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TRACED $*$-AUTONOMOUS CATEGORIES ARE COMPACT CLOSED

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ABSTRACT. We show that any traced $*$-autonomous category is compact closed.

1. Introduction

Suppose that $\mathbb{C} = (\mathbb{C}, I, \otimes, \multimap, \perp, Tr)$ is a traced $*$-autonomous category; here we understand that a $*$-autonomous category is a symmetric monoidal closed category $(\mathbb{C}, I, \otimes, \multimap)$ (we write $A \multimap B$ for the internal hom from A to B) equipped with a dualizing object $\perp$ [Barr, 1979], and that the trace

$$Tr^X_{A,B} : \mathbb{C}(A \otimes X, B \otimes X) \longrightarrow \mathbb{C}(A, B)$$

is given on the symmetric monoidal structure in the sense of Joyal, Street and Verity [Joyal *et al.*, 1996] (rather than the trace for linearly distributive categories with MIX by Blute, Cockett and Seely [Blute *et al.*, 2000]).

In $\mathbb{C}$, we have the trace of the evaluation map

$$\frac{(X \multimap (\perp \otimes X)) \otimes X \ \xrightarrow{ev}\ \perp \otimes X}{X \multimap (\perp \otimes X) \ \xrightarrow{Tr^X ev}\ \perp}$$

which gives rise to a morphism $t_X : I \longrightarrow X \otimes (X \multimap I)$ via the isomorphism

$$(X \multimap (\perp \otimes X)) \multimap \perp \ \simeq\ X \otimes (X \multimap I).$$

It is then natural to ask if t_X satisfies the equations

$$X \xrightarrow{t_X \otimes X} X \otimes (X \multimap I) \otimes X \xrightarrow{X \otimes ev_{X,I}} X \quad = \quad id_X \tag{1}$$

and

$$X \multimap I \xrightarrow{(X \multimap I) \otimes t_X} (X \multimap I) \otimes X \otimes (X \multimap I) \xrightarrow{ev_{X,I} \otimes (X \multimap I)} X \multimap I \quad = \quad id_{X \multimap I} \tag{2}$$

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which mean that $X \multimap I$ is a (left) dual of X, hence $\mathbb{C}$ is compact closed [Kelly and Laplaza, 1980]. Below we see that this is the case.

Before proceeding to the proof, let us explain how we came across this observation. In the sequel, we write $\widehat{f} : A \longrightarrow B \multimap C$ for the transpose of $f : A \otimes B \longrightarrow C$ in a symmetric monoidal closed category. In a traced symmetric monoidal closed category, we have a family of morphisms

$$\tau^X_B = Tr^X_{X \multimap (B\otimes X),B}(ev_{X,B\otimes X}) : X \multimap (B \otimes X) \longrightarrow B.$$

It is easy to see that τ's are sufficient to determine trace of any $f : A \otimes X \longrightarrow B \otimes X$ as

$$Tr^X_{A,B} f \;=\; A \xrightarrow{\widehat{f}} X \multimap (B \otimes X) \xrightarrow{\tau^X_B} B.$$

In the case of traced $*$-autonomous categories, we can further restrict our attention to τ's with $B = \bot$, from which τ^X_B for any B is recovered as

$$\begin{array}{rcl} X \multimap (B \otimes X) & \simeq & X \multimap ((B \parr \bot) \otimes X) \\ & \xrightarrow{X \multimap \delta} & X \multimap (B \parr (\bot \otimes X)) \\ & \simeq & B \parr (X \multimap (\bot \otimes X)) \\ & \xrightarrow{B \parr \tau^X_\bot} & B \parr \bot \\ & \simeq & B \end{array}$$

where δ denotes linear distributivity [Cockett and Seely, 1997].

Therefore, giving $\tau^X_\bot : (X \multimap (\bot \otimes X)) \longrightarrow \bot$ for each X is enough to determine a trace. With some efforts of spelling out how the trace can be recovered directly from $\tau^X_\bot$'s, we noticed that $\tau^X_\bot$ actually determines the unit map $t_X : I \longrightarrow X \otimes (X \multimap I)$ of the duality between X and $X \multimap I$.

To make the proof short and readable, we use some basic results on (enriched) extraordinary natural transformations of Eilenberg and Kelly [Eilenberg and Kelly, 1966, Kelly, 1982], though it is also possible to derive the result by direct calculation from scratch.

2. A Characterization of Compact Closedness

There are many ways of characterizing compact closed categories as special symmetric monoidal closed categories [Day, 1977, Kelly and Laplaza, 1980]. In our development, the following characterization turns out to be useful:

2.1. PROPOSITION. *Suppose that $\mathbb{C}$ is a symmetric monoidal closed category such that there is an extraordinary $\mathbb{C}$-natural transformation $t_X : I \longrightarrow X \otimes (X \multimap I)$ with t_I invertible. Then $\mathbb{C}$ is compact closed.*

Proof. Recall that, for a symmetric monoidal closed category $\mathbb{V}$, $\mathbb{V}$-categories $\mathbb{A}$, $\mathbb{B}$, a $\mathbb{V}$-functor $F : \mathbb{A}^{\mathrm{op}} \otimes \mathbb{A} \to \mathbb{B}$ and an object B of $\mathbb{B}$, a family of morphisms $\alpha_X : B \longrightarrow F(X, X)$ is said to be an extraordinary $\mathbb{V}$-natural transformation [Kelly, 1982] when the following diagram commutes for all X, X'.

$$\begin{tikzcd}
& \mathbb{A}(X, X') \arrow[dl, "F(X,-)"'] \arrow[dr, "F(-,X')"] & \\
\mathbb{B}(F(X,X), F(X,X')) \arrow[dr, "\mathbb{B}(\alpha_X, id)"'] & & \mathbb{B}(F(X',X'), F(X,X')) \arrow[dl, "\mathbb{B}(\alpha_{X'}, id)"] \\
& \mathbb{B}(B, F(X,X')) &
\end{tikzcd}$$

In the proposition, the assumption that $t_X : I \longrightarrow X \otimes (X \multimap I)$ is extraordinally $\mathbb{C}$-natural means that the following diagram commutes for all X and X':

$$\begin{tikzcd}
& X \multimap X' \arrow[dl, "(X \multimap X') \otimes t_X"'] \arrow[dr, "t_{X'} \otimes (X \multimap X')"] & \\
(X \multimap X') \otimes X \otimes (X \multimap I) \arrow[dr, "ev \otimes (X \multimap I)"'] & & X' \otimes (X' \multimap I) \otimes (X \multimap X') \arrow[dl, "X' \otimes comp"] \\
& X' \otimes (X \multimap I) &
\end{tikzcd}$$

By letting X be I in the diagram above, we see that (modulo some obvious simplifications)

$$X' \xrightarrow{t_{X'} \otimes X'} X' \otimes (X' \multimap I) \otimes X' \xrightarrow{X' \otimes ev} X'$$

agrees with

$$X' \xrightarrow{X' \otimes t_I} X' \otimes I \otimes (I \multimap I) \xrightarrow{\simeq} X'.$$

Similarly, by letting X' be I in the diagram, we have that

$$X \multimap I \xrightarrow{(X \multimap I) \otimes t_X} (X \multimap I) \otimes X \otimes (X \multimap I) \xrightarrow{ev \otimes (X \multimap I)} X \multimap I$$

agrees with

$$X \multimap I \xrightarrow{t_I \otimes (X \multimap I)} I \otimes (I \multimap I) \otimes (X \multimap I) \xrightarrow{\simeq} X \multimap I.$$

Hence

$$t'_X \quad = \quad I \xrightarrow{\simeq} I \otimes (I \multimap I) \xrightarrow{t_I^{-1}} I \xrightarrow{t_X} X \otimes (X \multimap I)$$

satisfies the equations (1) and (2) for making $X \multimap I$ a left dual of X. ∎

Note that, in the proof above, t'_X agrees with t_X if $t_I : I \longrightarrow I \otimes (I \multimap I)$ itself is the canonical isomorphism from I to $I \otimes (I \multimap I)$.

2.2. Remark. In Proposition 2.1, the assumption that t_I is invertible cannot be dropped. For instance, consider the category $\omega\mathbf{Cppo}_\perp$ of pointed ω-complete partial orders and strict ω-continuous functions. $\omega\mathbf{Cppo}_\perp$ is symmetric monoidal closed, with Sierpinski space as the unit object, smash products as tensor and strict function spaces as internal hom. In $\omega\mathbf{Cppo}_\perp$, there is an extraordinary $\omega\mathbf{Cppo}_\perp$-natural transformation $t_X : I \longrightarrow X \otimes (X \multimap I)$ given by the constant functions returning the least element. However, t_I is not invertible, and $\omega\mathbf{Cppo}_\perp$ is not compact closed.

3. Proof of the Main Result

As in the introduction, let us define

$$\tau_B^X = Tr^X_{X \multimap (B\otimes X),B}(ev_{X,B\otimes X}) : X \multimap (B \otimes X) \longrightarrow B$$

in traced symmetric monoidal closed categories.

3.1. Lemma. *In a traced symmetric monoidal closed category $\mathbb{C}$ with an object B,*

$$\widehat{\tau_B^X} : I \;\longrightarrow\; (X \multimap (B \otimes X)) \multimap B$$

is extraordinary $\mathbb{C}$-natural in X.

Proof. The extranaturality amounts to the commutativity of

$$\begin{tikzcd}
& (X \multimap X') \otimes (X' \multimap (B \otimes X)) \arrow[dl, "id \otimes \widehat{\tau_B^X}"'] \arrow[dr, "id \otimes \widehat{\tau_B^{X'}}"] & \\
(X \multimap X') \otimes (X' \multimap (B \otimes X)) \otimes ((X \multimap (B \otimes X)) \multimap B) \arrow[d, "comp \otimes id"'] & & (X \multimap X') \otimes (X' \multimap (B \otimes X)) \otimes ((X' \multimap (B \otimes X')) \multimap B) \arrow[d, "comp \otimes id"] \\
(X \multimap (B \otimes X)) \otimes ((X \multimap (B \otimes X)) \multimap B) \arrow[dr, "ev"'] & & (X' \multimap (B \otimes X')) \otimes ((X' \multimap (B \otimes X')) \multimap B) \arrow[dl, "ev"] \\
& B &
\end{tikzcd}$$

which is a consequence of the sliding property (dinaturality) of trace. ∎

3.2. Lemma. *In a $*$-autonomous category $\mathbb{C}$, the isomorphism*

$$\varphi_{X,Y} : (X \multimap (\bot \otimes Y)) \multimap \bot \xrightarrow{\simeq} X \otimes (Y \multimap I)$$

given by

$$\begin{array}{rcl}(X \multimap (\bot \otimes Y)) \multimap \bot & \simeq & (X \multimap (((\bot \otimes Y) \multimap \bot) \multimap \bot)) \multimap \bot \\ & \simeq & (X \multimap ((Y \multimap (\bot \multimap \bot)) \multimap \bot)) \multimap \bot \\ & \simeq & (X \multimap ((Y \multimap I) \multimap \bot)) \multimap \bot \\ & \simeq & ((X \otimes (Y \multimap I)) \multimap \bot) \multimap \bot \\ & \simeq & X \otimes (Y \multimap I)\end{array}$$

is $\mathbb{C}$-natural in X and Y.

Proof. Each isomorphism involved in φ is $\mathbb{C}$-natural. ∎

3.3. Lemma. *[Eilenberg and Kelly, 1966, Kelly, 1982] Assume that $\mathbb{V}$ is a symmetric monoidal closed category. Let $\mathbb{A}$, $\mathbb{B}$ be $\mathbb{V}$-categories and G, H be $\mathbb{V}$-functors of the form $\mathbb{A}^{\mathrm{op}} \otimes \mathbb{A} \to \mathbb{B}$ and suppose that B is an object of $\mathbb{B}$. If $\alpha_X : B \to G(X,X)$ is extraordinary $\mathbb{V}$-natural in X and $\beta_{X,Y} : G(X,Y) \to H(X,Y)$ is $\mathbb{V}$-natural in X and Y, then*

$$\beta_{X,X} \circ \alpha_X : B \to H(X,X)$$

is extraordinary $\mathbb{V}$-natural in X.

3.4. Corollary. *In a traced $*$-autonomous category $\mathbb{C}$, $t_X = \varphi_{X,X} \circ \widehat{\tau_{\bot}^{X}} : I \longrightarrow X \otimes (X \multimap I)$ is extraordinary $\mathbb{C}$-natural in X.* □

3.5. Lemma. *$t_I : I \longrightarrow I \otimes (I \multimap I)$ agrees with the canonical isomorphism from I to $I \otimes (I \multimap I)$.*

Proof. A consequence of the vanishing property of trace. ∎

Putting Proposition 2.1, Corollary 3.4 and Lemma 3.5 together, we obtain our main result.

3.6. Theorem. *Any traced $*$-autonomous category is compact closed.*

It is possible that a compact closed category is equipped with a dualizing object which is not isomorphic to the unit object (and par not isomorphic to tensor). For instance, the linearly ordered set of integers $\mathbb{Z}$ is compact closed with unit $I = 0$ and tensor $X \otimes Y = X + Y$ and duality $X^* = -X$, while any element of $\mathbb{Z}$ serves as a dualizing object. (The same can be done for any partially ordered Abelian group regarded as a compact closed poset.)

Since a compact closed category has a unique trace (cf. [Hasegawa, 2009]), we have:

3.7. Theorem. *To give a traced $*$-autonomous category is to give a compact closed category with a dualizing object.*

Note that a dualizing object in a compact closed category is just an object $\bot$ such that the unit morphism $I \longrightarrow \bot \otimes \bot^*$ is invertible, cf. the Abelian group example above.

4. On Linear Distributivity

In a compact closed category with a dualizing object $\perp$, linear distributivity [Cockett and Seely, 1997] on the $*$-autonomous structure is invertible. To see this, recall that the linear distributivity $\delta : (A \parr B) \otimes C \longrightarrow A \parr (B \otimes C)$ in a $*$-autonomous category (regarded as a symmetric linearly distributive category with negation) amounts to the canonical morphism $(A^\perp \multimap B) \otimes C \longrightarrow A^\perp \multimap (B \otimes C)$ which is just the associativity isomorphism $((A^\perp)^* \otimes B) \otimes C \simeq (A^\perp)^* \otimes (B \otimes C)$ in a compact closed category.

Conversely, a $*$-autonomous category with invertible linear distributivity is compact closed. We have

$$A \multimap B \simeq A^\perp \parr B \simeq A^\perp \parr (I \otimes B) \overset{\delta^{-1}}{\simeq} (A^\perp \parr I) \otimes B \simeq (A \multimap I) \otimes B$$

In particular, the canonical map $(A \multimap I) \otimes A \longrightarrow A \multimap A$ is invertible, and it follows that the category is compact closed (cf. [Day, 1977]).

Together with Theorem 3.7, we have that the following three structures are essentially the same:

- a traced $*$-autonomous category,
- a compact closed category equipped with a dualizing object, and
- a $*$-autonomous category with invertible linear distributivity.

4.1. Remark. As noted in [Cockett and Seely, 1997], in a symmetric linearly distributive category with invertible linear distributivity and also equipped with a tensor-inverse of $\perp$ (an object $\perp^*$ such that there is an isomorphism $I \simeq \perp \otimes \perp^*$ subject to a coherence axiom), the par $A \parr B$ is isomorphic to the "$\perp$-shifted tensor" $A \otimes \perp^* \otimes B$. This is the case for $*$-autonomous categories with invertible linear distributivity (equivalently: traced $*$-autonomous categories, or compact closed categories with a dualizing object), in which $\perp^* = \perp \multimap I$ serves as a tensor-inverse of $\perp$ and we have $A \parr B \simeq A \otimes (\perp \multimap I) \otimes B$.

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