tag{6.20}$$

In particular, for a monoid M in $\mathscr{F}_h$, using Proposition 2.4, we obtain a monoid isomorphism

$$\text{end mod } M \cong M \tag{6.21}$$

which is in fact the counit for (6.19), confirming that mod is an equivalence on homs.

6.6. Proposition. *The* 2*-functor* end *in (6.19) is strong monoidal.*

Proof. The isomorphism (6.20) gives

$$\begin{aligned} \text{end } (\mathscr{A} \circ \mathscr{B})((A,B),-) &\cong (\mathscr{A} \circ \mathscr{B})((A,B),(A,B)) \\ &\cong \mathscr{A}(A,A) \circ \mathscr{B}(B,B) \\ &\cong \text{end } \mathscr{A}(A,-) \circ \text{end } \mathscr{B}(B,-) \end{aligned}$$

and

$$\text{end } \overline{\mathbf{1}}(\bullet,-) \cong \overline{\mathbf{1}}(\bullet,\bullet) \cong \mathbf{1}.$$

■

As previously remarked, a monoid in $(\text{Mon } \mathscr{F}_h)^{\text{op}}$ is precisely a bimonoid in $\mathscr{F}$; see Definition 3.6. Since Mon $\mathscr{F}$ has discrete homs, these monoids are the same as pseudomonoids. The biadjunction (6.18) determines a biadjunction

$$(\text{Bimon } \mathscr{F}_h)^{\text{op}} \underset{\text{mod}}{\overset{\text{end}}{\underset{\longrightarrow}{\overset{\longleftarrow}{\bot}}}} \text{Mon}_{\text{ps}}(\mathscr{F}\text{-Cat} \downarrow^{\text{ps}}_{\text{rep}} \mathscr{F}). \tag{6.22}$$

A pseudomonoid in $\mathscr{F}$-Cat $\downarrow \mathscr{F}$ is a monoidal $\mathscr{F}_h$-category $\mathscr{A}$ together with a strong monoidal $\mathscr{F}_h$-functor $U : \mathscr{A} \longrightarrow \mathscr{F}_h$ (where $\mathscr{F}_h$ has $\hat{\circ}$ as the monoidal structure).

This leads to the following lifting to the duoidal setting of a result attributed to Bodo Pareigis (see [22], [20] and [21]).

6.7. Theorem. *For a horizontal monoid M in a duoidal $\mathscr{V}$-category $\mathscr{F}$, bimonoid structures on M are in bijection with isomorphism classes of monoidal structures on $\mathscr{F}^{*M}$ such that $U_M : \mathscr{F}^{*M} \longrightarrow \mathscr{F}$ is strong monoidal into the vertical structure on $\mathscr{F}$.*

Proof. For any horizontal monoid M in $\mathscr{F}$ we (in the order they appear) have (6.18), Proposition 6.6 and (6.21) giving

$$\begin{aligned} & (\mathscr{F}\text{-Cat} \downarrow^{\text{ps}} \mathscr{F})(\text{mod } M \circ \text{mod } M, \text{mod } M) \\ \cong\ & (\text{Mon } \mathscr{F})(M, \text{end } (\text{mod } M \circ \text{mod } M)) \\ \cong\ & (\text{Mon } \mathscr{F})(M, \text{end mod } M \circ \text{end mod } M)) \\ \cong\ & (\text{Mon } \mathscr{F})(M, M \circ M) \end{aligned}$$

and

$$(\mathscr{F}\text{-Cat} \downarrow^{\text{ps}} \mathscr{F})(\ulcorner \mathbf{1} \urcorner, \text{mod } M) \simeq (\text{Mon } \mathscr{F})(M, \mathbf{1}).$$

By Proposition 6.6, each bimonoid structure on M yields a pseudomonoid structure on mod M; and each pseudomonoid structure on mod M yields a bimonoid structure on end mod $M \cong M$. The above equivalences give the bijection of the Theorem. ■

7. Hopf bimonoids

We have seen that a bimonoid M in a duoidal $\mathscr{V}$-category $\mathscr{F}$ leads to a monoidal $\mathscr{F}_h$-category $\mathscr{F}^{*M}$ of right M-modules. In this section, we are interested in when $\mathscr{F}^{*M}$ is closed. We lean heavily on papers [4] and [5].

A few preliminaries from [25] adapted to $\mathscr{F}_h$-categories are required. For an $\mathscr{F}_h$-category $\mathscr{A}$, a *right $\mathscr{A}$-module* $W : \mathcal{J} \longrightarrow\!\!\!\!\!\rightarrow \mathscr{A}$ is a family of objects WA of $\mathscr{F}$ indexed by the objects A of $\mathscr{A}$ and a family

$$W_{AB} : WA * \mathscr{A}(B, A) \longrightarrow WB$$

of morphisms of $\mathscr{F}$ indexed by pairs of objects A, B of $\mathscr{A}$, satisfying the action conditions. For modules $W, W' : \mathcal{J} \longrightarrow\!\!\!\!\!\rightarrow \mathscr{A}$, define $[W, W']$ to be the limit as below when it exists in $\mathscr{F}$.

$$\begin{array}{ccc} [WA, W'A] & \xrightarrow{-*\mathscr{A}(B,A)} & [WA * \mathscr{A}(B, A), W'A * \mathscr{A}(B, A)] \\ & \nearrow & \downarrow{\scriptstyle [1, W'_{AB}]} \\ [W, W'] & & \\ & \searrow & \\ [WB, W'B] & \xrightarrow[[W_{AB},1]]{} & [WA * \mathscr{A}(B, A), W'B] \end{array} \tag{7.1}$$

7.1. Example. A monoid M in $\mathscr{F}_h$ can be regarded as a one object $\mathscr{F}_h$-category. A right M-module $A : \mathcal{J} \longrightarrow\!\!\!\!\!\rightarrow M$ is precisely an object of $\mathscr{F}^{*M}$.

7.2. Example. For any $\mathscr{F}_h$-functor $S : \mathscr{A} \longrightarrow \mathscr{X}$ and object X of $\mathscr{X}$, we obtain a right $\mathscr{A}$-module $\mathscr{X}(S, X) : \mathcal{J} \longrightarrow\!\!\!\!\!\rightarrow \mathscr{A}$ defined by the objects $\mathscr{X}(SA, X)$ of $\mathscr{F}$ and the morphisms

$$\mathscr{X}(SA, X) * \mathscr{A}(B, A) \xrightarrow{1*S_{BA}} \mathscr{X}(SA, X) * \mathscr{X}(SB, SA) \xrightarrow{comp} \mathscr{X}(SB, X) .$$

Recall from [25] that the *colimit* $\mathrm{colim}(W, S)$ *of* $S : \mathscr{A} \longrightarrow \mathscr{X}$ *weighted by* $W : \mathcal{J} \longrightarrow\!\!\!\!\!\rightarrow \mathscr{A}$ is an object of $\mathscr{X}$ for which there is an $\mathscr{F}_h$-natural isomorphism

$$\mathscr{X}(\mathrm{colim}(W, S), X) \cong [W, \mathscr{X}(S, X)] \tag{7.2}$$

By Yoneda, such an isomorphism is induced by the module morphism

$$\lambda : W \longrightarrow \mathscr{X}(S, \mathrm{colim}(W, S)) . \tag{7.3}$$

The $\mathscr{F}_h$-functor $S : \mathscr{A} \longrightarrow \mathscr{X}$ is *dense* when $\lambda = 1 : \mathscr{X}(S, Y) \longrightarrow \mathscr{X}(S, Y)$ induces

$$\mathrm{colim}(\mathscr{X}(S, Y), S) \cong Y \tag{7.4}$$

for all Y in $\mathscr{X}$.

7.3. PROPOSITION. *The $\mathscr{F}_h$-functor $\ulcorner J \urcorner : \mathcal{J} \longrightarrow \mathscr{F}_h$ is dense.*

PROOF. The right-hand side of (7.2) becomes $[[J,Y],[J,X]]$ which is isomorphic to $[Y,X]$. So

$$\mathrm{colim}([J,Y],J) \cong Y,$$

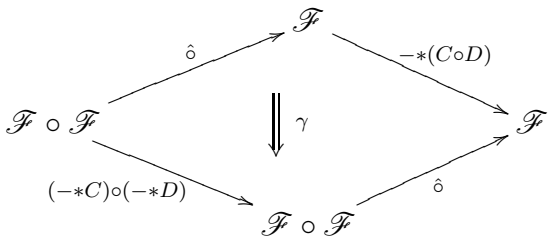
which is (7.4) in this case. ∎

Another element of our analysis is to recast the middle-of-four interchange morphisms as a 2-cell in $\mathscr{F}_h$-Cat.

7.4. PROPOSITION. *The family of morphisms*

$$\gamma : (X \circ Y) * (C \circ D) \longrightarrow (X * C) \circ (Y * D)$$

defines an $\mathscr{F}_h$-natural transformation



for all objects C and D of $\mathscr{F}$.

PROOF. Regard the commutative diagram

$$\begin{array}{ccc}
([X,U]\circ[Y,V]) * ((X*C)\circ(Y*D)) & \xleftarrow{1*\gamma} & ([X,U]\circ[Y,V]) * (X\circ Y) * (C\circ D) \\
\downarrow{\scriptstyle \gamma} & (3.3) & \downarrow{\scriptstyle \gamma*1} \\
([X,U]*X*C)\circ([Y,V]*Y*D) & \xleftarrow[\gamma]{} & (([U,X]*X)\circ([Y,V]*Y)) * (C\circ D) \\
\downarrow{\scriptstyle (ev*C)\circ(ev*D)} & \textit{naturality} & \downarrow{\scriptstyle (ev\circ ev)*1} \\
(U*C)\circ(V*D) & \xleftarrow[\gamma]{} & (U\circ V) * (C\circ D)
\end{array}$$

in which we have written as if $*$ were strict. ∎

7.5. PROPOSITION. *Suppose $\theta : F \Longrightarrow G : \mathscr{X} \longrightarrow \mathscr{Y}$ is an $\mathscr{F}_h$-natural transformation between $\mathscr{F}_h$-functors F and G which preserve colimits weighted by $W : \mathcal{J} \longrightarrow\!\!\!\!\!\rightarrow \mathscr{A}$. If each $\theta_{SA} : FSA \longrightarrow GSA$ is invertible then so is*

$$\theta_{\mathrm{colim}(W,S)} : F\,\mathrm{colim}(W,S) \longrightarrow G\,\mathrm{colim}(W,S)\,.$$

PROOF.

$$\begin{array}{ccc}
F\,\mathrm{colim}(W,S) & \xrightarrow{\theta_{\mathrm{colim}(W,S)}} & G\,\mathrm{colim}(W,S) \\
\downarrow{\scriptstyle \cong} & & \downarrow{\scriptstyle \cong} \\
\mathrm{colim}(W,FS) & \xrightarrow[\theta_{\mathrm{colim}(1,\theta_S)}]{} & \mathrm{colim}(W,GS).
\end{array}$$

∎

7.6. Definition. For a bimonoid M in a duoidal category $\mathscr{F}$, the composite v_ℓ:

$$(J \circ M) * M \xrightarrow{1*\delta} (J \circ M) * (M \circ M) \xrightarrow{\gamma} (J * M) \circ (M * M) \xrightarrow{\ell\circ\mu} M \circ M$$

is called the *left fusion morphism*. The composite v_r:

$$(M \circ J) * M \xrightarrow{1*\delta} (M \circ J) * (M \circ M) \xrightarrow{\gamma} (M * M) \circ (J * M) \xrightarrow{\mu\circ\ell} M \circ M$$

is called the *right fusion morphism*. We call M *left Hopf* when v_ℓ is invertible and *right Hopf* when v_r is invertible. We call M *Hopf* when both v_ℓ and v_r are invertible.

Suppose $\mathscr{A}$ and $\mathscr{X}$ are monoidal $\mathscr{F}_h$-categories and $U : \mathscr{A} \longrightarrow \mathscr{X}$ is a monoidal $\mathscr{F}_h$-functor. Writing $\circ$ for the tensor and $\mathbf{1}$ for the tensor unit, we must have morphisms

$$\varphi : UA \circ UB \longrightarrow U(A \circ B) \qquad \text{and} \qquad \varphi_0 : \mathbf{1} \longrightarrow U\mathbf{1}$$

satisfying the usual Eilenberg-Kelly [10] conditions. Suppose $\mathscr{A}$ and $\mathscr{X}$ are left closed and write $\ell om(A, B)$ and $\ell om(X, Y)$ for the left homs. As pointed out by Eilenberg-Kelly, the monoidal structure φ, φ_0 is in bijection with *left closed structure*

$$\varphi^\ell : U\ell om(A, B) \circ UB \longrightarrow U\ell om(UA, UB) \qquad \text{and} \qquad \varphi_0 : \mathbf{1} \longrightarrow U\mathbf{1},$$

where φ^ℓ corresponds under the adjunction to the composite

$$U\, hom(A, B) \circ UA \xrightarrow{\varphi} U(hom(A, B) \circ A) \xrightarrow{U_{ev}} UB$$

Following [9], we say U is *strong left closed* when both φ^ℓ and φ_0 are invertible.

Recall from [4] (and [5] for the enriched situation) that the Eilenberg-Moore (enriched) category for an opmonoidal monad T on $\mathscr{X}$ is left closed and the forgetful $U_T : \mathscr{X}^T \longrightarrow \mathscr{X}$ is strong left closed if and only if T is "left Hopf". The monad T is *left Hopf* when the *left fusion morphism*

$$v_\ell(X, Y) : T(X \circ TY) \xrightarrow{\varphi} TX \circ T^2Y \xrightarrow{1\circ\mu} TX \circ TY \tag{7.5}$$

is invertible for all X and Y. It is *right Hopf* when the *right fusion morphism*

$$v_r(X, Y) : T(TX \circ Y) \xrightarrow{\varphi} T^2X \circ TY \xrightarrow{\mu\circ1} TX \circ TY \tag{7.6}$$

is invertible.

In particular, for a bimonoid M in $\mathscr{F}$, taking $T = - * M$, we see that $v_\ell(X, Y)$ is the composite

$$\begin{array}{ccc}
(X \circ (Y * M)) * M & \xrightarrow{1*\delta} & (X \circ (Y * M)) * (M \circ M) \\
 & & \downarrow{\scriptstyle \gamma} \\
\downarrow{\scriptstyle v_\ell(X,Y)} & & (X * M) \circ ((Y * M) * M) \\
 & & \downarrow{\scriptstyle \cong\ 1\circ a} \\
(X * M) \circ (Y * M) & \xleftarrow[1\circ(1*\mu)]{} & (X * M) \circ (Y * (M * M))
\end{array} \tag{7.7}$$

and that $v_r(X,Y)$ is

$$\begin{array}{ccc} ((X*M)\circ Y)*M & \xrightarrow{1*\delta} & ((X*M)\circ Y)*(M\circ M) \\ \Big\downarrow{\scriptstyle v_r(X,Y)} & & \Big\downarrow{\scriptstyle \gamma} \\ & & ((X*M)*M)\circ(Y*M) \\ & & \Big\downarrow{\scriptstyle \cong\, a\circ 1} \\ (X*M)\circ(Y*M) & \xleftarrow[(1*\mu)\circ 1]{} & (X*(M*M))\circ(Y*M). \end{array} \tag{7.8}$$

Recall from Section 5 that, when $\mathscr{F}$ is horizontally left closed, not only does it become an $\mathscr{F}_h$-category, it becomes a pseudomonoid in $\mathscr{F}_h$-Cat using the tensor $\hat{\circ}$. That is, $(\mathscr{F}, \hat{\circ}, \ulcorner \mathbf{1} \urcorner)$ is a monoidal $\mathscr{F}_h$-category.

We are interested in when $(\mathscr{F}, \hat{\circ}, \ulcorner \mathbf{1} \urcorner)$ is closed and when the closed structure lifts to $\mathscr{F}^{*M}$ for a bimonoid M in $\mathscr{F}$.

7.7. Proposition. *The monoidal $\mathscr{F}_h$-category $(\mathscr{F}, \hat{\circ}, \ulcorner \mathbf{1} \urcorner)$ is closed if and only if*

(i) $\mathscr{F}_v$ is a closed monoidal $\mathscr{V}$-category, and

(ii) there exist $\mathscr{V}$-natural isomorphisms

$$X \circ (W * Y) \cong W * (X \circ Y) \cong (W * X) \circ Y.$$

Proof. To say $(\mathscr{F}, \hat{\circ}, \ulcorner \mathbf{1} \urcorner)$ is left closed is to say we have a "left hom" $\ell om(X,Y)$ and an $\mathscr{F}_h$-natural isomorphism

$$[X \circ Y, Z] \cong [X, \ell om(Y,Z)].$$

By Yoneda, this amounts to a $\mathscr{V}$-natural isomorphism

$$\mathscr{F}(W, [X \circ Y, Z]) \cong \mathscr{F}(W, [X, \ell om(Y,Z)]).$$

Since $[\cdot,\cdot]$ is the horizontal left hom for $\mathscr{F}$, this amounts to

$$\mathscr{F}(W * (X \circ Y), Z) \cong \mathscr{F}(W * X, \ell om(Y,Z)). \tag{7.9}$$

Taking $W = J$, we obtain

$$\mathscr{F}(X \circ Y, Z) \cong \mathscr{F}(X, \ell om(Y,Z)),$$

showing that ℓom is a left hom for $\mathscr{F}_v$ as a monoidal $\mathscr{V}$-category. So (i) is implied. Now we have this, we can rewrite (7.9) as

$$\mathscr{F}(W * (X \circ Y), Z) \cong \mathscr{F}((W * X) \circ Y, Z)$$

which, again by Yoneda, is equivalent to

$$W * (X \circ Y) \cong (W * X) \circ Y. \tag{7.10}$$

Similarly, to say $(\mathscr{F}, \hat{\circ}, \mathbf{1})$ is right closed means

$$[X \circ Y, Z] \cong [Y, rom(X, Z)],$$

which means

$$\mathscr{F}(W * (X \circ Y), Z) \cong \mathscr{F}(W * Y, rom(X, Z)).$$

Taking $W = J$, we see that rom is a right hom for $\mathscr{F}_v$, and this leads to

$$W * (X \circ Y) \cong X \circ (W * Y). \tag{7.11}$$

This completes the proof. ∎

7.8. Remark. Under the condition of Proposition 7.7, it follows that the $\mathscr{F}_h$-functors

$$- * X, \quad - \circ X, \quad X \circ - \quad : \mathscr{F}_h \longrightarrow \mathscr{F}_h$$

all preserve weighted colimits.

7.9. Proposition. *For any duoidal $\mathscr{V}$-category $\mathscr{F}$, condition (ii) of Proposition 7.7 is equivalent to*

(ii)′ there exist $\mathscr{V}$-natural isomorphisms

$$X * (J \circ Y) \cong X \circ Y \cong Y * (X \circ J). \tag{7.12}$$

Proof. (ii)$\Longrightarrow$(ii)′ The second isomorphism of (ii)′ comes from the first isomorphism of (ii) with $Y = J$ and W replaced by Y. The first isomorphism of (ii)′ comes from the second isomorphism of (ii) with $X = J$ and W replaced by X.
(ii)′ $\Longrightarrow$ (ii) Using (ii)′, we have

$$\begin{aligned}
X \circ (W * Y) &\cong (W * Y) * (X \circ J) \\
&\cong W * (Y * (X \circ J)) \\
&\cong W * (X \circ Y), \text{ and} \\
(W * X) \circ Y &\cong (W * X) * (J \circ Y) \\
&\cong W * (X * (J \circ Y)) \\
&\cong W * (X \circ Y).
\end{aligned}$$

∎

7.10. Theorem. *Suppose $\mathscr{F}$ is a duoidal $\mathscr{V}$-category which is horizontally left closed, has equalizers, and satisfies condition (ii)$'$ of Proposition 7.9. Suppose M is a bimonoid in $\mathscr{F}$ and regard $\mathscr{F}^{*M}$ as a monoidal $\mathscr{F}_h$-category as in Theorem 6.7. The following conditions are equivalent:*

(i) M is a (left, right) Hopf bimonoid;

(ii) $- * M$ is a (left, right) Hopf opmonoidal monad on $\mathscr{F}_h$.

If $\mathscr{F}_v$ is a closed monoidal $\mathscr{V}$-category then these conditions are also equivalent to

(iii) $\mathscr{F}^{*M}$ is (left, right) closed and $U_M : \mathscr{F}^{*M} \longrightarrow \mathscr{F}_h$ is strong (left, right) closed.

Proof. (ii) $\Longleftrightarrow$ (iii) under the extra condition on $\mathscr{F}_v$ by [4] as extended by [5].
(ii) $\Longrightarrow$ (i) by taking $X = Y = J$ in (7.7), we see that $v_\ell(X, Y) = v_\ell$.
(i) $\Longrightarrow$ (ii) Proposition 7.9 (ii)$'$ and associativity of $*$ yield the isomorphisms

$$X \circ (Y * J) \cong Y * (X \circ J),$$

$$(Y * J) \circ X \cong Y * (J \circ X), \text{ and}$$

$$(Y * J) * X \cong Y * (J * X),$$

showing that $X \circ -$, $- \circ X$ and $- * X$ preserve the canonical weighted colimit of Proposition 7.3 (since $\mathrm{colim}(W, S) \cong W * S$ when $S : \mathcal{J} \longrightarrow \mathscr{F}_h$).

Using Proposition 7.4, we see that $v_\ell(X, Y)$ is an $\mathscr{F}_h$-natural transformation, in the variables X and Y, between two $\mathscr{F}_h$-functors that preserve weighted colimits of the form

$$\mathrm{colim}(Z, J) \cong Z * J \cong Z.$$

By Proposition 7.5, $v_\ell(X, Y)$ is invertible if $v_\ell(J, J) = v_\ell$ is. ∎

7.11. Example. Any braided closed monoidal $\mathscr{V}$-category $\mathscr{F}$, regarded as duoidal by taking both $*$ and $\circ$ to be the monoidal structure given on $\mathscr{F}$, is an example satisfying the conditions of Proposition 7.7.

7.12. Remark. One reading of Proposition 7.9 (ii)$'$ is that, to know $\circ$ we only need to know $*$ and either $J \circ -$ or $- \circ J$. Proposition 7.7 (ii) also yields

$$Y \circ (W * \mathbf{1}) \cong W * Y \cong (W * \mathbf{1}) \circ Y \tag{7.13}$$

showing that to know $*$ we only need to know $\circ$ and $- * \mathbf{1}$. From (7.12) we deduce

$$\mathbf{1} * (J \circ X) \cong X \cong \mathbf{1} * (X \circ J) \tag{7.14}$$

and from (7.13) we deduce

$$J \circ (X * \mathbf{1}) \cong X \cong (X * \mathbf{1}) \circ J \tag{7.15}$$

showing each of the composites

$$\mathscr{F} \xrightarrow{-\circ J} \mathscr{F} \xrightarrow{\mathbf{1}*-} \mathscr{F}, \qquad \mathscr{F} \xrightarrow{J\circ-} \mathscr{F} \xrightarrow{\mathbf{1}*-} \mathscr{F}, \tag{7.16}$$

$$\mathscr{F} \xrightarrow{-*\mathbf{1}} \mathscr{F} \xrightarrow{J\circ-} \mathscr{F}, \qquad \mathscr{F} \xrightarrow{-*\mathbf{1}} \mathscr{F} \xrightarrow{-\circ J} \mathscr{F}$$

to be isomorphic to the identity $\mathscr{V}$-functor of $\mathscr{F}$. From the first and last of these we see that $-\circ J$ is an equivalence and

$$\mathbf{1} * - \cong - * \mathbf{1} \tag{7.17}$$

both sides being inverse equivalences for $-\circ J$. From the second of (7.16) it then follows that $\mathbf{1} * -$ is an inverse equivalence for $J \circ -$. Consequently

$$J \circ - \cong - \circ J. \tag{7.18}$$

8. Warped monoidal structures

Let $\mathscr{A} = (\mathscr{A}, \otimes, I)$ be a monoidal category. The considerations at the end of Section 7 suggest the possibility of defining a tensor product on $\mathscr{A}$ of the form

$$A \,\square\, B = TA \otimes B$$

for some suitable functor $T : \mathscr{A} \longrightarrow \mathscr{A}$. In the case of Section 7, the functor T was actually an equivalence but we will not assume that here in the first instance.

A *warping of* $\mathscr{A}$ consists of the following data:

(a) a functor $T : \mathscr{A} \longrightarrow \mathscr{A}$;

(b) an object K of $\mathscr{A}$;

(c) a natural isomorphism

$$v_{A,B} : T(TA \otimes B) \longrightarrow TA \otimes TB\ ;$$

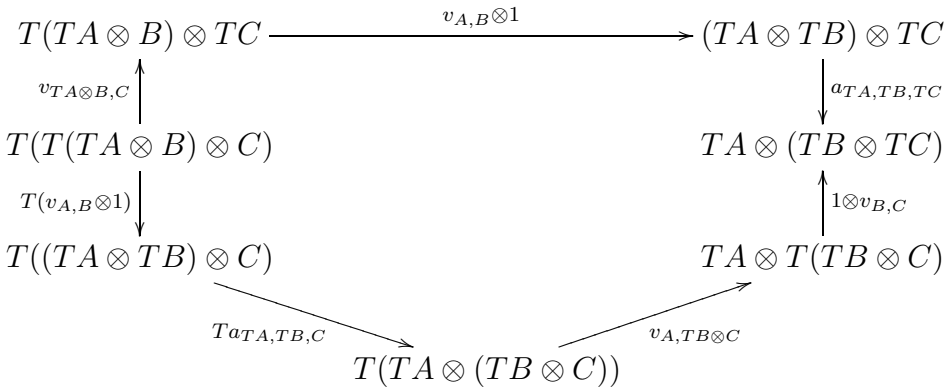
(d) an isomorphism

$$v_0 : TK \longrightarrow I\ ;\ \text{and}$$

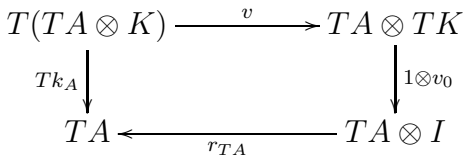
(e) a natural isomorphism

$$k_A : TA \otimes K \longrightarrow A\ ;$$

such that the following diagrams commute.



(8.1)



(8.2)

8.1. Remark. Diagram (8.1) is a generalized fusion equation in the sense of [24].

8.2. Remark. If $T : \mathscr{A} \longrightarrow \mathscr{A}$ is essentially surjective on objects and fully-faithful on isomorphisms then all we need to build it up to a warping is $v_{A,B}$ as in (c) satisfying (8.1). For K and v_0 exist by essential surjectivity and k_A is defined by (8.2).

8.3. Proposition. *A warping of $\mathscr{A}$ determines a monoidal structure on $\mathscr{A}$ defined by the tensor product*

$$A \,\square\, B = TA \otimes B$$

with unit object K and coherence isomorphisms

$$\alpha : T(TA \otimes B) \otimes C \xrightarrow{v \otimes 1} (TA \otimes TB) \otimes C \xrightarrow{a} TA \otimes (TB \otimes C)$$

$$\ell : TK \xrightarrow{v_0 \otimes 1} I \otimes B \xrightarrow{\ell} TK \otimes B$$

$$r : TA \otimes K \xrightarrow{k} A\,.$$

Proof. The pentagon condition for $\square$ is obtained from (8.1) by applying $- \otimes D$. Similarly, the unit triangle is obtained from (8.2) by applying $- \otimes B$. ∎

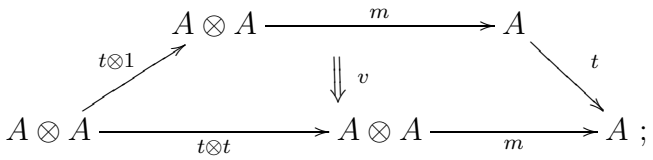
In investigating when $\otimes$ and $\square$ together formed a duoidal structure on $\mathscr{A}$, we realized we could use a lifting of Proposition 8.3 to a monoidal bicategory $\mathscr{M}$. We now describe this lifted version. The duoidal structure formed by $\otimes$ and $\square$ will be explained in an example.

A *warping of a monoidale* $A = (A, m, i)$ in a monoidal bicategory $\mathscr{M}$ consists of

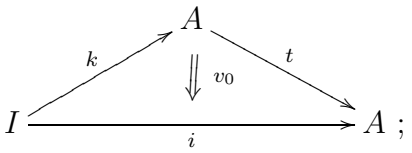
(a) a morphism $t : A \longrightarrow A$;

(b) a morphism $k : I \longrightarrow A$;

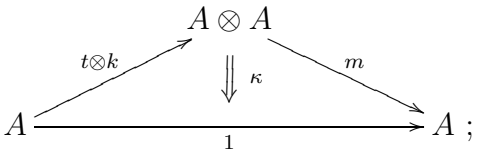
(c) an invertible 2-cell



(d) an invertible 2-cell

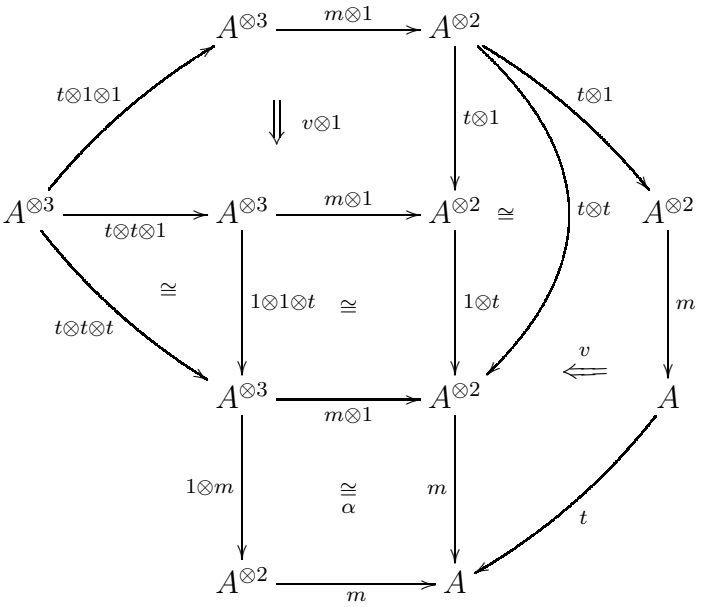


(e) an invertible 2-cell



satisfying

(8.3)


$A^{\otimes 3}$
$m\otimes 1$
$A^{\otimes 2}$
$t\otimes 1\otimes 1$
$v\otimes 1$
$t\otimes 1$
$t\otimes 1$
$A^{\otimes 3}$
$m\otimes 1$
$A^{\otimes 2}$
$t\otimes t$
$A^{\otimes 2}$
$t\otimes 1\otimes 1$
$\cong$
$t\otimes t\otimes t$
$1\otimes 1\otimes t$
$\cong$
$1\otimes t$
m
v
$A^{\otimes 3}$
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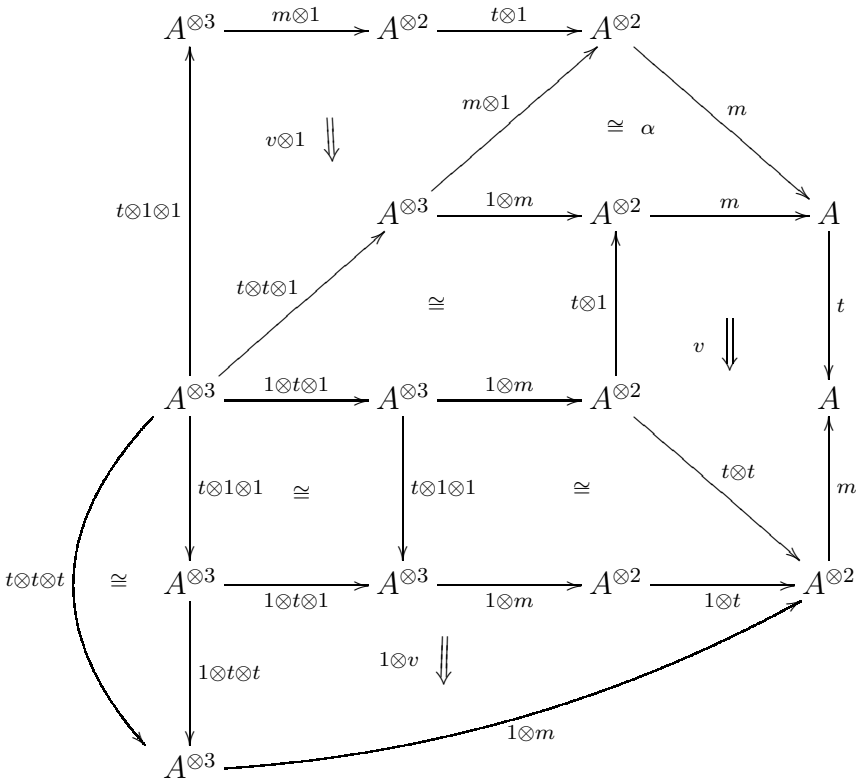
$=$


$A^{\otimes 3}$
$m\otimes 1$
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$t\otimes 1$
$A^{\otimes 2}$
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$t\otimes t\otimes t$
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$A^{\otimes 3}$
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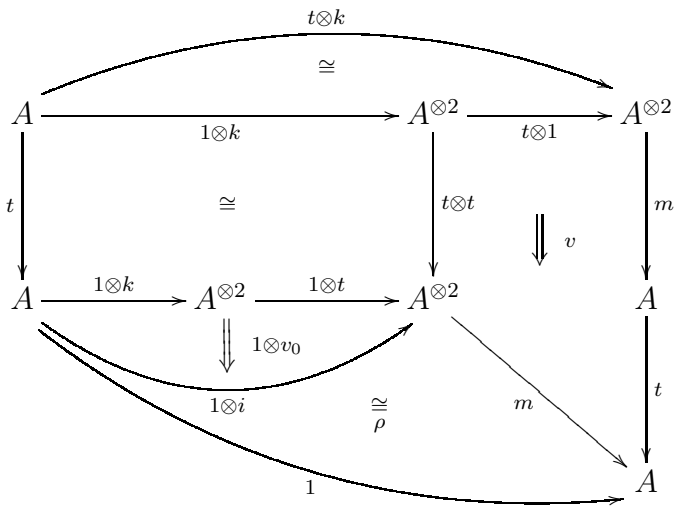
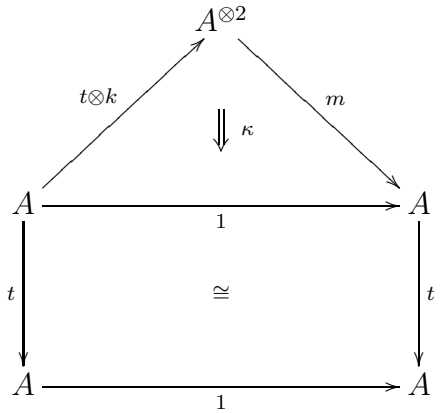
$A^{\otimes 3}$
$1\otimes m$
$A^{\otimes 2}$
$1\otimes t$
$A^{\otimes 2}$
$1\otimes t\otimes t$
$1\otimes v$
$1\otimes m$
$A^{\otimes 3}$

and



(8.4)

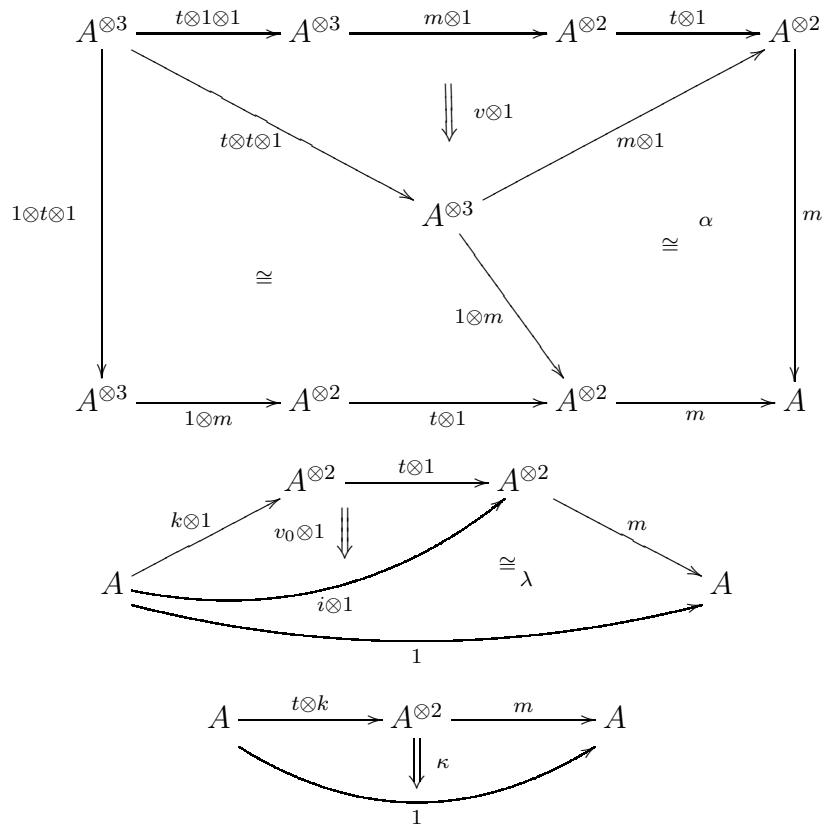
$=$



8.4. Proposition. *A warping of a monoidale A determines a monoidale structure on A defined by*

$$ {}_tm : A \otimes A \xrightarrow{t\otimes 1} A \otimes A \xrightarrow{m} A $$

$$ I \xrightarrow{k} A $$



PROOF. Conditions (8.3) and (8.4) yield the two axioms for a monoidale $(A, {}_tm, k)$. ∎

8.5. EXAMPLE. Suppose $\mathscr{F}$ is a duoidal $\mathscr{V}$-category satisfying the second isomorphism of (7.13). Define a $\mathscr{V}$-functor $T : \mathscr{F} \longrightarrow \mathscr{F}$ by

$$T = - * \mathbf{1}\,.$$

The horizontal right unit isomorphism gives

$$T(J) = J * \mathbf{1} \cong \mathbf{1}$$

and (7.13) gives

$$\begin{aligned} T(TA \circ B) &= ((A * \mathbf{1}) \circ B) * \mathbf{1} \\ &\cong (A * B) * \mathbf{1} \\ &\cong A * (B * \mathbf{1}) \\ &\cong (A * \mathbf{1}) \circ (B * \mathbf{1}) \\ &= TA \circ TB\,. \end{aligned}$$

Finally, we have

$$\begin{aligned} TA \circ J &= (A * \mathbf{1}) \circ J \\ &\cong A * J \\ &\cong A\,. \end{aligned}$$

This gives an example of a warping in $\mathscr{M} = \mathscr{V}$-Cat of the monoidale (= monoidal $\mathscr{V}$-category) $\mathscr{F}_v$. Proposition 8.4 gives back $\mathscr{F}_h$.

8.6. Example. Consider the case of $\mathscr{M} = \mathrm{Mon}(\mathscr{V}\text{-Cat})$. A monoidale is a duoidal $\mathscr{V}$-category $(\mathscr{F}_h, \circ, \mathbf{1})$. A warping of this monoidale consists of a monoidal $\mathscr{V}$-functor $T : \mathscr{F}_h \longrightarrow \mathscr{F}_h$, a monoid K in $\mathscr{F}_h$, a horizontally monoidal $\mathscr{V}$-natural isomorphism $v : T(TA \circ B) \cong TA \circ TB$, a horizontal monoid isomorphism $v_0 : TK \cong \mathbf{1}$, and a horizontally monoidal $\mathscr{V}$-natural isomorphism $k : TA \circ K \cong A$, subject to the two conditions. Proposition 8.4 gives the recipe for obtaining a duoidal $\mathscr{V}$-category $(\mathscr{F}_h, (T-) \circ -, K)$. In particular, take $\mathscr{V} =$ Set and consider a lax braided monoidal category $\mathscr{A} = (\mathscr{A}, \otimes, I, c)$ as a duoidal category; the lax braiding gives the monoidal structure on $\otimes : \mathscr{A} \times \mathscr{A} \longrightarrow \mathscr{A}$. A warping consists of a monoidal functor $T : \mathscr{A} \longrightarrow \mathscr{A}$, a monoid K in $\mathscr{A}$, a monoidal natural $v : T(TA \otimes B) \cong TA \otimes TB$, a monoid isomorphism $v_0 : TK \cong I$, and a monoidal natural $k : TA \otimes K \cong A$, satisfying the conditions (8.1) and (8.2). Proposition 8.4 then shows that the recipe of Proposition 8.3 yields a duoidal category $(\mathscr{A}, \otimes, I, \square, K)$.

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