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TIGHTLY BOUNDED COMPLETIONS

MARTA BUNGE

Abstract. By a 'completion' on a 2-category $\mathscr{K}$ we mean here an idempotent pseudomonad on $\mathscr{K}$. We are particularly interested in pseudomonads that arise from KZ-doctrines. Motivated by a question of Lawvere, we compare the Cauchy completion [23], defined in the setting of **V**-Cat for **V** a symmetric monoidal closed category, with the Grothendieck completion [7], defined in the setting of **S**-Indexed Cat for **S** a topos. To this end we introduce a unified setting ('indexed enriched category theory') in which to formulate and study certain properties of KZ-doctrines. We find that, whereas all of the KZ-doctrines that are relevant to this discussion (Karoubi, Cauchy, Stack, Grothendieck) may be regarded as 'bounded', only the Cauchy and the Grothendieck completions are 'tightly bounded' – two notions that we introduce and study in this paper. Tightly bounded KZ-doctrines are shown to be idempotent. We also show, in a different approach to answering the motivating question, that the Cauchy completion (defined using 'distributors' [2]) and the Grothendieck completion (defined using 'generalized functors' [21]) are actually equivalent constructions[1].

Introduction

The aim of this paper, motivated by a question of Lawvere, is to identify what is common to the Cauchy completion [23], the Stack completion [11, 7], and related constructions, such as the Karoubi envelope [13, 7] and the Grothendieck completion [7].

KZ-doctrines have alternatively been called 'lax idempotent monads' in the literature [18, 20], but I shall stick here to the original terminology. A KZ-doctrine [19] on a 2-category $\mathscr{K}$ is a special sort of pseudomonad on $\mathscr{K}$ that is not always intuitively a completion. For instance, adding coproducts freely is part of a KZ-adjointness on Cat. Another non-example of completion is the KZ-doctrine known as the 'symmetric monad' [10] on a suitable 2-category $\mathscr{K}$. In view of these cases, we shall restrict our consideration to those KZ-doctrines which, as pseudomonads, are idempotent – that is, for which the multiplication is an equivalence. Such pseudomonads shall be referred to here as 'completions'.

For a proper comparison between the Cauchy and the Grothendieck completions, we must first unify their settings. On the one hand, the Grothendieck completion takes place

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[1]Lecture [9] given at the Octoberfest, Montréal, November 27-28, 2012.

in the setting of **S**-indexed categories [24]. On the other hand, the Cauchy completion is given in the setting of categories enriched over a symmetric monoidal closed category **V** [12, 17]. We are therefore led into merging these two settings into a 2-category $\mathrm{Cat}_{(\mathbf{S},\mathbf{V})}$ of '**S**-indexed **V**-categories', for **V** an '**S**-indexed symmetric monoidal closed category'.

Roughly speaking, a KZ-doctrine **M** on $\mathrm{Cat}_{(\mathbf{S},\mathbf{V})}$ is called 'bounded' if, as a pseudomonad, it can be carved out from the Yoneda embedding, and is such that **V** is orthogonal to the unit of **M**. However, bounded KZ-doctrines need not be completions, as the example of the free addition of coproducts shows.

Certain completions are better than others. For instance, the stack completion of a (small) groupoid in a topos **S** is of interest precisely because it provides a way of retrieving the given groupoid from its classifying topos up to weak equivalence [14, 15, 8]. The generalization of the stack construction from (small) groupoids to arbitrary (small) categories is meaningful [11], but it need no longer have this property. This defect is repaired by the more general Grothendieck completion [15, 7].

Another instance of a construction whose main interest was originally of a similar nature ('Morita equivalence theorem') is that of the Karoubi envelope of a small additive category [13]. The Karoubi envelope loses this property when carried out in the context of enriched **V**-category theory where the coproducts in **V** need not be disjoint. This defect is repaired by the more general Cauchy completion [23].

These considerations led us to single out those bounded KZ-doctrines that are 'tightly bounded', in the sense of satisfying a special form of Morita equivalence. We prove that every tightly bounded KZ-doctrine is an idempotent pseudomonad, hence a completion. The KZ-doctrine property is crucially used in this connection : this is the main reason why we do not deal with just arbitrary pseudomonads.

We prove that the Grothendieck KZ-doctrine (which is bounded as a composite of two commuting bounded KZ-doctrines) is in fact tightly bounded. The Cauchy completion involves **V**-distributors instead of **V**-generalized functors. For this reason we need to relativize the passage from distributors to generalized functors [2, 21, 1, 4]. With it, we easily derive that the Cauchy and the Grothendieck completions are equivalent constructions in the indexed enriched setting. In particular, we are in a position to better understand the relationship of the Cauchy completion to the Karoubi envelope – two constructions that are often wrongly identified in the literature. Indeed, we show that, in general, the Cauchy completion is not just the splitting of idempotents.

1. KZ-Completions

Let $\mathscr{K}$ be a 2-category. The notion of a KZ-doctrine [19] on $\mathscr{K}$ is a special sort of pseudomonad $\langle M, \delta, \mu \rangle$ on $\mathscr{K}$ that is 'property-like', in the sense that, for its algebras, structure is adjoint to units, so that, in particular, the so called structure may instead be regarded as a property.

We recall that lax adjointness, lax monads and their algebras were introduced in [6] in connection with what we called "families of coherently closed Kan extensions" therein. In the work of [20], they go further and show that the lax monads induced by such families in [6] are KZ-doctrines – which they relabel as "lax idempotent monads" following [18]. We see no reason to adopt the newer terminology [18, 20] here – in fact, the original one is better suited to our purpose, which is to single out the actual idempotent (in the sense of equivalence) KZ-doctrines.

1.1. Definition. A KZ-doctrine in a 2-category $\mathscr{B}$ is a pseudomonad $\langle M, \delta, \mu\rangle$ that satisfies the conditions

$$M(\delta_B) \dashv \mu_B \dashv \delta_{M(B)} \; . \tag{1}$$

for each object B in $\mathscr{B}$. A KZ-doctrine is said to be *fully faithful* if for every object B of $\mathscr{B}$, the unit

$$\delta_B : B \to M(B)$$

is a full and faithful 1-cell.

A KZ-doctrine on a 2-category $\mathscr{B}$ often arises explicitly from a special sort of lax adjointness [6] known as 'KZ-adjointness'. A KZ-adjointness

$$F : \mathscr{A} \to \mathscr{B} \dashv G : \mathscr{B} \to \mathscr{A}$$

between 2-categories has units $\delta_B : B \to GF(B)$ and counits $\varepsilon_A : FG(A) \to A$ such that

$$F(\delta_B) \dashv \varepsilon_F(B); G(\varepsilon_A) \dashv \delta_{G(A)}.$$

The induced pseudomonad $\langle M, \delta, \mu\rangle$ is a KZ-doctrine.

1.2. Examples.

1. The free addition of finite coproducts to a small category is a simple instance of a KZ-doctrine on Cat. It arises from a KZ-adjointness $F \dashv U$, where

$$U : \mathrm{Cat}^{\oplus} \to \mathrm{Cat}$$

is the forgetful 2-functor. In this case, the K-adjointness $F \dashv U$ is monadic. The construction is meaningful for arbitrary categories.

2. Denote by $\mathrm{Top}_{\mathbf{S}}$ the 2-category whose objects are toposes bounded over a base topos $\mathbf{S}$, whose 1-cells are inverse images of geometric morphisms over $\mathbf{S}$, and whose 2-cells are natural isomorphisms between inverse images of geometric morphisms. We may forget that a 1-cell between toposes preserves finite limits and leave the rest unchanged. The resulting 2-category is denoted $\mathrm{Dist}_{\mathbf{S}}$ since a 1-cell $\mathscr{E} \to \mathscr{F}$ is precisely an $\mathscr{F}$-valued Lawvere distribution on $\mathscr{E}$ [21]. The forgetful 2-functor

$$U : \mathrm{Top}_{\mathbf{S}} \to \mathrm{Dist}_{\mathbf{S}}$$

has a right 2-adjoint $U \dashv \Sigma$. It induces a pseudomonad $\langle M, \delta, \mu\rangle$ on $\mathrm{Top}_{\mathbf{S}}$. The adjoint pair $U \dashv \Sigma$ is a KZ-adjointness, hence the induced pseudomonad is a KZ-doctrine, called 'the symmetric monad" [10]. The symmetric monad is fully faithful but not idempotent.

Although ‘property-like’, the constructions of Examples 1.2 do not feel like completions precisely because they are not idempotent.

1.3. Definition. By a *completion* on a 2-category $\mathscr{K}$ we mean a fully faithful KZ-doctrine $\langle M, \delta, \mu \rangle$ on $\mathscr{K}$ that is idempotent, that is, for which $\mu : M^2 \to M$ is an equivalence 2-cell.

1.4. Examples.

1. The Karoubi envelope of a category (in *Set*) as a universal splitting of the idempotents in it is briefly described in [13] as an exercise. The same construction can be carried out [7] for a category in a topos $\mathbf{S}$ and may be regarded as a KZ-doctrine on $\mathrm{Cat}(\mathbf{S})$. Its unit $v_{\mathbb{C}} : \mathbb{C} \to \hat{\mathbb{C}}$ evaluated at $\mathbb{C}$ is fully faithful. Furthermore, as shown therein, it has the following properties: (1) Idempotents of $\mathbb{C}$ split in $\hat{\mathbb{C}}$ via $v_{\mathbb{C}} : \mathbb{C} \to \hat{\mathbb{C}}$. (2) Idempotents split in $\mathbb{C}$ if and only if $v_{\mathbb{C}} : \mathbb{C} \to \hat{\mathbb{C}}$ is an equivalence of categories. It follows that the Karoubi envelope is a fully faithful idempotent KZ-monad, hence a completion in the sense of Definition 1.3.

2. The stack completion [11] $u_{\mathscr{A}} : \mathscr{A} \to \tilde{\mathscr{A}}$ (of an $\mathbf{S}$-indexed category $\mathscr{A}$) is defined in two equivalent forms, one of which characterizes it as a solution to the problem of universally inverting weak equivalent functors. Any $\mathbf{S}$-indexed natural transformation $u_{\mathscr{A}} : \mathscr{A} \to \tilde{\mathscr{A}}$, where $\tilde{\mathscr{A}}$ is a stack, provides such a solution. Such a construction exists in this form [7]. It follows that the stack completion can be regarded as a fully faithful idempotent KZ-doctrine on $\mathbf{S}$-Indexed Cat, that is, as a completion in the sense of Definition 1.3.

2. Indexed Enriched Category Theory

This section introduces a 2-category $\mathrm{Cat}_{(\mathbf{S},\mathbf{V})}$ for the purpose of stating, in a unified way, properties of KZ-doctrines on it that can then be tested in any of its models –in particular in $\mathbf{S}$-Indexed Cat and $\mathbf{V}$-Cat.

We assume familiarity with $\mathbf{V}$-category theory [12, 5, 17] for $\langle V, \otimes, Z \rangle$ a symmetric monoidal closed category, and with $\mathbf{S}$-Indexed category theory [24], for $\mathbf{S}$ an arbitrary base topos.

2.1. Definition. Let $\mathbf{S}$ be an *elementary topos*. An $\mathbf{S}$*-indexed monoidal category* $\boldsymbol{V}$ is given by the following data:

1. $\mathbf{V}$ is an $\mathbf{S}$-indexed category.
2. There are given $\mathbf{S}$-indexed functors

$$\otimes : \mathbf{V} \times \mathbf{V} \longrightarrow \mathbf{V}$$

and

$$Z : \mathbf{1} \longrightarrow \mathbf{V}$$

3. together with $\mathbf{S}$-indexed natural isomorphisms

$$a : \otimes \times \mathrm{id}_{\mathbf{V}} \longrightarrow \mathrm{id}_{\mathbf{V}} \times \otimes$$

(modulo the canonical isomorphism

$$(\mathbf{V} \times \mathbf{V}) \times \mathbf{V} \longrightarrow \mathbf{V} \times (\mathbf{V} \times \mathbf{V})$$

as $\mathbf{S}$-indexed categories),

$$l : \mathrm{id}_{\mathbf{V}} \otimes Z \longrightarrow \mathrm{id}_{\mathbf{V}}$$

$$r : Z \otimes \mathrm{id}_{\mathbf{V}} \longrightarrow \mathrm{id}_{\mathbf{V}}$$

4. satisfying the (analogues of the) associativity and unitary axioms MC1 and MC2 [5, 17] for a monoidal category.

In particular, for each $I \in \mathbf{S}$, $\langle \mathbf{V}^I, \otimes^I, Z^I \rangle$ is a monoidal category and, for each morphism $\alpha : J \longrightarrow I$ in $\mathbf{S}$, the transition functor $\alpha^\star : \mathbf{V}^I \longrightarrow \mathbf{V}^J$ is a strong monoidal functor.

We say that $\mathbf{V}$ is a *symmetric monoidal closed* $\mathbf{S}$-indexed category if,

- for each $I \in \mathbf{S}$, the monoidal category $\langle \mathbf{V}^I, \otimes^I, Z^I \rangle$ is both symmetric and closed, and
- for each $\alpha : J \longrightarrow I$ in $\mathbf{S}$, the strong monoidal functor $\alpha^\star : \mathbf{V}^I \longrightarrow \mathbf{V}^J$ is symmetric and closed.

2.2. Remark. With the data for an $\mathbf{S}$-indexed symmetric monoidal closed category $\mathbf{V}$ in Definition 2.1 is associated a pseudofunctor

$$\mathbf{V} : \mathbf{S}^{\mathrm{op}} \longrightarrow \mathrm{SMCCat}$$

where SMCCat is the usual 2-category of symmetric monoidal closed categories, symmetric monoidal closed functors, and monoidal natural transformations. In turn, this is equivalently given by a (symmetric monoidal closed) fibration

$$\mathscr{V} \longrightarrow \mathbf{S}.$$

2.3. Assumption. *In what follows,* $\mathbf{S}$ *is assumed to be an elementary topos and* $\boldsymbol{V}$ *an* $\mathbf{S}$*-indexed symmetric monoidal closed category in the sense of Definition 2.1.*

2.4. Definition. For $\mathbf{V}$ an $\mathbf{S}$-indexed monoidal category, we give the data for a 2-category

$$\mathrm{Cat}_{(\mathbf{S},\mathbf{V})}$$

of $\mathbf{S}$-*indexed* $\boldsymbol{V}$-*categories,* $\mathbf{S}$-*indexed* $\boldsymbol{V}$-*functors, and 2-cells*, as follows:

- **Objects.** An $\mathbf{S}$-*indexed* $\boldsymbol{V}$-*category* consists of an $\mathbf{S}$-indexed category $\mathscr{A}$ such that for each object I of $\mathbf{S}$, $\mathscr{A}^I$ is a $\mathbf{V}^I$-category, and for each morphism $\alpha : J \to I$ of $\mathbf{S}$, $\alpha^\star : \mathscr{A}^I \to \mathscr{A}^I$ preserves the structures by means of coherent natural isomorphisms

$$\varphi^{\alpha}{}_{A,B} : \alpha^\star(\mathscr{A}^I(A,B)) \to \mathscr{A}^J(\alpha^\star A, \alpha^\star B).$$

- **1-cells.** An $\mathbf{S}$-*indexed* $\boldsymbol{V}$-*functor* consists of an $\mathbf{S}$-indexed functor $T : \mathscr{A} \to \mathscr{B}$ such that for each object I of $\mathbf{S}$, $T^I : \mathscr{A}^I \to \mathscr{B}^I$ is a $\mathbf{V}^I$-functor, and for each morphism $\alpha : J \to I$ in $\mathbf{S}$, the square below

$$\begin{tikzcd}
\alpha^\star(\mathscr{A}^I(A,A'))) \arrow[r, "\alpha^\star(T^I_{A,A'})"] \arrow[d, "\varphi^\alpha_{A,A'}"'] & \alpha^\star(\mathscr{B}^I(T^I(A),T^I(A')) \arrow[d, "\tilde{\varphi}^\alpha_{T(A),T(A')}"] \\
\mathscr{A}^J(\alpha^\star(A),\alpha^\star(A')) \arrow[r, "T^J_{\alpha^\star A,\alpha^\star A'}"'] & \mathscr{B}^J(T^J(\alpha^\star(A),T^J(\alpha^\star(A')))
\end{tikzcd}$$

  where the right vertical arrow is the composite of

$$\varphi^\alpha_{TA,TA'} : \alpha^\star(\mathscr{B}^I(T^I A, T^I A')) \to \mathscr{B}^J(\alpha^\star(T^I A), \alpha^\star(T^I A'))$$

  with isomorphisms deriving from the $\mathbf{V}$-functor structure of T, commutes.

- **2-cells.** An $\mathbf{S}$-*indexed* $\boldsymbol{V}$-*natural transformation* consists of an $\mathbf{S}$-indexed natural transformation $\eta : T_1 \to T_2$, such that for each $I \in \mathbf{S}$, $\eta^I : T_1^I \to T_2^I$ is a $\mathbf{V}$-natural transformation, such that for each $\alpha : J \to I$ in $\mathbf{S}$, the following identity

$$\alpha^\star \cdot \eta^I = \eta^J \cdot \alpha^\star$$

  holds after inserting isomorphisms deriving from the $\mathbf{V}$-functor structures of T_1 and T_2.

2.5. Theorem. The data given in Definitions 2.1 and 2.4 define a 2-category, which will be denoted $\mathrm{Cat}_{(\mathbf{S},\mathbf{V})}$

Proof. The verifications that the data of Definition 2.4 constitute a 2-category are straightforward on the basis of [24] and [17] and left to the reader. ■

2.6. PROPOSITION.

1. The pair $(\mathbf{S}, \mathbf{V}) = (\mathbf{S}, \mathbf{S})$, for $\mathbf{S}$ a topos, where $\mathbf{S}$ is $\mathbf{S}$-indexed in the usual way [24, 11], satisfies the conditions of Definition 2.1. The 2-category $\mathrm{Cat}_{(\mathbf{S},\mathbf{S})}$ in the sense of Definition 2.4 is equivalent to the 2-category $\mathbf{S}$-Indexed Cat [24].

2. The pair $(\mathit{Set}, \mathbf{V})$, for $\mathbf{V} = \langle V, \otimes, Z\rangle$ a symmetric monoidal closed complete and cocomplete category with $\mathrm{Hom}(Z, -) : \mathbf{V} \to \mathit{Set}$ faithful, satisfies the conditions of Definition 2.1. The 2-category $\mathrm{Cat}_{(\mathbf{S},\mathbf{V})}$ is equivalent to the 2-category $\mathbf{V}$-Cat [17].

PROOF. Straightforward.

■

2.7. PROPOSITION. *Let* $\langle \boldsymbol{V}, \otimes, Z\rangle$ *be an* $\mathbf{S}$*-indexed symmetric monoidal closed category. Then* $\boldsymbol{V}$ *is an* $\mathbf{S}$*-indexed* $\boldsymbol{V}$*-category in a natural way.*

PROOF. $\mathbf{V}$ is an $\mathbf{S}$-indexed category. Furthermore, for each $I \in \mathbf{S}$, $\mathbf{V}^I$ is closed, hence a $\mathbf{V}^I$-category. For each $\alpha : J \to I \in \mathbf{S}$, $\alpha^* : \mathbf{V}^I \to \mathbf{V}^J$ is (monoidal) closed, so that we have coherent natural isomorphisms

$$\varphi^{\alpha}{}_{A,B} : \alpha^\star(\mathbf{V}^I(A, B)) \to \mathbf{V}^J(\alpha^\star A, \alpha^\star B).$$

■

Recall [17] that, for a symmetric monoidal closed category $\mathbf{V} = \langle V, \otimes, Z\rangle$, a $\mathbf{V}$-category $\mathscr{A}$ is said to be small if $\mathrm{Obj}(\mathscr{A})$ is a small set. On the other hand, a small $\mathbf{S}$-indexed category [24], for $\mathbf{S}$ a topos, is the externalization of an internal category in $\mathbf{S}$. We need a notion of smallness for an $\mathbf{S}$-indexed $\mathbf{V}$-category $\mathscr{A}$ that has these two as instances. As in the case of a topos $\mathbf{S}$, such a notion can be given internally in $\mathbf{V}$ and then extended to an $\mathbf{S}$-indexed $\mathbf{V}$-category.

2.8. DEFINITION. Let $\langle \mathbf{V}, \otimes, Z\rangle$ be an $\mathbf{S}$-indexed symmetric monoidal closed category. A *small* $\boldsymbol{V}$*-category* $\mathbb{C}$ consists of

- An object $C_0 \in \mathbf{S}$.
- An object $C_1 \in \mathbf{V}^{C_0 \times C_0}$.
- A morphism $u : Z^{C_0} \longrightarrow C_1$ in $\mathscr{V}$ over $\mathrm{diag} : C_0 \longrightarrow C_0 \times C_0$ in $\mathbf{S}$.
- A morphism $C_1 \otimes_{C_0} C_1 \longrightarrow C_1$ in $\mathscr{V}$ over $\pi_{02} : C_0 \times C_o \times C_0 \longrightarrow C_0 \times C_0$, such that
- the following diagrams in $\mathscr{V}$

$$\begin{tikzcd}
C_1 \otimes_{C_0} (C_1 \otimes_{C_0} C_1) \arrow[r, "(m\otimes_{C_0}\mathrm{id})\cdot \mathrm{a}"] \arrow[d, "\mathrm{id}\otimes_{C_0} m"'] & C_1 \otimes_{C_0} C_1 \arrow[d, "m"] \\
C_1 \otimes_{C_0} C_1 \arrow[r, "m"'] & C_1
\end{tikzcd}$$

where

$$\mathrm{a} = \mathrm{a}_{C_1C_1C_1} : C_1 \otimes_{C_0} (C_1 \otimes_{C_0} C_1) \longrightarrow (C_1 \otimes_{C_0} C_1) \otimes_{C_0} C_1$$

is the isomorphism that is part of the data for the monoidal category $\mathbf{V}^{C_0 \times C_0 \times C_0}$,

$$\begin{tikzcd} Z^{C_0} \otimes_{C_0} C_1 \arrow[r, "u \otimes_{C_0} \mathrm{id}"] \arrow[dr, "r"'] & C_1 \otimes_{C_0} C_1 \arrow[d, "m"] \\ & C_1 \end{tikzcd}$$

and

$$\begin{tikzcd} C_1 \otimes_{C_0} Z^{C_0} \arrow[r, "\mathrm{id} \otimes_{C_0} u"] \arrow[dr, "l"'] & C_1 \otimes_{C_0} C_1 \arrow[d, "m"] \\ & C_1 \end{tikzcd}$$

commute.

Similar definitions can be stated for the notions of a small $\mathbf{V}$-functor $F : \mathbb{C} \longrightarrow \mathbb{D}$ between small $\mathbf{V}$-categories, and for a $\mathbf{V}$-natural transformation $\eta : F \longrightarrow G$ between small $\mathbf{V}$-functors from $\mathbb{C}$ to $\mathbb{D}$.

2.9. Remarks.

- Along the lines of [24], one can then form the *externalization* of a small $\mathbf{V}$-category $\mathbb{C}$ (respectively, of a small $\mathbf{V}$-functor $F : \mathbb{C} \longrightarrow \mathbb{D}$ and of a $\mathbf{V}$-natural transformation $\eta : F \longrightarrow G$). These are, respectively, an $\mathbf{S}$-indexed $\mathbf{V}$-category denoted $[\mathbb{C}]$, an $\mathbf{S}$-indexed $\mathbf{V}$-functor denoted $[F]$, and an $\mathbf{S}$-indexed natural transformation denoted $[\eta]$.

- Let $\mathbb{C}$ be a small $\mathbf{V}$-category. The opposite of a small $\mathbf{V}$-category $\mathbb{C}$ is a small $\mathbf{V}$-category $\mathbb{C}^{\mathrm{op}}$. Its externalization is equivalent to $[\mathbb{C}]^{\mathrm{op}}$.

- Let $\mathbb{C}$ be a small $\mathbf{V}$-category and $[\mathbb{C}]$ its externalization. Denote by $\mathbf{V}^{\mathbb{C}^{\mathrm{op}}}$ the $\mathbf{S}$-indexed $\mathbf{V}$-category of $\mathbf{S}$-indexed functors $[\mathbb{C}]^{\mathrm{op}} \longrightarrow \mathbf{V}$ and $\mathbf{S}$-indexed $\mathbf{V}$-natural transformations between such.

- For each morphism $\alpha : J \longrightarrow I$,

$$\alpha^\star : (\mathbf{V}^{\mathbb{C}^{\mathrm{op}}})^I \longrightarrow (\mathbf{V}^{\mathbb{C}^{\mathrm{op}}})^J$$

preserves the structure by coherent isomorphisms

$$\varphi^\alpha{}_{F,G} : \alpha^\star((\mathbf{V}^{\mathbb{C}^{\mathrm{op}}})^I(F,G)) \longrightarrow (\mathbf{V}^{\mathbb{D}^{\mathrm{op}}})^J(\alpha^\star F, \alpha^\star G).$$

2.10. Theorem. *There is an* $\mathbf{S}$*-indexed* $\boldsymbol{V}$*-functor*

$$\mathrm{HOM}_{\mathbb{C}} : \mathbb{C}^{\mathrm{op}} \times \mathbb{C} \longrightarrow \boldsymbol{V}$$

whose transpose

$$\mathrm{Y}_{\mathbb{C}} : \mathbb{C} \longrightarrow \boldsymbol{V}^{\mathbb{C}^{\mathrm{op}}}$$

is a fully faithful $\mathbf{S}$*-indexed* $\boldsymbol{V}$*-functor. In this context, an appropriate name for it is* the $\mathbf{S}$-indexed Yoneda $\mathbf{V}$-embedding *(at* $\mathbb{C}$*).*

Proof. An $\mathbf{S}$-indexed functor $F : \mathscr{A} \longrightarrow \mathscr{B}$ is fully faithful if for each $I \in \mathbf{S}$,

$$F^I : \mathscr{A}^I \longrightarrow \mathscr{B}^I$$

is fully faithful. For each $I \in \mathbf{S}$, the Yoneda $\mathbf{V}^I$-functor

$$\mathrm{Y}_{\mathbb{C}^I} : \mathbb{C}^I \longrightarrow (\mathbf{V}^I)^{(\mathbb{C}^I)^{\mathrm{op}}}$$

is fully faithful. ∎

2.11. Definition. Let $(\mathbf{S}, \mathbf{V})$ be a pair consisting of a topos $\mathbf{S}$ and of an $\mathbf{S}$-indexed symmetric monoidal closed category $\mathbf{V}$, and let $\mathbf{M} = \langle M, \delta, \mu\rangle$ be a KZ-doctrine on $\mathrm{Cat}_{(\mathbf{S},\mathbf{V})}$. We say that $\boldsymbol{M}$ *is small for* $(\mathbf{S}, \boldsymbol{V})$ if, for any small $\mathbf{V}$-category $\mathbb{C}$, $M(\mathbb{C})$ is small. If the KZ-doctrine $\mathbf{M}$ is a completion of type P, for a given property P of $\mathbf{S}$-indexed $\mathbf{V}$-categories, then we say instead, in this case, that *P-completions are small for* $(\mathbf{S}, \boldsymbol{V})$.

2.12. Examples.

- Let $(\mathbf{S}, \mathbf{V})$ be a pair with $\mathbf{S}$ any topos and $\mathbf{V}$ any $\mathbf{S}$-indexed symmetric monoidal closed regular category. Then the Karoubi envelope (universal splitting of idempotents) is small for $(\mathbf{S}, \mathbf{V})$.
- An example of a small category in an arbitrary topos $\mathbf{S}$ whose stack completion is not small, given by Joyal, is mentioned by Lawvere [22]. However, it follows from Lemma 8.35 of [16] that, for any Grothendieck topos $\mathbf{S}$, the stack completion is small for the pair $(\mathbf{S}, \mathbf{S})$.
- It is shown in [3] that the Cauchy completion of a small $\mathbf{V}$-category need not be small. This is the case for the small $\mathbf{V}$-category $\mathbb{Z}$, for $\mathbf{V}$ the (symmetric monoidal) closed category of suplattices, which is monoidal with the usual tensor product and unit object $Z = 2$. On the other hand, the Cauchy completion is small for the pair $(\mathit{Set}, \mathbf{V})$, where $\mathbf{V} = \mathrm{Mod}(R)$ for R a commutative ring with identity.

3. Bounded KZ-doctrines

Let $\mathbf{S}$ be a topos and $\mathbf{V}$ an $\mathbf{S}$-indexed symmetric monoidal closed category.

3.1. Definition. Let $\mathbf{M} = \langle M, \delta, \mu \rangle$ be a KZ-doctrine on $\mathrm{Cat}_{(\mathbf{S},\mathbf{V})}$. We say that $\mathbf{M}$ is *bounded* if the following conditions hold.

1. $\mathbf{M}$ is small for the pair $(\mathbf{S}, \mathbf{V})$ in the sense of Definition 2.11.

2. $\mathbf{M}$ is a fully faithful KZ-doctrine in the sense of Definition 1.1.

3. For any small $\mathbf{V}$-category $\mathbb{C}$, there is a factorization

$$\mathbb{C} \xrightarrow{\delta_{\mathbb{C}}} M(\mathbb{C}) \hookrightarrow \mathbf{V}^{\mathbb{C}^{\mathrm{op}}}$$

   of the Yoneda embedding $Y_{\mathbb{C}} : \mathbb{C} \to \mathbf{V}^{\mathbb{C}^{\mathrm{op}}}$ in $\mathrm{Cat}_{(\mathbf{S},\mathbf{V})}$, where the second factor is an inclusion.

4. $\mathbf{V}$ is orthogonal to the opposite of the unit of the pseudomonad $\mathbf{M}$ in the sense that, for any small $\mathbf{V}$-category $\mathbb{C}$,

$$\mathbf{V}^{\delta_{\mathbb{C}}^{\mathrm{op}}} : \mathbf{V}^{M(\mathbb{C})^{\mathrm{op}}} \to \mathbf{V}^{\mathbb{C}^{\mathrm{op}}}$$

   is an equivalence.

3.2. Definition. An object $\varphi : \mathbb{C} \to \mathbb{D}$ of the category $\mathrm{Hom}(\mathbb{C}, \mathbb{D})$, of $\mathbf{V}$-functors from $\mathbb{C}$ to $\mathbb{D}$ and $\mathbf{V}$-natural transformations, where $\mathbb{C}$ and $\mathbb{D}$ are small $\mathbf{V}$-categories in the sense of Definition 2.8, is said to be *an M-equivalence* if $M(\varphi) : M(\mathbb{C}) \to M(\mathbb{D})$ is an equivalence. Denote by

$$\mathrm{Equiv}_M(\mathbb{C}, \mathbb{D})$$

the full subcategory of $\mathrm{Hom}(\mathbb{C}, \mathbb{D})$ whose objects are the M-equivalences.

3.3. Proposition. *Let $\boldsymbol{M} = \langle M, \delta, \mu \rangle$ be a bounded KZ-doctrine on a 2-category* $\mathrm{Cat}_{(\mathbf{S},\boldsymbol{V})}$. *Then, for small $\boldsymbol{V}$-categories $\mathbb{C}$ and $\mathbb{D}$, the restriction of the functor*

$$\boldsymbol{V}^{(-)^{\mathrm{op}}} : \mathrm{Hom}(\mathbb{C}, \mathbb{D}) \to \mathrm{Hom}(\boldsymbol{V}^{\mathbb{D}^{\mathrm{op}}}, \boldsymbol{V}^{\mathbb{C}^{\mathrm{op}}})$$

to $\mathrm{Equiv}_M(\mathbb{C}, \mathbb{D})$ *is a well defined functor*

$$\boldsymbol{V}^{(-)^{\mathrm{op}}} : \mathrm{Equiv}_M(\mathbb{C}, \mathbb{D}) \to \mathrm{Equiv}(\boldsymbol{V}^{\mathbb{D}^{\mathrm{op}}}, \boldsymbol{V}^{\mathbb{C}^{\mathrm{op}}})$$

Proof. Let $\varphi : \mathbb{C} \to \mathbb{D}$ be any M-equivalence 1-cell in $\mathrm{Cat}_{(\mathbf{S},\mathbf{V})}$. Consider the commutative square

$$\begin{tikzcd}
\mathbf{V}^{M(\mathbb{D})^{\mathrm{op}}} \arrow[r, "\mathbf{V}^{M(\varphi)^{\mathrm{op}}}"] \arrow[d, "\mathbf{V}^{(\delta_{\mathbb{D}}{}^{\mathrm{op}})}"'] & \mathbf{V}^{M(\mathbb{C})^{\mathrm{op}}} \arrow[d, "\mathbf{V}^{(\delta_{\mathbb{C}}{}^{\mathrm{op}})}"] \\
\mathbf{V}^{\mathbb{D}^{\mathrm{op}}} \arrow[r, "\mathbf{V}^{\varphi^{\mathrm{op}}}"'] & \mathbf{V}^{\mathbb{C}^{\mathrm{op}}}
\end{tikzcd}$$

Since $\mathbf{M}$ is bounded, the two vertical arrows are equivalences. Therefore, if $M(\varphi) : M(\mathbb{C}) \to M(\mathbb{D})$ is an equivalence, so is $\mathbf{V}^{M(\varphi)^{\mathrm{op}}}$, and then so is $\mathbf{V}^{\varphi^{\mathrm{op}}}$. ∎

3.4. Definition. A bounded small KZ-doctrine $\mathbf{M} = \langle M, \delta, \mu \rangle$ on $\mathrm{Cat}_{(\mathbf{S},\mathbf{V})}$ is said to be *tightly bounded* if, for every pair $\mathbb{C}$, $\mathbb{D}$ of small $\mathbf{V}$-categories, the functor

$$\mathbf{V}^{(-)^{\mathrm{op}}} : \mathrm{Equiv}_M(\mathbb{C}, \mathbb{D}) \to \mathrm{Equiv}(\mathbf{V}^{\mathbb{D}^{\mathrm{op}}}, \mathbf{V}^{\mathbb{C}^{\mathrm{op}}})$$

is an equivalence.

3.5. Theorem. *Let $\boldsymbol{M} = \langle M, \delta, \mu \rangle$ be a tightly bounded KZ-doctrine on a 2-category $\mathrm{Cat}_{(\mathbf{S},\boldsymbol{V})}$. Assume that $\boldsymbol{M}$ is small for the pair $(\mathbf{S}, \boldsymbol{V})$. Then, $\boldsymbol{M}$ is idempotent when restricted to small $\boldsymbol{V}$-categories, hence a completion.*

Proof. We wish to show that 1-cell

$$\mu : M^2 \to M$$

is an equivalence. Let $\mathbb{C}$ be a small $\mathbf{V}$-category.

1. Since the KZ-doctrine is bounded,

$$\mathbf{V}^{\delta_{\mathbb{C}^{\mathrm{op}}}} : \mathbf{V}^{M(\mathbb{C})^{\mathrm{op}}} \to \mathbf{V}^{\mathbb{C}^{\mathrm{op}}}$$

   is an equivalence for any small $\mathbf{V}$-category $\mathbb{C}$.

2. Since furthermore the KZ-doctrine $\mathbf{M}$ is assumed tightly bounded, it follows from the above that

$$M(\delta_{\mathbb{C}}) : M(\mathbb{C}) \to M^2(\mathbb{C})$$

   is an equivalence.

3. Since $\mathbf{M} = \langle M, \delta, \mu \rangle$ is a locally fully faithful KZ-doctrine,

$$M(\delta_{\mathbb{C}}) \dashv \mu_{\mathbb{C}} \dashv \delta_{M(\mathbb{C})}$$

   where the counit

$$\epsilon : \delta_{M(\mathbb{C})} \cdot \mu_{\mathbb{C}} \to \mathrm{id}$$

   an equivalence –equivalently, with the unit

$$\eta : \mathrm{id} \to \mu_{\mathbb{C}} \cdot M(\delta_{\mathbb{C}})$$

   an equivalence, it follows that $\mu_{\mathbb{C}} : M^2(\mathbb{C}) \to M(\mathbb{C})$ is an equivalence.

■

3.6. Definition. Let $\mathbf{M} = \langle M, \delta, \mu \rangle$ and $\mathbf{N} = \langle N, \kappa, \nu \rangle$ be pseudomonads on a 2-category $\mathscr{K}$. We say that $\boldsymbol{M}$ *and* $\boldsymbol{N}$ *commute* if there exists an iso 1-cell

$$\gamma : MN \to NM$$

such that the following diagrams

1.

$$\begin{tikzcd} & \mathrm{id} \arrow[dl, "(\delta N)\cdot\kappa"'] \arrow[dr, "(\kappa M)\cdot\delta"] & \\ MN \arrow[rr, "\gamma"] & & NM \end{tikzcd}$$

2.

$$\begin{tikzcd} MMNN \arrow[r, "\bar{\gamma}"] \arrow[d, "\mu\cdot\nu"'] & NNMM \arrow[d, "\nu\cdot\mu"] \\ MN \arrow[r, "\gamma"'] & NM \end{tikzcd}$$

commute, where $\bar{\gamma}$ is the obvious composite of various combinations of γ, M and N.

commute.

3.7. Proposition. *Let* $\boldsymbol{M} = \langle M, \delta, \mu \rangle$ *and* $\boldsymbol{N} = \langle N, \kappa, \nu \rangle$ *be bounded KZ-doctrines (respectively, bounded completions) on* $\mathrm{Cat}_{(\mathbf{S}, \boldsymbol{V})}$. *Assume furthermore that* $\boldsymbol{M}$ *and* $\boldsymbol{N}$ *are small for the pair* $(\mathbf{S}, \boldsymbol{V})$. *If* $\boldsymbol{M}$ *and* $\boldsymbol{N}$ *commute with each in the sense of Definition 3.6, then the composite* $\boldsymbol{N}\dot{\boldsymbol{M}}$ *is a bounded KZ-doctrine (respectively a bounded completion).*

Proof.

1. Denote by $\mathbf{T} = \langle T, u, m \rangle$ the composite pseudomonad, that is, for each object $\mathscr{A}$ of $\mathscr{K}$, $T(\mathscr{A}) = NM(\mathscr{A})$, $u_{\mathscr{A}} = \kappa_{M(\mathscr{A})} \cdot \delta_{\mathscr{A}}$, and $m_{\mathscr{A}} = N(\mu_{\mathscr{A}}) \cdot \nu_{M^2(\mathscr{A})} \cdot \gamma$.

   We wish to show that, since each of $\mathbf{M}$ and $\mathbf{N}$ are KZ-doctrines, so is $\mathbf{T}$. In other words, we wish to establish the relationships

$$T(u_{\mathscr{A}}) \dashv m_{\mathscr{A}} \dashv u_{T(\mathscr{A})}$$

   for every $\mathscr{A}$.

   We know that

$$\mu(\delta_{\mathscr{A}}) \dashv \mu_{\mathscr{A}} \dashv \delta_{M(\mathscr{A})}$$

   and

$$N(\kappa_{\mathscr{A}}) \dashv \nu_{\mathscr{A}} \dashv \kappa_{N(\mathscr{A})}.$$

It follows that

$$N(\kappa_{M^2(\mathscr{A})}) \cdot NM(\delta_{\mathscr{A}}) \dashv N(\mu_{\mathscr{A}}) \cdot \nu_{M^2(\mathscr{A})}$$

which, using the commutativity relations, states that $T(u_{\mathscr{A}}) \dashv m_{\mathscr{A}}$.

It also follows that

$$N(\mu_{\mathscr{A}}) \cdot \nu_{M^2(\mathscr{A})} \dashv \kappa_{NM^2(\mathscr{A})} \cdot N(\delta_{M(\mathscr{A})})$$

which, using the commutativity relations, says that $m_{\mathscr{A}} \dashv u_{T(\mathscr{A})}$.

2. The commutativity of **M** with **N** implies that the composite KZ-doctrine $\mathbf{N}\dot{\mathbf{M}}$ is fully faithful since each one is.

3. Since **V** is orthogonal to each of the opposites of the units $\delta_{\mathbb{C}}$ and $\kappa_{\mathbb{C}}$, for $\mathbb{C}$ a small **V**-category, then **V** is also orthogonal to the opposite of the composite

$$\mathbb{C} \xrightarrow{\delta_{\mathbb{C}}} M(\mathbb{C}) \xrightarrow{\kappa_{M(\mathbb{C})}} NM(\mathbb{C}).$$

4. Furthermore, if each of **M**, **N** is idempotent, so is the composite.

∎

3.8. Remark. There is a tempting comparison with the usual formulation of Morita equivalence [7] for a KZ-doctrine $\mathbf{M} = \langle M, \delta, \mu \rangle$ on **S**-Indexed Cat. The latter states that two small **S**-indexed categories $\mathbb{C}$ and $\mathbb{D}$ are 'Morita equivalent' (in the sense that there is an equivalence $\mathbf{S}^{\mathbb{C}^{\mathrm{op}}} \cong \mathbf{S}^{\mathbb{D}^{\mathrm{op}}}$) if and only if $M(\mathbb{C})$ and $\mathrm{M}(\mathbb{D})$ are equivalent. However, in this formulation – unlike the case for tightly bounded KZ-doctrines, it does not follow that the KZ-doctrine **M** is idempotent.

4. The Grothendieck Completion

In this section we work within $\mathrm{Cat}_{(\mathbf{S},\mathbf{S})}$, for **S** a topos. In the first part, we review the Karoubi envelope and the stack completion on **S**-Indexed Cat as the principal ingredients of the Grothendieck completion. The constructions themselves and most of their properties (stated here without proof) are the contents of [7]. The main purpose of this section is to prove that both of these constructions are bounded completions in the sense of Definition 3.1. It will follow from this that the Grothendieck completion is also bounded and, in fact, tightly bounded.

The *Karoubi envelope* of a category $\mathbb{C}$ in **S** is constructed as a universal solution

$$v_{\mathbb{C}} : \mathbb{C} \to \widehat{\mathbb{C}}$$

to the problem of universally splitting the idempotents [7] by internalizing the analogous case applied to a small additive category $\mathbb{C}$ [13].

4.1. THEOREM. *The Karoubi envelope is a bounded completion on* $\mathrm{Cat}_{(\mathbf{S},\mathbf{S})}$.

PROOF.

1. We have already remarked in Examples 2.12 that the Karoubi envelope is small for any pair $(\mathbf{S}, \mathbf{V})$ in the sense of Definition 2.11.

2. The unit $v_{\mathbb{C}} : \mathbb{C} \to \widehat{\mathbb{C}}$ is an embedding. Hence, the Karoubi envelope is a fully faithful KZ-doctrine.

3. The embedding of a small $\mathbf{V}$-category $\mathbb{C}$ into its Karoubi envelope is equivalent to the embedding
$$\mathrm{yon}_{\mathbb{C}} : \mathbb{C} \to \mathrm{RRep}(\mathbf{V}^{\mathbb{C}^{\mathrm{op}}})$$
where $\mathrm{RRep}(\mathbf{V}^{\mathbb{C}^{\mathrm{op}}})$ is the full $\mathbf{S}$-indexed $\mathbf{V}$-subcategory of $\mathbf{V}^{\mathbb{C}^{\mathrm{op}}}$ whose objects are the retracts of the representables.

4. The Karoubi envelope is a completion. A small $\mathbf{V}$-category $\mathbb{C}$ has split idempotents if and only if $v_{\mathbb{C}} : \mathbb{C} \to \widehat{\mathbb{C}}$ is an equivalence. Further, idempotents split in $\widehat{\mathbb{C}}$. Therefore $v_{\widehat{\mathbb{C}}} : \widehat{\mathbb{C}} \to \widehat{\widehat{\mathbb{C}}}$ is an equivalence.

5. That $\mathbf{S}$ is orthogonal to the unit $v_{\mathbb{C}} : \mathbb{C} \to \hat{\mathbb{C}}$ is shown exactly as in [7] (Proposition 3.3) by remarking that what is used of the topos assumption is that it be a regular category. Notice now that
$$(\widehat{\mathbb{C}})^{\mathrm{op}} \cong \widehat{\mathbb{C}^{\mathrm{op}}}.$$
It follows from this that $\mathbf{S}$ is also orthogonal to the opposite of the unit $v_{\mathbb{C}} : \mathbb{C} \to \widehat{\mathbb{C}}$, again because $\mathbf{S}$ is a regular category.

∎

We recall some definitions from [11, 7]

4.2. DEFINITION. Let $F : \mathscr{B} \longrightarrow \mathscr{C}$ be an $\mathbf{S}$-indexed functor between $\mathbf{S}$-indexed categories. F is said to be a *weak equivalence* if the following conditions hold.

1. (locally essentially surjective) For each object I of $\mathbf{S}$, $F^I : \mathscr{B}^I \longrightarrow \mathscr{C}^I$, and an object c in $\mathscr{C}^I$, there exists an epimorphism $e : J \twoheadrightarrow I$ in $\mathbf{S}$, an object b in $\mathscr{B}^J$, and an isomorphism $\theta : F^J(b) \to e^*(c)$.

2. (fully faithful) For all objects I of $\mathbf{S}$, and for all objects x, x' of $\mathscr{B}^I$, the morphism
$$Hom_{\mathscr{B}^I}(x, x') \xrightarrow{F_{x,x'}} Hom_{\mathscr{C}^I}(Fx, Fx')$$
is an isomorphism in $\mathscr{B}^J$.

4.3. Definition. An **S**-indexed category $\mathscr{A}$ is a *stack* if for every weak equivalence **S**-indexed functor $F : \mathscr{B} \to \mathscr{C}$, the **S**-indexed functor $\mathscr{A}^F : \mathscr{A}^{\mathscr{C}} \to \mathscr{A}^{\mathscr{B}}$ is an equivalence of **S**-indexed categories.

4.4. Remark. For $\mathscr{A}$ and $\mathscr{C}$ both **S**-indexed categories, the expression $\mathscr{A}^{\mathscr{C}}$ denotes the **S**-indexed category of **S**-indexed functors from $\mathscr{C}$ to $\mathscr{A}$ and **S**-indexed natural transformations. It is well defined [24]. In view of the merging of **S**-Indexed Cat with **V**-Cat for **V** a symmetric monoidal closed category, we are interested in restricting our attention to exponentials by small **S**-indexed categories. This in fact makes no restriction on the notion of a stack, as Proposition (2.2) [11] shows, since it is enough to consider those weak equivalence functors of the form

$$F_e : \mathbb{J}_e \to \mathbb{I}$$

for each regular epimorphism $e : J \to I$ in **S**, where $\mathbb{J}_e$ is the 1-kernel of $e : J \twoheadrightarrow I$.

4.5. Definition. Let $\mathbb{C}$ be a small **S**-category, that is, a category internal to **S**. An object F of $(\mathbf{S}^{\mathbb{C}^{\mathrm{op}}})^I$ is said to be *locally representable* if there exists an epimorphism

$$e : J \twoheadrightarrow I$$

in **S**, an object c of $\mathbb{C}^J$, and an **S**-indexed natural isomorphism

$$\theta : e^{\star}(F) \to \mathbb{C}(-, c)$$

in $(\mathbf{S}^{\mathbb{C}^{\mathrm{op}}})^J$.

The *stack completion* $u_{\mathbb{C}} : \mathbb{C} \to \widetilde{\mathbb{C}}$ of a small **S**-indexed category $\mathbb{C}$, as constructed in [7] using results of [11], has as unit the embedding

$$u_{\mathbb{C}} : \mathbb{C} \to \mathrm{LocRep}(\mathbf{S}^{\mathbb{C}^{\mathrm{op}}})$$

into the full **S**-indexed subcategory of $\mathbf{S}^{\mathbb{C}^{\mathrm{op}}}$ whose objects in each fiber are those X that are locally representable in the sense of Definition 4.5.

4.6. Theorem. *For* **S** *a Grothendieck topos, the stack completion on* $\mathrm{Cat}_{(\mathbf{S},\mathbf{S})}$ *is a bounded completion.*

Proof.

1. It follows easily from its defining property – to wit, 'inverting' the weak equivalence **S**-indexed functors, that the stack completion is a KZ-doctrine.

2. As we observed in Examples 2.12, if **S** is a Grothendieck topos, then the pair $(\mathbf{S}, \mathbf{S})$ satisfies the axiom of small stack completions. Indeed, for any small **S**-indexed category $\mathbb{C}$, the stack completion $\mathrm{LocRep}(\mathbf{S}^{\mathbb{C}^{\mathrm{op}}})$ is a small **S**-indexed category. This is on account of the existence of a generating family.

3. The unit $u_{\mathbb{C}} : \mathbb{C} \rightarrow \widetilde{\mathbb{C}}$ is a weak equivalence $\mathbf{S}$-indexed functor. In fact, this property characterizes it, as any weak equivalence $\mathbf{S}$-indexed functor $f : \mathbb{C} \rightarrow \mathbb{D}$, with $\mathbb{D}$ a stack, is (up to equivalence) the stack completion of $\mathbb{C}$.

4. Since weak equivalence $\mathbf{S}$-indexed functors compose, the stack completion is idempotent, thus a completion in the sense of Definition 1.3.

5. From the above follows that also its opposite

$$u_{\mathbb{C}^{\mathrm{op}}} : \mathbb{C}^{\mathrm{op}} \rightarrow (\widetilde{\mathbb{C}})^{\mathrm{op}}$$

is a weak equivalence $\mathbf{S}$-indexed functor.

6. Any topos $\mathbf{S}$ is an $\mathbf{S}$-stack as shown in [11], and therefore so is $\mathbf{S}^{\mathbb{C}^{\mathrm{op}}}$. Hence $\mathbf{S}$ is orthogonal to (both the unit and) the opposite of the unit.

■

4.7. Proposition. *If* $\mathbf{S}$ *is a Grothendieck topos, the stack completion on* $\mathrm{Grpds}_{(\mathbf{S},\mathbf{S})}$ *is tightly bounded.*

Proof. This special form of [7] (Theorem 5.2) is given in [8] in connection with topos cohomology.

We can now prove, based mostly on results established in [7], that the Grothendieck completion on $\mathrm{Cat}_{(\mathbf{S},\mathbf{S})}$, for $\mathbf{S}$ a Grothendieck topos, is tightly bounded. In fact, this example has been our motivation for the very definition of this notion.

4.8. Definition. Let $\mathbf{S}$ be a topos and Let $\mathbb{C}$ be a small $\mathbf{S}$-indexed category.

1. An $\mathbf{S}$-*point of the topos* $\mathbf{S}^{\mathbb{C}^{\mathrm{op}}}$ is a geometric morphism

$$\varphi : \mathbf{S} \longrightarrow \mathbf{S}^{\mathbb{C}^{\mathrm{op}}}$$

over $\mathbf{S}$.

2. Denote by $\mathrm{Points}_{\mathbf{S}}(\mathbf{S}^{\mathbb{C}^{\mathrm{op}}})$ the $\mathbf{S}$-indexed category whose fiber at $I \in \mathbf{S}$ is the category whose objects are the geometric morphisms $\mathbf{S}/I \rightarrow (\mathbf{S}/I)^{\mathbb{C}^{\mathrm{op}}}$ over $\mathbf{S}$, and whose morphisms are the usual 2-cells between geometric morphisms.

3. An $\mathbf{S}$-point φ of $\mathbf{S}^{\mathbb{C}^{\mathrm{op}}}$ is said to be $\mathbf{S}$-*essential* if $\varphi^{\star}$ has an $\mathbf{S}$-indexed left adjoint $\varphi_{!} \dashv \varphi^{\star}$. Denote by $\mathrm{EssPoints}_{\mathbf{S}}(\mathbf{S}^{\mathbb{C}^{\mathrm{op}}})$ the full $\mathbf{S}$-indexed subcategory of $\mathrm{Points}_{\mathbf{S}}(\mathbf{S}^{\mathbb{C}^{\mathrm{op}}})$ whose objects are the essential points.

4.9. Definition. By the *Grothendieck completion*

$$w_{\mathbb{C}} : \mathbb{C} \to \mathscr{G}(\mathbb{C})$$

of a small **S**-indexed category $\mathbb{C}$ we mean here the canonical **S**-indexed functor

$$w_{\mathbb{C}} : \mathbb{C} \to \mathrm{EssPoints}_{\mathbf{S}}(\mathbf{S}^{\mathbb{C}^{\mathrm{op}}})$$

that assigns, to an object c of $\mathbb{C}$, the 3-tuple

$$F_c \dashv E_c \dashv G_c,$$

where E_c is evaluation at c.

4.10. Remarks. A justification for the terminology 'Grothendieck completion' comes from the following observations.

- Let G be a *groupoid* in **S** and $\mathscr{B}(G)$ its classifying topos. There is [15] an *equivalence* γ_G in the commutative diagram

$$\begin{tikzcd} & \mathrm{Tors}^1{}_{\mathbf{S}}(G) \arrow[dd, "\gamma_G"] \\ G \arrow[ur, "u_{\mathbb{G}}"] \arrow[dr, "w_G"'] & \\ & \mathrm{Points}_{\mathbf{S}}(\mathscr{B}(G)) \end{tikzcd} \tag{2}$$

- Let $\mathbb{C}$ be a *category* in **S**. There is a diagram of **S**-indexed functors

$$\begin{tikzcd} & \mathrm{LocRep}(\mathbf{S}^{\mathbb{C}^{\mathrm{op}}}) \arrow[dd, "\gamma_{\mathbb{C}}"] \\ \mathbb{C} \arrow[ur, "u_{\mathbb{C}}"] \arrow[dr, "w_{\mathbb{C}}"'] & \\ & \mathrm{EssPoints}_{\mathbf{S}}(\mathbf{S}^{\mathbb{C}^{\mathrm{op}}}) \end{tikzcd} \tag{3}$$

  The upper arrow $u_{\mathbb{C}}$ is the stack completion of $\mathbb{C}$. The lower arrow $w_{\mathbb{C}}$ is what we have called the *Grothendieck completion* of $\mathbb{C}$ in Definition 4.9. As shown in [7], the Grothendieck completion of $\mathbb{C}$ is equivalent to the stack completion of the Karoubi envelope of $\mathbb{C}$. This is what induces the unique vertical arrow $\gamma_{\mathbb{C}}$ which is identified with $u_{\tilde{\mathbb{C}}}$ and which, in this case, need not be an equivalence.

- Diagram 3 reduces to Diagram 2 in case $\mathbb{C}$ is a groupoid.

- In Diagram 3, $\mathrm{LocRep}(\mathbf{S}^{\mathbb{C}^{\mathrm{op}}})$ is the correct generalization of the category of G-torsors when $\mathbb{C}$ is not necessarily a groupoid G, but $\mathscr{B}(\mathbb{C}) = \mathbf{S}^{\mathbb{C}^{\mathrm{op}}}$ is not its classifying topos. By an instance of Diaconescu's theorem [16], $\mathbf{S}^{\mathbb{C}^{\mathrm{op}}}$ classifies $\mathbf{S}$-valued flat presheaves on $\mathbb{C}$.

It is shown in [7] that the Grothendieck completion $w_{\mathbb{C}} : \mathbb{C} \to \mathscr{G}(\mathbb{C})$ admits two different factorizations. With $w_{\mathbb{C}} : \mathbb{C} \to \mathscr{G}(\mathbb{C})$ as the diagonal, the two factorizations in question are exhibited in the commutative square below

$$\begin{tikzcd}
\mathbb{C} \arrow[r, "u_{\mathbb{C}}"] \arrow[d, "v_{\mathbb{C}}"'] & \widehat{\mathbb{C}} \arrow[d, "v_{\widehat{\mathbb{C}}}"] \\
\widetilde{\mathbb{C}} \arrow[r, "u_{\widetilde{\mathbb{C}}}"'] & \mathscr{G}(\mathbb{C})
\end{tikzcd}$$

4.11. Theorem. *For* $\mathbf{S}$ *a Grothendieck topos, the Grothendieck completion is a bounded completion.*

Proof. The proof relies on Proposition 3.7 since the Grothendieck completion is the composite of two commuting bounded completions on $\mathbf{S}$-Indexed Cat – to wit, the Karoubi and the stack completions. ∎

We shall now prove that – unlike either of its components in general, the Grothendieck completion is tightly bounded. This requires a characterization of it in terms of 'atoms' in the sense of [4, 5, 7].

4.12. Definition. An *atom* of a cocomplete $\mathbf{S}$-indexed category $\mathscr{A}$ is any object A of $\mathscr{A}$ such that the $\mathbf{S}$-indexed functor

$$\mathrm{Hom}(A, -) : \mathscr{A} \to \mathbf{S}$$

preserves $\mathbf{S}$-indexed colimits, that is, coequalizers and $\mathbf{S}$-indexed coproducts. Denote by

$$\mathrm{Atoms}(\mathscr{A})$$

the full $\mathbf{S}$-indexed subcategory of $\mathscr{A}$ determined by its atoms.

Let $\mathbb{C}$ be a small $\mathbf{S}$-indexed category, that is, a category object in $\mathbf{S}$. The topos $\mathbf{S}^{\mathbb{C}^{\mathrm{op}}}$ is $\mathbf{S}$-bounded via an adjoint pair $\Delta \dashv \Gamma : \mathbf{S}^{\mathbb{C}^{\mathrm{op}}} \to \mathbf{S}$, where Γ indicates taking global sections. From this follows that $\mathbf{S}^{\mathbb{C}^{\mathrm{op}}}$ can be indexed via

$$(\mathbf{S}^{\mathbb{C}^{\mathrm{op}}})^I = \mathbf{S}^{\mathbb{C}^{\mathrm{op}}}/\Delta(I)$$

with change of base functors defined in the usual way [24].

4.13. Proposition. *The* $\mathbf{S}$*-indexing of the category* $\mathbf{S}^{\mathbb{C}^{\mathrm{op}}}$ *restricts to the full* $\mathbf{S}$*-indexed subcategory* $\mathrm{Atoms}(\mathbf{S}^{\mathbb{C}^{\mathrm{op}}})$.

Proof. Change of base along any 1-cell $\alpha : K \to I$ has a (left and a) right adjoint satisfying the Beck-Chevalley condition. ∎

4.14. Theorem. *There is a canonical equivalence of* $\mathbf{S}$*-indexed categories*

$$\Theta : \mathscr{G}(\mathbb{C}) \cong \mathrm{Atoms}(\mathbf{S}^{\mathbb{C}^{\mathrm{op}}})$$

which commutes with the inclusions of the representables.

Proof. We only sketch the proof here, but see [7]. The assignments are as follows. Given any $\mathbf{S}$-essential point $f : \mathbf{S} \to \mathbf{S}^{\mathbb{C}^{\mathrm{op}}}$, the $\mathbf{S}$-adjoint pair $f_! \dashv f^\star$ is equivalent to one of the form

$$(-) \otimes A \dashv \mathrm{Hom}(A, -)$$

and the $\mathbf{S}$-adjoint pair $f^\star \dashv f_\star$ shows that A is an atom of $\mathbf{S}^{\mathbb{C}^{\mathrm{op}}}$. Conversely, any atom A of $\mathbf{S}^{\mathbb{C}^{\mathrm{op}}}$ gives rise to an $\mathbf{S}$-essential point of $\mathbf{S}^{\mathbb{C}^{\mathrm{op}}}$. ∎

4.15. Proposition.

1. *Every representable* $\mathbb{C}(-, c) : \mathbb{C}^{\mathrm{op}} \to \mathbf{S}$ *is an atom of* $\mathbf{S}^{\mathbb{C}^{\mathrm{op}}}$.
2. *Every retract of a small coproduct of representables is an atom of* $\mathbf{S}^{\mathbb{C}^{\mathrm{op}}}$.
3. *Every atom of* $\mathbf{S}^{\mathbb{C}^{\mathrm{op}}}$ *is locally the retract of a small coproduct of representables.*

Proof.

1. For any object X of $\mathbf{S}^{\mathbb{C}^{\mathrm{op}}}$ there is an isomorphism

   $$\mathrm{Hom}(\mathbb{C}(-, c), X) \cong E_c(X),$$

   natural in X. The evaluation functor $E_c : \mathbf{S}^{\mathbb{C}^{\mathrm{op}}} \to \mathbf{S}$ preserves all small colimits.
2. If R is a retract of a small coproduct of representables in $\mathbf{S}^{\mathbb{C}^{\mathrm{op}}}$, then $\mathrm{Hom}(R, -)$ is obtained from $\mathrm{Hom}(\mathbb{C}(-, c), -)$ by a coequalizer diagram involving small coproducts. By the commutativity between colimits, $\mathrm{Hom}(R, -) : \mathbf{S}^{\mathbb{C}^{\mathrm{op}}} \to \mathbf{S}$ preserves all small colimits.
3. Let $\xi : X \to \Delta(I)$ be an atom of $(\mathbf{S}^{\mathbb{C}^{\mathrm{op}}})^I$ for $I \in \mathbf{S}$, and let

   $$p_X : \Delta(J_X) \times_{\Delta(C_0)} C_1 \twoheadrightarrow X$$

   be the canonical presentation of X as a colimit of representables over $\Delta(I)$.

   Since $\mathrm{Hom}(\xi, -)$ preserves (regular) epimorphisms, we obtain an epimorphism

   $$\mathrm{Hom}(X, \Delta(J_X) \times_{\Delta(C_0)} C_1) \twoheadrightarrow \mathrm{Hom}(X, X)$$

   over $\Delta(I)$.

   Since $\mathrm{Hom}(\xi, -)$ preserves $\mathbf{S}$-indexed coproducts, we have an isomorphism

   $$J_X \times_{C_0} \mathrm{Hom}(X, C_1) \cong \mathrm{Hom}(X, \Delta(J_X) \times_{\Delta(C_0)} C_1)$$

Call $q_X : J_X \times_{C_0} \mathrm{Hom}(X, C_1) \twoheadrightarrow \mathrm{Hom}(X, X)$ the composite of the above two.

Consider the morphism $I \rightarrow \mathrm{Hom}(X, X)$ which picks up the identity on ξ, that is, the identity on X over $\Delta(I)$. Consider the pullback

$$\begin{tikzcd} K \arrow[r, two heads, "\bar{q}"] \arrow[d, "r"'] & I \arrow[d, "\mathrm{id}_X"] \\ J_X \times_{C_0} \mathrm{Hom}(X, C_1) \arrow[r, two heads, "q"'] & \mathrm{Hom}(X, X) \end{tikzcd}$$

in $\mathbf{S}$. The top horizontal arrow is an epimorphism since $\mathbf{S}$ is a regular category. This shows that ξ is locally the retract of a small coproduct of representables.

∎

The following is now all we need to show our desired result.

4.16. Theorem. *Let there be given an* $\mathbf{S}$*-indexed (strong) equivalence* $\Phi : \mathbf{S}^{\mathbb{D}^{\mathrm{op}}} \rightarrow \mathbf{S}^{\mathbb{C}^{\mathrm{op}}}$*. Then,* Φ *restricts to an* $\mathbf{S}$*-indexed (strong) equivalence* $\bar{\Phi} : \mathrm{Atoms}(\mathbf{S}^{\mathbb{D}^{\mathrm{op}}}) \rightarrow \mathrm{Atoms}(\mathbf{S}^{\mathbb{C}^{\mathrm{op}}})$*, depicted in a commutative square*

$$\begin{tikzcd} \mathrm{Atoms}(\mathbf{S}^{\mathbb{D}^{\mathrm{op}}}) \arrow[r, "\bar{\Phi}"] \arrow[d] & \mathrm{Atoms}(\mathbf{S}^{\mathbb{C}^{\mathrm{op}}}) \arrow[d] \\ \mathbf{S}^{\mathbb{D}^{\mathrm{op}}} \arrow[r, "\Phi"'] & \mathbf{S}^{\mathbb{C}^{\mathrm{op}}} \end{tikzcd}$$

where the vertical arrows are inclusions.

Proof. Straightforward. ∎

4.17. Corollary. *The Grothendieck completion is tightly bounded.*

Proof. Since the Grothendieck completion is bounded by Theorem 4.11, it remains to verify the condition of Definition 3.4. In turn, this follows readily from Theorem 4.16 applied to an equivalence of the form $\mathbf{S}^{\varphi^{\mathrm{op}}} : \mathbf{S}^{\mathbb{D}^{\mathrm{op}}} \rightarrow \mathbf{S}^{\mathbb{C}^{\mathrm{op}}}$ for a given $\varphi : \mathbb{C} \rightarrow \mathbb{D}$. ∎

4.18. Remark. To the natural question of whether the results of this section extend to an arbitrary pair $(\mathbf{S}, \mathbf{V})$, where $\mathbf{V}$ is an $\mathbf{S}$-indexed symmetric monoidal category not necessarily the topos $\mathbf{S}$ with is cartesian closed structure, the answer is that *a priori* this is not the case. For instance, a property of $\mathbf{S}$ that is crucially used in theory of stacks [11] is that, since $\mathbf{S}$ is a topos, it is a regular – in fact, an exact category. Perhaps suitably adding this as a condition on $\mathbf{V}$ will be sufficient for developing a good theory of $\mathbf{V}$-stacks over $\mathbf{S}$. We leave this as an open question.

5. The Cauchy completion

In this section we work within $\mathrm{Cat}_{(Set,\mathbf{V})}$, where $\langle \mathbf{V}, \otimes, Z\rangle$ a symmetric monoidal closed category. Moreover, we will assume that$\langle \mathbf{V}, \otimes, Z\rangle$ be a (symmetric monoidal) closed category with $\mathrm{Hom}_{\mathbf{V}} : \mathbf{V} \to Set$ faithful, and such that $\mathbf{V}$ is complete and cocomplete [12, 5, 17].

The *Cauchy completion* of a $\mathbf{V}$-category is introduced in [23] (see also [3, 17]). It is the purpose of this section to show that the Cauchy completion, for a suitable $\mathbf{V}$ such that the Cauchy completion is small for $(Set, \mathbf{V})$, is bounded – in fact, tightly bounded. As this construction is not immediately seen as being 'carved out of Yoneda', we prove it explicitly.

Let $\mathbb{C}$ and $\mathbb{D}$ be small $\mathbf{V}$-categories. We recall [2] that a ***V**-distributor* F from $\mathbb{C}$ to $\mathbb{D}$, denoted $F : \mathbb{D} \mapsto \mathbb{C}$, is $\mathbf{V}$-functor $F : \mathbb{C} \longrightarrow \mathbf{V}^{\mathbb{D}^{\mathrm{op}}}$. Denote by

$$\mathrm{Dist}_{\mathbf{V}}(\mathbb{D}, \mathbb{C})$$

the category of $\mathbf{V}$-distributors from $\mathbb{D}$ to $\mathbb{C}$ and $\mathbf{V}$-natural transformations between them.

5.1. Definition. Denote by $\mathrm{AdjDist}_{\boldsymbol{V}}(\mathbb{D}, \mathbb{C})$ the full subcategory of $\mathrm{Dist}_{\mathbf{V}}(\mathbb{D}, \mathbb{C})$ determined by the $\mathbf{V}$-valued distributors $F : \mathbb{D} \mapsto \mathbb{C}$ with a $\mathbf{V}$-distributor right adjoint $G : \mathbb{C} \mapsto \mathbb{D}$.

5.2. Remark. Let $\langle \mathbf{V}, \otimes, Z\rangle$ be a symmetric monoidal closed category. Recall that $\mathbb{Z}$ denotes the $\mathbf{V}$-category with one object o, and with $\mathrm{Hom}_{\mathbb{Z}}(o, o) = Z$. There are natural equivalences $\mathbb{C} \cong \mathrm{Hom}_{\mathbf{V}}(\mathbb{Z}, \mathbb{C})$ and $\mathbf{V}^{\mathbb{C}^{\mathrm{op}}} \cong \mathrm{Dist}_{\mathbf{V}}(\mathbb{Z}, \mathbb{C})$ for any small $\mathbf{V}$-category $\mathbb{C}$.

5.3. Proposition. *There is a commutative diagram*

$$\begin{tikzcd}
\mathbb{C} \arrow[r, "\mathrm{Y}_{\mathbb{C}}"] \arrow[d] & \boldsymbol{V}^{\mathbb{C}^{\mathrm{op}}} \arrow[d] \\
\mathrm{Hom}_{\boldsymbol{V}}(\mathbb{Z}, \mathbb{C}) \arrow[r, "\Phi_{\mathbb{Z},\mathbb{C}}"'] & \mathrm{Dist}_{\boldsymbol{V}}(\mathbb{Z}, \mathbb{C})
\end{tikzcd}$$

Proof. To a $\mathbf{V}$-functor $f : \mathbb{Z} \to \mathbb{C}$, $\Phi_{\mathbb{Z},\mathbb{C}}$ assigns the $\mathbf{V}$-distributor $f_\star : \mathbb{Z} \mapsto \mathbb{C}$ such that $f_\star(c, o) = \mathbb{C}(c, f(o))$. Indeed, there is $f_\star \dashv f^\star$ where $f : \mathbb{C} \mapsto \mathbb{Z}$ is such that $f^\star(o, c) = \mathbb{C}(f(o), c)$. ∎

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There is an $\mathbf{S}$-indexed functor

$$\Psi_{\mathbb{D},\mathbb{C}} : \mathrm{Hom}_{\mathbf{V}}(\mathbb{D}, \mathbb{C}) \to \mathrm{Dist}_{\mathbf{V}}(\mathbb{D}, \mathbb{C})$$

whose fiber at an object I of $\mathbf{S}$ is defined as follows. To a $\mathbf{V}$-functor $f : \mathbb{D} \to \mathbb{C}$, Ψ^I assigns, to the I-fiber $f^I : \mathbb{D}^I \to \mathbb{C}^I$, the $\mathbf{V}$-distributor $(f^I)_\star : \mathbb{D}^I \mapsto \mathbb{C}^I$ such that

$(f^I)_\star(c,d) = \mathrm{Hom}_{\mathbb{D}^I}(c, f^I(d))$. To an $\mathbf{S}$-indexed $\mathbf{V}$-natural transformation $\alpha : f \to f'$, Ψ^I assigns, to α^I the obvious morphism ${\alpha^I}_\star : {f^I}_\star \to {f'^I}_\star$ of $\mathbf{V}$-distributors. It is easy to check that ${\alpha^I}_\star$ is a natural transformation.

Given any $\mathbf{S}$-indexed $\mathbf{V}$-functor $f : \mathbb{D} \longrightarrow \mathbb{C}$, the corresponding $\mathbf{S}$-indexed $\mathbf{V}$-distributor $f_\star : \mathbb{D} \mapsto \mathbb{C}$ has an $\mathbf{S}$-indexed $\mathbf{V}$-distributor right adjoint $f^\star : \mathbb{C} \mapsto \mathbb{D}$, defined so that, for I in $\mathbf{S}$, $f^I\star(d,c) = \mathrm{Hom}_{\mathbb{C}^I}(f^I(d), c)$.

We now recall the definition of Cauchy completion following [23] (see also [3, 17]).

5.4. Definition. The *Cauchy completion*

$$z_{\mathbb{C}} : \mathbb{C} \to \mathscr{C}(\mathbb{C})$$

of a small $\mathbf{V}$-category $\mathbb{C}$ is identified with the first factor in the factorization

$$\mathrm{Hom}_{\mathbf{V}}(\mathbb{Z}, \mathbb{C}) \xrightarrow{z_{\mathbb{C}}} \mathrm{AdjDist}_{\mathbf{V}}(\mathbb{Z}, \mathbb{C}) \hookrightarrow \mathrm{Dist}_{\mathbf{V}}(\mathbb{Z}, \mathbb{C})$$

of $\Phi_{\mathbb{Z},\mathbb{C}}$.

A small $\mathbf{V}$-category $\mathbb{C}$ is said to be *Cauchy complete* if $z_{\mathbb{C}} : \mathbb{C} \to \mathscr{C}(\mathbb{C})$ is an equivalence of $\mathbf{V}$-categories.

Assumption. We assume in what follows that $\mathbf{V}$ is a symmetric monoidal closed category such that the Cauchy completion is small for the pair $(\mathit{Set}, \mathbf{V})$.

5.5. Remark. The name 'Cauchy completion' arises from the following motivating example [23]. Let $\mathbf{V}$ be $\mathbb{R}_+$, the category whose objects are all non-negative reals (including ∞), as morphisms $a \longrightarrow v$ the greater-than-or-equal-to relations $a \geq v$, and as tensor $a \otimes u = a + u$. An $\mathbb{R}_+$-category is an arbitrary (generalized) metric space. The motivating result for the notion called *Cauchy completeness* in [23] is that a metric space $< X, d >$ is Cauchy complete in the usual sense (that is, all Cauchy sequences on X converge) if and only if $< X, d >$ is Cauchy complete as a $\mathbb{R}_+$-category.

5.6. Proposition. *The Cauchy completion of a small $\boldsymbol{V}$-category $\mathbb{C}$ is equivalent to the full $\boldsymbol{V}$-subcategory of $\boldsymbol{V}^{\mathbb{C}^{\mathrm{op}}}$ determined by the retracts of small coproducts of representables. In particular, there is an identification*

$$\mathscr{C}(\mathbb{C}) \cong \mathrm{Atoms}(\boldsymbol{V}^{\mathbb{C}^{\mathrm{op}}}).$$

5.7. Corollary. *The Cauchy completion on $\boldsymbol{V}$-Cat is tightly bounded. In particular, it is an idempotent KZ-doctrine when evaluated at small $\boldsymbol{V}$-categories, hence a completion in the sense of Definition 1.3.*

Proof. It follows from Proposition 5.3 that the Cauchy completion is fully faithful and carved out from the Yoneda embedding. The defining property of the Cauchy completion as a pseudomonad is that of having absolute colimits [3]. This implies, since *Set* is complete and cocomplete, that *Set* is orthogonal to the unit. It follows that it is bounded. To see that it is tightly bounded, we now use the same reasons as those employed for the Grothendieck completion, namely, Proposition 5.6 and Theorem 4.16 in the relative case. ∎

5.8. Remark.

1. For $\mathbf{V}$ the category of modules over a commutative ring R with unit, the Cauchy completion of the $\mathbf{V}$-category $\mathbf{Z}$, for Z the unit of $\mathbf{V}$ (which is just the ring R itself), is the category of finitely generated projective R-modules [3]. The proof relies on a well known characterization of finitely generated projective modules as retracts of finitely generated free modules. In particular, this shows that the Cauchy completion does not always reduce to the Karoubi envelope.

2. If in $\mathbf{V}$ coproducts are disjoint (and universal) then the Cauchy completion reduces to the Karoubi envelope.

3. If $\mathbf{V} = \mathbf{S}$ is a topos, regarded as a cartesian closed category, then, for $\mathbb{C}$ a category in $\mathbf{S}$, the atoms of $\mathbf{S}^{\mathbb{C}^{\mathrm{op}}}$ are those presheaves that are locally retracts of representables [7]. In this case, coproducts in $\mathbf{S}$ are disjoint, but the axiom of choice need not hold in $\mathbf{S}$. If the axiom of choice holds in $\mathbf{V} = \mathbf{S}$, then the Cauchy completion reduces to the Karoubi envelope.

5.9. Remark. The results of this section should hold for any pair $(\mathbf{S}, \mathbf{V})$, where $\mathbf{S}$ is an arbitrary topos and $\mathbf{V}$ is an $\mathbf{S}$-indexed symmetric monoidal category. Indeed, nothing in [17] seems to make an essential use of *Set* as a base topos.

We have now identified a property that is common to both the Grothendieck and the Cauchy completion, to wit, that they are both tightly bounded in their respective universes. There is another way to comparing them, and that is to do so just as *constructions*. The following theorem, which is the relative version of a theorem in [1, 4], makes this clear.

5.10. Definition. Let $\mathbb{C}$ and $\mathbb{D}$ be small $\mathbf{V}$-categories. By a *generalized $\boldsymbol{V}$-functor F from $\mathbb{D}$ to $\mathbb{C}$* we mean, following Lawvere [21], a cocontinuous $\mathbf{V}$-functor $F : \mathbf{V}^{\mathbb{D}^{\mathrm{op}}} \longrightarrow \mathbf{V}^{\mathbb{C}^{\mathrm{op}}}$. Denote by

$$\mathrm{Gen}_{\mathbf{V}}(\mathbb{D}, \mathbb{C})$$

the category of generalized $\mathbf{V}$-functors from $\mathbb{D}$ to $\mathbb{C}$ and $\mathbf{V}$-natural transformations between them.

5.11. Theorem. *Let $\mathbb{D}$ and $\mathbb{C}$ be small $\boldsymbol{V}$-categories. Then, the functor*

$$\Phi : \mathrm{Dist}_{\boldsymbol{V}}(\mathbb{D}, \mathbb{C}) \to \mathrm{Gen}_{\boldsymbol{V}}(\mathbb{D}, \mathbb{C})$$

which assigns, to a $\boldsymbol{V}$-distributor

$$F : \mathbb{D} \mapsto \mathbb{C},$$

the $\boldsymbol{V}$-functor

$$T_F : \boldsymbol{V}^{\mathbb{D}^{\mathrm{op}}} \longrightarrow \boldsymbol{V}^{\mathbb{C}^{\mathrm{op}}}$$

given by Kan extension of F along Yoneda, is an equivalence of categories.

PROOF. Given a $\mathbf{V}$-distributor $F : \mathbb{D} \longrightarrow \mathbf{V}^{\mathbb{C}^{\mathrm{op}}}$, one can extend it along Yoneda to a $\mathbf{V}$-functor $T_F = F \otimes (-) : \mathbf{V}^{\mathbb{D}^{\mathrm{op}}} \longrightarrow \mathbf{V}^{\mathbb{C}^{\mathrm{op}}}$. Recall that the value of $F \otimes (-)$ at a $\mathbf{V}$-functor $\varphi : \mathbb{D}^{\mathrm{op}} \rightarrow \mathbf{V}$ is the following end:

$$F \otimes \varphi = \int^{d} \mathrm{Hom}_{\mathbf{V}^{\mathbb{D}^{\mathrm{op}}}}(\mathbb{D}(-,d),\varphi) \otimes F(d)$$

$F \otimes (-)$ always has right $\mathbf{V}$-adjoint to wit, $\widehat{T_F} = \mathrm{Hom}(F,-)$. In particular, T_F is cocontinuous. This assignment extends to a functor

$$T_{(-)} : \mathrm{Dist}_{\mathbf{V}}(\mathbb{D},\mathbb{C}) \longrightarrow \mathrm{Hom}_{\mathbf{V}}(\mathbf{V}^{\mathbb{D}^{\mathrm{op}}}, \mathbf{V}^{\mathbb{C}^{\mathrm{op}}}).$$

Going backwards, any $\mathbf{V}$-functor $T : \mathbf{V}^{\mathbb{D}^{\mathrm{op}}} \longrightarrow \mathbf{V}^{\mathbb{C}^{\mathrm{op}}}$ can be "restricted along Yoneda", defining in particular a $\mathbf{V}$-distributor

$$F_T : \mathbb{D} \mapsto \mathbb{C}.$$

The condition that the $\mathbf{V}$-functor $T : \mathbf{V}^{\mathbb{D}^{\mathrm{op}}} \longrightarrow \mathbf{V}^{\mathbb{C}^{\mathrm{op}}}$ be cocontinuous, hence preserves ends, is used in the verification that $T_{F_T} \cong T$.

We have the canonical isomorphisms, natural in φ:

$$T_{F_T}(\varphi) = \int^{d} \mathrm{Hom}_{\mathbf{V}^{\mathbb{D}^{\mathrm{op}}}}(\mathbb{D}(-,d),\varphi) \otimes F_T(d)$$

$$= \int^{d} \mathrm{Hom}_{\mathbf{V}^{\mathbb{D}^{\mathrm{op}}}}(\mathbb{D}(-,d),\varphi) \otimes T(\mathbb{D}(-,d))$$

$$\cong T(\int^{d} \mathrm{Hom}_{\mathbf{V}^{\mathbb{D}^{\mathrm{op}}}}(\mathbb{D}(-,d),\varphi) \otimes \mathbb{D}(-,d)) \cong T(\varphi).$$

We also have the isomorphisms, natural in D, for any object D of $\mathbb{D}$:

$$F_{T_F}(d) = (T_F \circ \mathrm{yon})(d) = T_F(\mathbb{D}(-,d)) =$$

$$\int^{e} \mathrm{Hom}_{\mathbf{V}^{\mathbb{D}^{\mathrm{op}}}}(\mathbb{D}(-,e),\mathbb{D}(-,d)) \otimes F(e) \cong F(d).$$

The correspondence between $\mathbf{V}$-natural transformations on both sides is the obvious one. ∎

The composite of $\mathbf{V}$-distributors $F : \mathbb{D} \mapsto \mathbb{E}$ and $G : \mathbb{E} \mapsto \mathbb{C}$ is the $\mathbf{V}$-distributor $(G \circ F) : \mathbb{D} \mapsto \mathbb{C}$ defined so that

$$(G \circ F)(c,d) = \mathrm{colim}(G(c,e) \otimes F(e,d))$$

indexed by all morphisms $e \rightarrow e'$ in $\mathbb{E}$. Using this composition we may define $\mathbf{V}$-adjointness between $\mathbf{V}$-distributors in the usual way.

5.12. THEOREM. *The equivalence given by* Φ *in Theorem 5.11 restricts to an equivalence*

$$\mathrm{AdjDist}_{\boldsymbol{V}}(\mathbb{D},\mathbb{C}) \cong \mathrm{Ess}_{\mathbf{S}}\mathrm{Gen}_{\boldsymbol{V}}(\mathbb{D},\mathbb{C}).$$

PROOF. Let $F : \mathbb{D} \mapsto \mathbb{C}$ be an adjoint $\mathbf{V}$-distributor with $F \dashv G$. To F corresponds, by Theorem 5.11, $\Phi(F) : \mathbf{V}^{\mathbb{D}^{\mathrm{op}}} \longrightarrow \mathbf{V}^{\mathbb{C}^{\mathrm{op}}}$, that is, the adjoint pair $F \otimes (-) \dashv \mathrm{Hom}(F,-)$. Similarly, to $G : \mathbb{C} \mapsto \mathbb{D}$ corresponds, by the same theorem, $\Phi(G) : \mathbf{V}^{\mathbb{C}^{\mathrm{op}}} \longrightarrow \mathbf{V}^{\mathbb{D}^{\mathrm{op}}}$, that is the adjoint pair $G \otimes (-) \dashv \mathrm{Hom}(G,-)$.

It follows from $F \dashv G$ that $F \otimes (-) \dashv G \otimes (-)$. The unit $\eta : \mathrm{id} \longrightarrow G \otimes F$ induces $\bar{\eta} : \mathrm{id} \longrightarrow (G \otimes F) \cong (G \otimes (-)) \circ (F \otimes (-))$, and similarly for the counit $\bar{\epsilon} : (F \otimes (-)) \circ (G \otimes (-)) \longrightarrow \mathrm{id}$ induced by $\epsilon : F \otimes G \longrightarrow \mathrm{id}$. The verification of the adjunction equations for $(\bar{\eta}, \bar{\epsilon})$ follow from the corresponding ones for (η, ϵ). We conclude that there is a natural isomorphism $G \otimes (-) \cong \mathrm{Hom}(F,-)$. This in turn results in a sequence of $\mathbf{V}$-adjoints

$$F \otimes (-) \dashv \mathrm{Hom}(F,-) \cong G \otimes (-) \dashv \mathrm{Hom}(G,-),$$

so in particular an object of $\mathrm{Ess}_{\mathbf{S}}\mathrm{Gen}(\mathbb{D},\mathbb{C})$.

Conversely, given a sequence of $\mathbf{V}$-adjoints

$$T \dashv \hat{T} \dashv \hat{\hat{T}},$$

restricting $\hat{T}$ along Yoneda gives a $\mathbf{V}$-distributor $G : \mathbb{C} \mapsto \mathbb{D}$, whereas restricting T along Yoneda gives a $\mathbf{V}$-distributor $F : \mathbb{D} \mapsto \mathbb{C}$. By Theorem 5.11 there are identifications $\hat{T} \cong G \circ (-)$ and $T \cong F \circ (-)$. It follows that

$$F \otimes (-) \dashv G \otimes (-)$$

hence that $F \dashv G$. We leave the remaining verifications, which are routine, to the reader. ∎

6. Final remarks

It has been advocated by Lawvere [22] that the notion of a mixed sort of 'category' needed to be explored, namely, one for which the objects are parameterized by the objects of a topos, while the morphisms are parameterized by objects in an enriching monoidal closed category. It is therefore curious that such a theory had not yet been formally developed.

Independently of the above, we were led to defining a 2-category $\mathrm{Cat}_{(\mathbf{S},\mathbf{V})}$ (of 'indexed enriched categories') on account of our desire to compare a construction (the Grothendieck completion) that exists in **S**-Indexed Cat, for a topos **S**, with another (the Cauchy completion) that exists in **V**-Cat, for a closed monoidal category **V**.

However, what we introduce here is just a portion of such a theory of **S**-indexed **V**-categories, albeit one that is sufficient for our purposes – namely, to introduce the unifying notion of (tightly) bounded KZ-doctrines on any 2-category $\mathrm{Cat}_{(\mathbf{S},\mathbf{V})}$, and not just in either **S**-Indexed Cat or **V**-Cat.

A theory of indexed enriched categories seems desirable also for other applications – to wit, those which Lawvere had in mind when he proposed the merge [22]. One such would be to develop a theory of internal metric spaces in a petite topos as time-parameterized sets, suggested by Einstein but bypassed because of lack of sufficient mathematical machinery. Another would be to develop linear algebra in any topos, such as functional analysis in a gros topos. Specifically, moduli spaces such as the parameterizer for irreducible representations of a Lie group, have the structure, beyond a mere category on a set, of an S-object, where S expresses the smoothness of the group and of the functional analysis.

Having both answered the question that motivated this investigation, and introduced indexed enriched categories for this purpose, we end with the observation that the possible further applications of the latter are of sufficient importance to warrant a further exploration.

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Department of Mathematics and Statistics, McGill University,
Montreal, QC, Canada H3A 2K6

Email: marta.bunge@mcgill.ca

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