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# A BICATEGORICAL APPROACH TO STATIC MODULES

*Dedicated to J. Lambek on the occasion of his 75th birthday*

RENATO BETTI

ABSTRACT. The purpose of this paper is to indicate some bicategorical properties of ring theory. In this interaction, static modules are analyzed.

## Introduction

Categorical generalizations of ring theory usually start from the observation that rings are (one-object) categories enriched in the monoidal closed category **Ab** of abelian groups. In this setting, $R$-$S$-bimodules are profunctors $R \longrightarrow\!\!\!\!\!\!\!+\;\; S$, i.e. functors $R^{op} \times S \longrightarrow \mathbf{Ab}$, and their composition $R \overset{P}{\longrightarrow\!\!\!\!\!\!\!+}\;\; S \overset{Q}{\longrightarrow\!\!\!\!\!\!\!+}\;\; T$ is provided by the tensor product $Q \otimes_S P$ over the ring $S$. So, one has a distributive bicategory **Mod**, whose objects are the (non-commutative) rings, whose arrows are the profunctors and whose 2-cells are the module morphisms.

To say that **Mod** is a distributive bicategory is to say that it admits local colimits which distribute over composition on both sides. Moreover, **Mod** is biclosed (see e.g. [6]), in the sense that it admits right Kan extensions and right liftings: in other words, compositions with an arrow $P : R \longrightarrow\!\!\!\!\!\!\!+\;\; S$ are functors:

$$
{}_S\mathbf{Mod} \xrightarrow{- \otimes_S P} {}_R\mathbf{Mod}
$$

$$
\mathbf{Mod}_R \xrightarrow{P \otimes_R -} \mathbf{Mod}_S
$$

having right adjoints (here ${}_S\mathbf{Mod}$ denotes the category of left $S$-modules and similarly $\mathbf{Mod}_S$ is the category of right $S$-modules).

In particular the right Kan extension of the left $E$-module $N$ along the $E$-$R$-bimodule $P$ is the $R$-module $hom^E(P, N)$ of $E$-module morphisms $P \longrightarrow N$. Analogously, the right lifting of the right $R$-module $M$ along the $E$-$R$-bimodule $P$ is the right $E$-module $hom_R(P, M)$ of $R$-module morphisms $P \longrightarrow M$.

The main observation which relates **Mod** with a relevant property of modules was first formulated by Lawvere [6]: finitely generated projective modules are exactly those profunctors which, when regarded as arrows in **Mod**, have a right adjoint. To prove this

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fact one should consider the dual module $P^* = hom_R(P, R)$ of $P$. It is easy to see that the adjointness $P \dashv P^*$ has the evaluation $x \otimes f \longmapsto f(x)$ as counit $P \otimes_{\mathbf{Z}} P^* \longrightarrow R$ and the assignment of a so-called dual basis $\mathbf{Z} \longrightarrow P^* \otimes_R P$ as unit.

In this paper we want to concentrate on the fact that categorical notions often are directly connected to the notions of ring theory, in general allowing one to formulate problems and prove results in a more conceptual way. Here we consider properties relative to static and e-static modules (with respect to a given module $P$) i.e. relative to those modules $V$ for which the evaluation $P \otimes_E hom_R(P, V) \longrightarrow V$ is an isomorphism (or an epimorphism). The calculus of adjoints in a bicategory and its general properties are put in use to study properties of static and e-static modules including aspects of Morita theory and an extension of the Dade-Cline version of Clifford theory to one in general ring theory (see e.g. Cline [3], Dade [4], Alperin [1], Nauman [8]).

## 1. Static and e-static modules

Notations and the formulation of the problem are from Lambek [5] which regards static modules with respect to a projective module $P$.

Given a bimodule $P : E \longrightarrow\!\!\!\!\!\!\!+\;\; R$, consider the adjoint pair of functors:

$$\mathbf{Mod}_R \underset{P \otimes_E -}{\overset{hom_R(P,-)}{\rightleftarrows}} \mathbf{Mod}_E \tag{1}$$

$(P \otimes_E - \dashv hom_R(P, -))$.

By general properties of adjoint pairs, these restrict to an equivalence between the full subcategory $(Fix\ \epsilon)$ of $\mathbf{Mod}_R$ consisting of the modules $U$ whose counit

$$\epsilon_U : U \otimes_E hom_R(P, U) \longrightarrow U$$

is an isomorphism and the full subcategory $(Fix\ \eta)$ of $\mathbf{Mod}_E$ consisting of those modules $V$ whose unit $\eta_V : V \longrightarrow hom_R(P, P \otimes_E V)$ is an isomorphism.

In case $P$ is a right $R$-module and $E = hom_R(P, P)$ is its ring of endomorphisms, the modules in $(Fix\ \epsilon)$ are said to be *static* with respect to $P$. If $\epsilon_U$ is an epimorphism, $U$ is said to be *e-static* and $(Epi\ \epsilon)$ denotes the full subcategory of e-static modules. In [7], McMaster proves that e-static modules relative to a projective $P$ are exactly those modules which are cotorsionfree with respect to $P$.

Observe that the restriction of the adjunction (1) to the subcategory $(Epi\ \epsilon)$ gives an adjunction:

$$(Epi\ \epsilon) \rightleftarrows (Mono\ \eta)$$

where $(Mono\ \eta)$ is the full subcategory of $\mathbf{Mod}_E$ constituted by the modules $U$ with a monomorphic unit $\eta_U : U \longrightarrow hom_R(P, P \otimes_E U)$.

When $P$ is finitely generated and projective then $hom_R(P, -) \cong P^* \otimes_R -$, hence $V$ is a static module if and only if

$$P \otimes_E P^* \otimes_R V \cong V$$

i.e. $V$ equifies the trace ideal $\tau : P \otimes_E P^* \to R$.

1.1. THEOREM. *If $P$ is finitely generated and projective, then it is the equifier of its trace ideal.*

Proof. The module $P$ equifies the trace ideal $\tau : P \otimes_E P^* \longrightarrow R$ because $P^* \otimes_R P \cong 1_E$. Moreover, any static module $U$ factors uniquely up to isomorphism in the form $U \cong P \otimes_E hom_R(P, U)$ because $hom_R(P, R) \otimes_R U \cong hom_R(P, U)$. ∎

If moreover $P$ is a generator of $\mathbf{Mod}_R$, then the trace ideal $\tau$ is an isomorphism and any module in $\mathbf{Mod}_R$ is static. This is one of the main results of Morita theory, namely the assertion that in this case $\mathbf{Mod}_R$ and $\mathbf{Mod}_E$ are equivalent categories (i.e. $R$ and $E$ are Morita equivalent rings).

In general, for a bimodule $P : E \longrightarrow\!\!\!\!\!\!\mid\;\; R$ with $E = hom_R(P, P)$, one has:

1.2. THEOREM. *Any finitely generated projective $E$-module $V$ is in $(Fix\ \eta)$.*

Proof. It is enough to remind the reader that, in any bicategory with right liftings, the functor $hom_R(P, -)$ preserves composition with right adjoints.

In the case of $\mathbf{Mod}$, if $V$ has a right adjoint, for any $W$ in $\mathbf{Mod}_E$ one has the following sequence of bijections, natural in $W$:

$$\frac{\dfrac{W \longrightarrow hom_R(P, P) \otimes_E V}{P \otimes_E W \longrightarrow P \otimes_E V}}{W \longrightarrow hom_R(P, P \otimes_E V)}$$

∎

From the previous theorem, one has that the subcategory of finitely generated $E$-modules is contained in $(Fix\ \eta)$. Nauman [8] particularly considers this subcategory and, through his result ([8], theorem 3.7) one concludes:

1.3. COROLLARY. *A right $R$-module $U$ weakly divides $P$ in $\mathbf{Mod}_R$ (i.e. it is a direct summand of finitely many copies of $P$) if and only if it is of the form $P \otimes_E V$ for a finitely generated and projective $E$-module $V$.*

Let $(\ )'$ denote the functor which associates to any $R$-module $V$ the image of the evaluation:

$$\epsilon_V : P \otimes_E hom_R(P, V) \xrightarrow{e_V} V' \xrightarrow{m_V} V \tag{2}$$

(here $e_V$ is epi and $m_V$ is mono).

1.4. Theorem. *The functor ( )′ lands in (Epi $\epsilon$) and it is the right adjoint to the embedding (Epi $\epsilon$)* $\hookrightarrow$ $\mathbf{Mod}_R$.

Proof. First one proves that $V'$ is in (*Epi* $\epsilon$). Consider the diagram:

$$\begin{tikzcd} P \otimes_E hom_R(P, V') \arrow[r, "\epsilon_{V'}"] \arrow[d, "m'"'] & V' \arrow[d, "m_V"] \\ P \otimes_E hom_R(P, V) \arrow[r, "\epsilon_V"] & V \end{tikzcd}$$

Here, $m'$ denotes $P \otimes_E hom_R(P, m_V)$. Now, $hom_R(P, m_V)$ is mono because $hom_R(P, -)$ is a left exact functor. Moreover, under the bijection of $P \otimes_E - \dashv hom_R(P, -)$, the evaluation (2) corresponds to the composite:

$$hom_R(P, V) \xrightarrow{\widetilde{e_V}} hom_R(P, V') \xrightarrow{hom_R(P, m_V)} hom_R(P, V)$$

which is the identity of $hom_R(P, V)$. Hence $hom_R(P, m_V)$ is also epi. As a consequence $m'$ is an isomorphism and from

$$e_V \cdot (P \otimes_E hom_R(P, m_V)) = \epsilon_{V'}$$

one has that also $\epsilon_{V'}$ is epi.

For the adjointness, it is easy to prove that, if $U$ is in (*Epi* $\epsilon$), the universal property of images provides a natural bijection:

$$\frac{U \longrightarrow V'}{U \longrightarrow V}$$

given by composition with $m_V$. ∎

A known characterization of static modules is provided by Auslander equivalence [2] (see also Alperin [1]): a module $V$ is static with respect to $P$ if and only if there is a presentation (i.e. an exact sequence):

$$\Sigma_Y P \longrightarrow \Sigma_X P \longrightarrow V \longrightarrow 0$$

such that the application of the functor $hom_R(P, -)$ gives another exact sequence (here $X$ and $Y$ are sets and $\Sigma$ denotes coproduct). In case $P$ is projective, the presentation is enough, as proved directly by Lambek [5].

By substituting the presentation of $V$ with a condition on its generation by $P$, e-static modules can be characterized. Say that the $R$-module $V$ is *generated* by $P$ if there is an epimorphism $\Sigma_X P \longrightarrow V$. Moreover, say that the *generators are preserved* if the application of the functor $hom_R(P, -)$ gives another epimorphism $hom_R(P, \Sigma_X P) \longrightarrow hom_R(P, V)$.

1.5. THEOREM. *The $R$-module $V$ is in $(Epi\ \epsilon)$ if and only if it is generated by $P$ and the generators are preserved by $hom_R(P,-)$.*

Proof. First, observe that $(Epi\ \epsilon)$ is a coreflective subcategory of $\mathbf{Mod}_R$ and thus it is closed under colimits. Moreover the module $P$ is static because $E = hom_R(P,P)$.

Now suppose that $V$ is generated by $P$ and that generators are preserved by $hom_R(P,-)$. Tensoring the epimorphism $hom_R(P,\Sigma_X P) \longrightarrow hom_R(P,V)$ with $P$ gives another epimorphism, as tensoring is a right exact functor. So, one has the commutative diagram:

$$\begin{tikzcd} P\otimes_E hom_R(P,\Sigma_X P) \arrow[r] \arrow[d, "\epsilon_{\Sigma_X P}"'] & P\otimes_E hom_R(P,V) \arrow[d, "\epsilon_V"] \\ \Sigma_X P \arrow[r] & V \end{tikzcd}$$

where the horizontal arrows are epimorphism and also $\epsilon_{\Sigma_X P}$ is an isomorphism because $(Epi\ \epsilon)$ is closed under coproducts. Hence $\epsilon_V$ is an epimorphism.

Conversely, suppose that $\epsilon_V : P\otimes_E hom_R(P,V) \longrightarrow V$ is epi. Now, $hom_R(P,V)$ as a right $E$-module is a quotient of a free $E$-module, i.e. it is generated in the form $\Sigma_X E \to hom_R(P,V)$. By tensoring and composing with $\epsilon_V$, one has that $V$ is generated by $P$:

$$\Sigma_X P \cong P\otimes_E \Sigma_X E \longrightarrow P\otimes_E hom_R(P,V) \xrightarrow{\epsilon_V} V$$

Applying $hom_R(P,-)$ we get the commutative diagram:

$$\begin{tikzcd} \Sigma_X E \arrow[r, "\alpha"] \arrow[d, "\eta_{\Sigma_X E}"'] & hom_R(P,V) \arrow[d, "\eta_{hom_R(P,V)}"] \\ hom_R(P,P\otimes_E \Sigma_X E) \arrow[r, "\beta"] & hom_R(P,P\otimes_E hom_R(P,V)) \end{tikzcd}$$

where the arrow $\alpha$ is epi. Taking into account the triangular identity:

$$id = hom_R(P,\epsilon_V)\cdot \eta_{hom_R(P,V)}$$

one has that:

$$\alpha = hom_R(P,\epsilon_V)\cdot \beta \cdot \eta_{\Sigma_X E}$$

hence $hom_R(P,\epsilon_V)$ is epi and the generators of $V$ by $P$ are preserved. ∎

## 2. Application to Clifford theory

We shall now prove a general result about modules obtained by induction and restriction along a ring homomorphism.

In [3], Cline extended the classical Clifford theory of projective representations and stable modules by utilising rings and modules which are graded by a group. His main result can be stated as an equivalence of categories and was reobtained by Dade ([4], theorem 7.4) by using only the functors $Hom$ and $\otimes$ in a natural way. Restricting it to the essential case of group algebras, the equivalence regards the homomorphism $kH \longrightarrow kG$ induced by a normal subgroup $H$ of $G$ (here $G$ is a finite group, $k$ is a field and $kG$ is the group algebra). In this case, Dade's result [4] characterizes those $kG$-modules which are static when restricted to $kH$. Furthemore, Alperin [1] gives a generalization to non-normal subgroups $H$.

In general, the equivalence is relative to a given ring homomorphism $f : R \longrightarrow S$. This gives rise to a bimodule $f_* : R \longrightarrow\!\!\!\!\!\!\mapsto S$ and to a bimodule $f^* : S \longrightarrow\!\!\!\!\!\!\mapsto R$. These bimodules are given by $S$ itself, and it is easy to prove that they are adjoint arrows in **Mod**. Precisely: $f_* \dashv f^*$.

*Induction* and *restriction* functors along $f$ are given by the compositions with $f_*$ and $f^*$ respectively. These are adjoint functors $(f_* \otimes_R - \dashv f^* \otimes_S -)$:

$$\mathbf{Mod}_R \underset{f^* \otimes_S -}{\overset{f_* \otimes_R -}{\rightleftarrows}} \mathbf{Mod}_S$$

If $P : E \longrightarrow\!\!\!\!\!\!\mapsto R$ is such that $E = hom_R(P, P)$, consider the module $P' = f_* \otimes_R P$ induced by $P$ through $f$ as a module $F \longrightarrow\!\!\!\!\!\!\mapsto S$, where $F = hom_S(P', P')$. One has another ring homomorphism $i : E \longrightarrow F$, easily described in **Mod** by the canonical morphism of right liftings

$$hom_R(P, P) \longrightarrow hom_S(P', P')$$

The generalization of Cline's [3] and Dade's [4] works on Clifford theory referred to above regards those $S$-modules $U$ whose restriction $f^* \otimes_S U$ is static with respect to $P$. In [8], Nauman expresses it in a purely ring theoretical way:

2.1. THEOREM. [Nauman [8], theorem 5.5] *The restrictions of the additive functors $hom_S(P', -)$ and $P' \otimes_F -$ form an equivalence between the full additive subcategory of $\mathbf{Mod}_S$ having as objects all $S$-modules $U$ such that $f^* \otimes_S U$ weakly divides $P$ in $\mathbf{Mod}_R$ and the full additive subcategory of $\mathbf{Mod}_F$ whose objects are the $F$-modules $V$ such that $i^* \otimes V$ is finitely generated and projective.*

By the universal property of adjoint pairs one proves that:

$$P' \otimes_F i_* \cong f_* \otimes_R P \tag{3}$$

Adjointness is essential also in proving the following:

2.2. Theorem. *The restriction of the right $S$-module $U$ along $f$ is in $(Fix\ \epsilon)$ if and only if the restriction of $hom_S(P', U)$ along $i$ is in $(Fix\ \eta)$. In symbols:*

$$f^* \otimes_S U \in (Fix\ \epsilon) \Longleftrightarrow i^* \otimes_F hom_S(P', U) \in (Fix\ \eta)$$

Proof. For any right $S$-module $U$:

$$f^* \otimes_S U \in (Fix\ \epsilon) \Longleftrightarrow hom_R(P, f^* \otimes_S U) \in (Fix\ \eta)$$

Consider the following diagram. By (3) the square of left adjoints commutes up to isomorphisms, and thus so does the square of right adjoints:

$$\begin{tikzcd}
\mathbf{Mod}_R \arrow[r, shift left, "hom_R(P,-)"] \arrow[d, shift right, "f_* \otimes_R -"'] & \mathbf{Mod}_E \arrow[l, shift left, "P \otimes_E -"] \arrow[d, shift right, "i_* \otimes_E -"'] \\
\mathbf{Mod}_S \arrow[r, shift left, "hom_S(P',-)"] \arrow[u, shift right, "f^* \otimes_S -"'] & \mathbf{Mod}_F \arrow[l, shift left, "P' \otimes_F -"] \arrow[u, shift right, "i^* \otimes_F -"']
\end{tikzcd}$$

Hence:

$$hom_R(P, f^* \otimes_S U) \cong i^* \otimes_F hom_S(P', U)$$

■

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*Dipartimento di Matematica*
*Politecnico di Milano*
*Piazza L. da Vinci, 32*
*20133 Milano (Italy)*
Email: `renbet@mate.polimi.it`

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