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CENTRAL EXTENSIONS IN MAL'TSEV VARIETIES

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ABSTRACT. We show that every algebraically–central extension in a Mal'tsev variety — that is, every surjective homomorphism $f : A \longrightarrow B$ whose kernel–congruence is contained in the centre of A, as defined using the theory of commutators — is also a central extension in the sense of categorical Galois theory; this was previously known only for varieties of Ω-groups, while its converse is easily seen to hold for any congruence–modular variety.

1. Introduction

A number of classical results in homological algebra, which constitute *the theory of central extensions*, originally for groups, rings, Lie algebras, and so on, have undergone two wide generalizations : first, by Fröhlich's school, to the context of Ω-groups in the sense of P. Higgins [H] (see especially A. Fröhlich [F], A. S.-T. Lue [L], and J. Furtado-Coelho [F-C]); and then further still, to the context of exact categories, and so in particular to universal algebras, by the present authors [JK1]; there the definition of "central extension" is purely categorical — as is the formulation of the main result, which in fact belongs to *categorical Galois theory.*

The present paper continues the investigation we started in [JK3], aiming to establish a connexion between the categorical notion of central extension introduced in [JK1] and the *generalized commutator theory* which has been developed in universal algebra (see J.D.H. Smith [S], R. Freeze and R. KcKenzie [FM], G. Janelidze and M.C. Pedicchio [JP2], and the references therein). Already in [JK3] we have shown that every "categorically central" extension $f : A \longrightarrow B$ in a congruence-modular variety (that is, every central extension in the sense of [JK1]) is also "algebraically central", meaning that the commutator $[R, \mathbf{1}]$ is $\mathbf{0}$, where R is the kernel-congruence $A \times_B A = \{(x, y) \in A \times A | f(x) = f(y)\}$ of the surjective homomorphism f, while $\mathbf{0} = \mathbf{0}_A$ and $\mathbf{1} = \mathbf{1}_A$ are the smallest and the largest congruences on A, given by

$$\mathbf{0}_A = \{(a, a) | a \in A\} \subset A \times A = \mathbf{1}_A .$$

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For the converse, however, we had in [JK3] a (very simple) proof only for a variety of Ω-groups; and the main purpose of the present paper is to extend this result to any Mal'tsev variety. The proof we give uses the approach to commutators of [JP2] — although in a special case, going back to [J], [JP1], and previous work of M.C. Pedicchio.

2. Revision of commutators

Let $\mathbf{C}$ be a Mal'tsev (that is, a congruence–permutable) variety of universal algebras, and let p be a chosen Mal'tsev term in the theory of $\mathbf{C}$, so that we have the identities

$$p(x, y, y) = x = p(y, y, x) \ . \tag{2.1}$$

Recall the classical results:

2.1 Proposition. *(i) Every reflexive (homomorphic) relation in* $\mathbf{C}$ *is a congruence; (ii) the join* $R \vee S$ *of congruences* R *and* S *on* A *is* RS.

Following [JP2, Corollary 5.6], we introduce:

2.2 Definition. *For an object* A *of* $\mathbf{C}$ *and congruences* R *and* S *on* A*, the commutator* $[R, S]$ *is the smallest congruence on* A *such that the function*

$$\{(x, y, z) \in A^3 | (x, y) \in R \ \textit{and} \ (y, z) \in S\} \ \longrightarrow \ A/[R, S] \ , \tag{2.2}$$

sending (x, y, z) *to the* $[R, S]$*-class of* $p(x, y, z)$*, is a homomorphism of algebras (that is, a morphism in* $\mathbf{C}$*).*

It follows, for example, from the results of [JP2] that the commutator $[R, S]$ coincides with the "classical" one originally defined by J.D.H. Smith [S], and that it does not in fact depend on the choice of the Mal'tsev term p. Among its simple basic properties are the following:

$$[R, S] \leq R \wedge S \ , \tag{2.3}$$

$$[R, S] = [S, R] \ , \tag{2.4}$$

$$S \leq T \Rightarrow [R, S] \leq [R, T] \ , \tag{2.5}$$

$$[R, S \vee T] = [R, S] \vee [R, T] \ , \tag{2.6}$$

$$f[R, S] = [fR, fS] \ \text{ for surjective } \ f : A \longrightarrow B \ ; \tag{2.7}$$

here (2.5) is included only for emphasis, it being of course a consequence of (2.6), while in (2.7), fR denotes the ordinary image of R under $f \times f : A \times A \longrightarrow B \times B$, which (being reflexive) is in fact a congruence by Proposition 2.1 (i). It turns out that the commutator can be defined as *the largest operation* satisfying (2.3) and (2.7) : see [FM, Chapter III] for details.

2.3 Remark. It follows from Definition 2.2 and Proposition 2.1 (i) that $[R,S]$ can be described as the subalgebra of $A \times A$ generated by all pairs of the form (u,v), where

$$u = p(s(x_1, \cdots, x_n), s(y_1, \cdots, y_n), s(z_1, \cdots, z_n)) \;, \tag{2.8}$$

$$v = s(p(x_1, y_1, z_1), \cdots, p(x_n, y_n, z_n)) \;, \tag{2.9}$$

for some n-ary operator s in a fixed signature of $\mathbf{C}$ and for elements $x_1, \cdots, x_n, y_1, \cdots, y_n$, $z_1, \cdots, z_n$ of A having each (x_i, y_i) in R and each (y_i, z_i) in S.

3. The chief lemma

We devote this section to the proof of:

3.1 Lemma. *With* $\mathbf{C}$ *a Mal'tsev variety as above, let* $f : A \longrightarrow B$ *and* $g : B \longrightarrow A$ *be homomorphisms in* $\mathbf{C}$ *satisfying* $fg = 1_B$, *and let* R *be the kernel–congruence of* f. *Then if* $[R, \mathbf{1}_A] = \mathbf{0}_A$ *we also have* $R \wedge [\mathbf{1}_A, \mathbf{1}_A] = \mathbf{0}_A$.

We abbreviate $\mathbf{1}_A$ to $\mathbf{1}$ and $\mathbf{0}_A$ to $\mathbf{0}$. Supposing that $[R, \mathbf{1}] = \mathbf{0}$, we conclude from Remark 2.3 that, for any n-ary term s, the u of (2.8) and the v of (2.9) coincide if each (x_i, y_i) lies in R; that is :

$$\text{(2.8) and (2.9) coincide for any term } s \text{ if } f(x_i) = f(y_i) \text{ for each } i \;. \tag{3.1}$$

We now define a function $+ : A \times A \longrightarrow A$ by setting

$$u + v = p(u, gf(u), gf(v)) \;. \tag{3.2}$$

Then, taking (x_i, y_i, z_i) in (3.1) to be $(u_i, gf(u_i), gf(v_i))$, which we may do since $f(u_i) = fgf(u_i)$, and observing that $s(gf(u_1), \cdots, gf(u_n)) = gf(s(u_1, \cdots, u_n))$ because f and g are homomorphisms, and that similarly $s(gf(v_1), \cdots, gf(v_n)) = gf(s(v_1, \cdots, v_n))$, we get, using the definition (3.2),

$$s(u_1, \cdots, u_n) + s(v_1, \cdots, v_n) = s(u_1 + v_1, \cdots, v_n + v_n) \tag{3.3}$$

for any term s; that is, $+ : A \times A \longrightarrow A$ is a homomorphism of algebras.

The right side of (3.2) simplifies in some particular cases; for

$$g(a) + g(b) = p(g(a), gfg(a), gfg(b)) = p(g(a), g(a), g(b)) = g(b) \;, \tag{3.4}$$

using (2.1); while

$$u + v = u \text{ when } f(u) = f(v) \;, \tag{3.5}$$

since then $u + v = p(u, gf(u), gf(v)) = p(u, gf(u), gf(u)) = u$, using (2.1) again. Observe also that, since

$$f(p(u, gf(u), gf(v))) = p(f(u), fgf(u), fgf(v)) = p(f(u), f(u), f(v))$$

because f is a homomorphism, we have, using (2.1) yet again,

$$f(u+v) = f(v) \ . \tag{3.6}$$

Since (3.5) may be written as

$$u+v = u \ \text{ when } \ (u,v) \in R \ , \tag{3.7}$$

to complete the proof that $R \wedge [\mathbf{1},\mathbf{1}] = \mathbf{0}$ it suffices to show that

$$u+v = v \ \text{ when } \ (u,v) \in [\mathbf{1},\mathbf{1}] \ . \tag{3.8}$$

Moreover, since $+$ is a homomorphism as in (3.3), we need only show that $u+v=v$ for each (u,v) in a generating set for $[\mathbf{1},\mathbf{1}]$; and thus, by Remark 2.3, for the values (2.8) and (2.9) of u and v, where s is an n-ary operator of $\mathbf{C}$ and x_i, y_i, z_i are now arbitrary. Here, however, $u+v$ is

$$\begin{aligned} & p(s(x_i,\cdots,x_n), s(y_1,\cdots,y_n), s(z_i,\cdots,z_n)) + v \\ &= p(s(x_i,\cdots,x_n), s(y_1,\cdots,y_n), s(z_1,\cdots,z_n)) + p(v,v,v) \ \text{ by } \ (2.1) \\ &= p(s(x_1,\cdots,x_n)+v, s(y_1,\cdots,y_n)+v, s(z_1,\cdots,z_n)+v) \ , \end{aligned} \tag{3.9}$$

by (3.3) with p for s. Giving v its value from (2.9), we have for $s(x_1,\cdots,x_n)+v$ the value

$$\begin{aligned} & s(x_1,\cdots,x_n) + s(p(x_1,y_1,z_1),\cdots,p(x_n,y_n,z_n)) \\ &= s(x_1 + p(x_1,y_1,z_1),\cdots,x_n + p(x_n,y_n,z_n)) \ \text{ by } \ (3.3) \\ &= s(x_1',\cdots,x_n') \ \text{ say } \ ; \end{aligned}$$

and similarly for $s(y_1,\cdots,y_n)+v$ and $s(z_1,\cdots,z_n)+v$. Thus the value (3.9) of $u+v$ is

$$p(s(x_1',\cdots,x_n'), s(y_1',\cdots,y_n'), s(z_1',\cdots,z_n')) \ . \tag{3.10}$$

However $f(x_i') = f(x_i + p(x_i,y_i,z_i))$, which by (3.6) is $f(p(x_i,y_i,z_i))$; and similarly $f(y_i') = f(p(x_i,y_i,z_i))$; so that, by (3.1), the value (3.10) of $u+v$ is

$$s(p(x_1',y_1',z_1'),\cdots,p(x_n',y_n',z_n')) \ . \tag{3.11}$$

But $p(x_i',y_i',z_i')$ is

$$\begin{aligned} & p(x_i + p(x_i,y_i,z_i), y_i + p(x_i,y_i,z_i), z_i + p(x_i,y_i,z_i)) \\ &= p(x_i,y_i,z_i) + p(p(x_i,y_i,z_i), p(x_i,y_i,z_i), p(x_i,y_i,z_i)) \ \text{ by } \ (3.3) \\ &= p(x_i,y_i,z_i) + p(x_i,y_i,z_i) \ \text{ by } \ (2.1) \\ &= p(x_i,y_i,z_i) \ \text{ by the case } \ u=v \ \text{ of } \ (3.7) \ . \end{aligned}$$

So the value $u+v$ of (3.11) is indeed the value (2.9) of v; which completes the proof of Lemma 3.1.

4. Central extensions

We continue to suppose that $\mathbf{C}$ is a Mal'tsev variety as above, and we recall some elementary and well-known results about abelian algebras in $\mathbf{C}$, with a sketch of their derivations. Recall that an algebra A in $\mathbf{C}$ is said to be *abelian* when $[\mathbf{1}_A, \mathbf{1}_A] = \mathbf{0}_A$. From Remark 2.3 we get at once :

4.1 PROPOSITION. *The algebra A is abelian if and only if (2.8) and (2.9) coincide for every operation s and for all values of x_i, y_i, and z_i in A.*

It follows that the abelian algebras form a subvariety $\mathrm{Ab}(\mathbf{C})$. By Definition 2.2, $[\mathbf{1}_A, \mathbf{1}_A]$ is the smallest congruence R on A for which the composite of $p : A^3 \longrightarrow A$ with the canonical quotient–map $r : A \longrightarrow A/R$ is a homomorphism. This is equally the composite of $r^3 : A^3 \longrightarrow (A/R)^3$ and $p : (A/R)^3 \longrightarrow A/R$; and to say that this is a homomorphism is — because r^3 is a surjective homomorphism — just to say that $p : (A/R)^3 \longrightarrow A/R$ is a homomorphism, and hence to say that A/R is abelian.

Thus $[\mathbf{1}_A, \mathbf{1}_A]$ is the smallest congruence on A for which $A/[\mathbf{1}_A, \mathbf{1}_A]$ is abelian; accordingly the canonical quotient–map $r_A : A \longrightarrow A/[\mathbf{1}_A, \mathbf{1}_A]$ is the unit of the reflexion of $\mathbf{C}$ into $\mathrm{Ab}(\mathbf{C})$. For any homomorphism $f : A \longrightarrow B$ in $\mathbf{C}$, write $f_* : A/[\mathbf{1}_A, \mathbf{1}_A] \longrightarrow B/[\mathbf{1}_B, \mathbf{1}_B]$ for the induced homomorphism — the unique homomorphism satisfying

$$f_* r_A = r_B f \ . \tag{4.1}$$

The notion of central extension developed in [JK1] applies to a variety and a chosen subvariety, which for us are $\mathbf{C}$ and $\mathrm{Ab}(\mathbf{C})$; this latter is an "admissible" subcategory in the sense of [JK1] by [JK1, Theorem 3.4]. We recall the definitions to which this leads :

4.2 DEFINITION. *By an "extension" $f : A \longrightarrow B$ we mean a surjective homomorphism in $\mathbf{C}$. This extension (A, f) of B is said to be*

(a) *"trivial" if, in the square represented by (4.1), f is the pullback of f_* along $r_B : B \longrightarrow B/[\mathbf{1}_B, \mathbf{1}_B]$;*
(b) *"split by the surjective homomorphism $q : E \longrightarrow B$" if its pullback along q is trivial;*
(c) *"central" (categorically) if it is split by some surjective homomorphism $q : E \longrightarrow B$.*

On the other hand, we have agreed to call the extension $f : A \longrightarrow B$ *algebraically central* when its kernel–congruence R satisfies $[R, \mathbf{1}_A] = \mathbf{0}_A$. This nomenclature is related to the algebraic definition of the *centre* of an algebra A : namely, as the greatest congruence S on A for which we have $[S, \mathbf{1}_A] = \mathbf{0}_A$. Thus the extension $f : A \longrightarrow B$ is algebraically central when its kernel R is contained in the centre of A; and the algebra A is abelian when its centre is all of $A \times A$. The result foreshadowed in our Introduction is contained in:

4.3 THEOREM. *Let $f : A \longrightarrow B$ be a surjective homomorphism in a Mal'tsev variety* $\mathbf{C}$, *and let $R = A \times_B A$ be its kernel–congruence. Then the extension (A, f) of $B \cong A/R$ is*

(a) *trivial if and only if $R \wedge [\mathbf{1}_A, \mathbf{1}_A] = \mathbf{0}_A$;*
(b) *central if and only if it is algebraically central, meaning that $[R, \mathbf{1}_A] = \mathbf{0}_A$.*

Proof. (a) is a special case of [JK1, Proposition 4.2] and the "only if" part of (b) is proved in [JK3] (in the more general situation where $\mathbf{C}$ is any congruence–modular variety); so it remains only to prove the "if" part of (b). Suppose, therefore, that $f : A \longrightarrow B$ is algebraically central, and let $h : C \longrightarrow A$ be the pullback of $f : A \longrightarrow B$ along itself. Then h, as a pullback of the algebraically central f, is itself algebraically central, as was proved in [JK3]; and moreover $hk = 1$ for some homomorphism $k : A \longrightarrow C$. Applying Lemma 3.1 now with h in place of f, and using part (a) of the present theorem, we see that the extension (C, h) of A is trivial; so that the extension (A, f) of B is (categorically) central by Definition 4.2 (c).

4.4 REMARK. It follows from the results of [JK2] (which referred to a much more general context) that the category Centr(B) of central extensions of B is a reflective full subcategory of the category of *all* extensions of B. Now Theorem 4.3 allows us to express this reflexion explicitly, by a (well-known) formula extending the classical one for the variety of groups. Given an extension $f : A \longrightarrow B$ with kernel-congruence R, write $s : A \longrightarrow A' = A/[R, \mathbf{1}_A]$ for the canonical quotient–map, and write $f' : A' \longrightarrow B$ for the unique homomorphism having $f's = f$; then f' is the reflexion of f into the central extensions of B. To see this, first observe that f' is a central extension : indeed, s being surjective, the kernel–congruence R' of f' is sR, while $\mathbf{1}_{A'} = s\mathbf{1}_A$, so that (2.7) gives $[R', \mathbf{1}_{A'}] = s[R, \mathbf{1}_A] = \mathbf{0}_{A'}$. Now let $g : C \longrightarrow B$ be a central extension with kernel-congruence S, so that $[S, \mathbf{1}_C] = \mathbf{0}_C$, and let $t : A \longrightarrow C$ satisfy $gt = f$. Write $t = iq$ where $i : D \longrightarrow C$ is an injective homomorphism and $q : A \longrightarrow D$ a surjective one: we are to show that t, or equally q, factorizes though s; that is, that $q[R, \mathbf{1}_A] = \mathbf{0}_D$. However, since q is surjective, qR is the kernel-congruence T of gi, while $q\mathbf{1}_A = \mathbf{1}_D$, so that $q[R, \mathbf{1}_A] = [T, \mathbf{1}_D]$ by (2.7); and now $[S, \mathbf{1}_C] = \mathbf{0}_C$ gives $[T, \mathbf{1}_D] = \mathbf{0}_D$ using Remark 2.3.

5. Relations to homological algebra and universal algebra

Given a group B and an abelian group K, consider the short exact sequence of abelian groups

$$0 \longrightarrow \operatorname{Ext}(H_1(B), K) \longrightarrow H^2(B, K) \longrightarrow \operatorname{Hom}(H_2(B), K) \longrightarrow 0 , \tag{5.1}$$

which is an instance of a well-known universal coefficient theorem in homological algebra; here $H_n(B) = H_n(B, \mathbf{Z})$ for $n = 1$ or 2 and $H^2(B, K)$ are the appropriate homology and cohomology groups respectively. Classically $H^2(B, K)$ can be interpreted as the group of (isomorphism classes of) central extensions of B with the fixed kernel K, and (as observed in [JK1]) the image of $\operatorname{Ext}(H_1(B), K)$ corresponds precisely to the extensions that are trivial in the sense of Definition 4.2. Therefore we have:

5.1 Proposition. *The following conditions on a group B are equivalent:*

(a) *every central extension of B is trivial;*
(b) $H_2(B) = 0$.

Let us also mention, without going into details, that the same result can be obtained using the Hopf formula for $H_2(B)$ in terms of commutators, and what are called *weak universal central extensions.* Moreover, all of this extends to the context of A. S.-T. Lue's central extensions of Ω-groups; see [JK1], [JK3], and references there. On the other hand, not just for groups or Ω-groups, but in the much more general context of an arbitrary Mal'tsev variety $\mathbf{C}$, Theorem 4.3 yields:

5.2 Corollary. *The following conditions on an object B in $\mathbf{C}$ are equivalent:*

(a) *every central extension of B is trivial;*
(b) *for any object A and any congruence R on A making $A/R \cong B$, the following implication holds:*

$$[R, \mathbf{1}_A] = \mathbf{0}_A \quad \Rightarrow \quad R \wedge [\mathbf{1}_A, \mathbf{1}_A] = \mathbf{0}_A \ . \tag{5.2}$$

Such implications as (5.2) have been studied in universal algebra. Among various interesting results mentioned by R. Freeze and R. McKenzie [FM] is the fact that any residually–small congruence–modular variety satisfies the congruence identity

$$R \wedge [S, S] = [R \wedge S, S] \ . \tag{5.3}$$

For $S = \mathbf{1}_A$ this identity becomes

$$R \wedge [\mathbf{1}_A, \mathbf{1}_A] = [R, \mathbf{1}_A] \ , \tag{5.4}$$

and so the implication (5.2) follows from it. Therefore we can conclude that no object in a residually–small Mal'tsev variety has nontrivial central extensions.

These simple observations seem to indicate a deep relationship between the homological–algebraic and the universal–algebraic investigations. Another possible conclusion is that, in studying the congruence identities (including implications between identities), one should also consider the cases where one or more of the congruences R involved has $A \longrightarrow A/R$ a split epimorphism : for example the implication (5.2) will then hold for every A in any Mal'tsev variety, as follows from Theorem 4.3 and the fact that every central extension (A, f) with f a split epimorphism is trivial (see [JK1], Theorem 4.8 and Remark 4.9).

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