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ENRICHED LAWVERE THEORIES

Dedicated to Jim Lambek

JOHN POWER

ABSTRACT. We define the notion of enriched Lawvere theory, for enrichment over a monoidal biclosed category V that is locally finitely presentable as a closed category. We prove that the category of enriched Lawvere theories is equivalent to the category of finitary monads on V. Moreover, the V-category of models of a Lawvere V-theory is equivalent to the V-category of algebras for the corresponding V-monad. This all extends routinely to local presentability with respect to any regular cardinal. We finally consider the special case where V is Cat, and explain how the correspondence extends to pseudo maps of algebras.

1. Introduction

In seeking a general account of what have been called notions of computation [13], one may consider a finitary 2-monad T on Cat, and a T-algebra (A, a), then make a construction of a category B and an identity on objects functor $j : A \longrightarrow B$. In making that construction, one is allowed to use the structure on A determined by the 2-monad, and that is all. For instance, given the 2-monad for which an algebra is a small category with a monad on it, then one possible construction would be the Kleisli construction. For another example, given the 2-monad for which an algebra is a small monoidal category A together with a specified object S, then a possible construction is that for which $B(a, b)$ is defined to be $A(S \otimes a, S \otimes b)$, with the functor $j : A \longrightarrow B$ sending a map h to $S \otimes h$.

We need a precise statement of what we mean by saying that these constructions only use structure determined by the 2-monad. In general, one may obtain one such definition by asserting that $B(a, b)$ must be of the form $A(fa, gb)$ for endofunctors f and g on A generated by the 2-monad; similarly for defining composition in B. Thus we need to know what exactly we mean by endofunctors on A and natural transformations between them that are generated by the 2-monad. This paper is motivated by a desire to make such notions precise.

In order to make precise the notion of functor generated by a finitary 2-monad T on a T-algebra (A, a), we first generalise from consideration of a finitary 2-monad on Cat to a finitary V-monad on V for any monoidal biclosed category that is locally finitely

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presentable as a closed category: we shall make that definition precise in the next section. We move to this generality primarily for simplicity but also because the computing application will need it later, for instance considering not mere categories but categories enriched in $Poset$ or in the category of ω-cpo's.

To support our main definition, we prove a theorem: we define the notion of a Lawvere V-theory, and we prove that to give a Lawvere V-theory is equivalent to giving a finitary V-monad on V. So, for a finitary V-monad T on V, there is a Lawvere V-theory $\mathcal{L}(T)$ for which the V-category of models of $\mathcal{L}(T)$, which we define, is equivalent to the V-category T-Alg of algebras for the monad. Conversely, given any Lawvere V-theory $\mathcal{T}$, there is a corresponding finitary V-monad $M(\mathcal{T})$; and these constructions yield an equivalence between the category of finitary V-monads on V and that of Lawvere V-theories.

Once we have such a correspondence, we have the definition we seek: given a 2-monad T and a T-algebra (A, a), an endofunctor f on A is generated by T whenever it is in the image of $\mathcal{L}(T)$ in the model of $\mathcal{L}(T)$ determined by the algebra (A, a). Similarly, a natural transformation $\alpha : f \Rightarrow g$ is is said to be generated by T whenever it is the image of a 2-cell of $\mathcal{L}(T)$ in the model of $\mathcal{L}(T)$ determined by the algebra (A, a).

For cognoscenti of enriched categories, the definitions and results here, at least assuming our enrichment is over a symmetric monoidal category, should come as no great surprise. However, they are a little subtle. One's first guess for a definition of Lawvere V-theory may be a small V-category with finite products, subject to a condition asserting single-sortedness. But that is too crude: not only does it not yield our theorem relating Lawvere theories and finitary monads, as in the case for Set (see [1]), but it does not agree with the known and very useful equivalences between finitary monads and universal algebra as in [9], [6], and in application to computation in [11] and [10], with an account for computer scientists in [14]. So we consider something a little more subtle: if V is symmetric, we define a Lawvere V-theory to be a small V-category with finite cotensors, subject to a single-sortedness condition. A model is therefore a finite cotensor preserving V-functor into V. In order to extend to nonsymmetric V, as for instance if V is the category of small locally ordered categories with the lax Gray tensor product [11], we need to introduce a twist: so a Lawvere V-theory is a small V_t-category with finite cotensors, subject to a single-sortedness condition, and a model is a finite cotensor preserving V_t-functor into V_t: that allows us to speak of the V-category of models, and prove our equivalence with finitary V-monads on V and their V-categories of algebras. There has been considerable development of categories enriched in monoidal biclosed categories [4, 5, 6, 11] recently: the main problem with extending the usual theory is that functor V-categories need not exist in general; but functor V-categories with base V_t do, and that suffices for our purposes here. If one is only interested in the case of symmetric V here, one may simply ignore all subscripts t, and one will have correct statements; of course, exponentials, i.e., terms of the form $[X, -]_l$ and $[X, -]_r$, may also be abbreviated to $[X, -]$.

The special case that $V = Cat$ also contains a mild surprise. Given a 2-monad T on Cat, one's primary interest lies in pseudo maps of algebras, i.e., maps of algebras in which the structure need only be preserved up to coherent isomorphism rather than strictly. Such maps are analysed in [2], with an explanation for computer scientists in [12]. One might imagine that, extending the above correspondence in this case, pseudo maps of algebras would correspond exactly to pseudo natural transformations between the corresponding models of the corresponding Lawvere 2-theory. But the position is a little more delicate than that: pseudo natural transformations allow too much flexibility to yield a bijection. However, given a pair of algebras, one still has an equivalence between the category of pseudo maps of algebras between the two algebras and that of pseudo natural transformations between the corresponding models of the corresponding Lawvere 2-theory. Hence one has a biequivalence between $T\text{-}Alg_p$ and the 2-category of models of the Lawvere theory, with pseudo natural transformations. We explain that in Section 5.

I am surprised I have not been able to find our main results, in the case of symmetric V, in the literature on enriched categories. There has been work on enriched monads, for instance [9] and [6], and there has been work on enriched finite limit theories, primarily [8] and [5]. But the closest work to this of which I am aware is by Dubuc [3], and that does not account for the finitary nature of Lawvere theories, which is central for us. Of course, what we say here is not explicitly restricted to finitariness: one could easily extend everything to cardinality $< \kappa$ for any regular κ.

The paper is organized as follows. In Section 2, we recall the basic facts about enriched categories we shall use to define our terms. In Section 3, we define Lawvere V-theories and their models, and prove that every finitary V-monad on V gives rise to a Lawvere V-theory with the same V-category of models. And in Section 4, we give the converse, i.e., to each Lawvere V-theory, we discover a finitary V-monad with the same V-category of models. We also prove the equivalence between the categories of Lawvere V-theories and finitary V-monads on V. Finally, in Section 5, we explain the more fundamental pseudo maps of algebras when $V = Cat$, and show how they relate to maps of Lawvere 2-theories.

For symmetric V, the standard reference for all basic structures other than monads is Kelly's book [7]. For nonsymmetric V, a reasonable reference is [11], but the central results were proved in the more general setting of categories enriched in bicategories as in [4, 5, 6]. For 2-categories, the best relevant reference is [2], and for an explanation directed towards computer scientists, see [12].

2. Background on enriched categories

A monoidal category V is called *biclosed* if for every object X of V, both $- \otimes X \colon V_\circ \longrightarrow V_\circ$ and $X \otimes - \colon V_\circ \longrightarrow V_\circ$ have right adjoints, denoted $[X, -]_r$ and $[X, -]_l$ respectively. For monoidal biclosed locally small V, it is evident how to define *V-categories*, *V-functors* and *V-natural transformations*, yielding the 2-category $V\text{-}Cat$ of small V-categories. The category $V_\circ$ enriches to a V-category with hom given by $[X, Y]_r$. Note that $[X, Y]_r$ cannot be replaced by $[X, Y]_l$ here, using the usual conventions of Kelly's book [7]. One can speak of *representable* V-functors and there is an elegant theory of V-adjunctions, see for instance [4]. If $V_\circ$ is complete, then it is shown in [7] that $V - Cat$ is complete. So we can speak of the Eilenberg-Moore V-category for a V-monad. If V is also cocomplete, there is an elegant theory of limits and colimits in V-categories generalising the situation for symmetric monoidal closed V, see for instance [5]. Note that for a V-category C, there is a construction of what should clearly be called C^{op}, but C^{op} is a V_t-category, not a priori a V-category.

In general, there is no definition of a functor V-category. However, if V is complete, for a small V-category C, one does have a functor V-category of the form $[C^{op}, V_t]$, whose objects are V_t-functors from C^{op} into V_t, and with homs given by the usual construction for symmetric V. Details appear in [5], where it is denoted PC. This also agrees with Street's construction [16], but the latter is formulated in terms of modules.

A *V-monad* on a V-category C consists of a V-functor $T\colon C \longrightarrow C$ and V-natural transformations $\eta\colon 1 \Rightarrow T$ and $\mu\colon T^2 \Rightarrow T$ satisfying the usual three axioms. The Eilenberg-Moore V-category $T\text{-}Alg$ has as objects the $T_\circ$-algebras, where $T_\circ$ is the ordinary monad underlying T, on the ordinary category $C_\circ$ underlying C. Given algebras (A, a) and (B, b), the hom object $T\text{-}Alg((A, a), (B, b))$ is the equaliser in V of the diagram

$$\begin{tikzcd} C(A,B) \arrow[rr, "{C(a,B)}"] \arrow[dr, "T"'] & & C(T(A),B) \\ & C(T(A),T(B)) \arrow[ur, "{C(T(A),b)}"'] & \end{tikzcd} \tag{1}$$

Composition in $T\text{-}Alg$ is determined by that in C. Moreover, there is a forgetful V-functor $U\colon T\text{-}Alg \longrightarrow C$, which has a left adjoint. If C is complete, then so is $T\text{-}Alg$, and U preserves limits.

Given monads T and S on C, a *map of monads* from T to S is a V-natural transformation $\alpha : T \Rightarrow S$ that commutes with the unit and multiplication data of the monads. With the usual composition of V-natural transformations, this yields the ordinary category $Mnd(C)$ of monads on C.

A monoidal biclosed category V is called *locally finitely presentable as a closed category* if $V_\circ$ is locally finitely presentable, if the unit I of V is finitely presentable, and if $x \otimes y$ is finitely presentable whenever x and y are. Henceforth, all monoidal biclosed categories to which we refer will be assumed to be locally finitely presentable as closed categories.

A V-category C is said to have *finite tensors* if for every finitely presentable object x of V, and for every object A of C, $[x, C(A, -)]_r\colon C \longrightarrow V$ is representable, i.e., if there exists an object $x \otimes A$ of C together with a natural isomorphism $[x, C(A, -)]_r \cong C(x \otimes A, -)$. The V-category C has *finite cotensors* if it satisfies the dual condition that for every finitely presentable x in V and every A in C, the V_t-functor $[x, C(-, A)]_l\colon C^{op} \longrightarrow V_t$ is representable, where V_t is the dual of V; i.e., there exists an object A^x of C together with a natural isomorphism $[x, C(-, A)]_l \cong C(-, A^x)$. A V-category C is *cocomplete* whenever $C_\circ$ is cocomplete, C has finite tensors, and $x \otimes -\colon C_\circ \longrightarrow C_\circ$ preserves colimits for all finitely presentable x. A cocomplete V-category C is *locally finitely presentable* if

$C_\circ$ is locally finitely presentable, C has finite cotensors, and $(-)^x: C_\circ \longrightarrow C_\circ$ preserves filtered colimits for all finitely presentable x. In general, a V-functor is called *finitary* if the underlying ordinary functor is so, i.e., if it preserves filtered colimits. We therefore denote the full subcategory of $Mnd(C)$ determined by the finitary monads on C by $Mnd_f(C)$. In [6], finitary monads on an lfp V-category are characterized in terms of algebraic structure.

For any V that is locally finitely presentable as a closed category, V is necessarily a locally finitely presentable V-category. The cotensor A^X is given by $[X, A]_l$. Given a small finitely cotensored V_t-category C, we write $FC(C, V_t)$ for the full sub-V-category of $[C, V_t]$ determined by those V-functors that preserve finite cotensors. It follows from Freyd's adjoint functor theorem, and from the fact that the inclusion preserves cotensors, that $FC(C, V_t)$ is a full reflective sub-V-category of $[C, V_t]$, i.e., the inclusion has a left adjoint. It follows immediately from the definitions that the inclusion is also finitary.

Note that, in all that follows, if V is symmetric, one may simply disregard all subscripts t, and one will have correct statements, with correct proofs: if V is symmetric, then the symmetric monoidal category V_t is isomorphic to V, and of course $[X, -]_l$ and $[X, -]_r$ agree.

3. Enriched Lawvere theories

We seek a definition of a Lawvere V-theory. In order to validate that definition, we shall prove that the V-category of models of a Lawvere V-theory is finitarily monadic over V; and for any finitary V-monad T on V, the V-category T-Alg is the V-category of models of a Lawvere theory. More elegantly, we shall establish an equivalence between the ordinary category of finitary V-monads on V and the category of Lawvere V-theories.

First we shall give our definition of Lawvere V-theory. Observe that the full subcategory V_f of V determined by (the isomorphism classes of) the finitary objects of V has finite tensors given as in V and is small.

3.1. Definition. *A* Lawvere V-theory *consists of a small V_t-category $\mathcal{T}$ with finite cotensors, together with an identity on objects strictly finite cotensor preserving V_t-functor $\iota : V_f^{op} \longrightarrow \mathcal{T}$.*

It is immediate that if $V = Set$, this definition agrees with the classical one. We extend the usual convention for Set by informally referring to $\mathcal{T}$ as a Lawvere V-theory, leaving the identity on objects V_t-functor implicit. A map of Lawvere V-theories from $\mathcal{T}$ to $\mathcal{S}$ is a strict finite cotensor preserving V_t-functor from $\mathcal{T}$ to $\mathcal{S}$ that commutes with the V_t-functors from V_f^{op}. Together with the usual composition of V_t-functors, this yields a category we denote by Law_V.

We now extend the usual definition of a model of a Lawvere theory.

3.2. Definition. *A* model *of a Lawvere V-theory $\mathcal{T}$ is a finite cotensor preserving V_t-functor from $\mathcal{T}$ to V_t. The* V-category of models of $\mathcal{T}$ *is $FC(\mathcal{T}, V_t)$, the full sub-V-category of $[\mathcal{T}, V_t]$ determined by the finite cotensor preserving V_t-functors.*

3.3. EXAMPLE. Let T be a V-monad on V. Then, one can speak of the Kleisli V-category $Kl(T)$ for T, and there is a V-functor $j : V \longrightarrow Kl(T)$ that is the identity on objects, has a right adjoint, and satisfies the usual universal property of Kleisli constructions, as explained in Street's article [15]. Since j has a right adjoint, it preserves tensors. So if we restrict j to the finitely presentable objects of V, we have an identity on objects (strictly) finite tensor preserving V-functor from V_f to the full sub-V-category $Kl(T)_f$ of $Kl(T)$ determined by the objects of V_f. Dualising, we have a Lawvere V-theory, which we denote by $\mathcal{L}(T)$. This construction extends to a functor $\mathcal{L} : Mnd(V) \longrightarrow Law_V$. Since our primary interest is in finitary V-monads, we also use $\mathcal{L}$ to denote its restriction to $Mnd_f(V)$.

3.4. THEOREM. *Let T be a finitary V-monad on V. Then $FC(\mathcal{L}(T), V_t)$ is equivalent to T-Alg.*

Proof. There is a comparison V-functor from $Kl(T)$ to T-Alg, and it preserves tensors since the canonical functors from V into each of $Kl(T)$ and T-Alg do. By composition with the inclusion of $Kl(T)_f$ into $Kl(T)$, we have a V-functor $c : Kl(T)_f \longrightarrow T\text{-}Alg$, and this yields a V-functor $\tilde{c} : T\text{-}Alg \longrightarrow [Kl(T)_f^{op}, V_t]$ sending a T-algebra (A, a) to the V_t-functor $T\text{-}Alg(c(-), (A, a))$. This V-functor factors through $FC(Kl(T)_f^{op}, V_t)$ since c preserves finite tensors and since representables preserve finite cotensors. Moreover, it is fully faithful since every object of T-Alg is a canonical colimit of a diagram in $Kl(T)_f$. So it remains to show that $\tilde{c}$ is essentially surjective.

Suppose $h : Kl(T)_f^{op} \longrightarrow V_t$ preserves finite cotensors. Let $A = h(I)$, where I is the unit of V. The behaviour of h on all objects is fully determined by its behaviour on I since h preserves finite cotensors and every object of $Kl(T)_f^{op}$, i.e., every object of V_f^{op}, is a finite cotensor of I. The behaviour of h on homs gives, for each finitely presentable x, a map in V from $Kl(T)(I, x)$ to $[A^x, A]_l$, or equivalently, since cotensors in V are given by a left exponential, and by the usual properties of maps in Kleisli categories and maps from units, a map in V from $[x, A]_l \otimes Tx$ to A. This must all be natural in x, thus, by finitariness of T, we have a map $a : TA \longrightarrow A$. Functoriality of h, together with the V_t-functor from V_f^{op} into $Kl(T)^{op}$, force a to be an algebra map. It is routine to verify that $\tilde{c}((A, a))$ is isomorphic to h. ∎

4. The converse

In this section, we start by proving a converse to Theorem 3.4. This involves constructing a finitary monad $M(\mathcal{T})$ from a Lawvere V-theory $\mathcal{T}$. Having proved that result, we establish that $\mathcal{L}$ together with M form an equivalence of categories between the categories of finitary V-monads on V and Lawvere V-theories, for any monoidal biclosed V that is locally finitely presentable as a closed category.

First we recall Beck's monadicity theorem. Given a functor $U : C \longrightarrow D$, a pair of arrows $h_1, h_2 : X \longrightarrow Y$ in C is called a *U-split coequaliser pair* if there exist arrows $h : UY \longrightarrow Z$, $i : Z \longrightarrow UY$, and $j : UY \longrightarrow UX$ in D such that $h \cdot Uh_1 = h \cdot Uh_2$, i

splits h, j splits h_1, and $Uh_2 \cdot j = i \cdot h$. It follows that h is a coequaliser of Uh_1 and Uh_2, and that coequaliser is preserved by all functors. Beck's theorem says

4.1. THEOREM. *A functor $U : C \longrightarrow D$ is monadic if and only if*

- *U has a left adjoint*
- *U reflects isomorphisms, and*
- *C has and U preserves coequalisers of U-split coequaliser pairs.*

For a detailed account of Beck's theorem, see Barr and Wells' book [1]. Now we are ready to prove our main result.

4.2. THEOREM. *Let $\mathcal{T}$ be an arbitrary Lawvere V-theory. Then there is a finitary monad $M(\mathcal{T})$ on V such that $M(\mathcal{T}) - Alg$ is equivalent to $FC(\mathcal{T}, V_t)$.*

Proof. Recall from the definition of Lawvere V-theory, we have a finite cotensor preserving V_t-functor from V_f^{op} to $\mathcal{T}$. By composition, this yields a V-functor $U : FC(\mathcal{T}, V_t) \longrightarrow FC(V_f^{op}, V_t)$, but the latter is equivalent to V: one proof of this is by applying Theorem 3.4 to the identity monad. The V-functor U is given by evaluation at I. Moreover, it has a left adjoint since the inclusion of $FC(\mathcal{T}, V_t)$ into $[\mathcal{T}, V_t]$ has a left adjoint, as mentioned in Section 2, and since evaluation functors from V-presheaf categories have left adjoints given by tensors. Moreover, it is finitary since $(-)^x$ preserves filtered colimits for finitely presentable x. It remains to show that U is monadic.

We shall apply Beck's monadicity theorem. It is routine to verify that U reflects isomorphisms, and one can readily calculate that U preserves the coequalisers of U-split coequaliser pairs: since $FC(\mathcal{T}, V_t)$ is a full reflective sub-V-category of $[\mathcal{T}, V_t]$, it is cocomplete; and the required coequaliser lifts to all objects as they are all finite cotensors of the unit I, so one need merely apply the associated cotensor to the given map and its splitting. That proves monadicity of the underlying ordinary functor of U; extending that to the enriched functor follows immediately from the fact that U respects cotensors. ∎

Our construction M extends routinely to a functor $M : Law_V \longrightarrow Mnd_f(V)$. Recall from the previous section that we also have a functor $\mathcal{L} : Mnd_f(V) \longrightarrow Law_V$. In fact we have

4.3. THEOREM. *The functors $M : Law_V \longrightarrow Mnd_f(V)$ and $\mathcal{L} : Mnd_f(V) \longrightarrow Law_V$ form an equivalence of categories.*

Proof. Given a finitary V-monad T on V, it follows from Theorem 3.4 and the construction of M that $M\mathcal{L}(T)$ is isomorphic to T. For the converse, given a Lawvere V-theory $\mathcal{T}$, we need to study the construction of $M(\mathcal{T})$. It is given by taking the left adjoint to the functor from $FC(\mathcal{T}, V_t)$ to V given by evaluation at the unit I. The left adjoint, applied to a finitely presentable object x of V, yields the representable functor $\mathcal{T}(x, -) : \mathcal{T} \longrightarrow V_t$: this follows since x is a finite tensor of I, and by preservation of finite cotensors. Now applying Yoneda yields the result that $\mathcal{L}M(\mathcal{T})$ is isomorphic to $\mathcal{T}$. Thus we are done. ∎

5. 2-Monads on Cat

We now restrict our attention to the case that $V = Cat$. So we consider finitary 2-monads on Cat. An example is that there is a finitary 2-monad for which the algebras are small monoidal categories with a specified object S, as mentioned in the introduction. Another finitary 2-monad on Cat has an algebra given by a small category with a monad on it, again as in the introduction. There are many variants of these examples: see [2] or [12] for many more examples and more detail.

When one does have a 2-monad T, the maps of primary interest are the *pseudo* maps of algebras. These correspond to the usual notion of structure preserving functor. They are defined as follows.

5.1. Definition. *Given T-algebras (A, a) and (B, b), a* pseudo *map of algebras from (A, a) to (B, b) consists of a functor $f : A \longrightarrow B$ and a natural isomorphism*

$$\begin{tikzcd}
TA \arrow[r, "Tf"] \arrow[d, "a"'] \arrow[dr, phantom, "\Downarrow\bar{f}"] & TB \arrow[d, "b"] \\
A \arrow[r, "f"'] & B
\end{tikzcd}$$

such that

$$\begin{tikzcd}
T^2A \arrow[r, "T^2f"] \arrow[d, "Ta"'] \arrow[dr, phantom, "\Downarrow T\bar{f}"] & TB \arrow[d, "Tb"] \\
TA \arrow[r, "Tf"] \arrow[d, "a"'] \arrow[dr, phantom, "\Downarrow\bar{f}"] & TB \arrow[d, "b"] \\
A \arrow[r, "f"'] & B
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
T^2A \arrow[r, "T^2f"] \arrow[d, "\mu_A"'] & TB \arrow[d, "\mu_B"] \\
TA \arrow[r, "Tf"] \arrow[d, "a"'] \arrow[dr, phantom, "\Downarrow\bar{f}"] & TB \arrow[d, "b"] \\
A \arrow[r, "f"'] & B
\end{tikzcd}$$

and

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[dd, "id_A"'] & B \arrow[dd, "id_B"] & & A \arrow[r, "f"] \arrow[d, "\eta_A"'] & B \arrow[d, "\eta_B"] \\
 & & = & TA \arrow[r, "Tf"] \arrow[d, "a"'] \arrow[dr, phantom, "\Downarrow \bar{f}"] & TB \arrow[d, "b"] \\
A \arrow[r, "f"'] & B & & A \arrow[r, "f"'] & B
\end{tikzcd}$$

A 2-cell *between pseudo maps* $(f, \bar{f})$ *and* $(g, \bar{g})$ *is a natural transformation between* f *and* g *that respects* $\bar{f}$ *and* $\bar{g}$.

By using the composition of Cat, it follows that T-algebras, pseudo maps of T-algebras, and 2-cells form a 2-category, which we denote by $T\text{-}Alg_p$. This agrees with the notation in some of the relevant literature, and this situation has undergone extensive study, in particular in [2]; and for an account directed towards computer scientists, see [12].

Similarly, given a small 2-category C, one has the notion of pseudo natural transformation between 2-functors $h, k : C \longrightarrow Cat$: one has isomorphisms where the definition of natural transformation has commuting squares, and those isomorphisms are subject to two coherence conditions, expressing coherence with respect to composition and identities in C. Thus, for any Lawvere 2-theory $\mathcal{T}$, we have the 2-category $FC_p(\mathcal{T}, Cat)$, given by finite cotensor preserving 2-functors, pseudo natural transformations, and the evident 2-cells. If a finitary 2-monad T corresponds to the Lawvere 2-theory $\mathcal{T}$, one might guess that the 2-equivalence between $T\text{-}Alg$ and $FC(\mathcal{T}, Cat)$ would extend to a 2-equivalence between $T\text{-}Alg_p$ and $FC_p(\mathcal{T}, Cat)$, but it does not!

5.2. Example. Let T be the identity 2-monad on Cat. Then $T\text{-}Alg_p$ is 2-equivalent to Cat. But $FC_p(\mathcal{T}, Cat)$ is not, as it contains more maps. First, all functors lie in $FC_p(\mathcal{T}, Cat)$ via the inclusion of Cat, which is 2-equivalent to $FC(\mathcal{T}, Cat)$, in $FC_p(\mathcal{T}, Cat)$. But also, one may vary any component of a pseudo natural transformation α, say α_2, by an isomorphism, leaving every other component fixed, and vary the structural isomorphisms of α by conjugation, and one still has a pseudo natural transformation.

So the correspondence we seek is a little more subtle. The notion of 2-equivalence between 2-categories is quite rare. More commonly, one has a *biequivalence*. This amounts to relaxing fully faithfulness to an equivalence on homcategories, and similarly for essential surjectivity. In our case, we already have essential surjectivity. So we could ask whether our 2-functor $\tilde{c} : T\text{-}Alg \longrightarrow FC(\mathcal{T}, Cat)$ extends to giving, for each pair of algebras $((A, a), (B, b))$, an equivalence between $T\text{-}Alg_p((A, a), (B, b))$ and the category of pseudo natural transformations between $\tilde{c}((A,a))$ and $\tilde{c}((B,b))$, hence a biequivalence of 2-categories. In fact, we have

5.3. THEOREM. *The 2-functor $\tilde{c} : T\text{-}Alg \longrightarrow FC(\mathcal{T}, Cat)$ extends to a biequivalence between $T\text{-}Alg_p$ and $FC_p(\mathcal{T}, Cat)$.*

Proof. If one routinely follows the argument for essential surjectivity of $\tilde{c}$ as in the proof of Theorem 3.4, and ones uses the 2-dimensional property of the colimit defining TA, one obtains a bijection between pseudo maps of T-algebras from (A,a) to (B,b), and those pseudo natural transformations between $\tilde{c}((A,a))$ and $\tilde{c}((B,b))$ that respect finite cotensors strictly, i.e., up to equality, not just isomorphism. Now, since cotensors are pseudo limits, they are automatically bilimits, and so every pseudo natural transformation is isomorphic to one that respects finite cotensors strictly. Local fully faithfulness is straightforward. ■

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Department of Computer Science
University of Edinburgh
King's Buildings
Edinburgh, EH9 3JZ
Scotland
Email: `ajp@dcs.ed.ac.uk`

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