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GEOMETRIC MORPHISMS OF REALIZABILITY TOPOSES

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ABSTRACT. We show that every geometric morphism between realizability toposes satisfies the condition that its inverse image commutes with the 'constant object' functors, which was assumed by John Longley in his pioneering study of such morphisms. We also provide the answer to something which was stated as an open problem on Jaap van Oosten's book on realizability toposes: if a subtopos of a realizability topos is (co)complete, it must be either the topos of sets or the degenerate topos. And we present a new and simpler condition equivalent to the notion of computational density for applicative morphisms of Schönfinkel algebras.

Introduction

In his thesis [4], John Longley studied what can be said about geometric morphisms between realizability toposes. He showed that, if such a morphism has the property that its inverse image functor commutes up to isomorphism with the 'constant object' functors usually denoted ∇, or equivalently restricts to a functor between the subcategories of assemblies, then it is induced by a particular type of relation (known as an *applicative morphism*) between the generating Schönfinkel algebras. He also showed that, among morphisms satisfying this condition, those whose direct image functors are regular (equivalently, preserve epimorphisms) correspond precisely to adjoint pairs of applicative morphisms. Subsequently, Pieter Hofstra and Jaap van Oosten [1] extended Longley's work by obtaining a characterization of those 'computationally dense' applicative morphisms which correspond to inverse image functors, thus removing the requirement for the direct image functor to preserve epimorphisms. However, their work still required the other assumption made by Longley. Details of this work may be found in van Oosten's book [5].

In this paper, we present a proof that the condition on inverse image functors assumed by Longley, and by Hofstra and van Oosten, is always satisfied. The proof has two parts: first we show that any localic morphism between realizability toposes satisfies Longley's condition, and then we show that all geometric morphisms between realizability toposes are localic. For good measure, we also present the answer to something which was stated as an open problem on page 235 of [5]: is **Set** the only non-degenerate Grothendieck topos which can occur as a (sheaf) subtopos of a realizability topos? (The (positive) answer to this question has been known (to the present author, at least) for many years, but never

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published; so it seems appropriate to include it here.) And we also present a simplification of the definition of computational density for applicative morphisms, which was recently discovered by the author.

The terminology and notation used in this paper is that of 'Sketches of an Elephant' [2]; but since a good deal of it is taken from the as-yet-unpublished third volume, we need to introduce it here. We use the term *Schönfinkel algebra* for what others call a partial combinatory algebra or PCA, in honour of M. Schönfinkel's astonishingly modern 1924 paper [7] where such structures were first studied. Given a Schönfinkel algebra Λ, we write $\mathbf{Eff}(\Lambda)$ for the realizability topos over Λ, and $\mathbf{Ass}(\Lambda)$ for its full subcategory of $\neg\neg$-separated objects, also called Λ-*valued assemblies.* (For simplicity, we shall deal only with 'ordinary' realizability toposes, and leave for subsequent investigation the question of how far our results may be extended to notions such as modified and relative realizability.) In this paper we shall not need a detailed description of the objects of $\mathbf{Eff}(\Lambda)$, but we shall be concerned a good deal with assemblies: an assembly is a pair (A, α), where A is a set and $\alpha \colon A \looparrowright \Lambda$ is an entire relation (i.e. one relating each element of A to at least one element of Λ). A morphism of assemblies $f \colon (A, \alpha) \to (B, \beta)$ is a function between their underlying sets which can be *tracked* by some element λ of Λ, in the sense that whenever we have $\alpha(a, \mu)$ (i.e. a and μ are related by α) then $\lambda\mu$ is defined and $\beta(f(a), \lambda\mu)$. We write ∇A for the assembly obtained by equipping A with the relation in which every element of A is related to every element of Λ; these objects are the $\neg\neg$-sheaves in $\mathbf{Eff}(\Lambda)$, and they form a category isomorphic to $\mathbf{Set}$. The left adjoint of $\nabla \colon \mathbf{Set} \to \mathbf{Eff}(\Lambda)$ (the associated $\neg\neg$-sheaf functor) is simply the functor represented by the terminal object (and its restriction to $\mathbf{Ass}(\Lambda)$ is the underlying-set functor); we shall denote it by Γ.

Given $\lambda \in \Lambda$, by the λth *caucus* of an assembly (A, α) we mean the set of all elements of A which are related to λ by α. Clearly, any morphism of assemblies must map caucuses into caucuses, from which it follows easily that (A, α) is isomorphic to its associated sheaf ∇A iff there is at least one caucus which contains all the members of A. We call such assemblies *corrupt*. At the other extreme, an assembly is *modest* if no caucus has more than one member; we write $\overline{\Lambda}$ for the modest assembly whose underlying set is Λ, with the identity relation (i.e. its λth caucus is $\{\lambda\}$).

We assume that the reader is familiar with the Heyting prealgebra operations on the power-set $P\Lambda$ of a Schönfinkel algebra Λ. As usual, we shall say that a family of implications $((f(i) \Rightarrow g(i)) \mid i \in I)$ is *uniformly realizable* if the corresponding subsets of Λ have an element in common.

We should mention that we shall feel free in this paper to assume classical logic, including the axiom of choice when necessary, in the topos of sets.

1. Complete Subtoposes of Realizability Toposes

We recall that sheaf subtoposes of a topos $\mathcal{E}$ correspond bijectively to *local operators* on $\mathcal{E}$, that is to morphisms $j \colon \Omega \to \Omega$ satisfying $jj = j$, $j\top = \top$ and $j\wedge = \wedge(j \times j)$ (cf. [2], A4.4.1). Local operators on realizability toposes were first studied by Andy Pitts in his thesis [6]. He showed that they may be characterized as follows:

1.1. PROPOSITION. *Local operators on* $\mathbf{Eff}(\Lambda)$ *correspond to equivalence classes of functions* $\mathbf{j}\colon P\Lambda \to P\Lambda$ *such that* $((p \Rightarrow q) \Rightarrow (\mathbf{j}(p) \Rightarrow \mathbf{j}(q))$, $(p \Rightarrow \mathbf{j}(p))$ *and* $(\mathbf{j}(\mathbf{j}(p)) \Rightarrow \mathbf{j}(p))$ *are uniformly realizable, two such functions* $\mathbf{j}$ *and* $\mathbf{k}$ *being considered equivalent if* $(\mathbf{j}(p) \Leftrightarrow \mathbf{k}(p))$ *is uniformly realizable. Moreover, given two such functions* $\mathbf{j}$ *and* $\mathbf{k}$*, the associated local operators* j *and* k *satisfy* $j \leq k$ *iff* $(\mathbf{j}(p) \Rightarrow \mathbf{k}(p))$ *is uniformly realizable.*

We omit the proof, which is straightforward verification. However, we mention that, at least for subobjects of objects of the form ∇A, the closure operation on subobjects corresponding to j has a simple description in terms of $\mathbf{j}$: the j-closure of $(A, \alpha) \rightarrowtail \nabla A$ is simply $(A, \mathbf{j}\alpha) \rightarrowtail \nabla A$ (here we are regarding α as a function $A \to P\Lambda$ rather than a relation $A \looparrowright \Lambda$).

1.2. PROPOSITION. *Let* Λ *be a Schönfinkel algebra, let* j *be a local operator on* $\mathbf{Eff}(\Lambda)$*, and let* $\mathbf{j}\colon P\Lambda \to P\Lambda$ *be a function corresponding to it as in* 1.1. *Then the following conditions are equivalent:*

(i) $\mathbf{sh}_j(\mathbf{Eff}(\Lambda))$ *is Boolean.*

(ii) $\neg\neg \leq j$ *in* $\mathbf{Lop}(\mathbf{Eff}(\Lambda))$.

(iii) $\bigcap\{\mathbf{j}(\{\lambda\}) \mid \lambda \in \Lambda\}$ *is inhabited.*

(iv) *The associated* j*-sheaf of* $\overline{\Lambda}$ *is corrupt.*

PROOF. (i)$\Leftrightarrow$(ii): Since $\mathbf{Eff}(\Lambda)$ is two-valued, it has only two closed subtoposes, namely itself and the degenerate topos; so, by the characterization of Boolean subtoposes in [2], A4.5.21, its only Boolean subtoposes are $\mathbf{sh}_{\neg\neg}(\mathbf{Eff}(\Lambda)) \simeq \mathbf{Set}$ and the degenerate topos. And these are precisely the subtoposes for which $\neg\neg \leq j$.

(ii)$\Leftrightarrow$(iii): It is easily seen that $\neg\neg$ corresponds to the mapping $\mathbf{n}\colon P\Lambda \to P\Lambda$ sending $\emptyset$ to itself and every inhabited subset to Λ (constructively, $\mathbf{n}(p) = \{\lambda \in \Lambda \mid p \text{ is inhabited}\}$). So by 1.1 we have $\neg\neg \leq j$ iff $(\mathbf{n}(p) \Rightarrow \mathbf{j}(p))$ is uniformly realizable, iff $\bigcap\{\mathbf{j}(p) \mid p \text{ is inhabited}\}$ is inhabited. This clearly implies (iii); but conversely if μ uniformly realizes $((p \Rightarrow q) \Rightarrow (\mathbf{j}(p) \Rightarrow \mathbf{j}(q)))$, and $\nu \in \mathbf{j}(\{\lambda\})$ for all λ, then $\mu \mathsf{I} \nu \in \mathbf{j}(q)$ for all inhabited $q \subseteq \Lambda$, where I denotes the identity combinator.

(iii)$\Leftrightarrow$(iv): Unless $\mathbf{sh}_j(\mathbf{Eff}(\Lambda))$ is degenerate, $\nabla\Lambda$ is a j-sheaf; so the associated j-sheaf of $\overline{\Lambda}$ is its j-closure as a subobject of $\nabla\Lambda$. But the latter is simply the assembly $(\Lambda, \mathbf{j}\{\})$; so condition (iii) is precisely the assertion that this assembly is corrupt. ∎

Clearly, if j satisfies the conditions of 1.2, then $\mathbf{sh}_j(\mathbf{Eff}(\Lambda))$ is a Grothendieck topos (it is either $\mathbf{Set}$ or the degenerate topos). We now show that the converse holds.

1.3. PROPOSITION. *For a subtopos* $\mathcal{E}$ *of* $\mathbf{Eff}(\Lambda)$*, the following are equivalent:*

(i) *The local operator* j *corresponding to* $\mathcal{E}$ *satisfies the conditions of* 1.2.

(ii) $\mathcal{E}$ *is a Grothendieck topos.*

(iii) $\mathcal{E}$ *is complete.*

(iv) $\mathcal{E}$ *is cocomplete.*

(v) $\mathcal{E}$ *admits a geometric morphism to* **Set**.

PROOF. Recall that a Grothendieck topos is complete and cocomplete. A topos admitting a geometric morphism to **Set** need not be (co)complete ([2], A2.1.7), but it does have arbitrary set-indexed copowers ([2], A4.1.9), and conversely any locally small topos with set-indexed copowers admits such a morphism. (Of course, any subtopos of $\mathbf{Eff}(\Lambda)$ is locally small.) In fact, to show that properties (ii–v) imply (i), all we shall require is the existence of set-indexed copowers in $\mathcal{E}$.

Suppose condition (i) fails. Then the associated j-sheaf $L\overline{\Lambda}$ of $\overline{\Lambda}$ is not corrupt, by 1.2(iv). Suppose the coproduct C of Λ copies of 1 exists in $\mathcal{E}$; then the exponential $(L\overline{\Lambda})^C$ is a Λ-fold product of copies of $L\overline{\Lambda}$. It is an assembly, since $\mathbf{Ass}(\Lambda)$ is an exponential ideal in $\mathbf{Eff}(\Lambda)$ by [2], A4.4.3(ii), and we can identify its underlying set with Λ^Λ since the underlying-set functor $\mathbf{Ass}(\Lambda) \to \mathbf{Set}$ is representable. For each $\lambda \in \Lambda$, the λth product projection $\Lambda^\Lambda \to \Lambda$ is trackable as a morphism $(L\overline{\Lambda})^C \to L\overline{\Lambda}$; let μ_λ be an element of Λ which tracks it. Since $L\overline{\Lambda}$ is not corrupt, we may choose $f(\lambda) \in \Lambda$ which is not in its $(\mu_\lambda\lambda)$th caucus. (If $\mu_\lambda\lambda$ is undefined, we take $f(\lambda)$ to be any element of Λ.) Now the function $f\colon \Lambda \to \Lambda$ cannot belong to any caucus of $(L\overline{\Lambda})^C$, which yields the desired contradiction. ■

We remark that, although we used the axiom of choice to define the function f in the above proof, it is not needed when the underlying set of Λ can be well-ordered (for example, when Λ is countable).

1.4. COROLLARY. *Let $\mathcal{E}$ be a topos admitting a geometric morphism to* **Set**, *and let Λ be a Schönfinkel algebra. Then, up to isomorphism, the only geometric morphism $f\colon \mathcal{E} \to \mathbf{Eff}(\Lambda)$ is the composite $\mathcal{E} \to \mathbf{Set} \to \mathbf{Eff}(\Lambda)$, where the second factor is the inclusion of $\neg\neg$-sheaves.*

PROOF. Consider the image (in the surjection–inclusion sense) of f: it is comonadic over $\mathcal{E}$, and so inherits set-indexed copowers from the latter. Hence by 1.3 it must be either **Set** or the degenerate topos; but the latter case can only occur if $\mathcal{E}$ itself is degenerate. And the morphism $\mathcal{E} \to \mathbf{Set}$ is unique up to isomorphism, by [2], A4.1.9. ■

2. All Morphisms Satisfy Longley's Condition

In this section, we assume given two Schönfinkel algebras Λ and M and a geometric morphism $f\colon \mathbf{Eff}(\mathrm{M}) \to \mathbf{Eff}(\Lambda)$. We begin with a simple consequence of the last result of the previous section:

2.1. LEMMA. *There is a (bi-)pullback square of toposes and geometric morphisms*

$$\begin{tikzcd} \mathbf{Set} \arrow[r, "1"] \arrow[d] & \mathbf{Set} \arrow[d] \\ \mathbf{Eff}(\mathrm{M}) \arrow[r, "f"] & \mathbf{Eff}(\Lambda) \end{tikzcd}$$

where the vertical arrows are inclusions of $\neg\neg$*-sheaves.*

PROOF. The composite $\mathbf{Set} \to \mathbf{Eff}(\mathrm{M}) \to \mathbf{Eff}(\Lambda)$ must be the inclusion of $\mathbf{sh}_{\neg\neg}(\mathbf{Eff}(\Lambda))$, by 1.4; so the square commutes. But also, if we have a commutative square

$$\begin{tikzcd} \mathcal{E} \arrow[r] \arrow[d] & \mathbf{Set} \arrow[d] \\ \mathbf{Eff}(\mathrm{M}) \arrow[r, "f"] & \mathbf{Eff}(\Lambda) \end{tikzcd}$$

then 1.4 ensures that the left vertical edge of this square factors uniquely through that of the square in the statement. ∎

The commutativity of the square in the statement of 2.1 ensures that we have natural isomorphisms $f_*\nabla' \cong \nabla$ and $\Gamma' f^* \cong \Gamma$, where we are writing ∇' and Γ' for the 'constant objects functor' of $\mathbf{Eff}(\mathrm{M})$ and its left adjoint. Also, since Γ and Γ' are representable and f^* preserves the terminal object, it is immediate from the adjunction $(f^* \dashv f_*)$ that we have $\Gamma f_* \cong \Gamma'$. But it is not obvious whether we necessarily have $f^*\nabla \cong \nabla'$.

In this connection, the following equivalences were known to Longley [4], and can also be found in [5].

2.2. LEMMA. *For a geometric morphism* $f\colon \mathbf{Eff}(\mathrm{M}) \to \mathbf{Eff}(\Lambda)$*, the following are equivalent:*

(i) *The Beck–Chevalley condition* $f^*\nabla \cong \nabla'$ *holds.*

(ii) *The weak Beck–Chevalley condition holds; i.e. the canonical natural transformation* $f^*\nabla \to \nabla'$ *is monic.*

(iii) f^* *restricts to a functor* $\mathbf{Ass}(\Lambda) \to \mathbf{Ass}(\mathrm{M})$.

PROOF. (i) $\Rightarrow$ (ii) is trivial, and (ii) $\Rightarrow$ (iii) is easy since the assemblies are, up to isomorphism, exactly the subobjects in $\mathbf{Eff}(\Lambda)$ of objects of the form ∇A (and f^* preserves monomorphisms). For (iii) $\Rightarrow$ (i), by the remarks above, we know that the restriction of f^* to assemblies commutes up to isomorphism with the underlying-set functors. (In fact we may as well assume that it commutes 'on the nose' with these functors, since we can 'transport the structure' of each $f^*(A,\alpha)$ across the isomorphism from A to the underlying set of this M-valued assembly to obtain a functor, naturally isomorphic to f^*, which preserves underlying sets strictly.) Given a nonempty set A, let B be a set whose cardinality is greater than card $(A \times \mathrm{M})$; then at least one caucus of the assembly $f^*\nabla B$ must have cardinality greater than card A. We can find a surjection $h\colon B \twoheadrightarrow A$ mapping this subset of B onto the whole of A. Now h must be trackable as a morphism of assemblies $f^*\nabla B \to f^*\nabla A$, from which it follows that $f^*\nabla A$ is corrupt, and hence isomorphic to $\nabla' A$. ∎

Longley showed that any geometric morphism satisfying these conditions is completely determined by a suitable entire relation $\theta\colon \Lambda \nrightarrow \mathrm{M}$, which he called an 'applicative morphism' of Schönfinkel algebras: specifically, we obtain θ by setting $f^*(\overline{\Lambda}) = (\Lambda, \theta)$. The relations θ which give rise to geometric morphisms in this way were characterized by Hofstra and van Oosten [1] as the applicative morphisms satisfying a further condition which they called 'computational density' (see section 3 below). But they left open the question whether any geometric morphism between realizability toposes might fail to satisfy the conditions of 2.2.

To answer this, let us recall the notion of localic morphism: a geometric morphism $f\colon \mathcal{F} \to \mathcal{E}$ is said to be *localic* if every object of $\mathcal{F}$ is a subquotient (i.e. a subobject of a quotient, or equivalently a quotient of a subobject) of one of the form f^*A, for some $A \in \mathrm{ob}\,\mathcal{E}$. We may now add a fourth equivalent condition to those of 2.2:

2.3. Lemma. *A geometric morphism satisfies the conditions of* 2.2 *iff it is localic.*

Proof. One direction is easy, since every object of $\mathbf{Eff}(\mathrm{M})$ is (a quotient of an assembly, and hence) a subquotient of one of the form $\nabla' A$. Conversely, suppose f is localic. Then we can express any $\nabla' A$ as a subquotient of some object in the image of f^* — and in fact, since f^* preserves monomorphisms and epimorphisms, as a subquotient of some $f^*\nabla B$. But $\nabla' A$ is projective in $\mathbf{Eff}(\mathrm{M})$, by [5], 3.2.7, so we may actually find a monomorphism $\nabla' A \rightarrowtail f^*\nabla B$. Applying the functor Γ', we obtain an injection $A \rightarrowtail B$, and hence a monomorphism $f^*\nabla A \rightarrowtail f^*\nabla B$. Now a straightforward diagram-chase shows that the triangle

$$\begin{tikzcd}
f^*\nabla A \arrow[r] \arrow[d, tail] & \nabla' A \arrow[dl] \\
f^*\nabla B &
\end{tikzcd}$$

commutes, where the top edge is the composite of the isomorphism $f^*\nabla A \cong f^*f_*\nabla' A$ with the counit of $(f^* \dashv f_*)$; so in particular this composite is monic. ∎

To complete the proof, we need:

2.4. Lemma. *Any geometric morphism between realizability toposes is localic.*

PROOF. Given a morphism $f\colon \mathbf{Eff}(\mathrm{M}) \to \mathbf{Eff}(\Lambda)$, we may form its hyperconnected–localic factorization $\mathbf{Eff}(\mathrm{M}) \to \mathcal{E} \to \mathbf{Eff}(\Lambda)$, as defined in [2], A4.6.5. We cannot assume that $\mathcal{E}$ is a realizability topos; but we know that it is a coreflective subcategory of $\mathbf{Eff}(\mathrm{M})$, and that the counit of the coreflection is monic. So if we write ∇'' for the composite direct image functor $\mathbf{Set} \to \mathbf{Eff}(\mathrm{M}) \to \mathcal{E}$, then $\nabla''A$ is a subobject of $\nabla'A$ and hence an assembly; and it contains all the points of $\nabla'A$, since 1 is in $\mathcal{E}$, so we may take its underlying set to be A. We may now argue just as in the proof of (iii) ⇒ (i) in 2.2 to show that $\nabla''A$ is corrupt, and hence isomorphic to $\nabla'A$. But $\mathcal{E}$ is closed under subobjects and quotients in $\mathbf{Eff}(\mathrm{M})$, so it must be the whole of the latter; i.e. f is localic. ∎

3. Computational Density

We recall that Longley defined an *applicative morphism* $\theta\colon \Lambda \nrightarrow \mathrm{M}$ of Schönfinkel algebras to be an entire relation which has a *witness* $\tau \in \mathrm{M}$ such that, if $\theta(\lambda, \mu)$ and $\theta(\lambda', \mu')$ hold and $\lambda\lambda'$ is defined in Λ, then $\tau\mu\mu'$ is defined in M and $\theta(\lambda\lambda', \tau\mu\mu')$ holds. He also defined the notion of inequality between applicative morphisms: if $\phi\colon \Lambda \nrightarrow \mathrm{M}$ is another such morphism, we say $\theta \leq \phi$ if there exists $\rho \in \mathrm{M}$ (a *witness* for the inequality) such that $\rho\mu$ is defined and $\phi(\lambda, \rho\mu)$ holds whenever $\theta(\lambda, \mu)$ holds. Schönfinkel algebras, applicative morphisms and inequalities form a locally preordered 2-category $\mathsf{Schön}$.

The Hofstra–van Oosten definition of computational density was originally formulated for morphisms of ordered Schönfinkel algebras; but, when interpreted for algebras in which the order is discrete, it reduces to the following.

3.1. DEFINITION. *An applicative morphism $\theta\colon \Lambda \nrightarrow \mathrm{M}$ is* computationally dense *if there exists $\sigma \in \mathrm{M}$ and a function $s\colon \mathrm{M} \to \Lambda$ such that, for all $\mu \in \mathrm{M}$ and all $\lambda \in \Lambda$, if $\mu\lambda'$ is defined for all λ' such that $\theta(\lambda, \lambda')$, then $s(\mu)\lambda$ is defined and, for all ν such that $\theta(s(\mu)\lambda, \nu)$, we have $\sigma\nu = \mu\lambda'$ for some λ' satisfying $\theta(\lambda, \lambda')$.*

(Actually, this is a slight simplification of the original definition, which would replace the function s by an entire relation $\mathrm{M} \nrightarrow \Lambda$. Provided we assume the axiom of choice in **Set**, any such relation will of course contain (the graph of) a function for which the condition as stated above will hold; readers who prefer not to assume choice should note that the proof of the following lemma continues to work if both s and the function r are replaced by entire relations.)

In the following proof, we shall write D, P_1 and P_2 for the pairing and unpairing combinators of any Schönfinkel algebra, and B, E and K for the combinator versions of the λ-terms $\lambda xyz.x(yz)$, $\lambda xy.yx$ and $\lambda xy.x$ respectively.

3.2. LEMMA. *For an applicative morphism $\theta\colon \Lambda \nrightarrow \mathrm{M}$, the following are equivalent:*

(i) *θ is computationally dense.*

(ii) *There exists a function $r\colon \mathrm{M} \to \Lambda$ and an element $\rho \in \mathrm{M}$ such that for all $\mu, \mu' \in \mathrm{M}$, if $\theta(r(\mu), \mu')$ holds, then $\rho\mu' = \mu$.*

PROOF. If (i) holds, we define $r(\mu) = s(\mathsf{K}\mu)$ and set $\rho = \mathsf{B}\sigma(\mathsf{B}(\mathsf{E}\nu)\tau)$ for some ν in the image of θ. Then $r(\mu)\lambda$ is defined for all $\lambda \in \Lambda$, and if we choose λ such that $\theta(\lambda, \nu)$, then $\theta(r(\mu), \mu')$ implies $\theta(r(\mu)\lambda, \tau\mu'\nu)$, so that

$$\rho\mu' = \sigma(\tau\mu'\nu) = \mathsf{K}\mu\lambda' = \mu$$

for some λ' such that $\theta(\lambda, \lambda')$.

Conversely, if (ii) holds, we define $s(\mu) = \mathsf{D}r(\mu)$, and set $\sigma = \mathsf{S}(\mathsf{B}\rho(\tau\pi_1))(\tau\pi_2)$, where τ is a witness for θ, and π_1 and π_2 are θ-relatives of the unpairing combinators of Λ. Clearly $s(\mu)\lambda$ is defined for all $\lambda \in \Lambda$; and if $\theta(s(\mu)\lambda, \nu)$ holds then $\tau\pi_1\nu$ and $\tau\pi_2\nu$ are θ-relatives of $r(\mu)$ and λ respectively. So $\rho(\tau\pi_1\nu) = \mu$, and hence $\sigma\nu = \mu(\tau\pi_2\nu)$. ∎

Since condition (ii) does not *prima facie* have much to do with computation, it seems more appropriate to call an applicative morphism *quasi-surjective* if it satisfies this condition. We note that it holds whenever there is a morphism $\phi\colon \mathrm{M} \nrightarrow \Lambda$ such that $\theta\phi \leq 1_{\mathrm{M}}$: we take r to be any function whose graph is contained in ϕ, and ρ to be a witness for the inequality. In particular, this holds when θ has a right adjoint; this corresponds to the case when the geometric morphism induced by θ is *exact* in Longley's sense, i.e. its direct image functor is regular. But it also holds when θ has a *left* adjoint for which the unit of the adjunction is an isomorphism; thus we obtain

3.3. COROLLARY. *Any coreflection in* $\mathsf{Sch\ddot{o}n}$ *gives rise to a local geometric morphism between the corresponding realizability toposes (that is, an adjoint pair of geometric morphisms of which the left adjoint is an inclusion).* ∎

Before proving Lemma 3.2, the author had observed that Corollary 3.3 may also be deduced from Corollary 3.3 of [3]. It implies that, for example, the applicative morphisms γ and ι described on pages 28–9 of [5] induce a local geometric morphism $\mathbf{Eff}(\mathcal{P}\omega) \to \mathbf{Eff}(\mathcal{K}_2)$.

Putting together the results of this paper with those of Longley and of Hofstra and van Oosten, we may now conclude:

3.4. THEOREM. *The assignment* $\Lambda \mapsto \mathbf{Eff}(\Lambda)$ *defines a full embedding of* 2*-categories* $\mathsf{Sch\ddot{o}n}_{\mathrm{qs}}^{\mathrm{op}} \to \mathsf{Top}$, *where* $\mathsf{Sch\ddot{o}n}_{\mathrm{qs}}$ *is the* 2*-category of Schönfinkel algebras, quasi-surjective applicative morphisms and inequalities between them.* ∎

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