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PURE MORPHISMS OF COMMUTATIVE RINGS ARE EFFECTIVE DESCENT MORPHISMS FOR MODULES – A NEW PROOF

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ABSTRACT. The purpose of this paper is to give a new proof of the Joyal-Tierney theorem (unpublished), which asserts that a morphism $f : R \to S$ of commutative rings is an effective descent morphism for modules if and only if f is pure as a morphism of R-modules.

Let R be a commutative ring with unit and $R-\mathrm{mod}$ the category of R-modules. Since, for any R-module M, the group $C(M) = \mathrm{Hom}_{Ab}(M, \mathbb{Q}/\mathbb{Z})$ (where Ab is the category of abelian groups and $\mathbb{Q}/\mathbb{Z}$ is the rational circle abelian group) becomes an R-module with the action of R on $C(M)$ by $(rf)(m) = f(rm)$, we can define a functor $C : (R-\mathrm{mod})^{op} \to R-\mathrm{mod}$, given by $C(M) = \mathrm{Hom}_{Ab}(M, \mathbb{Q}/\mathbb{Z})$. Since the abelian group $\mathbb{Q}/\mathbb{Z}$ is an injective cogenerator in the category of abelian groups (see, for example, [1]), the functor C is exact and reflects isomorphisms. We say that a morphism $f : M \to M'$ of R-modules is pure if for any R-module N,

$$1_N \otimes_R f : N \otimes_R M \to N \otimes_R M'$$

is monic. Let $f : M \to M'$ be a morphism of R-modules. The next lemma follows from the commutativity of the diagram

$$\begin{tikzcd}
\mathrm{Hom}_R(C(M), C(M')) \arrow[r, "\mathrm{Hom}_R(C(M),C(f))"] \arrow[d, "\approx"] & \mathrm{Hom}_R(C(M, C(M))) \arrow[d, "\approx"] \\
C(C(M) \otimes_R M') \arrow[r, "C(1_{C(M)} \otimes_R f)"'] & C(C(M) \otimes_R M),
\end{tikzcd}$$

where the vertical morphisms are the canonical isomorphisms.

1. LEMMA. *Let $f : M \to M'$ be a morphism of R-modules. The following conditions are equivalent:*

(a) f is a pure morphism of R-modules.

(b) $C(f)$ is a split epimorphism of R-modules.

Let $f : R \to S$ be a morphism of commutative rings. Recall that a descent datum on an object $M \in \mathrm{Ob}(S-\mathrm{mod})$ can be described as an S-module morphism $\theta : M \to S \otimes_R M$ such that θ makes

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$$\begin{tikzcd} M \arrow[r, "\theta"] \arrow[dr, equal] & S \otimes_R M \arrow[d, "\mu"] \\ & M \end{tikzcd}$$

and

$$\begin{tikzcd} M \arrow[r, "\theta"] \arrow[d, "\theta"'] & S \otimes_R M \arrow[d, "1 \otimes_R \theta"] \\ S \otimes_R M \arrow[r, "1 \otimes_R i_M"'] & S \otimes_R S \otimes_R M \end{tikzcd}$$

commutative, where μ denotes the S-module structure on M, and $i_M : M \to S \otimes_R M$ is an R-morphism given by $i_M(m) = 1 \otimes_R m$.

Let $\mathrm{Des}(f)$ denote the category of pairs (M, θ), θ descent datum on $M \in \mathrm{Ob}(S-\mathrm{mod})$, in which morphisms $(M, \theta) \to (M', \theta')$ are just morphisms $g : M \to M'$ in $S-\mathrm{mod}$ which commute with descent data in the obvious sense (see, for example, [2]).

Any object $f^*(M) = (S \otimes_R M, i_M)$, $M \in \mathrm{Ob}(R-\mathrm{mod})$ can be equipped with descent data in a canonical way, and this gives rise to a commutative diagram

$$\begin{tikzcd} R-\mathrm{mod} \arrow[r, "f^*"] \arrow[dr, "S \otimes_R -"'] & \mathrm{Des}(f) \arrow[d, "U"] \\ & S-\mathrm{mod}, \end{tikzcd}$$

where U is the forgetful functor. f is said to be a descent morphism if f^* is full and faithful, and an effective descent morphism if f^* is an equivalence.

The functor f^* has a right adjoint f_* which is defined by requiring that

$$\begin{tikzcd} f_*(M, \theta) \arrow[r, "e"] & M \arrow[r, shift left, "\theta"] \arrow[r, shift right, "i_M"'] & S \otimes_R M \end{tikzcd}$$

is an equalizer of S-modules for each $(M, \theta) \in \mathrm{Ob}(\mathrm{Des}(f))$. The counit of this adjunction is defined by $\delta_M = \mu(1 \otimes_R e)$. The unit $\epsilon_M : M \to f_* f^*(M)$ is obtained from the diagram

$$\begin{tikzcd} M \arrow[dr, "i_M"] \arrow[dd, dashed, "\varepsilon_M"] & & \\ & S \otimes_R M \arrow[r, shift left, "1_S \otimes_R i_M"] \arrow[r, shift right, "i_{S \otimes_R M}"'] & S \otimes_R S \otimes_R M. \\ f_* f^*(M) \arrow[ur] & & \end{tikzcd}$$

It does exist because i_M equalizes the two morphisms on the right hand side. From the description of ϵ and δ we obtain immediately the two following propositions.

2. Proposition. *$f : R \to S$ is a descent morphism if and only if f is pure as a morphism of R-modules.*

3. Proposition. *A descent morphism f is effective if and only if $S \otimes_R -$ preserves the equalizer*

$$\begin{tikzcd} f_*(M,\theta) \arrow[r, "e"] & M \arrow[r, shift left, "\theta"] \arrow[r, shift right, "i_M"'] & S \otimes_R M \end{tikzcd}$$

for each $(M,\theta) \in Ob(Des(f))$.

Let $f : R \to S$ be pure as a morphism of R-modules. Then by Lemma 1 there is an R-module morphism $g : C(R) \to C(S)$ such that $c(f)g = 1_{C(R)}$.

If $(M,\theta) \in \mathrm{Ob}(\mathrm{Des}(f))$, then we have a commutative diagram

$$\begin{tikzcd} f_*(M,\theta) \arrow[r, "e"] & M \arrow[r, shift left, "\theta"] \arrow[r, shift right, "i_M"'] \arrow[d, "i_M"'] & S \otimes_R M \arrow[d, "i_{S\otimes_R M}"] \\ M \arrow[r, "\theta"] & S \otimes_R M \arrow[r, shift left, "1_S \otimes_R \theta"] \arrow[r, shift right, "1_S \otimes_R i_M"'] & S \otimes_R S \otimes_R M, \end{tikzcd}$$

in which the rows are equalizer diagrams.

Applying the functor C to this diagram, we obtain the commutative diagram

$$\begin{tikzcd} C(S \otimes_R S \otimes_R M) \arrow[r, shift left, "C(1_S \otimes_R \theta)"] \arrow[r, shift right, "C(1_S \otimes_R i_M)"'] \arrow[d, "C(i_{S\otimes_R M})"'] & C(S \otimes_R M) \arrow[r, "C(\theta)"] \arrow[d, "C(i_M)"] & C(M) \\ C(S \otimes_R M) \arrow[r, shift left, "C(\theta)"] \arrow[r, shift right, "C(i_M)"'] & C(M) \arrow[r, "C(e)"'] & C(f_*(M,\theta)), \end{tikzcd}$$

in which the rows are coequalizer diagrams. Now, since for any R-module P we have the isomorphism of functors $C(P \otimes_R -) \to \mathrm{Hom}_R(-, C(P))$, we obtain the commutative diagram

$$\begin{tikzcd} \mathrm{Hom}_R(S \otimes_R M, C(S)) \arrow[r, shift left, "-\circ\theta"] \arrow[r, shift right, "-\circ i_M"'] \arrow[d, shift left, "-\circ C(f)"] & \mathrm{Hom}_R(M, C(S)) \arrow[d, shift left, "-\circ C(f)"] \\ \mathrm{Hom}_R(S \otimes_R M, C(R)) \arrow[r, shift left, "-\circ\theta"] \arrow[r, shift right, "-\circ i_M"'] \arrow[u, shift left, "-\circ g"] & \mathrm{Hom}_R(M, C(R)) \arrow[r, "-\circ e"] \arrow[u, shift left, "-\circ g"] & \mathrm{Hom}_R(f_*(M,\theta), C(R)). \end{tikzcd}$$

Applying the above isomorphism of functors backwards, we deduce that there are R-morphisms h and h', such that the diagram

$$\begin{tikzcd} C(S \otimes_R S \otimes_R M) \arrow[r, shift left, "C(1_S \otimes_R \theta)"] \arrow[r, shift right, "C(1_S \otimes_R i_M)"'] \arrow[d, shift left, "C(i_{S\otimes_R M})"] & C(S \otimes_R M) \arrow[r, "C(\theta)"] \arrow[d, shift left, "C(i_M)"] & C(M) \\ C(S \otimes_R M) \arrow[r, shift left, "C(\theta)"] \arrow[r, shift right, "C(i_M)"'] \arrow[u, shift left, "h'"] & C(M) \arrow[r, "C(e)"] \arrow[u, shift left, "h"] & C(f_*(M,\theta)) \arrow[u, dashed, "k"] \end{tikzcd}$$

commutes. Since both left hand side squares commute, there is an R-morphism $k : C(f_*(M,\theta) \to C(M)$ such that $C(\theta)h = kC(e)$. It means that the bottom row becomes a split coequalizer diagram [3] in the category of R-modules,which is split by the morphisms

$$C(f_*(M,\theta)) \xrightarrow{k} C(M) \xrightarrow{h} C(S \otimes_R M).$$

Since split coequalizers are preserved by any functor, its image under the functor $\mathrm{Hom}(S,-)$ is a coequalizer diagram. So

$$\mathrm{Hom}_R(S, C(S \otimes_R M)) \underset{-\circ C(i_M)}{\overset{-\circ C(\theta)}{\rightrightarrows}} \mathrm{Hom}_R(S, C(M)) \xrightarrow{-\circ C(e)} \mathrm{Hom}_R(S, f_*(M,\theta))$$

is a coequalizer diagram, and hence so is

$$C(S \otimes_R S \otimes_R M) \underset{C(1_S \otimes_R i_M)}{\overset{C(1_S \otimes_R \theta)}{\rightrightarrows}} C(S \otimes_R M) \xrightarrow{C(1_S \otimes_R e)} C(S \otimes_R f_*(M,\theta)).$$

The functor C is exact and reflects isomorphisms. Therefore it also reflects coequalizers. It follows that

$$S \otimes_R f_*(M,\theta) \xrightarrow{1_S \otimes_R e} S \otimes_R M \underset{1_S \otimes_R i_M}{\overset{1_S \otimes_R \theta}{\rightrightarrows}} S \otimes_R S \otimes_R M$$

is an equalizer. But

$$M \xrightarrow{\theta} S \otimes_R M \underset{1_S \otimes_R i_M}{\overset{1_S \otimes_R \theta}{\rightrightarrows}} S \otimes_R S \otimes_R M$$

is also an equalizer diagram. Thus we have an isomorphism $S \otimes_R f_*(M,\theta) \to M$.

We obtain

4. THEOREM. *$f : R \to S$ is an effective descent morphism for modules if and only if f is pure as a morphism of R-modules.*

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