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SOLUTION MANIFOLDS FOR SYSTEMS OF DIFFERENTIAL EQUATIONS

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ABSTRACT. This paper defines a solution manifold and a stable submanifold for a system of differential equations. Although we eventually work in the smooth topos, the first two sections do not mention topos theory and should be of interest to non-topos theorists. The paper characterizes solutions in terms of barriers to growth and defines solutions in what are called filter rings (characterized as C^∞-reduced rings in a paper of Moerdijk and Reyes). We examine standardization, stabilization, perturbation, change of variables, non-standard solutions, strange attractors and cycles at infinity.

Introduction

We explore what is meant by a solution of a system of differential equations. Although the approach is based on solving equations in the smooth topos, the material in the first two sections does not depend on topos theory and should be of independent interest. One of our results analyzes limit cycles of autonomous differential equations. In section 3, we define a solution manifold and a stabilization operation. Section 4 contains examples.

Conceptually, this paper is related to [3] which dealt with differential equations for a single function. We have had to change our technical approach considerably to accommodate systems of equations. So the reading of [3] is not a prerequisite for this paper.

In section 1, we characterize solutions of a system of differential equations as n-tuples of functions which respect certain "barriers to growth". This fact enables us, in section 2, to use barriers to define solutions in filter rings (the C^∞-reduced rings of [5], [6]). Solutions in a filter ring are, in effect, parameterized solutions, such as solutions of parameterized differential equations, or solutions parameterized by initial conditions, see Theorem 2.12 and its corollary. The filter sometimes gives us non-standard real parameters, and transfinite cycles as in Theorem 2.20. Filter ring solutions often reflect the behavior of nearby solutions which accounts for their effectiveness in examining issues such as stability.

In section 3, we examine solutions in a smooth topos. The solution manifold is defined as the subobject of $(\mathbf{R}^n)^{\mathbf{R}}$ where the barrier conditions are satisfied and where $\mathbf{R}$ is (the manifold corresponding to) the real line. We work in the topos of sheaves on the category $\mathbf{V}$ of filter rings, and use the finite open cover topology, see [4]. Our approach allows

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us to define the submanifolds of stable and of asymptotically stable solutions and the quotient manifold of standard solutions. For most of our theorems, we need to impose a boundedness condition due to the fact that $\mathbf{R}$ is not Archimedean in this topos.

NOTATION. We often use $x = (x_1, \ldots, x_n)$ to denote an element of $\mathbf{R}^n$. In this case, $\|x\|$ denotes $\mathrm{Max}\{|x_i|\}$, the ℓ_∞ norm. (Although any reasonable norm will usually do.)

1. Barriers for Systems of Differential Equations

We consider systems of the following type:

$$(*)\ \frac{dx_i}{dt} = W_i(t, x_1, \ldots, x_n) \text{ for } i = 1, \ldots, n$$

We assume that each W_i is a continuous, real-valued function defined on all of $\mathbf{R} \times \mathbf{R}^n$. Letting $x = (x_1, \ldots, x_n)$ and $W = (W_1, \ldots, W_n)$, system $(*)$ becomes:

$$(*)\ \frac{dx}{dt} = W(t, x)$$

We say that f is a *solution of* $(*)$ iff f is an n-tuple of differentiable functions from $\mathbf{R}$ to $\mathbf{R}$ such that, for $1 \leq i \leq n$, we have $f_i'(t) = W_i(t, f(t))$ for all $t \in \mathbf{R}$.

Our approach is to study solutions of $(*)$ by finding barriers to their growth.

1.1. DEFINITION. *Let $(*)$ be as above. Then a C^∞-function $B(t,x) = B(t, x_1, \ldots, x_n)$ is a* barrier function for $(*)$ over $[a,b]$ *if $a < b$ and if whenever $B(t,x) = 0$ for $t \in [a,b]$ then $\partial B/\partial t + \sum(\partial B/\partial x_i)W_i < 0$ at the point (t,x).*

1.2. LEMMA. *Let $g(t)$ be a differentiable function on $[a,b]$ (where $a < b$) with the property that if $g(c) = 0$ for any $c \in [a,b]$ then $g'(c) < 0$. It follows that if $g(a) \leq 0$ then $g(b) < 0$.*

Proof. Case 1: Assume $g(a) < 0$. Suppose $g(b) \geq 0$. Let c be the smallest element of $[a,b]$ for which $g(c) = 0$. By choice of c, we have $g(t) < 0$ for $a \leq t < c$ which implies that $g'(c) \geq 0$ by calculating $g'(c)$ as t approaches c from the left. This contradicts the hypothesis that $g'(c) < 0$ since $g(c) = 0$.
Case 2: Assume $g(a) = 0$. Then, by hypothesis, $g'(a) < 0$ so there clearly exists a_0 with $a < a_0 < b$ such that $g(a_0) < 0$. The argument given in case 1, applied to the interval $[a_0, b]$, now leads to the result that $g(b) < 0$. ∎

1.3. DEFINITION. *A system $f = (f_1, \ldots, f_n)$ satisfies the* strong barrier condition for $(*)$ *if whenever $B(t,x)$ is a barrier over some $[a,b]$ then $B(a, f(a)) > 0$ or $B(b, f(b)) < 0$. Also f satisfies the* weak barrier condition for $(*)$ *if under the same assumptions on B, $B(a, f(a)) \geq 0$ or $B(b, f(b)) \leq 0$.*

1.4. LEMMA. *If $f = (f_1, \ldots, f_n)$ is a solution of $(*)$ then f satisfies the strong barrier condition for $(*)$.*

Proof. Let $B(t,x)$ be a barrier over $[a,b]$ and define $g(t) = B(t, f(t))$. Then apply Lemma 1.2 to g which directly leads to the result. ∎

1.5. Theorem. *Let $f = (f_1, \ldots, f_n)$ be an n-tuple of functions from $\mathbf{R}$ to $\mathbf{R}$ (with no differentiability or even continuity assumed). Then the following are equivalent:*

(1) Each f_i is differentiable and f is a solution of $(*)$.

(2) f satisfies the strong barrier condition for $(*)$.

(3) f satisfies the weak barrier condition for $(*)$.

Proof. (1)⇒(2): By Lemma 1.4.
(2)⇒(3): Obvious.
(3)⇒(1): Let t_0 be given and let $x^0 = f(t_0)$ where $x^0 = (x_1^0, \ldots, x_n^0)$ and $x_i^0 = f_i(t_0)$. Let $m = W(t_0, x^0)$, with $m = (m_1, \ldots, m_n)$. We need to show that $f_i'(t_0)$ exists and equals m_i for $i = 1, \ldots, n$.

Let $\epsilon > 0$ be given. Choose an open neighborhood U of (t_0, x^0) in $\mathbf{R} \times \mathbf{R}^n$ such that for $(t, x) \in U$, with $x = (x_1, \ldots, x_n)$, we have, for $i = 1, \ldots, n$:

$$(m_i - \epsilon) < W_i(t, x) < (m_i + \epsilon)$$

Choose $\delta > 0$ so that $(t, x) \in U$ whenever $|t - t_0| < \delta$ and $|x_i - x_i^0| < \delta$ for all i. Let η be any real number with $0 < \eta \leq \delta/2$. Let $h_i : \mathbf{R} \to \mathbf{R}$ be the straight line function with slope $m_i + \epsilon$ for which $h_i(t_0) = x_i^0 + \eta$. Let $\ell_i : \mathbf{R} \to \mathbf{R}$ be the straight line function with slope $m_i - \epsilon$ for which $\ell_i(t_0) = x_i^0 - \eta$. Choose $t_1 > t_0$ such that whenever $t \in [t_0, t_1]$ then $\ell_i(t)$ and $h_i(t)$ are within δ of x_i^0. (It suffices to do this for $\eta = \delta/2$.) It follows that if $x = (x_1, \ldots, x_n)$ and $\ell_i(t) \leq x_i \leq h_i(t)$ and $t \in [t_0, t_1]$ then $(t, x) \in U$. Now, using the notation $\exp(r)$ for e^r, we define:

$$b_i(t, x_i) = \exp[K(h_i(t) - x_i)(\ell_i(t) - x_i)]$$

where $K > 0$ is to be chosen. To continue the proof, we need:

1.6. Lemma. *Using the construction in the above proof, and assuming $t \in [t_0, t_1]$ and $x = (x_1, \ldots, x_n)$, we have:*

(1) If $b_i(t, x_i) \leq 1$, then $\ell_i(t) \leq x_i \leq h_i(t)$.

(2) If $b_i(t, x_i) \leq 1$ for all i, then $\partial b_i/\partial t + \partial b_i/\partial x_i W_i < 0$.

(3) $B(t, x) = (\sum b_i(t, x_i)^2) - 1$ is a barrier function for $(*)$ over $[t_0, t_1]$.

Proof. (1) If $b_i(t, x_i) \leq 1$, then $(h_i(t) - x_i)(\ell_i(t) - x_i) \leq 0$ which implies $\ell_i(t) \leq x_i \leq h_i(t)$.

(2) If $b_i(t, x_i) \leq 1$ for all i, then, by (1), we have $\ell_i(t) \leq x_i \leq h_i(t)$, for all i, and by choice of t_1, we see that $(t, x) \in U$. Now let $W_i(t, x) = \overline{m_i}$. then $(m_i - \epsilon) < \overline{m_i} < (m_i + \epsilon)$, by definition of U. We readily find that:

$$\begin{aligned} \partial b_i/\partial t &= K b_i[(m_i + \epsilon)(\ell_i(t) - x_i) + (m_i - \epsilon)(h_i(t) - x_i)] \\ \partial b_i/\partial x_i &= K b_i[-(\ell_i(t) - x_i) - (h_i(t) - x_i)] \end{aligned}$$

So, collecting these terms and using $\overline{m_i} = W_i(t, x)$, we get:

$$\partial b_i/\partial t + \partial b_i/\partial x_i W_i = K b_i[\ell_i(t)(m_i + \epsilon - \overline{m_i}) + h_i(t)(m_i - \epsilon - \overline{m_i}) + x_i(2\overline{m_i} - 2m_i)]$$

Since $m_i - \epsilon < \overline{m_i} < m_i + \epsilon$, we see that the coefficient of $\ell_i(t)$ is positive while the coefficient of $h_i(t)$ is negative. Also $Kb_i > 0$. But, by (1), $\ell_i(t) \le x_i \le h_i(t)$ (and at least one of these inequalities is strict, $\ell_i(t) < x_i \le h_i(t)$ or $\ell_i(t) \le x_i < h_i(t)$.) So, if we replace $\ell_i(t)$ and $h_i(t)$ by x_i, we get a strictly larger expression, which, as can be readily seen, simplifies to 0:

$$\partial b_i/\partial t + \partial b_i/\partial x_i W_i < 0$$

(3) Let $B(t,x) = (\sum b_i(t,x_i)^2) - 1$. Suppose $t \in [t_0, t_1]$ and that $B(t,x) = 0$. Then, clearly, $b_i(t,x_i) \le 1$ for all i, so $(t,x) \in U$. We calculate that:

$$\partial B/\partial t + \sum \partial B/\partial x_i W_i = \sum 2b_i[\partial b_i/\partial t + \partial b_i/\partial x_i W_i] < 0$$

(as follows from (2).) This shows that B is a barrier function for $(*)$. ∎

Proof of theorem 1.5, continued. By the lemma, $B(t,x)$ is a barrier function over $[t_0, t_1]$ for any positive choice of K. By choosing K sufficiently large, we can make each $b_i(t_0, x_i^0)$ small enough so that $B(t_0, x^0) < 0$. Since f satisfies the weak barrier condition, it follows that $B(t_1, f(t_1)) \le 0$. But this means that each $b_i(t_1, f_i(t_1)) < 1$ so $f_i(t_1)$ is caught between $\ell_i(t_1)$ and $h_i(t_1)$. Moreover, this argument works as we let $\eta \to 0$ (noting that the same value of t_1 works for all η). We can also repeat the same argument for any $t \in [t_0, t_1]$. This is enough to show that the differential quotient $(f_i(t) - f_i(t_0))/(t - t_0)$ approaches m_i as t approaches t_0 from the right. An entirely analogous argument works for approaching t_0 from the left. (In fact, it could be formalized by systematically replacing t by $-t$, see the first part of the proof of 2.23, and then showing that we get no genuinely new barrier conditions on f). ∎

1.7. Proposition. *A pointwise limit of solutions of $(*)$ is again a solution of $(*)$. Moreover, if $\mathcal{S}$ is a family of solutions of $(*)$ and f is an n-tuple of functions such that for every a, b and $\epsilon > 0$, there exists $s \in \mathcal{S}$ with $\|s(a) - f(a)\| < \epsilon$, $\|s(b) - f(b)\| < \epsilon$ then f is a solution of $(*)$.*

Proof. Let $B(t,x)$ be any barrier function for $(*)$ over $[a,b]$. We can readily show that f satisfies the strong barrier condition at B by approximating f sufficiently closely by a member of $\mathcal{S}$ at a, b. ∎

2. Solutions in Filter Rings

To study non-standard solutions, stability and cycles at infinity, we use the barrier conditions to define solutions in what are called filter rings (or the reduced rings of [5], [6]).

By a *proper filter* on $\mathbf{R}^m$ we mean a non-empty collection, $\mathcal{F}$, of non-empty subsets of $\mathbf{R}^m$ closed under finite intersection and supersets (meaning that if $F_1, F_2 \in \mathcal{F}$ then $F_1 \cap F_2 \in \mathcal{F}$ and if $F \in \mathcal{F}, F \subseteq G$ then $G \in \mathcal{F}$). The collection of *all* subsets of $\mathbf{R}^m$ is called the *improper filter* on $\mathbf{R}^m$. We say that $\mathcal{B}$ is a *base* for $\mathcal{F}$, or that $\mathcal{B}$ *generates* $\mathcal{F}$, when $F \in \mathcal{F}$ iff there exists $B \in \mathcal{B}$ with $B \subseteq F$.

2.1. Definition. *By a* closed filter on $\mathbf{R}^m$ *we mean a filter with a base of closed subsets of* $\mathbf{R}^m$.

We let $C^\infty(\mathbf{R}^m)$ *denote the ring of all* C^∞*-functions from* $\mathbf{R}^m$ *to* $\mathbf{R}$*. If* $\mathcal{F}$ *is a closed filter on* $\mathbf{R}^m$*, then* $I(\mathcal{F})$ *is the ideal of all* $f \in C^\infty(\mathbf{R}^m)$ *which vanish on a member of* $\mathcal{F}$*.*

By a filter ring, *we mean a ring of the form* $C^\infty(\mathbf{R}^m)/I(\mathcal{F})$.

Note: If $\mathcal{F}$ is the improper filter, generated by the empty set, which is closed, then $C^\infty(\mathbf{R}^m)/I(\mathcal{F})$ will be the trivial ring consisting of just one element.

The point of defining a filter ring A is to be able to define what is meant by "a solution of $(*)$ with parameters in A". To do this we define the "ring of real-valued maps with parameters in A". Note that if $u = (u_1, \ldots, u_n)$ represents an n-tuple of generators of A then it is plausible that $\alpha(u,t)$ represents "a function of t parameterized by A ". For technical reasons, we sometimes need to impose a boundedness condition. (See also section 3 for a more theoretical approach to these definitions.)

2.2. Definition. *For* $A = C^\infty(\mathbf{R}^m)/I(\mathcal{F})$ *define* $\mathrm{Map}(A)$ *as* $C^\infty(\mathbf{R}^m \times \mathbf{R})/I(\pi^*\mathcal{F})$ *where* $\pi^*\mathcal{F}$ *is the closed filter generated by sets of the form* $F \times \mathbf{R}$ *for* $F \in \mathcal{F}$*. We also define* $\mathrm{Map}_0(A)$ *as the subring of* $\mathrm{Map}(A)$ *consisting of the "semi-bounded" elements, where* $w \in C^\infty(\mathbf{R}^m \times \mathbf{R})$ *is* semi-bounded with respect to $\mathcal{F}$ *if for every closed bounded interval* J*, there exists* $F \in \mathcal{F}$ *such that the restriction of* w *to* $F \times J$ *is bounded.*

If $\mathcal{F}$ *is understood, then* $C^\infty_{sb}(\mathbf{R}^m \times \mathbf{R})$ *denotes the subring of all maps in* $C^\infty(\mathbf{R}^m \times \mathbf{R})$ *which are semi-bounded with respect to* $\mathcal{F}$*.*

Also, n-$\mathrm{Map}(A)$ *is the product of* n *copies of* $\mathrm{Map}(A)$ *and* n-$\mathrm{Map}_0(A)$ *is the product of* n *copies of* $\mathrm{Map}_0(A)$*.*

Remark. If w denotes an element of $\mathrm{Map}_0(A)$, then, through intentional abuse of notation, we let w also denote a representative function in $C^\infty_{sb}(\mathbf{R}^m \times \mathbf{R})$.

C^∞-maps can be evaluated in filter rings. If $(w_1, \ldots, w_n)$ is a k-tuple of elements of the filter ring A, and if $\lambda \in C^\infty(\mathbf{R}^k)$, then $\lambda(w_1, \ldots, w_n)$ makes sense. For if we regard each w_i as a function (see above remark) then $\lambda(w_1, \ldots, w_n)$ is a well-defined composite function, and can readily be shown to represent an element of A. It remains to check that the indicated construction is independent of the actual functions used to represent the elements $(w_1, \ldots, w_n)$ of A, and this is straightforward.

(A ring for which C^∞-maps can be evaluated in a reasonable way is called a C^∞*-ring*. See [4] for details.)

A consequence of evaluating C^∞-functions is that we can use the barrier functions to define whether an element of $\mathrm{Map}(A)$ satisfies the weak and strong barrier conditions, and so be a "non-standard" solution of $(*)$.

2.3. Definition. *Let* A *be a filter ring. Then* $\sigma \in$ n-$\mathrm{Map}(A)$ *is an* A-solution of $(*)$ *if it satisfies the* strong barrier condition for $(*)$ *which means that whenever* $B(t,x)$ *is a barrier over some* $[a,b]$ *then there exists* $F \in \mathcal{F}$ *such that for all* $u \in F$ *either* $B(a, \sigma(u,a)) > 0$ *or* $B(b, \sigma(u,b)) < 0$*. We say that* σ *is a* semi-bounded solution *of* $(*)$ *if* σ *is a solution in* n-$\mathrm{Map}_0(A)$

The weak barrier condition for $(*)$ *is defined analogously, using* $\geq, \leq$ *instead of* $>, <$, *in the manner of Definition 1.3.*

We will show that the strong and weak barrier conditions are equivalent for semi-bounded solutions. First we need:

2.4. Definition. *A closed filter on* $\mathbf{R}^m$ *is* maximally closed *if it is a maximal element of the family of all proper closed filters on* $\mathbf{R}^m$.

Remark. Since every closed subset of $\mathbf{R}^m$ is a zero-set, the closed filters and maximally closed filters defined above coincide with the filters generated by the z-filters and the z-ultrafilters as defined in [1]. Note that a z-ultrafilter does not necessarily generate an ultrafilter, see [1], page 152.

Notation. If $\mathcal{F}$ is a proper filter on $\mathbf{R}^m$ and $f : \mathbf{R}^m \to \mathbf{R}$ then L, "the limit of f along $\mathcal{F}$", denoted by $L = \lim_{\mathcal{F}} f$, is defined by the condition that for each $\epsilon > 0$ there exist $F \in \mathcal{F}$ such that $|L - f(u)| < \epsilon$ for all $u \in F$. Clearly L is unique, if it exists.

2.5. Definition. *Let* $A = C^\infty(\mathbf{R}^m)/I(\mathcal{F})$ *be a non-trivial filter ring and let* $\sigma(u,t) \in \mathrm{Map}_0(A)$ *be given. Whenever* $\mathcal{M}$ *is a maximally closed extension of* $\mathcal{F}$, *we define* $\sigma_\mathcal{M}$ *so that, for each fixed* t, $\sigma_\mathcal{M}(t) = \lim_\mathcal{M} \sigma(u,t)$.

Similarly, if $\sigma = (\sigma_1, \ldots, \sigma_n)$, *then* $\sigma_\mathcal{M} = (\sigma_{1,\mathcal{M}}, \ldots, \sigma_{n,\mathcal{M}})$.

A straightforward compactness argument shows that $\sigma_\mathcal{M}(t)$ exists and is unique. Note that by semi-boundedness, there exists $F \in \mathcal{F}$ such that σ is bounded on $F \times \{t\}$. By the smooth Tietze theorem, we may as well assume that $\sigma(u,t)$ is bounded when restricted to $\mathbf{R}^m \times \{t\}$.

2.6. Lemma. *Let* $A = C^\infty(\mathbf{R}^m)/I(\mathcal{F})$ *be a non-trivial filter ring and assume that* $\sigma(u,t) \in$ n-$\mathrm{Map}_0(A)$ *satisfies the weak barrier condition for* $(*)$. *Then, whenever* $\mathcal{M}$ *is a maximally closed extension of* $\mathcal{F}$, *we have that* $\sigma_\mathcal{M}$ *is a solution of* $(*)$.

Proof. By Theorem 1.5, it suffices to show that $\sigma_\mathcal{M}$ satisfies the weak barrier condition. Let $B(t,x)$ be a barrier function over $[a,b]$ and suppose that neither $B(a,\sigma_\mathcal{M}(a)) \geq 0$ nor $B(b,\sigma_\mathcal{M}(b)) \leq 0$. So, $B(a,\sigma_\mathcal{M}(a)) < 0$ and $B(b,\sigma_\mathcal{M}(b)) > 0$. Since B is continuous, there exists a neighborhood L_1 of $\sigma_\mathcal{M}(a)$ (in $\mathbf{R}^n$) such that $B(a,x) < 0$ for $x \in L_1$, and a neighborhood L_2 of $\sigma_\mathcal{M}(b)$ such that $B(b,x) > 0$ for $x \in L_2$. By the convergence of $\sigma(u,a)$ to $\sigma_\mathcal{M}(a)$ we can find $U \in \mathcal{M}$ such that $\sigma(u,a) \in L_1$ for $u \in U$. Similarly we can find $V \in \mathcal{M}$ such that $\sigma(u,b) \in L_2$ for $u \in V$.. But $\sigma(u,t)$ satisfies the weak barrier condition, so there exists $F \in \mathcal{F}$ such that for $u \in F$ we have either $B(a,\sigma(u,a)) \geq 0$ or $B(b,\sigma(u,b)) \leq 0$. It follows that $U \cap V \cap F = \emptyset$ so $\emptyset \in \mathcal{M}$, which is a contradiction. ∎

2.7. Lemma. *Let* $A = C^\infty(\mathbf{R}^m)/I(\mathcal{F})$ *be a non-trivial filter ring and let* $\sigma(u,t)$ *be in* n-$\mathrm{Map}_0(A)$. *Assume that* $\sigma_\mathcal{M}$ *is a solution of* $(*)$ *whenever* $\mathcal{M}$ *is a maximally closed extension of* $\mathcal{F}$. *Then* $\sigma(u,t)$ *satisfies the strong barrier condition for* $(*)$.

Proof. Let $B(t,x)$ be a barrier function over $[a,b]$ and let:

$$E = \{u \in \mathbf{R}^m : \text{Either } B(a,\sigma(u,a)) > 0 \text{ or } B(b,\sigma(u,b)) < 0\}$$

We have to find an $F \in \mathcal{F}$ with $F \subseteq E$. Assume no such F exists. Then we can find a maximally closed extension $\mathcal{M}$ of $\mathcal{F}$ such that $E^c \in \mathcal{M}$ (where the closed set E^c is the complement of E). By hypothesis, $\sigma_{\mathcal{M}}$ is a solution of $(*)$ and, by Theorem 1.5, it satisfies the strong barrier condition so:

$$\text{Either } B(a, \sigma_{\mathcal{M}}(a)) > 0 \text{ or } B(b, \sigma_{\mathcal{M}}(b)) < 0$$

Case 1: Assume $B(a, \sigma_{\mathcal{M}}(a)) > 0$. By continuity, there exists a neighborhood L_1 of $\sigma_{\mathcal{M}}(a)$ with $B(a,x) > 0$ for every $x \in L_1$. Since $\sigma(u,a)$ converges to $\sigma_{\mathcal{M}}(a)$ there exists $U \in \mathcal{M}$ such with $\sigma(u,a) \in L_1$ for all $u \in U$. It follows that $U \cap E^c = \emptyset$, a contradiction.
Case 2: Assume $B(b, \sigma_{\mathcal{M}}(b)) < 0$. This case uses a similar contradiction. ∎

2.8. Proposition. *Let $A = C^\infty(\mathbf{R}^m)/I(\mathcal{F})$ be a non-trivial filter ring and let $\sigma(u,t)$ be an n-tuple of elements of $\mathrm{Map}_0(A)$. Then the following statements are equivalent:*

*(1) σ is an A-solution of $(*)$ (*i.e. *σ satisfies the strong barrier condition).*

(2) σ satisfies the weak barrier condition for $(*)$.

(3) Whenever $\mathcal{M}$ is a maximally closed extension of $\mathcal{F}$, then $\sigma_{\mathcal{M}}$ is a solution of $(*)$.

Proof. (1)⇒(2): Obvious.
(2)⇒(3): By Lemma 2.6.
(3)⇒(1): By Lemma 2.7. ∎

2.9. Definition. *Let $A = C^\infty(\mathbf{R}^m)/I(\mathcal{F})$. Then $\mathrm{I}_{\mathrm{PtCn}}$, the* ideal of pointwise convergence, *is the set of all $\alpha \in \mathrm{Map}_0(A)$ such that for every fixed real t_0 and every $\epsilon > 0$ there exists $F \in \mathcal{F}$ with $|\alpha(u,t_0)| < \epsilon$ whenever $u \in F$. It follows by semi-boundedness that $\mathrm{I}_{\mathrm{PtCn}}$ is an ideal of $\mathrm{Map}_0(A)$ and $\alpha \in \mathrm{I}_{\mathrm{PtCn}}$ iff $\alpha_{\mathcal{M}} = 0$ for every maximally closed extension $\mathcal{M}$ of $\mathcal{F}$.*

2.10. Proposition. *Assume $\sigma \in n\text{-}\mathrm{Map}_0(A)$ for some filter ring A. Then σ represents an A-solution of $(*)$ iff $\sigma + \alpha$ does for every n-tuple α of members of $\mathrm{I}_{\mathrm{PtCn}}$.*

Proof. Straightforward, using Proposition 2.8. ∎

Perturbed differential equations. A differential equation, $(*_u)$ $dx/dt = V(u,t,x)$, is said to be a perturbation of $(*)$ if $V(u,t,x)$ equals $W(t,x)$ for a particular value of the parameter u. More generally, we will allow $V(u,t,x)$ to approach $W(t,x)$ for a "limiting value" of u in the sense of the following definition. In this case, parameterized solutions of $dx/dt = V(u,t,x)$ not only approximate solutions of $(*)$, they actually form an A-solution of $(*)$ for suitable A.

2.11. Definition. *The differential equation $(*_u)$ $dx/dt = V(u,t,x)$, with parameter $u \in \mathbf{R}^m$ is a* perturbation *of $(*)$ with respect to a closed filter $\mathcal{F}$ on $\mathbf{R}^m$ if $W(t,x) = \lim_{\mathcal{F}} V(u,t,x)$ for fixed t and x.*

2.12. Theorem. *Suppose that $(*)$ is perturbed by an equation $(*_u)$ $dx/dt = V(u,t,x)$ with respect to $\mathcal{F}$ a filter on $\mathbf{R}^m$. If the semi-bounded $\sigma(u,t)$ is a parameterized solution of $(*_u)$, then σ represents an A-solution of $(*)$ where $A = C^\infty(\mathbf{R}^m)/I(\mathcal{F})$.*

Proof. Let $B(t,x)$ be a barrier function on $[a,b]$ for $(*)$. By semi-boundedness, we may as well assume that for some M, we have $\|\sigma(u,t)\| \leq M$ for $(u,t) \in \mathbf{R}^m \times [a,b]$. Let:

$$K = \{(t,x) : t \in [a,b], \|x\| \leq M, B(t,x) = 0\}$$

Clearly K is compact and, since $\partial B/\partial t + \sum \partial B/\partial x_i W_i < 0$ for all $(t,x) \in K$, there is a maximum value m of $\partial B/\partial t + \sum \partial B/\partial x_i W_i$ (on K) and m is negative. Since each $\partial B/\partial x_i$ is bounded on K there exists $\epsilon > 0$ such that if $\|V(u,t,x) - W(t,x)\| < \epsilon$ then $\partial B/\partial t + \sum \partial B/\partial x_i W_i < m/2 < 0$ on K. By hypothesis, each $(t_0, x_0) \in K$ has a neighborhood L with some $F \in \mathcal{F}$ for which $V(u,t,x)$ is within ϵ of $W(t,x)$ for $(t,x) \in L$ and $u \in F$. By covering K with finitely many of these L's and taking the corresponding finite intersection of the F's we can find $F_0 \in \mathcal{F}$ such that $V(u,t,x)$ is within ϵ of $W(t,x)$ for $(t,x) \in K$ and $u \in F_0$. Therefore, for each $u \in F_0$ we see that $B(t,x)$ is a barrier function for the perturbed equation $(*_u)$ and since $\sigma(u,t)$ is a solution of this equation, we see that σ satisfies the required barrier condition to be an A-solution of $(*)$. ∎

2.13. Corollary. *Let $\sigma(u,t)$ be a solution of $(*)$ for each fixed value of u. For example, $\sigma(u,t)$ might be a solution satisfying an initial condition which depends on u. Then $\sigma(u,t)$ represents an A-solution of $(*)$ whenever $\sigma(u,t) \in \mathrm{Map}_0(A)$.*

Proof. Let $V(u,t,x) = W(t,x)$ for all u. ∎

The associated map and non-standard solutions. As shown in section 1, a standard solution of $(*)$ is an n-tuple of functions $f = (f_1, \ldots, f_n)$, with $f_i : \mathbf{R} \to \mathbf{R}$, which satisfies the weak barrier conditions. This same definition can be interpreted in the internal language of a smooth topos and, as will be shown in the next section, the resulting notion relates to A-solutions.

One advantage of the internal definition is that we can discuss solutions of $(*)$ in which $\mathbf{R}$ is replaced by a non-standard version of the reals. We will illustrate this possibility by working with ultrapowers of $\mathbf{R}$, which can be used to show that some differential equations have non-standard cyclic-like solutions with transfinite periods.

2.14. Definition. *Let $\mathbf{N}$ denote the positive integers and let $\mathcal{U}$ be an ultrafilter on $\mathbf{N}$. We say that two sequences, $x, y \in \mathbf{R}^{\mathbf{N}}$ are equivalent modulo $\mathcal{U}$ iff $\{i : x_i = y_i\} \in \mathcal{U}$. Then $\mathbf{R}_{\mathcal{U}}$, the resulting set of equivalence classes is called an* ultrapower *of $\mathbf{R}$.*

The above are the only ultrapowers we will consider but, in general, the set $\mathbf{N}$ can be replaced by any index set. It is well-known that any ultrapower of $\mathbf{R}$ has all the relations and operations that $\mathbf{R}$ has and satisfies the same first-order properties. Such ultrapowers are also filter rings, as shown by:

2.15. Lemma. *Every ultrapower of $\mathbf{R}$, arising from an ultrafilter on $\mathbf{N}$, is a filter ring.*

Proof. The ultrafilter $\mathcal{U}$ on $\mathbf{N}$ is clearly the base of a closed filter on $\mathbf{R}$, which we will also denote by $\mathcal{U}$. It is then readily seen that $\mathbf{R}_{\mathcal{U}} = C^\infty(\mathbf{R})/I(\mathcal{U})$. ∎

2.16. Definition. *Let* $\mathbf{R}_{\mathcal{U}}$ *and* $\mathbf{R}_{\mathcal{V}}$ *be ultrapowers of* $\mathbf{R}$*. Then an* n*-tuple* $f = (f_1, \ldots, f_n)$ *with* $f_i : \mathbf{R}_{\mathcal{U}} \to \mathbf{R}_{\mathcal{V}}$ *is a* non-standard *solution of* $(*)$ *if the weak barrier conditions are satisfied.*

2.17. Definition. *If* $\sigma(u,t) \in \mathrm{Map}(A)$*, then the* associated map $\hat{\sigma} : A \to A$ *is defined so that if* $\alpha \in A$ *is represented by* $\alpha(u)$*, then* $\hat{\sigma}(\alpha)$ *is represented by* $\sigma(u, \alpha(u))$*. Similarly, if* $\sigma(u,t) \in n\text{-}\mathrm{Map}(A)$ *with* $\sigma = (\sigma_1, \ldots, \sigma_n)$ *, then* $\hat{\sigma} = (\hat{\sigma}_1, \ldots, \hat{\sigma}_n)$

2.18. Proposition. *Let* $\mathbf{R}_{\mathcal{U}}$ *be an ultrapower of* $\mathbf{R}$ *and let* $\sigma \in n\text{-}\mathrm{Map}(\mathbf{R}_{\mathcal{U}})$ *be given. Then* σ *is an* $\mathbf{R}_{\mathcal{U}}$*-solution of* $(*)$ *iff* $\hat{\sigma}$ *is a non-standard solution of* $(*)$*.*

Proof. σ is an $\mathbf{R}_{\mathcal{U}}$-solution iff there exists $U \in \mathcal{U}$ such that for $u \in U$ we have either $B(a, \sigma(u,a)) \geq 0$ or $B(b, \sigma(u,b)) \leq 0$. But since $\mathcal{U}$ is an ultrafilter it is a prime filter and this means that the above is true iff either $\{u | B(a, \sigma(u,a))\} \geq 0$ is in $\mathcal{U}$ or $\{u | B(b, \sigma(u,b))\} \leq 0$ is in $\mathcal{U}$. This is easily seen to be equivalent to saying that $\hat{\sigma}$ is a non-standard solution of $(*)$. (Note: Not every non-standard solution is the associated map of a member of n-$\mathrm{Map}(\mathbf{R}_{\mathcal{U}})$.) ∎

Autonomous equations and the pseudo-cycle at ∞. An equation of the form $dx/dt = W(x)$, where W depends only on x and not on t, is said to be *autonomous*. In this case, if f is a solution of $(*)$ then $f(u+t)$ is also a solution for any constant u.

We will assume that f is a solution of the autonomous equation $(*)$ and is *bounded* meaning that the set $\{f(t) : t \in \mathbf{R}\}$ is a bounded subset of $\mathbf{R}^n$. We aim to study the limit points $L^+(f)$, or points that $f(t)$ gets close to as $t \to \infty$ (see below).

We will investigate the behavior of $f(t)$ as $t \to \infty$ by, in effect, considering a solution of the form $f(\gamma + t)$ where γ is a transfinite constant. In order for this to make sense, we need to replace f by a non-standard solution with domain $\mathbf{R}_{\mathcal{U}}$ where $\mathcal{U}$ is a free (i.e. non-principal) ultrafilter, so that $\mathbf{R}_{\mathcal{U}}$ has transfinite elements.

Construction. (1) We are given $f : \mathbf{R} \to \mathbf{R}^n$ a solution with bounded range of the autonomous differential equation $(*)$. (As will be noted below, this construction extends to the case where we only assume that $\{f(t) | t > 0\}$ is a bounded subset of $\mathbf{R}^n$.)

(2) We assume that $(*)$ satisfies a *uniqueness condition* which means that any two ordinary solutions which agree at a single value of t must then agree at all (finite) values of t. Mild conditions on W, such as being a C^1 function, are sufficient to guarantee the uniqueness condition, see [2], page 542.

(3) We let $L^+(f)$ be the set of all points $P \in \mathbf{R}^n$ such that $f(t)$ comes arbitrarily close to P for arbitrarily large values of t. So $P \in L^+(f)$ iff for every $\epsilon > 0$ and every t_0 there exists $t > t_0$ such that $\|f(t) - P\| < \epsilon$. If $n = 2$, the Poincaré-Bendixson theorem says that, barring any limit equilibrium points, the set $L^+(f)$ is a cyclic orbit. But $L^+(f)$, for $n > 2$, can be a "strange attractor" with weird properties, see [2].

(4) We let $\mathcal{U}$ be any free ultrafilter on $\mathbf{N}$ and let $\mathbf{R}_{\mathcal{U}}$ be the resulting filter ring. By Corollary 2.13, we see that $\sigma(u,t) = f(u+t)$ is an $\mathbf{R}_{\mathcal{U}}$-solution of $(*)$.

(5) Let $\hat{\sigma}$ be the resulting non-standard solution of $(*)$ as in Proposition 2.18. Note that if $\tau \in \mathbf{R}_{\mathcal{U}}$ is represented by the sequence $(\tau_1, \ldots, \tau_i, \ldots)$ then $\hat{\sigma}(\tau)$ is represented by the sequence $(\sigma(1, \tau_1), \ldots, \sigma(i, \tau_i), \ldots)$.

(6) We let $\mathrm{Lim}_{\mathcal{U}}\hat{\sigma} : \mathbf{R}_{\mathcal{U}} \to \mathbf{R}^n$ be the function obtained by taking limits along $\mathcal{U}$. Since the range of f is bounded, this limit exists and is unique. Since limits preserve the weak barrier conditions, it follows that $\mathrm{Lim}_{\mathcal{U}}\hat{\sigma}$ is still a non-standard solution of $(*)$.

(7) We let $\mathbf{R}^+_{\mathcal{U}}$ denote the positive elements of $\mathbf{R}_{\mathcal{U}}$. We also let $\overline{f} : \mathbf{R}^+_{\mathcal{U}} \to \mathbf{R}^n$ be the restriction of $\mathrm{Lim}_{\mathcal{U}}\hat{\sigma}$ to $\mathbf{R}^+_{\mathcal{U}}$. Note that $\overline{f}$ is essentially the limit of $f(\gamma + t)$ where γ is represented by the identity sequence $(1, 2, \ldots, i, \ldots)$. In effect this map shows us what happens to $f(u + t)$ as u "goes to infinity" along $\mathcal{U}$.

2.19. Lemma. *Let $(*)$ and f be as above. Assume that $(*)$ satisfies the uniqueness condition. Let $\gamma = \{\gamma_k\}$ be such that $\gamma_k \to \infty$ (along $\mathcal{U}$) and let $\epsilon = \{\epsilon_k\}$ be such that $Lim_{\mathcal{U}}(\epsilon_k) = 0$. Then $Lim_{\mathcal{U}} f(\gamma_k) = Lim_{\mathcal{U}} f(\gamma_k + \epsilon_k)$.*

Proof. Since $f(t)$ is bounded for $t > 0$, we see that $f'(t) = W(f(t))$ is also bounded for $t > 0$. So there exists $M > 0$ such that $\|f'(t)\| < M$ for all $t > 0$. (Recall that we are using the ℓ_∞ norm on $\mathbf{R}^n$.) From the mean-value theorem we now get that $\|f(\gamma_k + \epsilon_k) - f(\gamma_k)\| < M\|\epsilon_k\|$ and the lemma follows. ∎

2.20. Theorem. *Let $(*)$ be an autonomous equation where each $W_i(x)$ is a C^∞-function. Let $f(t)$ be a bounded solution for $(*)$. Let $\overline{f} : \mathbf{R}^+_{\mathcal{U}} \to \mathbf{R}^n$ and $L^+(f)$ be as defined above. Then:*

(1) The range of $\overline{f}$ is precisely $L^+(f)$.

(2) $\overline{f}$ is "locally cyclic" in the sense that for each $\alpha \in \mathbf{R}^+_{\mathcal{U}}$ there exists $\rho \in \mathbf{R}^+_{\mathcal{U}}$ such that $\overline{f}(\alpha) = \overline{f}(\alpha + \rho)$.

(3) If $\overline{f}(\alpha) = \overline{f}(\alpha + \rho)$, then $\overline{f}(\beta) = \overline{f}(\beta + \rho)$ whenever $\beta - \alpha$ is bounded.

(4) $\overline{f}$ imposes no order on its range $L^+(f)$ in the sense that if $\overline{f}(\alpha) = P_1$ and P_2 is any other point in $L^+(f)$ then there exists $\rho \in \mathbf{R}^+_{\mathcal{U}}$ such that $\overline{f}(\alpha + \rho) = P_2$.

Proof. (1) It is obvious that the range of $\overline{f}$ is contained in $L^+(f)$. Suppose $P \in L^+(f)$. Then we can find $\{t_k\}$ such that P is the classical limit of $f(t_k)$ by choosing $t_k > k + 1$ with $f(t_k)$ within $1/k$ of P. Let $\tau = \{\tau_k\}$ where $\tau_k = t_k - k$ then $P = \overline{f}(\tau)$.

(2) Let $\alpha = \{\alpha_k\} \in \mathbf{R}^+_{\mathcal{U}}$ be given and let $\overline{f}(\alpha) = \mathrm{Lim}_{\mathcal{U}} f(k + \alpha_k) = P$. Clearly we can find an increasing sequence of positive integers m_k such that $\|P - f(m_k + \alpha_{m_k})\| < 1/k$. We can clearly further require that $m_k > \alpha_k + k$. Define $\rho = \{\rho_k\}$ where $\rho_k = (m_k + \alpha_{m_k}) - (k + \alpha_k)$. Then it easily follows that $\overline{f}(\alpha + \rho) = \overline{f}(\alpha)$.

(3) Assume $\overline{f}(\alpha + \rho) = \overline{f}(\alpha)$ and that $\beta - \alpha = \lambda = \{\lambda_k\}$ is bounded. This means that $\mathrm{Lim}_{\mathcal{U}}(\lambda_k) = t_0$ where $t_0 \in \mathbf{R}$. Write $\lambda_k = t_0 + \epsilon_k$ then $\mathrm{Lim}_{\mathcal{U}}(\epsilon_k) = 0$.

For each t, let $g(t) = \mathrm{Lim}_{\mathcal{U}} f(k + \alpha_k + t)$. Since, for fixed k, $f(\alpha_k + k + t)$ is a solution of $(*)$, it follows from Proposition 1.7 that $g(t)$ is a solution of $(*)$ (in the usual sense, as a map from $\mathbf{R}$ to $\mathbf{R}^n$). Similarly, let $h(t) = \mathrm{Lim}_{\mathcal{U}} f(k + \alpha_k + \rho_k + t)$. By the same argument, $h(t)$ is also a solution of $(*)$. But $g(0) = h(0)$. Since W is C^∞, it follows that $(*)$ satisfies the uniqueness condition, see [2], and therefore $g(t) = h(t)$, $\forall t \in \mathbf{R}$.

Now, $\overline{f}(\beta) = \mathrm{Lim}_{\mathcal{U}}(k + \beta_k) = \mathrm{Lim}_{\mathcal{U}}(k + \alpha_k + \lambda_k) = \mathrm{Lim}_{\mathcal{U}}(k + \alpha_k + t_0 + \epsilon_k) = \mathrm{Lim}_{\mathcal{U}}(k + \alpha_k + t_0)$ (by Lemma 2.19). But this is $g(t_0)$. A similar calculation shows that $\overline{f}(\beta + \rho) = h(t_0)$, so the proof follows from the fact that $g(t_0) = h(t_0)$.

(4) Since $\overline{f}(\alpha) = P_1$, we can write $P_1 = \mathrm{Lim}_{\mathcal{U}} f(k + \alpha_k)$. Since $P_2 \in L^+(f)$, we can, for each k, find $t_k > k + \alpha_k$ such that $f(t_k)$ is within $1/k$ of P_2. Let $\rho_k = t_k - (k + \alpha_k)$, then $\rho = \{\rho_k\}$ has the required property. ∎

Remarks. (1) Part (4) of the above theorem says, in effect, that "everything comes around again, infinitely often". This is illustrated in Example 4.8.

(2) If $W(t, x)$ is not sufficiently smooth, we can usually use C^∞-approximation and Theorem 2.12.

(3) The above theorem applies even if we only assume $\{f(t) : t > 0\}$ is bounded. In this case, let $\{f(t) : t > 0\}$ be contained in a closed ball B_1 in $\mathbf{R}^n$. Let B_2 be a larger closed ball such that B_1 is entirely contained in the interior of B_2. Then modify the definition of $W(x)$ so that it has the same values for $x \in B_1$ but is redefined on the boundary of B_2 so that as we trace $f(t)$ for negative values of t it cannot leave B_2. (Since t is going backwards, this means redefining W as, say, the unit vector normal to the surface of B_2 and pointing outward.) We can redefine W by the smooth Tietze theorem and f will exist (as it cannot "go off to ∞") and $f(t)$ will be unchanged for $t > 0$ as W is unchanged on B_1.

(4) Note that the above theorem has no hypothesis about the absence of equilibrium points, so, for $n = 2$, it applies even when Poincaré-Bendixson might not because of limiting equilibrium points. In this case, $L^+(f)$ may consist of equilibrium points and orbits between them (that is, orbits which tend to equilibria as $t \to \pm\infty$). The orbit based on $\mathbf{R}^+_{\mathcal{U}}$ hits all of these equilibria and traverses all of the orbits between them. This example shows there is no hope of proving that once $\overline{f}(\alpha) = \overline{f}(\alpha+\rho)$, then $\overline{f}(\beta) = \overline{f}(\beta+\rho)$ for *all* β. When the non-standard orbit hits an equilibrium point at, say, $\overline{f}(\alpha)$, then $\overline{f}(\alpha) = \overline{f}(\alpha+\rho)$ for any finite ρ, and, by (3) of the above theorem, any such finite ρ works as a period until sometime "infinitely later" when, by (4) of this theorem, the orbit must leave the equilibrium point to hit the other points in $L^+(f)$. See Example 4.8.

Change of variables. We conclude this section by examining the behavior of $(*)$ under a change of variables. We use the following notation:

(1) Members of $\mathbf{R}^n$ will be thought of as *column vectors*, that is, as $n \times 1$ matrices.

(2) If $f : \mathbf{R}^n \to \mathbf{R}^m$ is a C^1-function, then Df is the $m \times n$ matrix with $\partial f_i / \partial x_j$ in row i, column j. Recall that the chain rule says that if $h = fg$ then $Dh = Df Dg$ (matrix product).

(3) If $x : \mathbf{R} \to \mathbf{R}^n$ is a solution of $(*)$ and if W is regarded as a column vector of functions, then $(*)$ can be written in the form:

$$(*_1)\ Dx = W$$

These conventions make the following easier to state:

2.21. Theorem. *Let $(*_1)$ be as above and consider the change of the dependent variable from x to y suggested by the equations:*

$$y = \phi(x) \text{ and } x = \theta(y)$$

where ϕ, θ are C^∞-inverses of each other. Then $(*_1)$ is transformed into:

$$(*_2)\ Dy = D\phi W(t, \theta(y)).$$

For any filter ring A, there is a one-to-one correspondence between A-solutions of $(*_1)$ and of $(*_2)$ under which the $(*_1)$-solution $\sigma(u,t)$ corresponds to the $(*_2)$-solution $\phi(\sigma(u,t))$. This correspondence takes semi-bounded solutions to semi-bounded solutions.

Proof. It is readily shown that that $B(t,x)$ is a barrier function for $(*_1)$ iff $B(t, \theta y)$ is a barrier for $(*_2)$. The result then follows. The preservation of semi-bounded solutions is obvious and follows from Proposition 2.8 anyway. ∎

Having proven this theorem, we immediately use it to obtain a more general version:

2.22. Theorem. *More generally, let $(*_1)$ be as above and consider the change of the dependent variable from x to y suggested by the equations:*

$$y = \phi(t,x) \text{ and } x = \theta(t,y)$$

where ϕ, θ are C^∞-inverses of each other. Then $(*_1)$ is transformed into:

$$(*_2)\ Dy = \partial\phi/\partial t + D\phi W(t, \theta(y)).$$

For any filter ring A, there is a one-to-one correspondence between A-solutions of $(*_1)$ and of $(*_2)$ under which the $(*_1)$-solution $\sigma(u,t)$ corresponds to the $(*_2)$-solution $\phi(t, \sigma(u,t))$. This correspondence takes semi-bounded solutions to semi-bounded solutions.

Proof. This is really the same theorem as the above if we imagine using an additional variable x_0 which plays the role of t. So we redefine $W(x_0, x)$ as $W(t,x)$ and $W_0 = 1$. The equations of the previous theorem then reduce to the ones given above. We also have to impose an initial condition that $x_0(0) = 0$ and it is easily verified that the transformations preserve solutions which satisfy this condition.(Alternatively, we could argue as in the previous proof.) ∎

2.23. Theorem. *Let $(*_1)$ be $dx/dt = W(t,x)$ and consider the change of the independent variable from t to s suggested by the equations:*

$$s = \phi(t) \text{ and } t = \theta(s)$$

where ϕ, θ are C^∞-inverses of each other. Then $(*_1)$ is transformed into:

$$(*_2)\ dx/ds = W(\theta(s), x)\theta'(s).$$

For any filter ring A, there is a one-to-one correspondence between A-solutions of $(*_1)$ and of $(*_2)$ under which the $(*_1)$-solution $\sigma(u,t)$ corresponds to the $(*_2)$-solution $\sigma(u, \theta s))$. This correspondence takes semi-bounded solutions to semi-bounded solutions.

Proof. First, consider the transformation $s = -t$ and $t = -s$. Then $B(t, x)$ is a barrier function for $(*_1)$ over $[a, b]$ iff $-B(s, x)$ is a barrier function over $[-b, -a]$ for $(*_2)$ and the result follows in this special case. In general, notice that $\phi'(t)$ can never be 0, as ϕ has a differentiable inverse. We may as well assume that $\phi'(t) > 0$ for all t (otherwise, first change t to $-t$, using the above case.) It is then readily shown that $B(t, x)$ is a barrier function for $(*_1)$ over $[a, b]$ iff $B(\theta s, x)$ is a barrier for $(*_2)$ over $[\phi a, \phi b]$ (and conversely). The result follows. The preservation of semi-boundedness is obvious. ∎

3. Solution Manifolds in a Smooth Topos

Our goal is to define and examine the "manifold" of all solutions of the system $(*)$ using a generalized notion of manifold which, in effect, allows for non-standard solutions. Typically a smooth topos is regarded as a category of generalized manifolds. The advantage of working in a topos is that it has good categorical properties, such as the existence of power objects, M^N, which conceptually is the "manifold" of all smooth maps from M to N. Also, in a topos, we can define subobjects using the internal language, as discussed below.

By a smooth topos we mean a topos which fully contains the category $\mathcal{M}$ of C^∞-manifolds and smooth (i.e. C^∞) maps. We follow the approach in [4] and extend $\mathcal{M}$ in several stages. First, each manifold $M \in \mathcal{M}$ gives rise to $C^\infty(M)$, the finitely presented C^∞-ring of all smooth maps from M to $\mathbf{R}$. This embeds $\mathcal{M}$ fully into $\mathbf{V}$ the dual of f.g. reduced C^∞-rings (where "f.g." means finitely generated and "reduced", defined algebraically in [5], is equivalent to being a filter ring. See also [4] where these rings are described as being of the form $C^\infty(\mathbf{R}^m)/I$ where I is a C^∞-radical ideal.)

By PreSh($\mathbf{V}$) we mean the category of functors from $\mathbf{V}^{op}$ to **Sets**, equivalently, the category of functors from filter rings to **Sets**. Note that if $A = C^\infty(\mathbf{R}^m)/I(\mathcal{F})$ and $B = C^\infty(\mathbf{R}^k)/I(\mathcal{G})$, then a C^∞-homomorphism from $h : A \to B$ is given by a smooth map $\eta : \mathbf{R}^k \to \mathbf{R}^m$ for which $\eta^{-1}(F) \in \mathcal{G}$ whenever $F \in \mathcal{F}$. Then h is defined by $h(\alpha)(u) = \alpha(\eta(u))$, see [4].

A presheaf is a *sheaf with respect to the finite open cover topology* if it maps every covering sieve to a limit diagram (see [4], pages 350 and 364 for details). This topology is *subcanonical* which means that every filter ring A determines a sheaf $[A, \_]$, which is the representable hom functor which assigns hom$[A, B]$ to the filter ring B.. Following [4], we let $\mathcal{V}_{\text{fin}}$ denote the topos of sheaves. If we trace the embedding of $\mathcal{M}$ into $\mathcal{V}_{\text{fin}}$, we see that the real line $\mathbf{R}$ is mapped to the underlying set functor from filter rings to **Sets**.

The internal language. As with any topos, $\mathcal{V}_{\text{fin}}$ has an internal language, see [4] pages 353-361. We use this language to define subsheaves much as subsets can be defined by conditions. In terms of this language, $\mathbf{R}$ is a ring with a compatible order. This follows since for each object A, the set $\mathbf{R}(A) = A$ has such a structure, and this structure is preserved by the maps. The ring structure on A is obvious and, for $\alpha, \beta \in A$, represented by $\alpha(u), \beta(u)$, we define $\alpha < \beta$ iff $\{u : \alpha(u) < \beta(u)\} \in \mathcal{F}$.

We next define x, y in $\mathbf{R}$ to be *infinitesimally close* if for each ordinary positive integer n, we have $-1/n < (x - y) < 1/n$. It is important to note that here x, y are internal variables associated with the sheaf $\mathbf{R}$, while n is an external variable (or an *ordinary* integer). We could, alternatively, use the natural number object, but we will not go into that. In summary:

(1) The relation $<$ has been defined on $\mathbf{R}$. (It is a subsheaf of $\mathbf{R} \times \mathbf{R}$.)

(2) The relation of being infinitesimally close on $\mathbf{R}$ (another subsheaf of $\mathbf{R} \times \mathbf{R}$) has been defined by the conditions $-1/n < (x-y) < 1/n$ for each ordinary positive integer n. By a convenient abuse of notation, we sometimes write this condition as $|x-y| < 1/n$, even though there is no actual absolute value operation from the sheaf $\mathbf{R}$ to itself. Similarly, we define infinitesimally close on $(\mathbf{R})^n$ by using the projections to $\mathbf{R}$. The defining conditions can be abbreviated to $\|x - y\| < 1/n$ for each ordinary positive integer n.

(3) The object $\mathbf{R}$ in $\mathcal{V}_{\text{fin}}$ lacks nilpotents, so the elegant Kock-Lawvere definition of the derivative is not available. Instead we define solutions to $(*)$ using the barrier conditions. This is, in effect, a non-standard analysis approach and it is useful for analyzing stability when we move infinitesimally away from an equilibrium point. See Examples 4.1, 4.2.

3.1. Definition. *The* solution manifold *for* $(*)$*, denoted by* **Sol***, is defined internally as the subobject of all* $f \in (\mathbf{R}^n)^{\mathbf{R}}$ *for which* $B(a, f(a)) > 0$ *or* $B(b, f(b)) < 0$ *whenever* B *is an ordinary barrier function over* $[a, b]$ *for* $(*)$.

It will follow from the results of section 2 and the lemma below that **Sol** is the functor which assigns to A the set of all A-solutions of $(*)$. We need to describe the functor $\mathbf{R}^{\mathbf{R}}$ from filter rings to **Sets**. Let $A = C^\infty(\mathbf{R}^m)/I(\mathcal{F})$ be a filter ring. Then, by the Yoneda lemma, $\mathbf{R}^{\mathbf{R}}(A) = n.t.([A, \_], \mathbf{R}^{\mathbf{R}})$, where "$n.t.$" stands for the set of all natural transformations, and "=" means "naturally isomorphic".

3.2. Lemma. *Let* $A = C^\infty(\mathbf{R}^m)/I(\mathcal{F})$*. Then* $\mathbf{R}^{\mathbf{R}}(A) = \mathrm{Map}(A)$ *as defined in section 2. It follows that the functor* $(\mathbf{R}^n)^{\mathbf{R}}$ *is* n-$\mathrm{Map}(A)$ *and that* $\mathbf{Sol}(A)$ *is the set of all* A*-solutions of* $(*)$.

Proof. The main steps are that $\mathbf{R}^{\mathbf{R}}(A)$ is naturally isomorphic to $\mathrm{Map}(A)$ are:

$n.t.([A, \_], \mathbf{R}^{\mathbf{R}}) = n.t.(\mathbf{R} \times [A, \_], \mathbf{R}) = n.t.([\mathrm{Map}(A), \_], \mathbf{R}) = \mathrm{Map}(A)$.

The remaining details are straightforward, but note we do have to use a covering argument: Let $\sigma(u, t)$ be an A-solution for some A and let B be a barrier over $[a, b]$. Then $\{U, V\}$ is a finite cover where $U = \{u : B(a, \sigma(u, a)) > 0\}$ and $V = \{u : B(b, \sigma(u, b)) < 0\}$. We can now readily show that $\sigma \in \mathbf{Sol}(A)$. ■

One drawback of the topos $\mathcal{V}_{\text{fin}}$ is that the real line (i.e. the underlying set functor, or the object which corresponds to the manifold of reals) is not Archimedean. For technical reasons (perhaps because our techniques are not good enough) we need to deal with the "bounded reals".

3.3. Definition. *The* bounded reals*,* $\mathbf{R}_0$*, is defined internally as the subobject of all* $x \in \mathbf{R}$ *for which there exists an ordinary integer* n *with* $-n < x < n$.

3.4. Definition. *(1) The sheaf of* $\mathbf{R}_0$-preserving maps *is defined internally as the subsheaf of* $\mathbf{R}^{\mathbf{R}}$ *of all* $f \in \mathbf{R}^{\mathbf{R}}$ *for which* $f(t) \in \mathbf{R}_0$ *whenever* $t \in \mathbf{R}_0$.

(2) The submanifold of semi-bounded solutions *for* $(*)$, *denoted by* $\mathbf{Sol}_0$, *is defined internally as the subobject of all* $f \in \mathbf{Sol}$ *which are* $\mathbf{R}_0$*-preserving (meaning that all components of* f *are* $\mathbf{R}_0$*-preserving).*

(3) The standardization manifold *of* $(*)$, *denoted by* $\mathbf{Stnd}$, *is obtained internally from* $\mathbf{Sol}_0$ *by setting two solutions,* f *and* g, *to be equivalent if* $f(t)$ *is infinitesimally close to* $g(t)$ *for each ordinary real number* t.

We will show that the sheaf of $\mathbf{R}_0$-preserving maps coincides with the functor Map_0 of semi-bounded maps, that $\mathbf{Sol}_0$ is the functor which assigns to A the set of all semi-bounded A-solutions of $(*)$, and that $\mathbf{Stnd}(A) = \mathbf{Sol}_0(A)/\mathrm{I}_{\mathrm{PtCn}}$. To do this, we need to examine $\mathbf{R}_0$, $\mathbf{R}^{\mathbf{R}_0}$ and $(\mathbf{R})_0^{\mathbf{R}_0}$.

3.5. Definition. *Let* $\mathcal{J}$ *be the set of all closed, bounded intervals of the ordinary reals. Let* $J \in \mathcal{J}$ *be given. Let* A *be a* C^∞*-ring. Then* $a \in A$ *is* J-bounded *if* $\lambda(a) = 0$ *whenever* $\lambda \in C^\infty(\mathbf{R})$ *vanishes on* J. *We way that* $a \in A$ *is* bounded *if there exists* $J \in \mathcal{J}$ *for which* a *is* J*-bounded.*

3.6. Lemma. *The object* $\mathbf{R}_0$, *regarded as a functor from* $\mathbf{V}^{op}$ *to* $\mathbf{Sets}$, *has the following properties:*

(1) $\mathbf{R}_0(A)$ *is the set of all bounded elements of* A.

(2) $\mathbf{R}_0$ *is the filtered colimit of the intervals* J, *for* $J \in \mathcal{J}$. *(Recall that each such* J *is identified with the representable functor* $[C^\infty(J), \_]$*.)*

(3) Let $A = C^\infty(\mathbf{R}^m)/I(\mathcal{F})$ *and let* $J \in \mathcal{J}$. *Define* $A_J = C^\infty(\mathbf{R}^m \times J)/I(\pi^*\mathcal{F})$. *The functors* $\mathbf{R}_0 \times [A, \_]$ *and* $\mathrm{Colim}[A_J, \_]$ *are then naturally equivalent.*

Proof. (1) If $x \in A$ satisfies $-n < x < n$, then x is obviously $[-n, n]$-bounded. Conversely, if x is $[-m, m]$-bounded, for any $m > n$, then x satisfies $-n < x < n$. It is easy to show that the set of all bounded elements is functorial and a sheaf and the result readily follows.

(2) Each $x \in A$ is the image of the identity under a unique homomorphism $h : C^\infty(\mathbf{R}) \to A$. It follows that x is J-bounded iff $h(\lambda) = \lambda(x) = 0$ whenever λ is in the kernel of the restriction map from $C^\infty(\mathbf{R})$ to $C^\infty(J)$ and the result follows.

(3) $\mathbf{R}_0 \times [A, \_] = (\mathrm{Colim}[J, \_]) \times [A, \_] = \mathrm{Colim}([J, \_] \times [A, \_])$. It is easily shown that $[J, \_] \times [A, \_] = [A_J, \_]$ and the result follows. ∎

The following lemma describes $\mathbf{R}^{\mathbf{R}_0}$.

3.7. Lemma. *Let* $A = C^\infty(\mathbf{R}^m)/I(\mathcal{F})$. *An element of* $\mathbf{R}^{\mathbf{R}_0}(A)$ *is then represented by a filtered family* $\{\alpha_J : J \in \mathcal{J}\}$ *where* $\alpha_J \in C^\infty(\mathbf{R}^m \times J)$ *and where the family is filtered in the sense that if* $J \subseteq K$ *then there exists* $F \in \mathcal{F}$ *such that* $\alpha_J(u, t) = \alpha_K(u, t)$ *whenever* $u \in F$ *and* $t \in J$.

Moreover, if $\alpha \in \mathbf{R}^{\mathbf{R}}(A)$ *is represented by* $\alpha(u, t) \in C^\infty(\mathbf{R}^m \times \mathbf{R})$ *then the restriction of* α *to* $\mathbf{R}^{\mathbf{R}_0}$ *is represented by the family of restrictions,* $\{\alpha_J\}$, *of* α *to* $\mathbf{R}^m \times J$.

Proof. $\mathbf{R}^{\mathbf{R}_0}(A) = n.t.([A,\_],\mathbf{R}^{\mathbf{R}_0}) = n.t.(\mathbf{R}_0 \times [A,\_],\mathbf{R}) = n.t.(\mathrm{Colim}[A_J,\_],\mathbf{R}) = \lim\, n.t.([A_J,\_],\mathbf{R}) = \lim A_J$ which leads to the above results. ∎

The question arises as to when a filtered family $\{\alpha_J\}$ can be patched together to be equivalent to the restrictions of some $\alpha \in C^\infty(\mathbf{R}^m \times \mathbf{R})$. The answer is "always", as shown below. Although this proposition is not needed in our development, it seems interesting on its own.

3.8. Proposition. *The restriction map from $\mathbf{R}^{\mathbf{R}}$ to $\mathbf{R}^{\mathbf{R}_0}$ is onto (but not always one-to-one).*

Proof. Let $A = C^\infty(\mathbf{R}^m)/I(\mathcal{F})$ be given. Let $\alpha_J \in C^\infty(\mathbf{R}^m \times J)$ be filtered (as in the above lemma). We need to find $\alpha \in C^\infty(\mathbf{R}^m \times \mathbf{R})$ such that the restriction of α to $\mathbf{R}^m \times J$ is equivalent to α_J for each $J \in \mathcal{J}$. In other words, there must exist, for each J, an $F \in \mathcal{F}$ such that $\alpha(u,t) = \alpha_J(u,t)$ for all $u \in F$ and $t \in J$.

For each positive integer n, let $\alpha_n = \alpha_{[-n,n]}$. Let $F_n \in \mathcal{F}$ be a closed set such that $\alpha_n(u,t) = \alpha_{n+1}(u,t)$ for $u \in F_n$ and $|t| \le n$. We may as well assume that the F_n's are nested, or that $F_n \supseteq F_{n+1}$ for all n, otherwise replace F_n by $F_1 \cap \ldots \cap F_n$. We observe that if $n < m$ and $u \in F_m$ then $\alpha_n(u,t) = \alpha_m(u,t)$ for $|t| \le n$.

For each $u \in \mathbf{R}^m$ let $\deg(u) = \max\{k : u \in F_k\}$. If $u \notin F_1$ let $\deg(u) = 0$ and if $u \in F_k, \forall k$ let $\deg(u) = \infty$. (There need not be any u with $\deg(u) = \infty$.) Let:

$$Q = \{(u,t) \in \mathbf{R}^m \times \mathbf{R} : |t| + 2 \le \deg(u)\}$$

We claim that Q is closed. Suppose $(u_0,t_0) \notin Q$. We need to find a neighborhood of (u_0,t_0) which misses Q. Clearly $|t_0| + 2 > \deg(u_0)$ since $(u_0,t_0) \notin Q$. Let $n = \deg(u_0)$ then $u_0 \notin F_{n+1}$ and $|t_0| > n-2$. Since F_{n+1} is closed, the set $\{(u,t) : u \notin F_{n+1}, |t| > n-2\}$ is the required neighborhood of (u_0,t_0).

Now define $\alpha(u,t)$ for $(u,t) \in Q$ as $\alpha_n(u,t)$ provided that $u \in F_n$, $|t| \le n$. Such an n must exist, for example, $n = \min\{k : |t| < (k - 0.5)\}$, as can be readily verified. On the other hand, if $u \in F_n$, $|t| \le n$ and $u \in F_m$, $|t| \le m$ then $\alpha_n(u,t) = \alpha_m(u,t)$ because if $n < m$ this follows from the definition and nested property of F_m.

It remains to show that α can be smoothly extended from Q to all of $\mathbf{R}^m \times \mathbf{R}$, as it then readily follows that α has the required restrictions. By using partitions of unity, it suffices to find a neighborhood of each $(u,t) \in Q$ to which α can be smoothly extended. But given $(u,t) \in Q$ let $n = \min\{k : |t| < (k-0.5)\}$, let U be the set where $(n-1.4) < |t| < (n-0.5)$, then α on $Q \cap U$ extends smoothly to α_n on U.

That the restriction from $\mathbf{R}^{\mathbf{R}}$ to $\mathbf{R}^{\mathbf{R}_0}$ is not always one-to-one follows from the example in the following remark. ∎

Remark. Let $A = C^\infty(\mathbf{R})/I(\mathcal{F})$ where $\mathcal{F}$ is the "filter at infinity" meaning the filter generated by sets of the form $[u,\infty)$. We can readily find $\alpha, \beta \in \mathrm{Map}(A)$ having equivalent sets of restrictions to members of $\mathcal{J}$. For example, we can let α be 0 on precisely $\{(u,t) : |t| \le u\}$ while β is identically 0.

On the other hand, if $A = C^\infty(\mathbf{R}^m)$ or even $C^\infty(M)$ for M a classical C^∞-manifold, then restriction from $\mathbf{R}^{\mathbf{R}}(A)$ to $\mathbf{R}^{\mathbf{R}_0}(A)$ is one-to-one and onto, as is easily seen.

We need to describe $\mathbf{R}_0^{\mathbf{R}_0}$ to show that the semi-bounded maps correspond to those which are "$\mathbf{R}_0$ preserving". First we need:

3.9. Lemma. *Let $A = C^\infty(\mathbf{R}^m)/I(\mathcal{F})$ be a given filter ring. Then $\mathbf{R}_0^{\mathbf{R}_0}(A) \subseteq \mathbf{R}^{\mathbf{R}_0}(A)$ is represented by the filtered families $\{\alpha_J\} \in \mathbf{R}^{\mathbf{R}_0}(A)$ for which each α_J is bounded, meaning that for each J there exists $B > 0$ and $F \in \mathcal{F}$ such that $|\alpha_J(u)| < B$ for all $u \in F$.*

Proof. Using the same method as in the proofs of 3.2 and 3.7, we see that $\mathbf{R}_0^{\mathbf{R}_0}(A) = \lim \text{n.t.}([A_J, \_], \mathbf{R}_0) = \lim(\mathbf{R}_0(A_J))$ which readily leads to the stated result. ∎

3.10. Proposition. *$\alpha \in \mathbf{R}^{\mathbf{R}}(A)$ is $\mathbf{R}_0$-preserving iff α is represented by $\alpha(u,t) \in C^\infty(\mathbf{R}^n \times \mathbf{R})$ which is semi-bounded with respect to $\mathcal{F}$.*

Proof. Clearly, α is $\mathbf{R}_0$-preserving iff the restriction of α to $\mathbf{R}^{\mathbf{R}_0}(A)$ is actually in $\mathbf{R}_0^{\mathbf{R}_0}(A)$. But this restriction is represented by the set of restrictions, $\{\alpha_J\}$ of α to $\mathbf{R}^n \times J$ for $J \in \mathcal{J}$. It is readily shown that this set of restrictions satisfies the condition of the above lemma iff α is semi-bounded with respect to $\mathcal{F}$. ∎

Behavior near a given equilibrium point. In what follows, we assume that $(*)$ is autonomous, meaning it is of the form $dx/dt = W(x)$ where W depends only on x, not on t. We further assume that $\overline{x}$ is an *equilibrium point of* $(*)$ meaning that $W(\overline{x}) = 0$, so the function which is constantly equal to $\overline{x}$ is a solution of $(*)$. The stable solutions are those which, in the internal language, stay infinitesimally close to $\overline{x}$. We also define the asymptotically stable solutions, even though the definition does not seem readily expressible in the internal language.

3.11. Definition. *The* submanifold of stable solutions *of $(*)$, denoted by* **Stab**, *(we assume $\overline{x}$ and $(*)$ are understood) is defined internally as the subfunctor of $\mathbf{Sol}_0$ of all solutions f with $f(t)$ infinitesimally close to $\overline{x}$ for each ordinary non-negative t.*

3.12. Definition. *The* submanifold of asymptotically stable solutions *of $(*)$, denoted by* **Asym**, *is defined as the following subfunctor of* **Stab**: *If $A = C^\infty(\mathbf{R}^m)/I(\mathcal{F})$ then $\mathbf{Asym}(A)$ is the set of all $\sigma(u,t) \in \mathbf{Stab}(A)$ for which $\exists F \in \mathcal{F}$ having the property that $\forall\epsilon > 0$, $\exists t_0$ such that $u \in F$ and $t > t_0$ imply $\|\sigma(u,t) - \overline{x}\| < \epsilon$. It readily follows that* **Asym** *is functorial, and a sheaf.*

If $A = C^\infty(\mathbf{R}^m)/I(\mathcal{F})$, then $\mathbf{Stab}(A)$ is readily shown to be the set of all $\sigma(u,t) \in \mathbf{Sol}_0(A)$ such that for every $\epsilon > 0$, $\exists F \in \mathcal{F}$ with $\|\sigma(u,t) - \overline{x}\| < \epsilon$ whenever $u \in F$ and $t \geq 0$.

Classically, we assume that the smooth map $\phi(x_0, t)$ is the unique solution of $(*)$ with initial value x_0. Then $\overline{x}$ is a *stable equilibrium point* if for every $\epsilon > 0$ there is a $\delta > 0$ such that $\|x_0 - \overline{x}\| < \delta$ implies $\|\phi(x_0,t) - \overline{x}\| < \epsilon$ for all $t \geq 0$. If, in addition, there exists $r > 0$ such that $\|x_0 - \overline{x}\| < r$ implies $\|\phi(x_0,t) - \overline{x}\| \to 0$ as $t \to \infty$, then $\overline{x}$ is *asymptotically stable*. See [2]. The connection between these conditions is given by:

3.13. Lemma. *Let $\phi(x_0, t)$ be as above and let $A = C^\infty(\mathbf{R}^n)/I(\mathcal{F})$ where $\mathcal{F}$ is the filter of neighborhoods of $\overline{x}$. Note that $\phi(x_0, t)$ is clearly semi-bounded, as $\mathcal{F}$ has bounded members. Then:*

(1) $\phi(x_0, t) \in \mathbf{Stab}(A)$ iff $\overline{x}$ is a stable equilibrium point.

(2) $\phi(x_0, t) \in \mathbf{Asym}(A)$ iff $\overline{x}$ is asymptotically stable.

Proof. (1) Suppose $\overline{x}$ is stable. Let $\epsilon > 0$ be given. Then, by stability, there exists $\delta > 0$ such that $\|x_0 - \overline{x}\| < \delta$ implies that, for all $t \geq 0$, we have $\|\phi(x_0, t) - \overline{x}\| < \epsilon$. Let $F \in \mathcal{F}$ be the δ-neighborhood of $\overline{x}$ then F has the required property which shows that $\phi \in \mathbf{Stab}(A)$. The proof of the converse is similar.

(2) If $\overline{x}$ is asymptotically stable there exists $r > 0$ such that $\|x_0 - \overline{x}\| < r$ implies $\|\phi(x_0, t) - \overline{x}\| \to 0$ as $t \to \infty$. Let F be the r-neighborhood about $\overline{x}$ and then $F \in \mathcal{F}$ has the required property. Again, the converse is similar. ■

Stabilization. Given A and $\sigma \in \mathbf{Sol}_0(A)$, we find the "best extension" $\nu : A \to A_\sigma$ of A in which σ becomes stable, meaning that $\mathbf{Sol}_0(\nu)(\sigma) \in \mathbf{Stab}(A_\sigma)$.

3.14. Definition. *Given $\sigma \in \mathbf{Sol}_0(A)$, then $h : A \to B$* makes σ stable *if $\mathbf{Sol}_0(h)(\sigma) \in \mathbf{Stab}(B)$.*

3.15. Theorem. *Let $\sigma \in \mathbf{Sol}_0(A)$. There exists $\nu : A \to A_\sigma$ which makes σ stable and such that whenever $h : A \to B$ makes σ stable then h factors as $h = h_0\nu$ for a unique h_0.*

Proof. For each $\epsilon > 0$ let

$$E_\epsilon = \{u \in \mathbf{R}^m : \|\sigma(u, t) - \overline{x}\| \leq \epsilon, \forall t \geq 0\}$$

Let $\mathcal{F}_\sigma$ be the filter generated by adding all sets E_ϵ to $\mathcal{F}$. Let $A_\sigma = C^\infty(\mathbf{R}^m)/I(\mathcal{F}_\sigma)$ (If $\mathcal{F}_\sigma$ is the improper filter, then the C^∞-ring A_σ is trivial.) Let $\nu : A \to A_\sigma$ be the obvious homomorphism, (induced by the identity on $\mathbf{R}^m$). It is then readily shown that ν makes σ stable. Suppose $h : A \to B$ also makes σ stable, where $B = C^\infty(\mathbf{R}^k)/I(\mathcal{G})$. Let h be determined by a smooth map $\eta : \mathbf{R}^k \to \mathbf{R}^m$ for which $\eta^{-1}(F) \in \mathcal{G}$ whenever $F \in \mathcal{F}$. Then, as h makes σ stable, there exists $G_\epsilon \in \mathcal{G}$ such that $\forall t \geq 0, v \in G_\epsilon$ we have $\|\sigma(\eta v, t) - \overline{x}\| \leq \epsilon$. This implies that $G_\epsilon \subseteq \eta^{-1}(E_\epsilon)$ which shows that η defines a map $h_0 : A_\sigma \to B$. The remaining details are straightforward. ■

The topos of presheaves. The definition of **Sol** depended on the fact that we were using the finite open cover topology as indicated in the proof of Lemma 3.2. If we interpreted the same condition in the topos of presheaves, we would get a presheaf which would miss many solutions. But, even in this topos, we could define **Sol** by the condition that "If $B(a, f(a)) \leq 0$, then $B(b, f(b)) < 0$" (which need not be equivalent to "$B(a, f(a)) > 0$ or $B(b, f(b)) < 0$" because the internal logic is intuitionistic). The presheaf defined by the "If-then" condition would actually be the sheaf of solutions, **Sol**. The definitions of $\mathbf{R}$, $\mathbf{R}_0$, $\mathbf{R}^{\mathbf{R}}$, $\mathbf{R}^{\mathbf{R}_0}$, $\mathbf{R}_0^{\mathbf{R}_0}$, $\mathbf{Sol}_0$, **Stnd**, **Stab** can all be interpreted in the topos of presheaves and the presheaves so defined would coincide with the sheaves defined above.

4. Examples

This section fleshes out the theory with several examples. All of these examples are assumed to be equations for a single function (in other words, $n = 1$) unless the contrary is explicitly noted.

4.1. Example. *Comparison of barrier conditions with the internal derivative.*

There is an internal notion of a derivative (based on nilpotents) for maps in a smooth topos which contains the topos we are working in. Obviously, this notion can be used to define the object of solutions of $(*)$. In general, our definition, based on barrier conditions, allows for a larger solution object than the definition based on the internal derivative. But for C^∞-rings of the form $C^\infty(F)$, where $F \subseteq \mathbf{R}^m$ is a closed subset, the two definitions for the solution object agree. (This case includes rings of the form $C^\infty(M)$ for M a C^∞-manifold.) In these cases, the solution object at $C^\infty(F)$ contains the maps $\sigma(u,t) \in C^\infty(F \times \mathbf{R})$ which are *smooth parameterized solutions over* F in the sense that σ is smooth and for each fixed $u \in F$, we have $f(t) = \sigma(u,t)$ is a (smooth) classical solution of $(*)$. For other C^∞-rings, the internal definition still leads to the object of maps that can be represented by smooth parameterized solutions. The barrier conditions definition leads to a larger solution object as shown by the next example.

4.2. Example. *The differential equation $dx/dt = 0$.*

Classically, the solutions of this equation are the constant functions. In filter rings, the semi-bounded solutions are "nearly constant" in the sense indicated below. We will show that:

(1) If $g \in C^\infty(\mathbf{R})$ and $\sigma(u,t)$ is a solution of $dx/dt = 0$, then so is $g(\sigma(u,t))$.

(2) If $f \in C^\infty(\mathbf{R})$, then $B(t,x) = f(x) - t$ is a barrier function over *any* interval $[a,b]$, for which $a < b$.

(3) Internally, an $\mathbf{R}_0$-preserving function f is a solution of $dx/dt = 0$ iff, for every pair of ordinary reals a, b, $f(a)$ is infinitesimally close to $f(b)$. Externally the semi-bounded $\sigma(u,t)$ is a solution iff, for all a, b and $\epsilon > 0$, $\exists F \in \mathcal{F}$ such that $|\sigma(u,a) - \sigma(u,b)| < \epsilon$ for all $u \in F$.

(4) In general, if $\sigma(u,t)$ is a solution (not necessarily semi-bounded) then, given a, b, $\arctan \sigma(u,a)$ must be infinitesimally close to $\arctan \sigma(u,b)$ (for all u in some member of the filter) but not conversely. In fact $\sigma(u,t)$ can be infinitesimally close to a constant (even uniformly so) without being a solution. So the semi-bounded hypothesis (that $\sigma \in n\text{-}\mathrm{Map}_0(A)$) cannot be eliminated from Proposition 2.10.

Proof. (1) Clearly $B(t, g(x))$ is a barrier whenever $B(t,x)$ is and the result follows.

(2) Note that $\partial B/\partial t = -1 < 0$ which is more than enough.

(3) If σ satisfies the given condition, then for any maximal closed filter $\mathcal{M}$ which extends $\mathcal{F}$ it is clear that $\sigma_{\mathcal{M}}(a) = \sigma_{\mathcal{M}}(b)$ for all a, b. So $\sigma_{\mathcal{M}}$ is a solution of $(*)$ which implies that σ is an A-solution, by Proposition 2.8.

Conversely, if there exist a, b, ϵ such that $E = \{u : |\sigma(u,a) - \sigma(u,b)| < \epsilon\}$ is not in $\mathcal{F}$, then we can extend $\mathcal{F}$ to a maximally closed filter $\mathcal{M}$ which contains E^c, the closed set which is the complement of E. It follows that $|\sigma_{\mathcal{M}}(a) - \sigma_{\mathcal{M}}(b)| \geq \epsilon$ which shows that $\sigma_{\mathcal{M}}$ cannot be a solution of $(*)$, which contradicts Proposition 2.8.

(4) The first sentence follows from (1), using $g = \arctan$, and (3), as $\arctan \sigma(u,t)$ is obviously semi-bounded. As for the second part, let $A = C^\infty(\mathbf{R})/I(\mathcal{F})$ where $\mathcal{F}$ is the filter generated by sets of the form $[u, \infty)$ for $u \in \mathbf{R}$. Let $\theta(u,t) \in C^\infty(\mathbf{R} \times \mathbf{R})$ be such that $\theta(u,t) = u + (t/u)$ for $u \geq 1$. Then θ is not an A-solution: Let $B(t,x) = f(x) - t$, where $f \in C^\infty(\mathbf{R})$ is such that $f(n) < 0$ for all integers n and $f(n + (1/n)) > 1$ for all $n \geq 2$. Then $B(0, \theta(n, 0)) < 0$ and $B(1, \theta(n, 1)) > 0$ for all integers $n \geq 2$. So θ does not even satisfy the weak barrier condition. In fact, defining $\theta(u,t) = u + (sin\, t)/u$ we get a non-solution which is *uniformly* close to constants and to actual solutions. ∎

4.3. Example. *The differential equation $dy/dt = y$.*

This equation can be obtained from $dx/dt = 0$ by the transformation $y = e^t x$ and $x = e^{-t} y$, using Theorem 2.22. So, $\sigma(u,t)$ is a solution for $dy/dt = y$ iff $e^{-t}\sigma(u,t)$ is a solution for $dx/dt = 0$. The results of Example 4.2 can now be restated for this equation.

4.4. Example. *High curves, low curves and barrier functions.*

The notions of "high curve" and "low curve" are introduced in [3], using a different notation, and treating the equation $x' = W(t,x)$ for a single function $x(t)$ (so $n = 1$). We recall that $h(t)$ is a *high curve* (resp. $\ell(t)$ is a *low curve*) over $[a,b]$ iff $h'(t) > W(t, h(t))$ (resp. $\ell'(t) < W(t, \ell(t))$) for all $t \in [a,b]$. It follows that $B(t,x) = x - h(t)$ (resp. $B(t,x) = \ell(t) - x$) is a barrier function over $[a,b]$ (but not conversely.) Clearly $f(t)$ satisfies the corresponding growth conditions (for these barriers, or as in [3]) iff either $f(a) > h(a)$ or $f(b) < h(b)$ (resp. iff either $f(a) < \ell(a)$ or $f(b) > \ell(b)$). So if $f(t)$ is transfinite for all t (which can happen in a filter ring, if f is not semi-bounded) then f satisfies all conditions arising from high and low curves, which is quite different from, for example, the restrictions and non-solutions mentioned in (4) of Example 4.2. (We note that the definition of "solution", as given in [3], requires more than the satisfaction of each individual growth condition.)

4.5. Example. *Smooth, global systems.*

The system $dx/dt = W(t,x)$, where $W(t,x) = (W_1(t,x), \ldots, W_n(t,x))$, is *smooth and global* if each W_i is C^∞ and if there exists a global solution with any given initial value $(x_1(0), \ldots, x_n(0))$. In this case, as is well-known, there exists $\phi(c,t) = (\phi_1(c,t), \ldots, \phi_n(c,t))$ which is the unique solution with initial condition c, for $c \in \mathbf{R}^n$. Moreover, each ϕ_i is known to be C^∞. Then:

(1) Let $A = C^\infty(\mathbf{R}^m)/I(\mathcal{F})$ be any filter ring. Let $c(u) = (c_1(u), \ldots, c_n(u))$ represent an n-tuple of *bounded* members of A. Then $\phi(c(u), t)$ represents a semi-bounded A-solution.

(2) Conversely, for every semi-bounded A-solution σ, there is $c(u)$, as above, such that for every t and every $\epsilon > 0$ there exists $F \in \mathcal{F}$ such that $\|\sigma(u,t) - \phi(c(u), t)\| < \epsilon$ for all $u \in F$.

(3) $\mathbf{Stnd}(A)$ is equivalent to the set of all n-tuples of bounded members of A. So the manifold **Stnd** is isomorphic to $\mathbf{R}_0^n$. This indicates that the standardized solutions are parameterized by n-tuples of (bounded) reals, just as the classical solutions are.

Proof. (1) ϕ clearly satisfies the barrier conditions. Since ϕ is continuous, if we put a bound on the initial values c and if t varies in some bounded interval J, then the values of ϕ will be bounded. This shows that ϕ is semi-bounded.

(2) Let $\sigma(u,0) = (\sigma_1(u,0), \ldots, \sigma_n(u,0))$. Since σ is semi-bounded, there exists $F \in \mathcal{F}$ on which the maps $\sigma_i(u,0)$ are bounded, so we can define bounded maps $c(u)$ where $c_i(u) = \sigma_i(u,0)$ for $u \in F$. The condition saying that σ is within ϵ of ϕ must be met, otherwise, arguing as in the proof of (3) in Example 4.2, we can find a solution $\sigma_{\mathcal{M}}$ which has the same initial conditions as $\phi_{\mathcal{M}}$ but values differing by ϵ for some t.

(3) This follows from (2) when we factor out by $\mathrm{I}_{\mathrm{PtCn}}$ to get $\mathbf{Stnd}(A)$. ∎

4.6. Example. *The differential equation* $dx/dt = 2\sqrt{|x|}$.

In this case, the only smooth, classical solution is constantly 0. So, by Example 4.1, if $A = C^\infty(\mathbf{R})$ or $C^\infty(M)$ for a manifold M, then the only A-solution is identically 0. But, we can approximate $2\sqrt{|x|}$ by a smooth function $V(u,t,x)$ as $u \to \infty$. So if $A = C^\infty(\mathbf{R})/I(\mathcal{F})$ where $\mathcal{F}$ is generated by $\{[u,\infty)\}$, we can get any ordinary, non-smooth solution of $dx/dt = 2\sqrt{|x|}$ in the form $\sigma_{\mathcal{M}}$ by choosing V with care. We can also choose V so that we get distinct A-solutions with the same initial conditions.

4.7. Example. *The differential equation* $dx/dt = 1 + x^2$.

For this equation, $B(t,x) = (t/2) - \arctan(x)$ is a barrier function over any interval $[a,b]$. If $a < -\pi$ and $b > \pi$ then $B(a,\sigma(u,a)) < 0$ and $B(b,\sigma(u,b)) > 0$ for any σ. It follows that this equation has no (global) solutions. (A different approach in [3] allowed infinite solutions.) Note that $1 + x^2$ can be pointwise approximated by functions $V(x)$ for which $dx/dt = V(u,x)$ does have smooth global solutions. This shows that Theorem 2.12 cannot be extended from semi-bounded solutions to all solutions.

4.8. Example. *The system* $dr/dt = (1-r)^2$, $d\theta/dt = \cos^2\theta + 1 - r$.

Consider the solution f which starts at $r = 1/2, \theta = 0$. Then $r(t) = 1 - \frac{1}{t+2}$ approaches 1 and θ steadily increases to ∞, as $t \to \infty$. Think of (r,θ) as polar coordinates and rewrite the system in terms of rectangular coordinates (x,y), for $|r| \geq 1/3$. This region contains the orbit of f for $t > 0$. The resulting equations, defined smoothly on $r \geq 1/3$, can be smoothly extended to the whole plane without affecting $f(t)$ for $t \geq 0$. The limit point set, $L^+(f)$, is clearly the unit circle where $r = 1$. It has two equilibrium points, at $(r,\theta) = (1,\pi/2)$ and $(1, 3\pi/2)$. There are also two orbits between these points, both moving counter-clockwise, one along the right side of the unit circle and the other along the left. Both take an infinite amount of time to get from one equilibrium point to the other. The non-standard $\overline{f} : \mathbf{R}^+_{\mathcal{U}} \to \mathbf{R}^2$ makes $L^+(f)$ cyclic as it traverses the entire circle, over and over again, stopping for only an "ordinary eternity" at each equilibrium point before moving on.

4.9. Example. *Stabilization at* 0.

Consider the following differential equations for a single function $x(t)$. In each case, 0 is an equilibrium point and we consider the behavior of solutions near 0. Specifically, we consider A-solutions where $A = C^\infty(\mathbf{R})/I(\mathcal{F})$ and $\mathcal{F}$ is the filter of neighborhoods of 0. We let $\phi(u,t)$ be the unique solution with initial value u. (Such a C^∞-map, ϕ, exists in these cases.) We let $C^\infty(\mathbf{R})/I(\mathcal{G})$ be the stabilization of ϕ and we seek to describe $\mathcal{G}$.

(1) $dx/dt = -x$. In this case, ϕ is already stable, so $\mathcal{G} = \mathcal{F}$

(2) $dx/dt = x^3$. Here ϕ is not stable and $\mathcal{G}$ is the trivial filter generated by $\{0\}$. But, surprisingly, the functor **Stab** is non-trivial for this equation, see Example 4.11.

(3) $dx/dt = \sin^2 x$. In this case, ϕ is not stable but $\mathcal{G}$ is not trivial, as $\mathcal{G}$ is the filter generated by $(-\epsilon, 0)$. Note that a solution will stay near 0 if it starts just below 0. But if it starts just above 0, it moves towards the next equilibrium point, at $x = \pi$.

4.10. Example. *The system $dx/dt = -x$, $dy/dt = y$.*

Consider the only equilibrium point, at $(0,0)$. Let $(\phi_1(x_0, y_0, t), \phi_2(x_0, y_0, t))$ be the solution with initial value (x_0, y_0). Let $\mathcal{F}$ be the filter of neighborhoods of $(0,0)$ in $\mathbf{R}^2$. Let $A = C^\infty(\mathbf{R}^2)/I(\mathcal{F})$. As in the above example, let $A = C^\infty(\mathbf{R}^2)/I(\mathcal{G})$ be the stabilization of ϕ. Then $\mathcal{G}$ is generated by the intersections of members of $\mathcal{F}$ with the "x-axis". This reflects the idea that ϕ is stable only in the x-direction.

4.11. Example. *The Pitchfork Bifurcation.*

(I read about this interesting example in [2].) Consider the equation $dx/dt = x^3$ which has an unstable equilibrium point at $x = 0$. Consider also $dx/dt = ux + x^3$, which has equilibrium points when $x = 0$ or when $u = -x^2$ (which form a "pitchfork" shaped subset of the (u, x) plane).

Let $A = C^\infty(\mathbf{R}^2)/I(\mathcal{F})$ where $\mathcal{F}$ is the filter of neighborhoods of $(0,0)$. Let $\phi(u, x_0, t)$ be the solution of $dx/dt = ux + x^3$ with initial value x_0. Then, in view of Theorem 2.12, ϕ represents an A-solution of $dx/dt = x^3$. Note that $(0,0)$ is an unstable equilibrium point of $dx/dt = x^3$ but some of the equilibrium points of $dx/dt = ux + x^3$ are stable (when $u = -x^2$ for $u < 0$), and ϕ does not wander far from 0 if $u < -x_0^2$. If $\mathcal{G}$ is the filter which defines the stabilization of ϕ, then $\mathcal{G}$ is generated by the following sets, defined for $\epsilon > 0$:

$$\{(u,x) : |x| < \epsilon \text{ and } -\epsilon^2 < u < -x^2\} \cup \{(u,x) : x = 0 \text{ and } |u| < \epsilon\}$$

Note that **Stab** for $dx/dt = x^3$, with respect to the equilibrium point 0, contains this stabilization of ϕ, so it is not, as one might expect, trivial.

On the definition of semi-boundedness. Suppose $A = C^\infty(\mathbf{R}^m)/I(\mathcal{F})$ and let $\lambda \in C^\infty(\mathbf{R}^m \times \mathbf{R})$ represent a member of $\mathrm{Map}(A)$. We want to find out if the definition of "semi-bounded" can be simplified. Say that λ is *bounded on* $\mathcal{J}$ if λ is bounded on $\mathbf{R}^m \times J$ for every $J \in \mathcal{J}$. Then clearly λ is semi-bounded with respect to any filter. Also $\nu \in C^\infty(\mathbf{R}^m \times \mathbf{R})$ is semi-bounded with respect to $\mathcal{F}$ if there exists $F \in \mathcal{F}$ with $\nu = \lambda$ on $F \times \mathbf{R}$ where λ is bounded on $\mathcal{J}$. The converse however is false, as shown by the example below.

4.12. Example. *The definition of semi-bounded cannot be simplified, as discussed above.*

(I thank W. W. Comfort who pointed out the crucial role of P-points and non-P-points.) Let $\mathcal{U}$ be an ultrafilter on $\mathbf{N}$ which is a non-P-point of $\beta\mathbf{N} - \mathbf{N}$. See [1], particularly problems 4J,4K,4L,6S for details, and the existence of such points. It follows that $\mathcal{U}$ is an element of some G_δ which is not a neighborhood of $\mathcal{U}$. So we can find a sequence $\{F_n\}$ of members of $\mathcal{U}$ such that for no $F \in \mathcal{U}$ is the set-theoretic difference $F - F_n$ finite for all n.

We may as well assume that the $\{F_n\}$ are nested (with $F_n \supseteq F_{n+1}$ for all n) else replace each F_n by $F_1 \cap \ldots \cap F_n$. Also, we may as well assume that no integer lies in all F_n, else replace F_n by deleting the finite set $\{1, 2, \ldots, n\}$. For each $u \in \mathbf{N}$ let $\deg(u) = \max\{n : u \in F_n\}$. (Assume $F_0 = \mathbf{N}$).

Now define $\nu \in C^\infty(\mathbf{R} \times \mathbf{R})$ so that for $u \in \mathbf{N}$ we have $\nu(u, t) = t$ if $|t| \leq \deg(u)$ and $\nu(u, t) = t + u$ if $|t| \geq 1 + \deg(u)$. (Such a map ν can clearly be found, by the smooth Tietze theorem.) We claim that ν is semi-bounded with respect to $\mathcal{U}$, regarded as a filter on $\mathbf{R}$. Given a bounded interval J choose n so that $|t| \leq n$ for $t \in J$. Then, for $u \in F_n$ and $t \in J$, we see that $|\nu(u, t)| \leq n$ (for if $u \in F_n$, then $\deg(u) \geq n$, etc.

But if there exist $F \in \mathcal{U}$ and λ bounded on $\mathcal{J}$ such that $\nu = \lambda$ on $F \times \mathbf{R}$, then for each n, there exists B_n such that $\lambda(u, t) \leq B_n$ for all t with $|t| \leq n$ and all u. So, if $u \in F - F_n$ then it readily follows that $n + u \leq B_n$ so $u \leq B_n - n$ which shows that $F - F_n$ is finite, contradicting the choice of the $\{F_n\}$.

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