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BALANCED COALGEBROIDS

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Transmitted by L. Breen

ABSTRACT. A balanced coalgebroid is a $\mathcal{V}^{\mathrm{op}}$-category with extra structure ensuring that its category of representations is a balanced monoidal category. We show, in a sense to be made precise, that a balanced structure on a coalgebroid may be reconstructed from the corresponding structure on its category of representations. This includes the reconstruction of dual quasi-bialgebras, quasi-triangular dual quasi-bialgebras, and balanced quasi-triangular dual quasi-bialgebras; the latter of which is a quantum group when equipped with a compatible antipode. As an application we construct a balanced coalgebroid whose category of representations is equivalent to the symmetric monoidal category of chain complexes. The appendix provides the definitions of a braided monoidal bicategory and sylleptic monoidal bicategory.

Contents

1. Introduction

This article is the sequel to [McC99b] and reports on the results of the author's doctoral thesis [McC99a].

The classification and reconstruction of mathematical objects from their representations is a broad and significant field of mathematics. For example, a locally compact abelian group can be reconstructed from its character group, and this is the subject of

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Pontrjagin duality. Tannaka duality shows that a compact group can be reconstructed from its continuous representations in the category of finite dimensional, complex vector spaces. See [Che46] and [JS91] for a good exposition of Pontrjagin and Tannaka duality. Morita theory shows that a commutative ring is determined up to Morita equivalence by its category of modules; see for example [DI71]. More generally, an algebraic theory in the sense of Lawvere may be reconstructed from its semantics [Law63], and this is the subject of the celebrated theory of structure and semantics.

This article studies the reconstruction of *balanced coalgebroids.* A balanced coalgebroid equipped with a compatible antipode is a *quantum opgroupoid* [DS97] which generalizes the quantum groups of [Dri87]. The reconstruction theorem presented in this article includes the reconstruction of dual quasi-bialgebras, quasi-triangular dual quasi-bialgebras, and balanced quasi-triangular dual quasi-bialgebras, and improves on a number of results on the reconstruction of such objects; see for example [HO97, JS91, Maj92, Maj93, Par81, Par96, Sch92b, Sch92a].

The approach to the reconstruction of balanced coalgebroids taken in [McC99b] and in this article parallels the approach to the reconstruction of coalgebras, bialgebras, cobraided bialgebras and balanced bialgebras taken in Tannaka duality. The theorems of the latter may be loosely divided into three areas: the reconstruction of coalgebras from their categories of representations; the characterization of those categories which are equivalent to the category of representations of some coalgebra; and the reconstruction of extra structure on a coalgebra from the corresponding extra structure on its category of representations. We will discuss these aspects in turn.

For a coalgebra C in the category of vector spaces, a *representation of* C, or a C-*comodule*, is a vector space M equipped with a coassociative, counital coaction $\delta\colon M \to M \otimes C$. There is a category $\mathrm{Comod}(C)$ of representations of C, and a small category $\mathrm{Comod}_f(C)$ of finite dimensional representations of C, which has a forgetful functor $\omega\colon \mathrm{Comod}_f(C) \to \mathrm{Vect}_f$ into the category Vect_f of finite dimensional vector spaces. Conversely, given a small category $\mathcal{C}$ equipped with a functor $\sigma\colon \mathcal{C} \to \mathrm{Vect}_f$, one may form the coalgebra $\mathrm{End}^\vee(\sigma)$ in the category of vector spaces [JS91, Section 3]. A fundamental result [JS91, Section 6, Proposition 5] of Tannaka duality is that when σ is the forgetful functor $\omega\colon \mathrm{Comod}_f(C) \to \mathrm{Vect}_f$, the coalgebra $\mathrm{End}^\vee(\omega)$ is canonically isomorphic to C, and this has been called the *reconstruction theorem.*

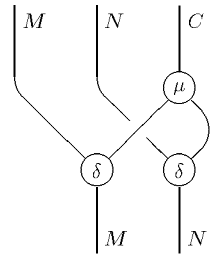
As coalgebras and their representations have been studied in categories more general than the category of vector spaces, variants of the reconstruction theorem have been considered for coalgebras in an arbitrary cocomplete braided monoidal category $\mathcal{V}$. In this more general case, a coalgebra cannot be reconstructed from its ordinary category of representations equipped with its forgetful functor, and so a more sensitive theory must be employed. Indeed, Pareigis [Par96] uses the theory of $\mathcal{V}$-*actegories* to facilitate reconstruction. If C is a coalgebra in $\mathcal{V}$, M is a C-comodule and V is an object of $\mathcal{V}$, then the arrow $V \otimes \delta\colon V \otimes M \to V \otimes M \otimes C$ equips $V \otimes M$ with the structure of a C-comodule. This is the value at the object (V, M) of a functor $\otimes\colon \mathcal{V} \times \mathrm{Comod}(C) \to \mathrm{Comod}(C)$ which is associative and unital up to coherent isomorphism. A category equipped with such an action is called a *$\mathcal{V}$-actegory*, and is an object of a 2-category $\mathcal{V}$-Act. Arrows of $\mathcal{V}$-Act are called *morphisms of $\mathcal{V}$-actegories*, and the forgetful functor $\omega\colon \mathrm{Comod}(\mathcal{C}) \to \mathcal{V}$ is an example of such. Conversely, given a morphism of $\mathcal{V}$-actegories $\sigma\colon \mathcal{C} \to \mathcal{V}$, there is a coalgebra $\mathrm{coend}_{\mathcal{V}}(\sigma)$ in $\mathcal{V}$, existing if a certain smallness condition is satisfied. As in the case of the preceding paragraph, when σ is the forgetful functor $\omega\colon \mathrm{Comod}(C) \to \mathcal{V}$, the coalgebra $\mathrm{coend}_{\mathcal{V}}(\omega)$ is canonically isomorphic to C [Par96, Corollary 4.3]. Similar results may be found in [Sch92b, Sch92a, Maj92].

Another generalization of the reconstruction theorem considers the *several-object case*. A one-object category is just a monoid and so one sometimes speaks loosely of categories as being several-object monoids. One may thus ask what should be several-object coalgebras; several possible definitions have been given in [DS97, CF94, BM94]. Day and Street [DS97] define a several-object coalgebra in $\mathcal{V}$ to be a $\mathcal{V}^{\mathrm{op}}$-enriched category, and indeed a $\mathcal{V}^{\mathrm{op}}$-category with exactly one object is a coalgebra in $\mathcal{V}$. The category of representations $\mathrm{Comod}(C)$ of a $\mathcal{V}^{\mathrm{op}}$-category C is also defined there, agreeing with the usual category of representations of a coalgebra in the one-object case. There is a family of forgetful functors $\omega_c\colon \mathrm{Comod}(C) \to \mathcal{V}$ indexed by the objects of C. It is observed in [McC99b] that as in the case for coalgebras, the category $\mathrm{Comod}(C)$ is a $\mathcal{V}$-actegory and the forgetful functors are morphisms of $\mathcal{V}$-actegories. Conversely, given a $\mathcal{V}$-actegory $\mathcal{A}$ and a family of morphisms of $\mathcal{V}$-actegories $\sigma_x\colon \mathcal{A} \to \mathcal{V}$, there is a $\mathcal{V}^{\mathrm{op}}$-category $E(\sigma)$, existing if a certain smallness condition is satisfied, and when the family σ_x is the family of forgetful functors $\omega_c\colon \mathrm{Comod}(C) \to \mathcal{V}$, the $\mathcal{V}^{\mathrm{op}}$-category $E(\omega)$ is canonically isomorphic to C [McC99b, Proposition 4.7]. Other results on the reconstruction of $\mathcal{V}^{\mathrm{op}}$-categories may be found in [DS97, Section 9].

We now recall results on the characterization of those categories which are equivalent to the category of representations of some coalgebra. If $\mathcal{C}$ is a small category and $\sigma\colon \mathcal{C} \to \mathrm{Vect}_f$ is a functor, then there is a canonical factorization of σ as a functor $\eta\colon \mathcal{C} \to \mathrm{Comod}_f(\mathrm{End}^{\vee}(\sigma))$ followed by the forgetful functor $\mathrm{Comod}_f(\mathrm{End}^{\vee}(\sigma)) \to \mathrm{Vect}_f$. Joyal and Street [JS91, Section 7, Theorem 3] show that η is an equivalence if and only if $\mathcal{C}$ is an abelian category and σ is an exact and faithful functor, and this has been called the *representation theorem*.

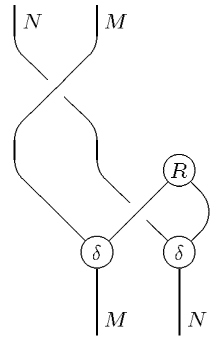
A several-object variant of the representation theorem is considered in [McC99b]. If $\mathcal{A}$ is a $\mathcal{V}$-actegory and $\sigma_x\colon \mathcal{A} \to \mathcal{V}$ is a family of morphisms of $\mathcal{V}$-actegories, then there is a canonical morphism of $\mathcal{V}$-actegories $\eta\colon \mathcal{A} \to \mathrm{Comod}(E(\sigma))$. Then η is an equivalence if and only if each σ_x has a right adjoint in the 2-category $\mathcal{V}$-Act, the underlying category of $\mathcal{A}$ has a certain class of limits, and each functor σ_x preserves and reflects these limits [McC99b, Theorem 5.11, Corollary 5.16].

We now recall results on the reconstruction of *extra structure* on a coalgebra from the corresponding extra structure on its category of representations. A *bialgebra* is a coalgebra C in the category of vector spaces equipped with coalgebra morphisms $\mu\colon C \otimes C \to C$ and $\eta\colon \mathbb{C} \to C$ that make the underlying vector space of C into an algebra. If C is a bialgebra, then it is well-known that $\mathrm{Comod}_f(C)$ is a monoidal category and the forgetful functor is a strong monoidal functor. In detail, if M and N are finite dimensional C-comodules then, using the string calculus of [JS93], the arrow



equips $M \otimes N$ with the structure of a C-comodule. Similarly, the unit $\eta\colon \mathbb{C} \to C$ equips the vector space $\mathbb{C}$ with a C-coaction. Joyal and Street [JS91, Section 8] show the converse is also true; that is, if $\sigma\colon \mathcal{C} \to \mathrm{Vect}_f$ is a strong monoidal functor out of a small monoidal category, then $\mathrm{End}^\vee(\sigma)$ is canonically a bialgebra. This procedure induces a bijection between bialgebra structures on a coalgebra C, and those monoidal structures on $\mathrm{Comod}_f(C)$ for which the forgetful functor is strong monoidal.

If C is a bialgebra, then a *cobraiding* for C is a linear map $R\colon C \otimes C \to \mathbb{C}$ satisfying various axioms. If C is a bialgebra equipped with a cobraiding then $\mathrm{Comod}_f(C)$ becomes a braided monoidal category: the braiding at a pair of finite dimensional C-comodules M and N is given by the following string diagram.



Again the converse is true, and there is a bijection between cobraidings on C and braidings on the monoidal category $\mathrm{Comod}_f(C)$ [JS91, Section 10, Propostion 3].

Majid [Maj92] considers a variant of the above reconstruction of extra structure motivated by the theory of quantum groups. A *dual quasi-bialgebra* is a coalgebra C in the category of vector spaces, equipped with arrows $\mu\colon C \otimes C \to C$ and $\eta\colon \mathbb{C} \to C$, along with an arrow $\Phi\colon C \otimes C \otimes C \to \mathbb{C}$, called the *Drinfeld associator*, satisfying seven axioms; notably, μ is not associative. If C is a dual quasi-bialgebra then $\mathrm{Comod}_f(C)$ is a monoidal category, however, the forgetful functor is a *multiplicative functor* rather than a strong monoidal functor. Majid also shows that the converse is true, so concluding that there is a bijection between dual quasi-bialgebra structures on a coalgebra C and those monoidal structures on $\mathrm{Comod}_f(C)$ for which the forgetful functor is multiplicative [Maj92, Theorem 2.8].

This article examines the reconstruction of certain extra structures on a $\mathcal{V}^{\mathrm{op}}$-category from the corresponding extra structures on its category of representations. These structures are called *pseudomonoidal, braided pseudomonoidal* and *balanced pseudomonoidal* structures, and the reconstruction theorem given includes the reconstruction of dual-quasi bialgebras, quasi-triangular dual-quasi bialgebras, and balanced dual-quasi bialgebras.

We proceed as follows. In Section 2, the definition of a pseudomonoid in a monoidal 2-category is given. A pseudomonoid monoidal 2-category $\mathrm{Comon}(\mathcal{V})$ of comonoids in $\mathcal{V}$ is a dual quasi-bialgebra, and a pseudomonoid in the monoidal 2-category $\mathrm{Cat}/\mathcal{V}$ of categories over $\mathcal{V}$ is a monoidal category equipped with a multiplicative functor. Other examples of pseudomonoids include bialgebras, quasi bialgebras, pseudomonads multiplicative functors and of course monoidal categories. For a monoidal 2-category $\mathcal{K}$, there is a 2-category $\mathrm{PsMon}(\mathcal{K})$ of pseudomonoids in $\mathcal{K}$ and a forgetful 2-functor $U\colon \mathrm{PsMon}(\mathcal{K}) \to \mathcal{K}$. If $F\colon \mathcal{K} \to \mathcal{L}$ is a weak monoidal 2-functor, then there is an induced 2-functor $\mathrm{PsMon}(F)\colon \mathrm{PsMon}(\mathcal{K}) \to \mathrm{PsMon}(\mathcal{L})$ and the following diagram commutes.

$$\begin{tikzcd} \mathrm{PsMon}(\mathcal{K}) \arrow[r, "\mathrm{PsMon}(F)"] \arrow[d, "U"'] & \mathrm{PsMon}(\mathcal{L}) \arrow[d, "U"] \\ \mathcal{K} \arrow[r, "F"'] & \mathcal{L} \end{tikzcd} \tag{1}$$

For example, there is a weak monoidal 2-functor $\mathrm{Comod}_f\colon \mathrm{Comon}(\mathcal{V}) \to \mathrm{Cat}/\mathrm{Vect}_f$ whose value on a comonoid C is its category of finite dimensional representations $\mathrm{Comod}_f(C)$ equipped with the forgetful functor $\omega\colon \mathrm{Comod}_f(C) \to \mathrm{Vect}_f$ [Str89]. The commutativity of (1) then states that if a comonoid C is equipped with the structure of a dual quasi-bialgebra, then $\mathrm{Comod}_f(C)$ is canonically a monoidal category and the forgetful functor is multiplicative. This article addresses the converse of this statement.

In Sections 3, 4 and 5, the definitions of braided pseudomonoid, symmetric pseudomonoid, and balanced pseudomonoid are given. These notions include, among others, that of a braided monoidal category, a symmetric monoidal category, a quasi-triangular quasi-bialgebra, and a quasi-triangular dual quasi-bialgebra. A balanced pseudomonoid in $\mathcal{V}^{\mathrm{op}}$-$\mathrm{Cat}^{\mathrm{op}}$ equipped with an antipode satisfying one axiom is a *quantum opgroupoid* [DS97], generalizing the quantum groups of [Dri87].

In Section 6, monoidally bi-fully-faithful weak monoidal 2-functors are defined, and it is shown that if $F\colon \mathcal{K} \to \mathcal{L}$ is such a 2-functor, then (1) is a bi-pullback in the 2-category of 2-categories, 2-functors and 2-natural transformations. This makes precise the idea that there is a bijection between pseudomonoid structures on an object A of $\mathcal{K}$ and on the object FA of $\mathcal{L}$.

Section 7 provides a novel example of a symmetric monoidal 2-category: $\mathcal{V}$-Act. Section 8 shows that the 2-category $\mathcal{V}$-Act$//\mathcal{V}$ studied in [McC99b] is also a symmetric monoidal 2-category. In Section 9, we recall the 2-functor $\mathrm{Comod}\colon \mathcal{V}^{\mathrm{op}}\text{-Cat} \to \mathcal{V}\text{-Act}//\mathcal{V}$ described in [McC99b] whose value on a $\mathcal{V}^{\mathrm{op}}$-category is its category of representations. We then show that this 2-functor may be equipped with the structure of a symmetric weak monoidal 2-functor. We show in Section 10 that it is monoidally bi-fully-faithful which allows us to conclude that the following diagram is a bi-pullback in the 2-category of 2-categories, 2-functors and 2-natural transformations.

$$\begin{tikzcd}
\mathrm{PsMon}(\mathcal{V}^{\mathrm{op}}\text{-}\mathrm{Cat}^{\mathrm{op}}) \arrow[r, "\mathrm{PsMon}(\mathrm{Comod})"] \arrow[d, "U"'] & \mathrm{PsMon}(\mathcal{V}\text{-}\mathrm{Act}//\mathcal{V}) \arrow[d, "U"] \\
\mathcal{V}^{\mathrm{op}}\text{-}\mathrm{Cat}^{\mathrm{op}} \arrow[r, "\mathrm{Comod}"'] & \mathcal{V}\text{-}\mathrm{Act}//\mathcal{V}
\end{tikzcd}$$

Similarly, the diagram

$$\begin{tikzcd}
\mathrm{BalPsMon}(\mathcal{V}^{\mathrm{op}}\text{-}\mathrm{Cat}^{\mathrm{op}}) \arrow[r, "\mathrm{BalPsMon}(\mathrm{Comod})"] \arrow[d, "U"'] & \mathrm{BalPsMon}(\mathcal{V}\text{-}\mathrm{Act}//\mathcal{V}) \arrow[d, "U"] \\
\mathcal{V}^{\mathrm{op}}\text{-}\mathrm{Cat}^{\mathrm{op}} \arrow[r, "\mathrm{Comod}"'] & \mathcal{V}\text{-}\mathrm{Act}//\mathcal{V}
\end{tikzcd}$$

is a bi-pullback, and this implies that balanced pseudomonoidal structures on a $\mathcal{V}^{\mathrm{op}}$-category are characterized by balanced pseudomonoidal structures on its category of representations. This allows us to construct a balanced coalgebra in Section 11 whose category of representations is equivalent to the symmetric monoidal category of chain complexes. This coalgebra was first described in [Par81].

There are two appendices to this article; the first defines braided monoidal bicategories and the second defines sylleptic monoidal bicategories.

Ordinary and enriched category theory are assumed throughout this article; as usual, [Mac71] and [Kel82] are references for these subjects respectively. Familiarity with 2-dimensional algebra is assumed and the reader may wish to consult [Bén67] or [KS74] for general theory on this subject; in particular we shall use the theory of mates [KS74]. We use the string calculus of [JS93], and the definition of a monoidal bicategory [GPS95]. The approach taken, and terminology used in this article follows [DS97] and [McC99b], the latter being heavily drawn upon.

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2. Pseudomonoids

In this section the definition of a *pseudomonoid* in a monoidal 2-category is provided. This definition mildly generalizes that of a pseudomonoid in a Gray-monoid [DS97, Section 3] and may be considered a categorification [BD98] of the definition of a monoid in a monoidal category. Motivating examples are provided.

Recall that a monoidal bicategory is a tricategory [GPS95, Section 2.2] with exactly one object and that a monoidal 2-category is a monoidal bicategory whose underlying bicategory is a 2-category [GPS95, Section 2.6]. A monoidal bicategory $(\mathcal{K}, \otimes, I, \mathsf{a}, \mathsf{l}, \mathsf{r}, \pi, \nu, \lambda, \rho)$ will be denoted simply by $\mathcal{K}$; this notation is that of [GPS95, Section 2.2], except that we denote the invertible modification μ of [GPS95, Section 2.2] by ν. We shall often write $\otimes$ as juxtaposition without comment, and leave unlabelled the 2-cells expressing the pseudonaturality of a, l and r. We shall restrict our attention to monoidal 2-categories, rather than the more general monoidal bicategories, although the latter are monoidally biequivalent to the former [GPS95, Section 3.7]. If $\mathcal{K}$ is a monoidal 2-category then the 2-categories $\mathcal{K}^{\mathrm{op}}$, $\mathcal{K}^{\mathrm{co}}$ and $\mathcal{K}^{\mathrm{rev}}$, obtained by reversing 1-cells, 2-cells and the tensor product respectively, are monoidal 2-categories in a canonical manner.

We now recall some well-known examples of monoidal 2-categories. A 2-category is called *locally discrete* if its only 2-cells are identities, and thus may be identified with a category. It is easy to see that a monoidal 2-category whose underlying 2-category is locally discrete may be identified with a monoidal category. A Gray-monoid [DS97, Section 1] may be considered to be a monoidal 2-category, and in fact every monoidal 2-category is biequivalent to a Gray-monoid [GPS95, Section 8.1]; indeed this is the coherence theorem for tricategories in the one-object case. For a 2-category $\mathcal{K}$, the 2-category $[\mathcal{K}, \mathcal{K}]$ of 2-functors, pseudonatural transformations and modifications from $\mathcal{K}$ to $\mathcal{K}$ is a Gray-monoid with composition as tensor product, a fortiori, a monoidal 2-category. If a 2-category admits finite products or finite bi-products, then it is a monoidal 2-category in a canonical manner. For a braided monoidal category $\mathcal{V}$, there is a 2-category $\mathcal{V}$-Cat of $\mathcal{V}$-enriched categories and the usual tensor product of $\mathcal{V}$-categories equips $\mathcal{V}$-Cat with the structure of monoidal 2-category [Kel82, Section 1.4]. Of course, the 2-category Cat of categories, functors and natural transformations is a monoidal 2-category by the previous two examples. For a braided monoidal category $\mathcal{V}$, the opposite category $\mathcal{V}^{\mathrm{op}}$ is a braided monoidal category with the same tensor product and the inverse of the associativity, unit and braiding isomorphisms, and thus the 2-category $\mathcal{V}^{\mathrm{op}}$-Cat of $\mathcal{V}^{\mathrm{op}}$-enriched categories is a monoidal 2-category. The full sub-2-categories of $\mathcal{V}$-Cat and $\mathcal{V}^{\mathrm{op}}$-$\mathrm{Cat}^{\mathrm{op}}$ consisting of those enriched categories with exactly one object are monoidal 2-categories, and are usually called the monoidal 2-categories of *monoids in* $\mathcal{V}$ and *comonoids in* $\mathcal{V}$ respectively; these shall be denoted by $\mathrm{Mon}(\mathcal{V})$ and $\mathrm{Comon}(\mathcal{V})$ respectively.

Street uses a monoidal 2-category $\mathrm{Cat}/\mathcal{V}$ in the analysis of the categories of representations of quantum groups [Str89]. The 2-category $\mathrm{Cat}/\mathcal{V}$ has objects (A, U) consisting of a category A equipped with a functor $U\colon A \to \mathcal{V}$. A 1-cell $(F, \varphi)\colon (A, U) \to (B, V)$ consists of a functor $F\colon A \to B$ and a natural isomorphism $\varphi\colon V \circ F \Rightarrow U$, and a 2-cell $\alpha\colon (F, \varphi) \Rightarrow (G, \psi)$ is simply a natural transformation $\alpha\colon F \Rightarrow G$. The evident compositions make $\mathrm{Cat}/\mathcal{V}$ into a 2-category. Given two objects (A, U) and (B, V) of $\mathrm{Cat}/\mathcal{V}$ define an object $(A, U) \otimes (B, V)$ of $\mathrm{Cat}/\mathcal{V}$ to be $(A \times B, \otimes \circ (U \times V))$ where $\otimes\colon \mathcal{V} \times \mathcal{V} \to \mathcal{V}$ is the tensor product on $\mathcal{V}$. This is the value at the pair $((A, U), (B, V))$ of a 2-functor $\otimes\colon \mathrm{Cat}/\mathcal{V} \times \mathrm{Cat}/\mathcal{V} \to \mathrm{Cat}/\mathcal{V}$. The unit for this tensor product is $(1, I)$ where 1 is the terminal category and $I\colon 1 \to \mathcal{V}$ is the functor whose value on the only object of 1 is the unit object I of $\mathcal{V}$. With the evident associativity and unit equivalences, and coherence modifications, $\mathrm{Cat}/\mathcal{V}$ is a monoidal 2-category. In Section 8 we shall provide a construction for monoidal 2-categories which includes $\mathrm{Cat}/\mathcal{V}$ as a special case.

For a symmetric monoidal category $\mathcal{V}$, the 2-category $\mathcal{V}$-Act of $\mathcal{V}$-actegories [McC99b, Section 3] maybe equipped with the structure of a monoidal 2-category. An object $\mathcal{A}$ of $\mathcal{V}$-Act consists of a category $\mathcal{A}$ equipped with a functor $\otimes\colon \mathcal{V}\times\mathcal{A}\to\mathcal{A}$ which is associative and unital up to coherent isomorphism. The tensor product of $\mathcal{V}$-actegories is analogous to the tensor product of modules over a commutative ring. This example is of pivotal importance to this article and a complete description is provided in Section 7.

For monoidal 2-categories $\mathcal{K}$ and $\mathcal{L}$ a *weak monoidal homomorphism* is a homomorphism of bicategories $T\colon \mathcal{K}\to\mathcal{L}$ equipped with pseudonatural transformations

$$
\begin{tikzcd}
\mathcal{K}^2 \arrow[r, "T^2"] \arrow[d, "\otimes"'] \arrow[dr, phantom, "\swarrow\chi"] & \mathcal{L}^2 \arrow[d, "\otimes"] \\
\mathcal{K} \arrow[r, "T"'] & \mathcal{L}
\end{tikzcd}
\qquad
\begin{tikzcd}
& 1 \arrow[dl, "I"'] \arrow[dr, "I"] \arrow[d, phantom, "\Leftarrow\iota"] & \\
\mathcal{K} \arrow[rr, "T"'] & {} & \mathcal{L}
\end{tikzcd}
$$

and invertible modifications

$$
\begin{tikzcd}
& \mathcal{L}^3 \arrow[rr, "\otimes 1"] \arrow[dr, phantom, "\Downarrow\chi 1"] & & \mathcal{L}^2 \arrow[dr, "\otimes"] \arrow[d, phantom, "\Downarrow\chi"] & \\
\mathcal{K}^3 \arrow[ur, "T^3"] \arrow[rr, "\otimes 1"] \arrow[dr, "1\otimes"'] & & \mathcal{L}^2 \arrow[ur, "T^2"] \arrow[dr, "\otimes"] \arrow[dl, phantom, "\swarrow\mathsf{a}"] & {} & \mathcal{L} \\
& \mathcal{K}^2 \arrow[rr, "\otimes"'] & & \mathcal{K} \arrow[ur, "T"'] &
\end{tikzcd}
\overset{\omega}{\Rrightarrow}
\begin{tikzcd}
& \mathcal{L}^3 \arrow[rr, "\otimes 1"] \arrow[dr, "1\otimes"] & & \mathcal{L}^2 \arrow[dr, "\otimes"] \arrow[dl, phantom, "\swarrow\mathsf{a}"] & \\
\mathcal{K}^3 \arrow[ur, "T^3"] \arrow[dr, "1\otimes"'] \arrow[r, phantom, "\Downarrow 1\chi"] & {} & \mathcal{L}^2 \arrow[rr, "\otimes"] \arrow[dr, phantom, "\Downarrow\chi"] & & \mathcal{L} \\
& \mathcal{K}^2 \arrow[ur, "T^2"] \arrow[rr, "\otimes"'] & & \mathcal{K} \arrow[ur, "T"'] &
\end{tikzcd}
$$

$$
\begin{tikzcd}
& \mathcal{L}^2 \arrow[dr, "\otimes"] & \\
\mathcal{L} \arrow[ur, "I1"] \arrow[r, phantom, "\overset{\iota 1}{\Rightarrow}"] & \mathcal{K}^2 \arrow[u, "T^2"] \arrow[dr, "\otimes"] \arrow[r, phantom, "\overset{\chi}{\Rightarrow}"] \arrow[d, phantom, "\Downarrow\mathsf{l}"] & \mathcal{L} \\
\mathcal{K} \arrow[u, "T"] \arrow[ur, "I1"] \arrow[rr, "1"'] & {} & \mathcal{K} \arrow[u, "T"']
\end{tikzcd}
\overset{\gamma}{\Rrightarrow}
\begin{tikzcd}
& \mathcal{L}^2 \arrow[dr, "\otimes"] \arrow[d, phantom, "\Downarrow\mathsf{l}"] & \\
\mathcal{L} \arrow[ur, "I1"] \arrow[rr, "1"] & {} & \mathcal{L} \\
\mathcal{K} \arrow[u, "T"] \arrow[rr, "1"'] & & \mathcal{K} \arrow[u, "T"']
\end{tikzcd}
$$

$$
\begin{tikzcd}
\mathcal{L} \arrow[rr, "1"] & & \mathcal{L} \\
\mathcal{K} \arrow[u, "T"] \arrow[rr, "1"] \arrow[dr, "1I"'] & {} \arrow[d, phantom, "\Downarrow\mathsf{r}"] & \mathcal{K} \arrow[u, "T"'] \\
& \mathcal{K}^2 \arrow[ur, "\otimes"'] &
\end{tikzcd}
\overset{\kappa}{\Rrightarrow}
\begin{tikzcd}
\mathcal{L} \arrow[rr, "1"] \arrow[dr, "1I"] & {} \arrow[d, phantom, "\Downarrow\mathsf{r}"] & \mathcal{L} \\
\mathcal{K} \arrow[u, "T"] \arrow[dr, "1I"'] \arrow[r, phantom, "\overset{1\iota}{\Rightarrow}"] & \mathcal{L}^2 \arrow[ur, "\otimes"] \arrow[r, phantom, "\overset{\chi}{\Rightarrow}"] & \mathcal{K} \arrow[u, "T"'] \\
& \mathcal{K}^2 \arrow[u, "T^2"] \arrow[ur, "\otimes"'] &
\end{tikzcd}
$$

subject to two coherence axioms [GPS95, Section 3.1]; this notation is that of [GPS95, Section 3.1] except the modification δ of [GPS95, Section 3.1] is denoted here by κ. *Opweak monoidal homomorphisms* may be similarly defined where the sense of the pseudonatural transformations χ and ι are reversed. A *weak monoidal 2-functor* is of course a weak monoidal homomorphism whose underlying homomorphism is a 2-functor. A *monoidal* *homomorphism* is a weak monoidal homomorphism where the pseudonatural transformations χ and ι are equivalences, and is sometimes referred to as a *strong* monoidal homomorphism. Note that if $T\colon \mathcal{K} \to \mathcal{L}$ is a weak monoidal homomorphism, then its underlying homomorphism is canonically equipped with the structure of a weak monoidal homomorphism $T^{\mathrm{rev}}\colon \mathcal{K}^{\mathrm{rev}} \to \mathcal{L}^{\mathrm{rev}}$.

If S and T are parallel weak monoidal homomorphisms then a *monoidal pseudonatural transformation* $\theta\colon S \to T$ consists of a pseudonatural transformation $\theta\colon S \to T$ equipped with invertible modifications

$$\begin{tikzcd}
\mathcal{K}^2 \arrow[r, bend left, "S^2"] \arrow[r, bend right, "T^2"'] \arrow[r, phantom, "\Downarrow\theta^2"] \arrow[d, "\otimes"'] & \mathcal{L}^2 \arrow[d, "\otimes"] \\
\mathcal{K} \arrow[r, bend right, "T"'] \arrow[ur, phantom, "\Downarrow\chi"] & \mathcal{L}
\end{tikzcd}
\overset{\theta_2}{\Longrightarrow}
\begin{tikzcd}
\mathcal{K}^2 \arrow[r, bend left, "S^2"] \arrow[d, "\otimes"'] \arrow[dr, phantom, "\Downarrow\chi"] & \mathcal{L}^2 \arrow[d, "\otimes"] \\
\mathcal{K} \arrow[r, bend left, "S"] \arrow[r, bend right, "T"'] \arrow[r, phantom, "\Downarrow\theta"] & \mathcal{L}
\end{tikzcd}$$

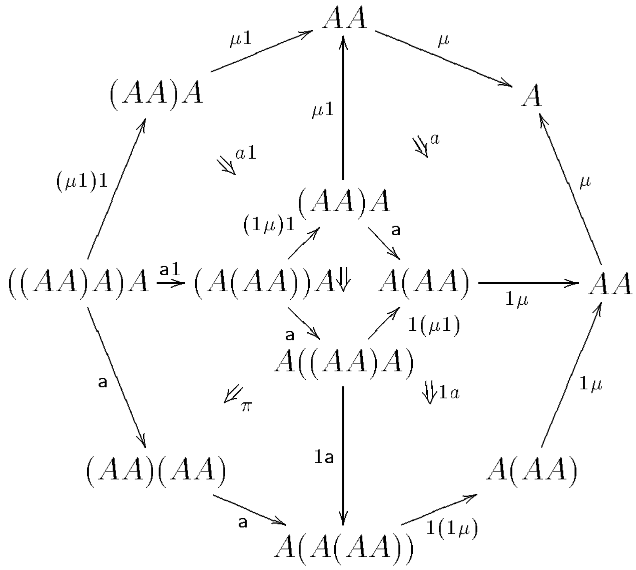
$$\begin{tikzcd}
\mathcal{K} \arrow[rr, bend left, "S"] \arrow[rr, bend right, "T"'] \arrow[rr, phantom, "\Downarrow\theta"] & & \mathcal{L} \\
& 1 \arrow[ul, "I"] \arrow[ur, "I"'] \arrow[u, phantom, "\iota\Downarrow"] &
\end{tikzcd}
\overset{\theta_0}{\Longrightarrow}
\begin{tikzcd}
\mathcal{K} \arrow[rr, bend left, "S"] & & \mathcal{L} \\
& 1 \arrow[ul, "I"] \arrow[ur, "I"'] \arrow[u, phantom, "\iota\Downarrow"] &
\end{tikzcd}$$

satisfying three coherence axioms [GPS95, Section 3.3]. A *monoidal modification* between parallel monoidal pseudonatural transformations is a modification satisfying two coherence axioms [GPS95, Section 3.3]. With the evident compositions, there is a 2-category $\mathrm{WMon}(\mathcal{K}, \mathcal{L})$ whose objects are weak monoidal homomorphisms, whose arrows are monoidal pseudonatural transformations and whose 2-cells are monoidal modifications from $\mathcal{K}$ to $\mathcal{L}$. When $\mathcal{K}$ and $\mathcal{L}$ are Gray-monoids, this 2-category agrees with the 2-category of weak monoidal homomorphisms given in [DS97, Section 1].

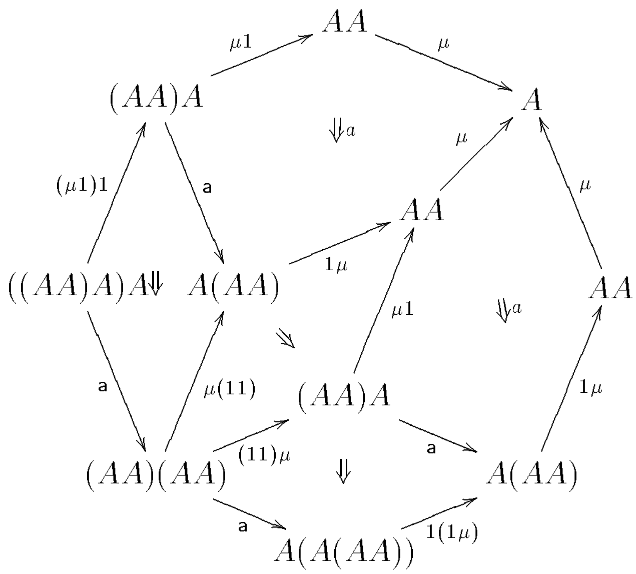
The terminal 2-category 1 is a monoidal 2-category in a unique way. For any monoidal 2-category $\mathcal{K}$, define $\mathrm{PsMon}(\mathcal{K})$ to be the full sub-2-category of $\mathrm{WMon}(1, \mathcal{K})$ whose objects consist of the weak monoidal 2-functors. An object of $\mathrm{PsMon}(\mathcal{K})$ is called a *pseudomonoid in* $\mathcal{K}$, and it amounts to an object A of $\mathcal{K}$ equipped with arrows $\mu\colon A \otimes A \to A$ and $\eta\colon I \to A$, called the *multiplication* and *unit* respectively, and invertible 2-cells,

$$\begin{tikzcd}
& A \otimes A \arrow[dr, "\mu"] & \\
(A \otimes A) \otimes A \arrow[ur, "\mu\otimes 1"] \arrow[d, "\mathsf{a}"'] & \Downarrow a & A \\
A \otimes (A \otimes A) \arrow[r, "1\otimes\mu"'] & A \otimes A \arrow[ur, "\mu"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
A \arrow[r, "\mathsf{r}"] \arrow[d, "1"'] \arrow[dr, phantom, "\overset{\Leftarrow}{r}"] & A \otimes I \arrow[d, "A\otimes\eta"] \\
A & A \otimes A \arrow[l, "\mu"]
\end{tikzcd}
\qquad
\begin{tikzcd}
& A \otimes A \arrow[d, "\mu"] \\
I \otimes A \arrow[ur, "\eta\otimes 1"] \arrow[dr, "\mathsf{l}"'] \arrow[r, phantom, "\Swarrow l"] & A
\end{tikzcd}$$

called the *associativity, left unit* and *right unit isomorphisms* respectively, such that the following two equations hold.



$$=$$



$$
\begin{tikzcd}
& & AA \arrow[dll, "r1"'] \arrow[drr, "1"] & & \\
(AI)A \arrow[d, "a"'] \arrow[drr, "(1\eta)1"] \arrow[rrrr, phantom, "\Swarrow r"] & & & & AA \arrow[dd, "\mu"] \\
A(IA) \arrow[d, "1"'] \arrow[drr, "1(\eta 1)"] \arrow[rr, phantom, "\Leftarrow"] & & (AA)A \arrow[urr, "\mu 1"'] \arrow[d, "a"] \arrow[drr, phantom, "\Swarrow a"] & & \\
A(IA) \arrow[drr, "1l"'] \arrow[rr, phantom, "\underset{l}{\Leftarrow}"] & & A(AA) \arrow[d, "1\mu"] & & A \\
& & AA \arrow[urr, "\mu"'] & &
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
& & AA \arrow[dll, "r1"'] \arrow[drr, "1"] \arrow[dddd, "1"] & & \\
(AI)A \arrow[d, "a"] & & & & AA \arrow[dd, "\mu"] \\
A(IA) \arrow[d, "1"] \arrow[rr, phantom, "\underset{a}{\Leftarrow}"] & & & & \\
A(IA) \arrow[drr, "1l"'] & & & & A \\
& & AA \arrow[urr, "\mu"'] & &
\end{tikzcd}
$$

Note that some canonical invertible 2-cells are omitted; for example, in the top left pentagon of the first diagram of the first axiom, we have written as if $\otimes \colon \mathcal{K} \times \mathcal{K} \to \mathcal{K}$ is a 2-functor. A pseudomonoid is called *strict* if a, r and l are identity 2-cells, and strict pseudomonoids are also called *monoids.* A pseudomonoid in the opposite monoidal 2-category $\mathcal{K}^{\mathrm{op}}$ is called a *pseudocomonoid in* $\mathcal{K}$ and a strict pseudomonoid in $\mathcal{K}^{\mathrm{op}}$ is called a *comonoid in* $\mathcal{K}$.

2.1. Example. *Bialgebras.* A monoid in the monoidal 2-category $\mathrm{Comon}(\mathcal{V})$ of comonoids in $\mathcal{V}$ is called a *bialgebra* in $\mathcal{V}$, and it amounts to a 5-tuple $(B, \delta, \varepsilon, \mu, \eta)$ such that (B, δ, ε) is a comonoid in $\mathcal{V}$, (B, μ, η) is a monoid in $\mathcal{V}$ and μ and η are comonoid morphisms. The fact that μ and η are comonoid morphisms is equivalent to the fact that δ and ε are monoid morphisms.

2.2. Example. *Quantum matrices.* We now provide an example of a bialgebra from the theory of quantum groups; this example is from [Str89]. Let n be a natural number, and q be a non-zero complex number. We shall describe the bialgebra $M_q(n)$ of *quantum* $n \times n$ *matrices.* Let $X = \{x_{ij} | i, j = 1, \ldots, n\}$ be a set of cardinality n^2, and let A be the free non-symmetric complex algebra generated by X. Let $M_q(n)$ be the quotient algebra of A by the ideal of generated by the following elements.

$$
\begin{array}{cl}
x_{ir}x_{jk} - x_{jk}x_{ir} & \text{for } i < j \text{ and } k < r \\
x_{ir}x_{jk} - x_{jk}x_{ir} - (q - q^{-1})x_{ik}x_{jr} & \text{for } i < j \text{ and } r < k \\
x_{ik}x_{jk} - qx_{jk}x_{ik} & \text{for } i < j \\
x_{ir}x_{ir} - qx_{ir}x_{ik} & \text{for } k < r
\end{array}
$$

The algebra $M_q(n)$ becomes a coalgebra when equipped with the comultiplication and counit

$$\delta(x_{ij}) = \textstyle\sum_{r=1}^{n} x_{ir} \otimes x_{rj} \qquad \varepsilon(x_{ij}) = \delta_{ij}$$

respectively, where δ_{ij} denotes the Kronecker delta. The multiplication and unit are coalgebra morphisms, and so $M_q(n)$ is a bialgebra. This bialgebra is called *the algebra of quantum matrices.* We shall see in Example 5.1 that $M_q(n)$ may be equipped with the structure of a *balanced coalgebra.*

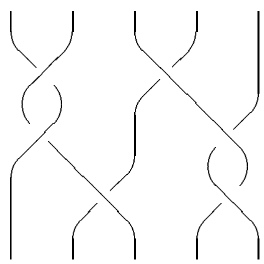
2.3. Example. *Quasi-bialgebras.* Drinfeld [Dri87, Dri89] defines a *quasi-bialgebra* to be an algebra A in the symmetric monoidal category of vector spaces equipped with arrows $\delta\colon A \to A \otimes A$ and $\varepsilon\colon A \to \mathbb{C}$, along with an arrow $\Phi\colon \mathbb{C} \to A$, called the *Drinfeld coassociator*, satisfying seven axioms; notably, δ is not coassociative. A quasi-bialgebra is precisely a pseudocomonoid in the monoidal 2-category Mon(Vect) such that the left and right unit isomorphisms are identities.

Dually, Majid [Maj92] defines *dual quasi-bialgebras*, and these are precisely pseudomonoids in the monoidal 2-category Comon(Vect) such that the left and right unit isomorphisms are identities.

2.4. Example. *Monoidal categories.* A pseudomonoid in the monoidal 2-category Cat is of course a monoidal category. In this context, the second axiom is sometimes called *Mac Lane's Pentagon* [Mac71, Chapter VII]. We now provide an example of a monoidal category pertinent to this article. Joyal and Street [JS93] define the braid category $\mathbb{B}$. The objects of $\mathbb{B}$ are natural numbers and the hom-sets are given by

$$\mathbb{B}(n,m) = \begin{cases} \mathbb{B}_n & \text{if } m = n; \\ \emptyset & \text{otherwise,} \end{cases}$$

where $\mathbb{B}_n$ is the n-th braid group [Art47]. Composition is given by multiplication in $\mathbb{B}_n$. An element of $\mathbb{B}_n$ is called a *braid on n strings* and may be visualised as an ambient isotopy class of *progressive* embeddings $[0,1] + \cdots + [0,1] \to \mathbb{R}^3$ with fixed end points. The following diagram is an example of a braid on 5 strings.



Composition may be visualised as putting one braid on top of another. The category $\mathbb{B}$ is equipped with a strict monoidal structure $\oplus\colon \mathbb{B}_n \times \mathbb{B}_m \to \mathbb{B}_{n+m}$ given by addition of braids, which may be visualised as placing braids next to each other.

2.5. Example. *Multiplicative functors.* Majid [Maj92] defines the notion of a *multiplicative functor*. If $\mathcal{W}$ and $\mathcal{V}$ are monoidal categories, then a multiplicative functor $F\colon \mathcal{W} \to \mathcal{V}$ is a functor $F\colon \mathcal{W} \to \mathcal{V}$ equipped with invertible natural transformations as in the following diagrams.

$$\begin{tikzcd} \mathcal{W}^2 \arrow[r, "F^2"] \arrow[d, "\otimes"'] \arrow[dr, phantom, "\swarrow\chi"] & \mathcal{V}^2 \arrow[d, "\otimes"] \\ \mathcal{W} \arrow[r, "F"'] & \mathcal{V} \end{tikzcd} \qquad \begin{tikzcd} & 1 \arrow[dl, "I"'] \arrow[dr, "I"] \arrow[d, phantom, "\Leftarrow\iota"] & \\ \mathcal{W} \arrow[rr, "F"'] & {} & \mathcal{V} \end{tikzcd}$$

These data are not subject to any coherence conditions. The 2-cell ι is required to be the identity in [Maj92]. A pseudomonoid in $\mathrm{Cat}/\mathcal{V}$ is precisely a monoidal category A equipped with a multiplicative functor $U\colon A \to \mathcal{V}$.

2.6. Example. *The unit of a monoidal 2-category.* The unit of a monoidal 2-category is a pseudomonoid in a canonical way. The multiplication $I \otimes I \to I$ is the left unit equivalence and the unit $I \to I$ is the identity. Of course the multiplication could also be taken to be the pseudoinverse of the right unit equivalence, and these pseudomonoids are isomorphic.

2.7. Example. *Pseudomonads.* A *pseudomonad* in a Gray-category T is exactly an object X of T equipped with a pseudomonoid in the monoidal 2-category $T(X,X)$. See [Lac98, Mar97] for theory of pseudomonads.

An arrow of $\mathrm{PsMon}(\mathcal{K})$ is called a *monoidal morphism*. A monoidal morphism from A to B amounts to an arrow $f\colon A \to B$ of $\mathcal{K}$ and 2-cells

$$\begin{tikzcd} A\otimes A \arrow[r, "f\otimes f"] \arrow[d, "\mu"'] \arrow[dr, phantom, "\swarrow\chi"] & B\otimes B \arrow[d, "\mu"] \\ A \arrow[r, "f"'] & B \end{tikzcd} \qquad \begin{tikzcd} & I \arrow[dl, "\eta"'] \arrow[dr, "\eta"] \arrow[d, phantom, "\Leftarrow\iota"] & \\ A \arrow[rr, "f"'] & & B \end{tikzcd}$$

satisfying three axioms axioms [GPS95, Section 3.3]. A monoidal morphism is said to be *strong* if χ and ι are invertible, and is said to be *strict* if χ and ι are the identity 2-cells. A 2-cell of $\mathrm{PsMon}(\mathcal{K})$ is called a *monoidal transformation* between parallel monoidal morphisms is a 2-cell of $\mathcal{K}$ satisfying two axioms [GPS95, Section 3.3].

Evaluation at the only object of 1 provides a 2-functor $U\colon \mathrm{PsMon}(\mathcal{K}) \to \mathcal{K}$ called the *forgetful 2-functor*. If $F\colon \mathcal{K} \to \mathcal{L}$ is a weak monoidal homomorphism, then there is an evident homomorphism $\mathrm{PsMon}(F)\colon \mathrm{PsMon}(\mathcal{K}) \to \mathrm{PsMon}(\mathcal{L})$ and the following diagram of 2-functors commutes; compare to [DS97, Proposition 5].

$$\begin{tikzcd} \mathrm{PsMon}(\mathcal{K}) \arrow[r, "\mathrm{PsMon}(F)"] \arrow[d, "U"'] & \mathrm{PsMon}(\mathcal{L}) \arrow[d, "U"] \\ \mathcal{K} \arrow[r, "F"'] & \mathcal{L} \end{tikzcd}$$

2.8. Example. *Multiplicative functors.* Let $\mathcal{V}$ be the symmetric monoidal category of vector spaces, and $\mathcal{V}_f$ the full subcategory of $\mathcal{V}$ consisting of the finite dimensional vector spaces. There is a 2-functor $\mathrm{Comod}_f\colon \mathrm{Comon}(\mathcal{V}) \to \mathrm{Cat}/\mathcal{V}_f$ whose value on a comonoid C in $\mathcal{V}$ is its category of finite dimensional representations $\mathrm{Comod}_f(C)$ equipped with the forgetful functor $\omega\colon \mathrm{Comod}_f(C) \to \mathcal{V}_f$. Street [Str89] equips this 2-functor with the structure of a weak monoidal 2-functor. By Examples 2.3 and 2.5, we obtain the result [Maj92, Theorem 2.2] that if B is a dual quasi-bialgebra then $\mathrm{Comod}_f(B)$ is a monoidal category and the forgetful functor $\omega\colon \mathrm{Comod}_f(B) \to \mathcal{V}$ is multiplicative.

Variants of the converse of Example 2.8 has been considered. Let $\mathcal{V}$ be the symmetric monoidal category of modules over a commutative ring, and $\mathcal{V}_f$ denote the full subcategory of $\mathcal{V}$ consisting of the finitely generated, projective modules. For a coalgebra C in $\mathcal{V}$, let $\mathrm{Comod}_f(C)$ denote the full subcategory of $\mathrm{Comod}(C)$ consisting of those C-comodules whose underlying module is an object of $\mathcal{V}_f$. Majid [Maj92, Theorem 2.2] shows that if $\mathcal{C}$ is a small category, and $\sigma\colon \mathcal{C} \to \mathcal{V}_f$ is a multiplicative functor, then there is a dual quasi-bialgebra C which is unique with the property that σ factorizes as a monoidal functor $\mathcal{C} \to \mathrm{Comod}_f(C)$ followed by the forgetful functor. Other results may be found in [JS91, Par81, Par96, Sch92b, Sch92a].

The converse of Example 2.8 is not true for an arbitrary braided monoidal category [McC99c, Proposition 2.6] and consideration of this problem is one of the main motivations of this article.

3. Braided pseudomonoids

In this section, the definition of a *braided pseudomonoid* in a braided monoidal 2-category is provided, generalizing that of a braided pseudomonoid in a braided Gray-monoid [DS97, Section 4]. First, the definition of a braided monoidal 2-category is given, generalizing that of a braided Gray-monoid [KV94, BN96, DS97, Cra98].

Recall that an equivalence in a 2-category $\mathcal{K}$ may be replaced with an adjoint equivalence. That is, if f is an equivalence in $\mathcal{K}$ then f has a right adjoint g with invertible counit and unit ε and η respectively. Of course, g is also the left adjoint of f with invertible unit and counit ε^{-1} and η^{-1} respectively. We call g the *adjoint pseudoinverse* of f.

Suppose $\mathcal{K}$ is a monoidal 2-category. Let $\sigma\colon \mathcal{K}^2 \to \mathcal{K}^2$ denote the symmetry 2-functor. Let a^* be the adjoint pseudoinverse of the associativity equivalence a in the 2-category $\mathrm{hom}(\mathcal{K}^3, \mathcal{K})$ of homomorphisms, pseudonatural transformations and modifications from $\mathcal{K}^3$ to $\mathcal{K}$. A *braiding* for $\mathcal{K}$ consists of a pseudonatural equivalence

$$\begin{tikzcd} \mathcal{K}^2 \arrow[rr, "\otimes"] \arrow[dr, "\sigma"'] & \arrow[d, phantom, "\Downarrow\rho"] & \mathcal{K} \\ & \mathcal{K}^2 \arrow[ur, "\otimes"'] & \end{tikzcd}$$

in $\mathrm{hom}(\mathcal{K}^2, \mathcal{K})$, together with invertible modifications R and S in $\mathrm{hom}(\mathcal{K}^3, \mathcal{K})$ whose components at an object (A, B, C) of $\mathcal{K}^3$ are exhibited in the following diagrams respectively,

$$\begin{tikzcd} & A(BC) \arrow[r, "\rho"] & (BC)A \arrow[dr, "\mathsf{a}"] & \\ (AB)C \arrow[ur, "\mathsf{a}"] \arrow[dr, "\rho C"'] & & \arrow[u, phantom, "\Downarrow R"] & B(CA) \\ & (BA)C \arrow[r, "\mathsf{a}"'] & B(AC) \arrow[ur, "B\rho"'] & \end{tikzcd}$$

$$\begin{tikzcd}[column sep=small]
 & (AB)C \arrow[r, "\rho"] & C(AB) \arrow[dr, "\mathsf{a}^*"] & \\
A(BC) \arrow[ur, "\mathsf{a}^*"] \arrow[dr, "A\rho"'] & & \Downarrow S & (CA)B \\
 & A(CB) \arrow[r, "\mathsf{a}^*"'] & (AC)B \arrow[ur, "\rho B"'] &
\end{tikzcd}$$

satisfying the four coherence axioms (BA1), (BA2), (BA3) and (BA4) of Appendix A. A *braided monoidal 2-category* is a monoidal 2-category equipped with a braiding. When $\mathcal{K}$ is in fact a Gray-monoid, the associativity equivalence a is the identity and so we may take a^* to be the identity also. In this case the definition of a braided monoidal 2-category agrees with that of a braided Gray-monoid as given in [BN96, DS97]. It differs from that given in [Cra98] in that we require no further axioms on the braiding of the unit, as [Cra98] does. If $\mathcal{K}$ is a braided monoidal category, then the monoidal 2-categories $\mathcal{K}^{\mathrm{op}}$, $\mathcal{K}^{\mathrm{co}}$ and $\mathcal{K}^{\mathrm{rev}}$, obtained by reversing 1-cells, 2-cells and the tensor product respectively, are equipped with a braiding in a canonical manner.

There are many well-known examples of braided monoidal 2-categories. Indeed every braided monoidal category may be considered to be a locally discrete braided monoidal 2-category. Every braided Gray-monoid may be considered to be a braided monoidal 2-category, although it is yet to be shown that every braided monoidal 2-category is braided monoidally biequivalent to a braided Gray-monoid. If a 2-category has finite products or finite bi-products then it is a braided monoidal 2-category in a canonical manner. For a braided monoidal category $\mathcal{V}$, one expects the monoidal 2-category $\mathcal{V}$-Cat to be a braided monoidal 2-category, however, for $\mathcal{V}$-categories A and B the natural map $A \otimes B \to B \otimes A$ is not even a $\mathcal{V}$-functor. When $\mathcal{V}$ is in fact symmetric it is a $\mathcal{V}$-functor and is the component at the pair (A, B) of a 2-natural isomorphism. With this 2-natural isomorphism, $\mathcal{V}$-Cat becomes a braided monoidal 2-category with the invertible modifications R and S taken to be identities. Since the opposite $\mathcal{V}^{\mathrm{op}}$ of a symmetric monoidal category is again symmetric, we deduce that the 2-category $\mathcal{V}^{\mathrm{op}}$-Cat is canonically braided when $\mathcal{V}$ is symmetric. The braidings on $\mathcal{V}$-Cat and $\mathcal{V}^{\mathrm{op}}$-Cat$^{\mathrm{op}}$ restrict to the full sub-2-categories Mon($\mathcal{V}$) and Comon($\mathcal{V}$) of monoids and comonoids in $\mathcal{V}$ respectively. The monoidal 2-category Cat/$\mathcal{V}$ is canonically braided when $\mathcal{V}$ is a braided monoidal category. The braiding $\rho\colon (A, U) \otimes (B, V) \to (B, V) \otimes (A, U)$ is (σ, φ) where $\sigma\colon A \times B \to B \times A$ is the usual symmetry on Cat and φ is given by the braiding on $\mathcal{V}$. The invertible modifications R and S are identities. This example will be generalized in Section 8. For a symmetric monoidal category $\mathcal{V}$ the 2-category $\mathcal{V}$-Act of $\mathcal{V}$-actegories is canonically equipped with a braiding; a description is provided in Section 7.

Now suppose $\mathcal{K}$ and $\mathcal{L}$ are braided monoidal 2-categories and $T\colon \mathcal{K} \to \mathcal{L}$ is a weak monoidal homomorphism. A *braiding* for T consist of an invertible modification

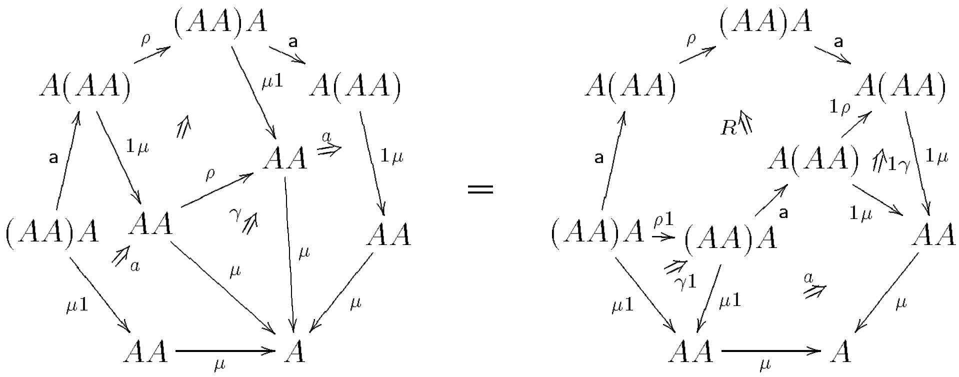
$$
\begin{tikzcd}
& \mathcal{L}^2 \arrow[dr, "\otimes"] & \\
\mathcal{L}^2 \arrow[ur, "\sigma"] & \mathcal{K}^2 \arrow[u, "T^2"] \arrow[dr, "\otimes"] \arrow[r, phantom, "\chi\Downarrow"] & \mathcal{L} \\
\mathcal{K}^2 \arrow[u, "T^2"] \arrow[ur, "\sigma"] \arrow[rr, "\otimes"'] \arrow[urr, phantom, "\Downarrow\rho"] & & \mathcal{K} \arrow[u, "T"']
\end{tikzcd}
\overset{u}{\Rrightarrow}
\begin{tikzcd}
& \mathcal{L}^2 \arrow[dr, "\otimes"] \arrow[d, phantom, "\Downarrow\rho"] & \\
\mathcal{L}^2 \arrow[ur, "\sigma"] \arrow[rr, "\otimes"'] & {} & \mathcal{L} \\
\mathcal{K}^2 \arrow[u, "T^2"] \arrow[rr, "\otimes"'] \arrow[urr, phantom, "\chi"] & & \mathcal{K} \arrow[u, "T"']
\end{tikzcd}
$$

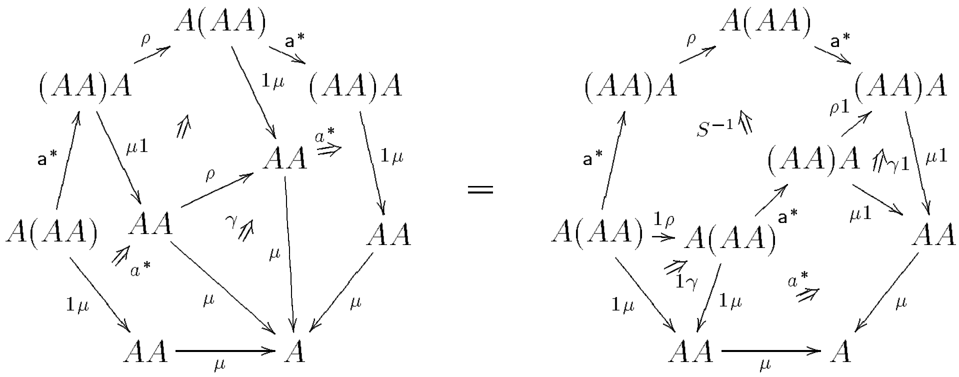
in $\hom(\mathcal{K}^2, \mathcal{L})$ subject to the axioms (BHA1) and (BHA2) of Appendix A. A *braided weak monoidal homomorphism* is a weak monoidal morphism equipped with a braiding. *Braided opweak monoidal homomorphisms* may be defined similarly. A *braided weak monoidal 2-functor* is of course a braided weak monoidal homomorphism whose underlying homomorphism is a 2-functor. A *braided monoidal pseudonatural transformation* between parallel braided weak monoidal homomorphisms is a monoidal pseudonatural transformation satisfying the axiom (BTA1) of Appendix A. A *braided monoidal modification* between parallel braided monoidal pseudonatural transformations is a simply a monoidal modification. With the evident compositions, there is a 2-category $\mathrm{BrWMon}(\mathcal{K}, \mathcal{L})$ whose objects are braided weak monoidal homomorphisms, whose 1-cells are braided monoidal pseudonatural transformations and whose 2-cells are braided monoidal modifications. When $\mathcal{K}$ and $\mathcal{L}$ are Gray-monoids, this 2-category agrees with the 2-category of braided weak monoidal homomorphisms given in [DS97].

The terminal 2-category 1 is a braided monoidal 2-category in a unique way. For any braided monoidal 2-category $\mathcal{K}$, define $\mathrm{BrPsMon}(\mathcal{K})$ to be the full sub-2-category of $\mathrm{BrWMon}(1, \mathcal{K})$ consisting of the braided weak monoidal 2-functors. An object of $\mathrm{BrPsMon}(\mathcal{K})$ is called a *braided pseudomonoid in* $\mathcal{K}$, and it amounts to a pseudomonoid A in $\mathcal{K}$ equipped with an invertible 2-cell,

$$
\begin{tikzcd}
A \otimes A \arrow[rr, "\rho"] \arrow[dr, "\mu"'] & \overset{\gamma}{\Rightarrow} & A \otimes A \arrow[dl, "\mu"] \\
& A &
\end{tikzcd}
$$

called the *braiding on A*, satisfying the following two axioms.





Here, the 2-cell a^* is defined using mates. Observe that some canonical isomorphisms are omitted; for example, $\otimes$ is written as a 2-functor in the diagrams on the right.

3.1. Example. *Commutative bialgebras.* Suppose B is a bialgebra in a symmetric monoidal category, whose underlying algebra is commutative. Then the identity 2-cell equips B with a braiding.

3.2. Example. *Dual quasi-triangular quasi-bialgebras.* Majid [Maj92] defines a dual quasi-triangular quasi-bialgebra to be a dual quasi-bialgebra C equipped with an arrow $R\colon C \otimes C \to \mathbb{C}$ satisfying four axioms. A braided pseudomonoid in Comon(Vect) is precisely a dual quasi-triangular quasi-bialgebra whose left and right unit isomorphisms are the identity 2-cells.

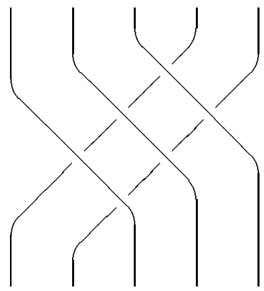
Dual quasi-triangular quasi-bialgebras have appeared under many names in the literature. For example, when the associativity isomorphism is the identity, they are called coquasi-triangular bialgebras in [Mon93, Sch92a], braided bialgebras in [LT91] and co-braided bialgebras in [Kas95]. When equipped with an antipode, they have been called quasi-quantum groups [Dri87]. At the risk of creating further confusion, the author prefers braided pseudomonoidal comonoids.

3.3. EXAMPLE. *Quantum matrices.* Recall the bialgebra $M_q(n)$ of Example 2.2. The linear map $R\colon M_q(n) \otimes M_q(n) \to \mathbb{C}$ defined by

$$R(x_{ij} \otimes x_{kl}) = \begin{cases} 1 & \text{for } i \neq k, j = k, \text{and } l = i; \\ q - q^{-1} & \text{for } i < k, j = i, \text{and } l = k; \\ q & \text{for } i = j = k = l; \\ 0 & \text{otherwise,} \end{cases}$$

equips $M_q(n)$ with a braiding [Str89].

3.4. EXAMPLE. *Braided monoidal categories.* A braided pseudomonoid in Cat is of course a braided monoidal category. The braid category $\mathbb{B}$ described in Example 2.4 is naturally equipped with a braiding. For objects n and m of $\mathbb{B}$, the braiding $c\colon n+m \to m+n$ may be visualised as passing m strings across n strings. For example, the diagram



is the braiding $c\colon 2 + 3 \to 3 + 2$.

3.5. EXAMPLE. *Braided monoidal categories equipped with a multiplicative functor.* For a braided monoidal category $\mathcal{V}$, a braided pseudomonoid in $\mathrm{Cat}/\mathcal{V}$ is precisely a braided monoidal A category equipped with a multiplicative functor $U\colon A \to \mathcal{V}$.

3.6. EXAMPLE. *The unit of a braided monoidal 2-category.* Example 2.6 shows that the unit of a monoidal 2-category $\mathcal{K}$ is a pseudomonoid in $\mathcal{K}$. When $\mathcal{K}$ is braided, this pseudomonoid is canonically equipped with a braiding.

An arrow of $\mathrm{BrPsMon}(\mathcal{K})$ is called a *braided monoidal morphism*, and it amounts to a monoidal morphism satisfying one axiom. A 2-cell of $\mathrm{BrPsMon}(\mathcal{K})$ is called a *braided monoidal transformation* and it is simply a monoidal transformation between the corresponding monoidal morphisms. There is a forgetful 2-functor $U\colon \mathrm{BrPsMon}(\mathcal{K}) \to \mathcal{K}$ and if $F\colon \mathcal{K} \to \mathcal{L}$ is a weak braided monoidal homomorphism then there is an evident homomorphism $\mathrm{BrPsMon}(F)\colon \mathrm{BrPsMon}(\mathcal{K}) \to \mathrm{BrPsMon}(\mathcal{L})$ and the following diagram commutes; compare to [DS97, Propositon 11].

$$\begin{tikzcd}
\mathrm{BrPsMon}(\mathcal{K}) \arrow[r, "\mathrm{BrPsMon}(F)"] \arrow[d, "U"'] & \mathrm{BrPsMon}(\mathcal{L}) \arrow[d, "U"] \\
\mathcal{K} \arrow[r, "F"'] & \mathcal{L}
\end{tikzcd}$$

3.7. Example. *Dual quasi-triangular quasi-bialgebras.* Let $\mathcal{V}$ be the symmetric monoidal category of vector spaces. Street shows that the 2-functor $\mathrm{Comod}_f\colon \mathrm{Comon}(\mathcal{V}) \to \mathrm{Cat}/\mathcal{V}_f$ is a weak braided monoidal 2-functor in a natural way [Str89]. By Examples 3.2 and 3.5, we obtain the result [Maj92, Theorem 2.8] that if B is a braided pseudomonoidal comonoid (that is to say, a quasi-triangular dual quasi-bialgebra) then $\mathrm{Comod}_f(B)$ is a braided monoidal category and the forgetful functor $\omega\colon \mathrm{Comod}_f(B) \to \mathcal{V}_f$ is multiplicative.

As for the case of pseudomonoids, variants of the converse of Example 3.7 have been considered. As at the conclusion of Section 2, let $\mathcal{V}$ be the symmetric monoidal category of modules over a commutative ring and let $\mathcal{V}_f$ be the full subcategory of $\mathcal{V}$ consisting of the finitely generated, projective modules. Majid [Maj92, Theorem 2.8] shows that if $\mathcal{C}$ is a braided monoidal category, and $\sigma\colon \mathcal{C} \to \mathcal{V}_f$ is a multiplicative functor, then the dual quasi-bialgebra C with the universal property described at the conclusion of Section 2, is equipped with a dual quasi-triangular structure. Other results may be found in [JS91, Par81, Par96, Sch92b, Sch92a].

4. Symmetric pseudomonoids

In this section, the definition of a *symmetric pseudomonoid* in a sylleptic monoidal 2-category is provided. First, the definition of a sylleptic monoidal 2-category is given, generalizing that of a sylleptic gray monoid [KV94, BN96, DS97, Cra98]. Then symmetric monoidal 2-categories are defined, and it is observed that the examples developed so far in this article are symmetric.

If $\mathcal{K}$ is a braided monoidal 2-category, then a *syllepsis* for $\mathcal{K}$ is an invertible modification

$$\begin{tikzcd} & \mathcal{K}^2 \arrow[dd, "\otimes"] \\ \mathcal{K}^2 \arrow[ur, bend left, "1"] \arrow[dr, bend right, "\otimes"'] & \\ & \mathcal{K} \end{tikzcd} \quad \overset{v}{\Rrightarrow} \quad \begin{tikzcd} & & \mathcal{K}^2 \arrow[dd, "\otimes"] \\ \mathcal{K}^2 \arrow[urr, bend left, "1"] \arrow[r, "\sigma"] \arrow[drr, bend right, "\otimes"'] & \mathcal{K}^2 \arrow[ur, "\sigma"] \arrow[dr, "\otimes"] \arrow[r, phantom, "\Downarrow\rho"] \arrow[d, phantom, "\Downarrow\rho"] & {} \\ & {} & \mathcal{K} \end{tikzcd}$$

in $\hom(\mathcal{K}^2, \mathcal{K})$, satisfying the axioms (SA1) and (SA2) of Appendix B. A *sylleptic monoidal 2-category* is a braided monoidal 2-category equipped with a syllepsis. When the underlying braided monoidal 2-category $\mathcal{K}$ is a braided gray monoid this definition agrees with that of [DS97]. If $\mathcal{K}$ is a sylleptic monoidal category, then the braided monoidal 2-categories $\mathcal{K}^{\mathrm{op}}$, $\mathcal{K}^{\mathrm{co}}$ and $\mathcal{K}^{\mathrm{rev}}$, obtained by reversing 1-cells, 2-cells and the tensor product respectively, are equipped with a syllepsis in a canonical manner.

The examples of braided monoidal 2-categories given in Section 3 give rise to examples of sylleptic monoidal 2-categories. The identity syllepsis makes a braided monoidal category $\mathcal{V}$ into a locally discrete sylleptic monoidal 2-category precisely when $\mathcal{V}$ is symmetric. If a 2-category has finite products or finite bi-products then it is a sylleptic monoidal

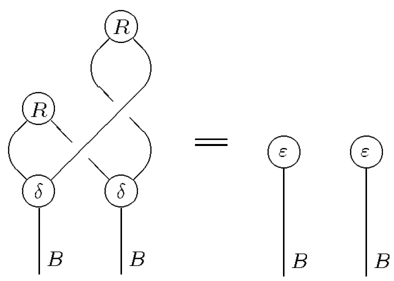
2-category in a canonical manner. For a symmetric monoidal category $\mathcal{V}$, the braided monoidal 2-category $\mathcal{V}$-Cat is sylleptic with identity syllepsis. Thus the braided monoidal 2-categories $\mathcal{V}^{\mathrm{op}}$-Cat, $\mathcal{V}^{\mathrm{op}}$-Cat$^{\mathrm{op}}$, Mon($\mathcal{V}$) and Comon($\mathcal{V}$) are also sylleptic with identity syllepsis. When $\mathcal{V}$ is a symmetric monoidal category, the identity modification is a syllepsis for the braided monoidal 2-category Cat/$\mathcal{V}$. Finally, the identity modification is a syllepsis for the braided monoidal 2-category $\mathcal{V}$-Act; this will be shown in Section 7.

Now suppose $\mathcal{K}$ and $\mathcal{L}$ are sylleptic monoidal 2-categories. A *sylleptic weak monoidal homomorphism* from $\mathcal{K}$ to $\mathcal{L}$ is a braided weak monoidal homomorphism satisfying the axiom (SHA1) of Appendix B. *Sylleptic opweak monoidal homomorphisms* may be similarly defined. A *sylleptic weak monoidal 2-functor* is of course a sylleptic weak monoidal homomorphism whose underlying homomorphism is a 2-functor. *Sylleptic monoidal pseudonatural transformations* and *sylleptic monoidal modifications* are simply braided monoidal pseudonatural transformations and braided monoidal modifications respectively. With the evident compositions, there is a 2-category SyllWMon($\mathcal{K},\mathcal{L}$) whose objects are sylleptic weak monoidal homomorphisms, whose arrows are sylleptic monoidal pseudonatural transformations and whose 2-cells are sylleptic monoidal modifications. It is a full sub-2-category of BrWMon($\mathcal{K},\mathcal{L}$).

The terminal 2-category 1 is a sylleptic monoidal 2-category in a unique way. For any sylleptic monoidal 2-category $\mathcal{K}$, define SymPsMon($\mathcal{K}$) to be the full sub-2-category of SyllWMon($1,\mathcal{K}$) consisting of the sylleptic weak monoidal 2-functors. An object of SymPsMon($\mathcal{K}$) is called a *symmetric pseudomonoid in* $\mathcal{K}$, and it amounts to a braided pseudomonoid A in $\mathcal{K}$ such that the following equation holds.

$$\begin{tikzcd} A\otimes A \arrow[r, "\rho"] \arrow[dr, "\mu"'] & A\otimes A \arrow[r, "\rho"] \arrow[d, "\mu"] \arrow[dl, phantom, "\overset{\gamma}{\Rightarrow}"] \arrow[dr, phantom, "\overset{\gamma}{\Rightarrow}"] & A\otimes A \arrow[dl, "\mu"] \\ & A & \end{tikzcd} \quad = \quad \begin{tikzcd} A\otimes A \arrow[r, "\rho"] \arrow[dr, "1"'] & A\otimes A \arrow[d, "\rho"] \arrow[dl, phantom, "\overset{v}{\Rightarrow}"] & \\ & A\otimes A \arrow[r, "\mu"] & A \end{tikzcd}$$

4.1. Example. *Cosymmetric bialgebras.* A *cosymmetric bialgebra* is a symmetric strict pseudomonoid in Comon($\mathcal{V}$). It amounts to a bialgebra B equipped with a braiding $R\colon B\otimes B \to I$ such that the following equations holds.



If B is a bialgebra whose underlying algebra is commutative, then the braiding for B given in Example 3.1 is a symmetry. In Section 11, a non-trivial example of a cosymmetric bialgebra will be given.

4.2. Example. *Symmetric monoidal categories.* A symmetric pseudomonoid in Cat is of course a symmetric monoidal category.

4.3. Example. *The unit of a sylleptic monoidal 2-category.* When $\mathcal{K}$ is a sylleptic monoidal 2-category, the unit braided pseudomonoid of Example 3.6 is a symmetric pseudomonoid.

An arrow of $\mathrm{SymPsMon}(\mathcal{K})$ is called a *symmetric monoidal morphism* and a 2-cell is called a *symmetric monoidal transformation*. There is an inclusion $\mathrm{SymPsMon}(\mathcal{K}) \to \mathrm{BrPsMon}(\mathcal{K})$ which is 2-fully-faithful. There is a *forgetful 2-functor* $U\colon \mathrm{SymPsMon}(\mathcal{K}) \to \mathcal{K}$, and if $F\colon \mathcal{K} \to \mathcal{L}$ is a weak sylleptic monoidal homomorphism, then there is an evident homomorphism $\mathrm{SymPsMon}(F)\colon \mathrm{SymPsMon}(\mathcal{K}) \to \mathrm{SymPsMon}(\mathcal{L})$ and the following diagram commutes.

$$\begin{tikzcd}
\mathrm{SymPsMon}(\mathcal{K}) \arrow[r, "\mathrm{SymPsMon}(F)"] \arrow[d, "U"'] & \mathrm{SymPsMon}(\mathcal{L}) \arrow[d, "U"] \\
\mathcal{K} \arrow[r, "F"'] & \mathcal{L}
\end{tikzcd}$$

A *symmetric monoidal 2-category* is a sylleptic monoidal 2-category satisfying the following equation for all objects A and B.

$$\begin{tikzcd}[column sep=small]
A\otimes B \arrow[rr, "1"] \arrow[dr, "\rho"'] & \arrow[d, phantom, "\Downarrow u"] & A\otimes B \arrow[r, "\rho"] & B\otimes A \\
& B\otimes A \arrow[ur, "\rho"'] & &
\end{tikzcd}
=
\begin{tikzcd}[column sep=small]
& A\otimes B \arrow[r, "\rho"] \arrow[d, phantom, "\Downarrow"] & B\otimes A \arrow[d, phantom, "\Downarrow v"] & \\
A\otimes B \arrow[ur, "1"] \arrow[r, "\rho"'] & B\otimes A \arrow[ur, "1"] \arrow[rr, "\rho"'] & & A\otimes B \arrow[ul, "\rho"']
\end{tikzcd}$$

All the examples of sylleptic monoidal 2-categories given above are in fact symmetric monoidal 2-categories. For symmetric monoidal 2-categories $\mathcal{K}$ and $\mathcal{L}$, there is a 2-category $\mathrm{SymWMon}(\mathcal{K}, \mathcal{L})$ which is equal to the 2-category $\mathrm{SyllWMon}(\mathcal{K}, \mathcal{L})$.

5. Balanced pseudomonoids

In this section we present the definition of a *balanced monoidal 2-category* and that of a *balanced pseudomonoid*. These are the main objects of study of this article, and generalize the balanced pseudomonoids in a balanced Gray-monoid of [DS97, Section 6].

When $\mathcal{K}$ is a braided monoidal 2-category the braiding on $\mathcal{K}$ equips the identity homomorphism with the structure of a monoidal homomorphism $S\colon \mathcal{K} \to \mathcal{K}^{\mathrm{rev}}$. Similarly, the adjoint pseudoinverse of the braiding on $\mathcal{K}$ equips the identity homomorphism with the structure of a monoidal homomorphism $T\colon \mathcal{K} \to \mathcal{K}^{\mathrm{rev}}$. These monoidal homomorphisms are pseudonatural in the sense that if $F\colon \mathcal{K} \to \mathcal{L}$ is a braided weak monoidal homomorphism, then there are canonical monoidal pseudonatural transformations

$$
\begin{tikzcd}
\mathcal{K} \arrow[r, "S"] \arrow[d, "F"'] \arrow[dr, phantom, "\Downarrow"] & \mathcal{K}^{\mathrm{rev}} \arrow[d, "F^{\mathrm{rev}}"] \\
\mathcal{K} \arrow[r, "S"'] & \mathcal{L}^{\mathrm{rev}}
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{K} \arrow[r, "T"] \arrow[d, "F"'] \arrow[dr, phantom, "\Downarrow"] & \mathcal{K}^{\mathrm{rev}} \arrow[d, "F^{\mathrm{rev}}"] \\
\mathcal{K} \arrow[r, "T"'] & \mathcal{L}^{\mathrm{rev}}
\end{tikzcd}
$$

which are isomorphisms in $\mathrm{WMon}(\mathcal{K},\mathcal{L})$. A *twist* for a braided monoidal category $\mathcal{K}$ is a monoidal pseudonatural equivalence $t\colon S \to T$ where S and T are described above. The data for a twist consists of an equivalence $t\colon A \to A$ for each object A of $\mathcal{K}$, along with invertible 2-cells as in the following diagrams, where f is any arrow of $\mathcal{K}$.

$$
\begin{tikzcd}
A \arrow[r, "t"] \arrow[d, "f"'] \arrow[dr, phantom, "\Nearrow"] & A \arrow[d, "f"] \\
B \arrow[r, "t"'] & B
\end{tikzcd}
\qquad
\begin{tikzcd}
A \otimes B \arrow[r, "t"] \arrow[d, "\rho"'] \arrow[dr, phantom, "\Nearrow"] & A \otimes B \\
B \otimes A \arrow[r, "t \otimes t"'] & B \otimes A \arrow[u, "\rho"']
\end{tikzcd}
\qquad
\begin{tikzcd}
I \arrow[r, "t"] \arrow[d, "1"'] \arrow[dr, phantom, "\Nearrow"] & I \arrow[d, "1"] \\
I \arrow[r, "t"'] & I
\end{tikzcd}
$$

A *balanced monoidal 2-category* is a braided monoidal 2-category equipped with a twist. As for the case of Gray-monoids [DS97, Section 6], the identity monoidal transformation with the syllepsis equips a sylleptic monoidal 2-category with a twist. All examples of balanced monoidal 2-categories considered in this article will be of this form.

A *balanced weak monoidal homomorphism* is a braided weak monoidal homomorphism $F\colon \mathcal{K} \to \mathcal{L}$ equipped with an invertible monoidal modification

$$
\begin{tikzcd}
\mathcal{K} \arrow[r, bend left, "S"] \arrow[r, bend right, "T"'] \arrow[r, phantom, "\Downarrow t"] \arrow[d, "F"'] \arrow[dr, phantom, "\Downarrow"] & \mathcal{K}^{\mathrm{rev}} \arrow[d, "F^{\mathrm{rev}}"] \\
\mathcal{K} \arrow[r, bend right, "T"'] & \mathcal{L}^{\mathrm{rev}}
\end{tikzcd}
\quad \overset{\tau}{\Rrightarrow} \quad
\begin{tikzcd}
\mathcal{K} \arrow[r, bend left, "S"] \arrow[d, "F"'] \arrow[dr, phantom, "\Downarrow"] & \mathcal{K}^{\mathrm{rev}} \arrow[d, "F^{\mathrm{rev}}"] \\
\mathcal{L} \arrow[r, bend left, "S"] \arrow[r, bend right, "T"'] \arrow[r, phantom, "\Downarrow t"] & \mathcal{L}^{\mathrm{rev}}
\end{tikzcd}
$$

in $\mathrm{WMon}(\mathcal{K},\mathcal{L})$. The data for τ consists of a family of invertible 2-cells $\tau_A\colon Ft_A \Rightarrow t_A$. If $F\colon \mathcal{K} \to \mathcal{L}$ is a sylleptic weak monoidal homomorphism, the for each object A of $\mathcal{K}$, the unit isomorphism $F1_A \Rightarrow 1_A$ equips F with the structure of a balanced weak monoidal homomorphism. If (F,τ) and (G,τ) are parallel balanced weak monoidal homomorphisms then a *balanced monoidal pseudonatural transformation* from (F,τ) to (G,τ) is a braided monoidal pseudonatural transformation $\theta\colon F \Rightarrow G$ such that

$$
\begin{tikzcd}
FA \arrow[r, bend left, "Ft"] \arrow[r, bend right, "t"'] \arrow[r, phantom, "\Downarrow \tau"] \arrow[d, "\theta"'] \arrow[dr, phantom, "\Downarrow \theta"] & FA \arrow[d, "\theta"] \\
GA \arrow[r, bend right, "t"'] & GA
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
FA \arrow[r, bend left, "Ft"] \arrow[d, "\theta"'] \arrow[dr, phantom, "\Downarrow \theta"] & FA \arrow[d, "\theta"] \\
GA \arrow[r, bend left, "Gt"] \arrow[r, bend right, "t"'] \arrow[r, phantom, "\Downarrow t"] & GA
\end{tikzcd}
$$

for all objects A of the domain. A *balanced monoidal modification* is simply a braided monoidal modification. With the evident compositions, there is a 2-category $\mathrm{BalWMon}(\mathcal{K}, \mathcal{L})$ whose objects are balanced weak monoidal homomorphisms, whose 1-cells are balanced monoidal pseudonatural transformations and whose 2-cells are balanced monoidal modifications. A *balanced weak monoidal 2-functor* is of course a balanced weak monoidal homomorphism whose underlying homomorphism is a 2-functor. For sylleptic monoidal 2-categories, there is a functor $\mathrm{SyllWMon}(\mathcal{K}, \mathcal{L}) \to \mathrm{BalWMon}(\mathcal{K}, \mathcal{L})$ whose value on objects is described above, and whose value on arrows and 2-cells is the identity.

The terminal 2-category 1 is a balanced monoidal 2-category in a unique way. For any balanced monoidal 2-category $\mathcal{K}$, define $\mathrm{BalPsMon}(\mathcal{K})$ to be the full sub-2-category of $\mathrm{BalWMon}(1, \mathcal{K})$ consisting of the balanced weak monoidal 2-functors. An object of $\mathrm{BalPsMon}(\mathcal{K})$ is called a *balanced pseudomonoid in* $\mathcal{K}$ and it amounts to a braided pseudomonoid A in $\mathcal{K}$ equipped with an invertible 2-cell

$$\begin{tikzcd} A \arrow[r, bend left, "1"{name=U}] \arrow[r, bend right, "1"'{name=D}] & A \arrow[from=U, to=D, phantom, "\Downarrow\tau"] \end{tikzcd}$$

called a *twist* such that two axioms hold. A 1-cell in $\mathrm{BalPsMon}(\mathcal{K})$ is called a *balanced monoidal morphism* and it amounts to a braided monoidal morphism satisfying one further axiom.

5.1. Example. *Quantum matrices.* Recall the braided bialgebra $M_q(n)$ of Example 3.3. The linear map $\tau\colon M_q(n) \to \mathbb{C}$ defined by $\tau(x_{ij}) = q^n \delta_{ij}$ equips $M_q(n)$ with the structure of a balanced monoid on Comon(Vect).

5.2. Example. *Balanced monoidal categories.* Joyal and Street [JS93, Section 6] define the notion of a balanced monoidal category, and a balanced pseudomonoid in Cat is of course a balanced monoidal category. The braid category $\mathbb{B}$ may be equipped with a twist. For each object n of $\mathbb{B}$, the twist τ_n can be viewed as taking the identity braid on n-strings with ends on two parallel rods, then twisting the top rod through one revolution anti-clockwise.

5.3. Example. *Symmetric pseudomonoids.* The identity 2-cell determines a twist for a braided pseudomonoid if an only if the braiding is a symmetry [JS93, Example 6.1].

5.4. Example. *Quantum groups.* Street [Str89] defines a *quantum group* to be a balanced pseudomonoid equipped with an *antipode* in the symmetric monoidal 2-category Comon(Vect), satisfying one further axiom. This notion includes the classical quantum groups of [Dri87]; see [Str89, Dri90, JS91, Kas95] for examples.

In light of Example 5.4, one might refer to a balanced pseudomonoid in $\mathrm{Comon}(\mathcal{V})$ as a *quantum comonoid* and a balanced pseudomonoid in $\mathcal{V}^{\mathrm{op}}\text{-}\mathrm{Cat}^{\mathrm{op}}$ as a *quantum* $\mathcal{V}^{\mathrm{op}}$-*category.* The latter and their categories of representations are the main object of study of this article.

6. Monoidally bi-fully-faithful homomorphisms

In this section *monoidally bi-fully-faithful weak monoidal homomorphisms* are defined. It is shown that if $F\colon \mathcal{K} \to \mathcal{L}$ is a monoidally bi-fully-faithful weak monoidal 2-functor, then

$$\begin{tikzcd} \mathrm{PsMon}(\mathcal{K}) \arrow[r, "\mathrm{PsMon}(F)"] \arrow[d, "U"'] & \mathrm{PsMon}(\mathcal{L}) \arrow[d, "U"] \\ \mathcal{K} \arrow[r, "F"'] & \mathcal{L} \end{tikzcd}$$

is a *bi-pullback* in the 2-category of 2-categories, 2-functors and 2-natural transformations. This makes makes precise the idea that there is essentially a bijection between pseudomonoid structures on an object A of $\mathcal{K}$ and pseudomonoid structures on FA. It will be shown that a corresponding statement is true for braided pseudomonoids, symmetric pseudomonoids and balanced pseudomonoids.

Suppose $T\colon \mathcal{K} \to \mathcal{L}$ is a weak monoidal homomorphism. Say T is *monoidally bi-fully-faithful* when it is bi-fully-faithful as a homomorphism, and the functors

$$\chi^*\colon \mathcal{L}(F(A \otimes B), FC) \longrightarrow \mathcal{L}(FA \otimes FB, FC)$$

$$\iota^*\colon \mathcal{L}(FI, FC) \longrightarrow \mathcal{L}(I, FC)$$

are equivalences for all objects A, B and C of $\mathcal{K}$. Clearly a (strong) monoidal homomorphism which is bi-fully-faithful is monoidally bi-fully-faithful.

We now recall the definition of a *bi-pullback* in a 2-category $\mathcal{K}$. Let D be the locally discrete 2-category whose underlying category is the free category on the graph $1 \to 2 \leftarrow 3$. Let $J\colon D \to \mathrm{Cat}$ be the constant 2-functor at the terminal category. Suppose that $R\colon D \to \mathcal{K}$ is a homomorphism of bicategories. Then a *bi-pullback of* R is a J-weighted bi-limit [Str80] of R, and it amounts to an object $\{J, R\}$ of $\mathcal{K}$ equipped with a pseudonatural equivalence $\mathcal{K}(A, \{J, R\}) \simeq \hom(D, \mathrm{Cat})(J, \mathcal{K}(A, R))$.

Any homomorphism $R\colon D \to \mathcal{K}$ is isomorphic to a 2-functor $R'\colon D \to \mathcal{K}$ and so may be identified with a diagram $k\colon K \to L \leftarrow M\colon m$ in $\mathcal{K}$; we thus assume R to be a 2-functor. For each object X of $\mathcal{K}$ define a category $\mathrm{Cone}(X, R)$ as follows. An object of $\mathrm{Cone}(X, R)$ is a triple (f, g, α), where $f\colon X \to K$ and $g\colon X \to M$ are arrows of $\mathcal{K}$ and $\alpha\colon m \circ f \Rightarrow k \circ g$ is an invertible 2-cell of $\mathcal{K}$. An arrow $(\beta_1, \beta_2)\colon (f, g, \alpha) \to (f', g', \alpha')$ is a pair of 2-cells $\beta_1\colon f \Rightarrow f'$ and $\beta_2\colon g \Rightarrow g'$ satisfying $(k \circ \beta_2)\alpha = (m \circ \beta_1)\alpha'$. The obvious composition makes $\mathrm{Cone}(X, R)$ into a category, and there is an evident 2-functor $\mathrm{Cone}(-, R)\colon \mathcal{K}^{\mathrm{op}} \to \mathrm{Cat}$ whose value on an object X of $\mathcal{K}$ is $\mathrm{Cone}(X, R)$. It is not difficult to construct a pseudonatural equivalence $\mathrm{Cone}(-, R) \simeq \hom(D, \mathrm{Cat})(J, \mathcal{K}(-, R))$ and so the bi-pullback of R is characterized by a bi-representation of $\mathrm{Cone}(-, R)\colon \mathcal{K}^{\mathrm{op}} \to \mathrm{Cat}$.

6.1. Proposition. *Suppose $F\colon \mathcal{K} \to \mathcal{L}$ is a monoidally bi-fully-faithful weak monoidal 2-functor. Then the following diagram is a bi-pullback in the 2-category of 2-categories,* *2-functors and 2-natural transformations.*

$$\begin{tikzcd} \mathrm{PsMon}(\mathcal{K}) \arrow[r, "\mathrm{PsMon}(F)"] \arrow[d, "U"'] & \mathrm{PsMon}(\mathcal{L}) \arrow[d, "U"] \\ \mathcal{K} \arrow[r, "F"'] & \mathcal{L} \end{tikzcd}$$

Proof. Suppose $P\colon \mathcal{X} \to \mathcal{K}$ and $Q\colon \mathcal{X} \to \mathrm{PsMon}(\mathcal{L})$ are 2-functors, and $\varphi\colon U \circ Q \Rightarrow F \circ P$ is an invertible 2-natural transformation. Then for all objects x of $\mathcal{X}$ the object Qx of $\mathcal{L}$ is equipped with the structure (Qx, μ, η, a, r, l) of a pseudomonoid in $\mathcal{L}$. Define arrows $\mu\colon FPx \otimes FPx \to FPx$ and $\eta\colon I \to FPx$ by the following diagrams respectively.

$$\begin{tikzcd} Qx \otimes Qx \arrow[d, "\mu"'] & FPx \otimes FPx \arrow[l, "\varphi^{-1} \otimes \varphi^{-1}"'] \arrow[d, "\mu"] \\ Qx \arrow[r, "\varphi"'] & FPx \end{tikzcd} \qquad \begin{tikzcd} & I \arrow[dl, "\eta"'] \arrow[dr, "\eta"] & \\ Qx \arrow[rr, "\varphi"'] & & FPx \end{tikzcd}$$

Similarly, there are invertible 2-cells a, r and l equipping FPx with the structure of a pseudomonoid in $\mathcal{L}$. Observe that there is an invertible 2-cell χ given by

$$\begin{tikzcd} Qx \otimes Qx \arrow[rr, "1"] \arrow[dr, "\varphi \otimes \varphi"'] & \arrow[d, phantom, "\Downarrow"] & Qx \otimes Qx \arrow[rr, "\mu"] & & Qx \arrow[dl, "\varphi"] \\ & FPx \otimes FPx \arrow[ur, "\varphi^{-1} \otimes \varphi^{-1}"'] \arrow[rr, "\mu"'] & & FPx & \end{tikzcd}$$

making $\varphi = (\varphi, \chi, 1)\colon Qx \to FPx$ is an isomorphism of pseudomonoids. Note that the 2-cell in the above diagram is not necessarily the identity as $\otimes\colon \mathcal{L} \times \mathcal{L} \to \mathcal{L}$ is a homomorphism and not necessarily a 2-functor. Now since

$$\mathcal{K}(Px \otimes Px, Px) \xrightarrow{F} \mathcal{L}(F(Px \otimes Px), FPx) \xrightarrow{\chi^*} \mathcal{L}(FPx \otimes FPx, FPx)$$

and

$$\mathcal{K}(I, Px) \xrightarrow{F} \mathcal{L}(FI, FPx) \xrightarrow{\iota^*} \mathcal{L}(I, FPx)$$

are equivalences, they are essentially surjective on objects, and so there exist arrows $\mu\colon Px \otimes Px \to Px$ and $\eta\colon I \to Px$ and invertible 2-cells exhibited by the following diagram.

$$\begin{tikzcd} & FPx \otimes FPx \arrow[dl, "\chi"'] \arrow[dr, "\mu"] \arrow[d, phantom, "\Leftarrow \alpha_1"] & \\ F(Px \otimes Px) \arrow[rr, "F\mu"'] & & FPx \end{tikzcd} \qquad \begin{tikzcd} & I \arrow[dl, "\iota"'] \arrow[dr, "\eta"] \arrow[d, phantom, "\Leftarrow \alpha_2"] & \\ FI \arrow[rr, "F\eta"'] & & FPx \end{tikzcd}$$

Next notice that the invertible modification γ which is given as part of the data of F (see Section 2) provides a natural isomorphism in the following diagram.

$$
\begin{tikzcd}
\mathcal{K}(Px,Px) \arrow[r, "\mathsf{l}^*"] \arrow[d, "F"'] & \mathcal{K}(I\otimes Px,Px) \arrow[d, "F"] \\
\mathcal{L}(FPx,FPx) \arrow[d, "\mathsf{l}^*"'] \arrow[r, phantom, "\Rightarrow"] & \mathcal{L}(F(I\otimes Px),FPx) \arrow[d, "\chi^*"] \\
\mathcal{L}(I\otimes FPx,FPx) & \mathcal{L}(FI\otimes FPx,FPx) \arrow[l, "(\iota\otimes 1)^*"]
\end{tikzcd}
$$

Since the left leg of this diagram is an equivalence, the right leg is, and since the functor $\mathsf{l}^*\colon \mathcal{K}(Px,Px)\to\mathcal{K}(I\otimes Px,Px)$ is an equivalence, the composite $(\iota\otimes 1)\circ\chi^*\circ F\colon \mathcal{K}(I\otimes Px,Px)\to\mathcal{L}(I\otimes FPx,FPx)$ is an equivalence. In particular, it is fully-faithful, so there exists a unique invertible 2-cell

$$
\begin{tikzcd}
 & I\otimes Px \arrow[dl, "\eta\otimes 1"'] \arrow[dr, "\mathsf{l}"] \arrow[d, phantom, "\Leftarrow l"] & \\
Px\otimes Px \arrow[rr, "\mu"'] & \, & Px
\end{tikzcd}
$$

such that $(\iota\otimes 1)\circ\chi\circ Fl$ is equal to the following diagram.

$$
\begin{tikzcd}
FI\otimes FPX \arrow[rrrr, "\chi"] & & & & F(I\otimes 1) \arrow[d, "F\mathsf{l}"] \\
I\otimes FPx \arrow[u, "\iota\otimes 1"] \arrow[rrrr, "\mathsf{l}"] \arrow[ddd, "\iota\otimes 1"'] \arrow[drr, "\eta\otimes 1"] \arrow[urrrr, phantom, "\gamma^{-1}\Uparrow"] & & & & FPx \\
 & \arrow[r, phantom, "\overset{\alpha_2\otimes 1}{\Rrightarrow}"] & FPx\otimes FPx \arrow[urr, "\mu"'] \arrow[ddrr, "\chi"] \arrow[u, phantom, "l\Uparrow"] \arrow[d, phantom, "\Uparrow"] & \arrow[r, phantom, "\underset{\alpha_1}{\Leftarrow}"] & \\
 & & \, & & \\
FI\otimes FPx \arrow[uurr, "F\eta\otimes 1"] \arrow[rr, "\chi"'] & & F(I\otimes Px) \arrow[rr, "F(\eta\otimes 1)"'] & & F(Px\otimes Px) \arrow[uuu, "F\mu"']
\end{tikzcd}
$$

Similarly, there is an associativity isomorphism a and a right unit isomorphism r, and it is not difficult to show that and (Px,μ,η,a,r,l) is a pseudomonoid in $\mathcal{K}$. Observe that the identity arrow $1\colon FPx\to FPx$ equipped with the 2-cells

$$
\begin{tikzcd}
FPx\otimes FPx \arrow[r, "1\otimes 1"] \arrow[dd, "\mu"'] & FPx\otimes FPx \arrow[d, "\chi"] \\
\arrow[r, phantom, "\overset{\alpha_1}{\swarrow}"] & F(Px\otimes Px) \arrow[d, "F\mu"] \\
FPx \arrow[r, "1"'] & FPx
\end{tikzcd}
\qquad
\begin{tikzcd}
I \arrow[r, "\eta"] \arrow[d, "\iota"'] \arrow[dr, phantom, "\overset{\alpha_2}{\swarrow}"] & FPx \arrow[d, "1"] \\
FI \arrow[r, "F\eta"'] & FPx
\end{tikzcd}
$$

described above, is a strong monoidal isomorphism $FPx \to \mathrm{PsMon}(F)(Px)$; denote this morphism by ψ.

Now suppose $f\colon x \to y$ is an arrow in $\mathcal{X}$. Thus $Qf\colon Qx \to Qy$ is equipped with the structure (Qf, χ, ι) of a monoidal morphism. There are 2-cells

$$\begin{tikzcd}
FPx \otimes FPx \arrow[rrr, "FPf \otimes FPf"] \arrow[dddd, "\mu"'] \arrow[dr, "\varphi^{-1}\otimes\varphi^{-1}"] & & & FPy \otimes FPy \arrow[dddd, "\mu"] \\
 & Qx \otimes Qx \arrow[r, "Qf"] \arrow[d, "\mu"'] \arrow[r, phantom, "\Downarrow", shift left=5] & Qy \otimes Qy \arrow[ur, "\varphi\otimes\varphi"] \arrow[d, "\mu"] & \\
 & Qx \arrow[r, "Qf"'] \arrow[ur, phantom, "\swarrow\chi"] & Qy \arrow[ddr, "\varphi"] & \\
 & & & \\
FPx \arrow[uur, "\varphi^{-1}"] \arrow[rrr, "FPf"'] \arrow[rrr, phantom, "\Downarrow", shift left=12] & & & FPy
\end{tikzcd}
\qquad
\begin{tikzcd}
 & & FPx \arrow[dddd, "FPf"] \\
 & Qx \arrow[ur, "\varphi"] \arrow[dd, "Qf"] \arrow[r, phantom, "\swarrow"] & \\
I \arrow[ur, "\eta"] \arrow[dr, "\eta"'] \arrow[uurr, bend left, "\eta"] \arrow[ddrr, bend right, "\eta"'] \arrow[r, phantom, "\Downarrow\iota"] & & \\
 & Qy \arrow[dr, "\varphi"] & \\
 & & FPy
\end{tikzcd}$$

making FPf into a monoidal morphism. Since

$$\mathcal{K}(Px \otimes Px, Py) \xrightarrow{F} \mathcal{L}(F(Px \otimes Px), FPy) \xrightarrow{\chi^*} \mathcal{L}(FPx \otimes FPx, FPy)$$

is an equivalence, it is fully-faithful, so there exists a unique 2-cell

$$\begin{tikzcd}
Px \otimes Px \arrow[r, "Pf \otimes Pf"] \arrow[d, "\mu"'] & Py \otimes Py \arrow[d, "\mu"] \\
Px \arrow[r, "Pf"'] \arrow[ur, phantom, "\swarrow\chi"] & Py
\end{tikzcd}$$

such that composing $F\chi$ with $\chi\colon FPx \otimes FPx \to F(Px \otimes Px)$ is the following 2-cell.

$$\begin{tikzcd}
 & FPx \otimes FPx \arrow[rr, "\chi"] \arrow[dl, "\chi"'] \arrow[dd, "\mu"] \arrow[dr, "FPf\otimes FPf"] & & F(Px \otimes Px) \arrow[dr, "F(Pf\otimes Pf)"] \arrow[dl, phantom, "\swarrow"] & \\
F(Px \otimes Px) \arrow[dr, "F\mu"'] \arrow[r, phantom, "\Leftarrow\alpha_1"] & & FPy \otimes FPy \arrow[rr, "\chi"] \arrow[dr, "\mu"] \arrow[dl, phantom, "\swarrow\chi"] & \arrow[d, phantom, "\Downarrow\alpha_1"] & F(Py \otimes Py) \arrow[dl, "F\mu"] \\
 & FPx \arrow[rr, "FPf"'] & & FPy &
\end{tikzcd}$$

Similarly, there is a 2-cell $\iota\colon Pf \circ \eta \Rightarrow \eta$, and (Pf, χ, ι) is a monoidal morphism. Finally if $\alpha\colon f \Rightarrow g$ is a 2-cell in $\mathcal{X}$, then $P\alpha\colon (Pf, \chi, \iota) \Rightarrow (Pg, \chi, \iota)$ is a monoidal transformation. This assignment respects compositions and so defines a 2-functor $R_\varphi = R_{(P,Q,\varphi)}\colon \mathcal{X} \to \mathrm{PsMon}(\mathcal{K})$. Clearly, the composite of R_φ with the forgetful 2-functor $U\colon \mathrm{PsMon}(\mathcal{K}) \to \mathcal{K}$ is $P\colon \mathcal{X} \to \mathcal{K}$. Also, for any object x of $\mathcal{X}$ there is a monoidal isomorphism $\psi \circ \varphi\colon Qx \to$

$FPx \to \mathrm{PsMon}(F)R_\varphi x$ which is the component at x of a 2-natural isomorphism $Q \Rightarrow \mathrm{PsMon}(F) \circ R_\varphi$. Note that the following equation holds.

$$\begin{tikzcd}
 & & \mathrm{PsMon}(\mathcal{L}) \arrow[dr, "U"] & & & & \mathrm{PsMon}(\mathcal{L}) \arrow[dr, "U"] & \\
\mathcal{X} \arrow[urr, bend left, "Q"] \arrow[r, "R_\varphi"] \arrow[drr, bend right, "P"'] \arrow[ur, phantom, "\Downarrow"] & \mathrm{PsMon}(\mathcal{K}) \arrow[ur] \arrow[dr, "U"] & & \mathcal{L} & = & \mathcal{X} \arrow[ur, "Q"] \arrow[dr, "P"'] \arrow[rr, phantom, "\Downarrow\varphi"] & & \mathcal{L} \\
 & & \mathcal{K} \arrow[ur, "F"'] & & & & \mathcal{K} \arrow[ur, "F"'] &
\end{tikzcd}$$

Similarly, if (P, Q, φ) and (P', Q', φ') are objects of $\mathrm{Cone}(\mathcal{X}, R)$, and $(\beta_1, \beta_2) \colon (P, Q, \varphi) \to (P', Q', \varphi')$ is an arrow, then there exists a unique 2-natural transformation $R_{(\beta_1,\beta_2)} \colon R_\varphi \Rightarrow R_{\varphi'}$ satisfying $U \circ R_{(\beta_1,\beta_2)} = \beta_1$ and $\mathrm{PsMon}(F) \circ R_{(\beta_1,\beta_2)} = \beta_2$. It follows that

$$\begin{tikzcd}
\mathrm{PsMon}(\mathcal{K}) \arrow[r, "\mathrm{PsMon}(F)"] \arrow[d, "U"'] & \mathrm{PsMon}(\mathcal{L}) \arrow[d, "U"] \\
\mathcal{K} \arrow[r, "F"'] & \mathcal{L}
\end{tikzcd}$$

is bi-universal among cones to the diagram $F \colon \mathcal{K} \to \mathcal{L} \leftarrow \mathrm{PsMon}(\mathcal{L}) \colon U$, and is hence a bi-pullback. ■

In a manner similar to the proof of Proposition 6.1, the following Proposition may be proved.

6.2. Proposition. *Suppose that $F \colon \mathcal{K} \to \mathcal{L}$ is a sylleptic weak monoidal 2-functor, whose underlying weak monoidal 2-functor is monoidally bi-fully-faithful. Then the following diagrams are bi-pullbacks in the 2-category of 2-categories, 2-functors and 2-natural transformations.*

$$\begin{tikzcd}
\mathrm{BrPsMon}(\mathcal{K}) \arrow[r, "\mathrm{BrPsMon}(F)"] \arrow[d, "U"'] & \mathrm{BrPsMon}(\mathcal{L}) \arrow[d, "U"] \\
\mathcal{K} \arrow[r, "F"'] & \mathcal{L}
\end{tikzcd}$$

$$\begin{tikzcd}
\mathrm{SymPsMon}(\mathcal{K}) \arrow[r, "\mathrm{SymPsMon}(F)"] \arrow[d, "U"'] & \mathrm{SymPsMon}(\mathcal{L}) \arrow[d, "U"] \\
\mathcal{K} \arrow[r, "F"'] & \mathcal{L}
\end{tikzcd}$$

$$\begin{tikzcd}
\mathrm{BalPsMon}(\mathcal{K}) \arrow[r, "\mathrm{BalPsMon}(F)"] \arrow[d, "U"'] & \mathrm{BalPsMon}(\mathcal{L}) \arrow[d, "U"] \\
\mathcal{K} \arrow[r, "F"'] & \mathcal{L}
\end{tikzcd}$$

■

7. The symmetric monoidal 2-category $\mathcal{V}$-Act

In this section, the symmetric monoidal 2-category $\mathcal{V}$-Act of $\mathcal{V}$-actegories for a symmetric monoidal category $\mathcal{V}$ is defined. Objects of $\mathcal{V}$-Act are called $\mathcal{V}$*-actegories*, and are categories $\mathcal{A}$ equipped with an action $\otimes\colon \mathcal{V}\times\mathcal{A}\to\mathcal{A}$, which is associative and unital up to coherent isomorphism. Thus $\mathcal{V}$-actegories may be considered to be a categorification of an R-module for a commutative ring R. Indeed the tensor product of $\mathcal{V}$-actegories given here is analogous to the tensor product of R-modules. There is another description of the monoidal structure using the tricategory of bicategories and bimodules, however, this description is inconvenient for the applications needed in this article. For the remainder of this article, $\mathcal{V}$ denotes a *symmetric* monoidal category with symmetry isomorphism $c\colon X\otimes Y\to Y\otimes X$. Detailed proofs in this section are often omitted, however, these may be found in [McC99a].

For a monoidal category $\mathcal{V}$, the 2-category $\mathcal{V}$-Act is the 2-category $\hom(\Sigma\mathcal{V},\mathrm{Cat})$ of homomorphisms, pseudonatural transformations and modifications from the suspension $\Sigma\mathcal{V}$ of $\mathcal{V}$ to Cat; see [McC99b, Section 3]. An object $\mathcal{A}$ of $\mathcal{V}$-Act is called a *(left)* $\mathcal{V}$*-actegory*, and it amounts to a category $\mathcal{A}$ equipped with a functor $\otimes\colon\mathcal{V}\times\mathcal{A}\to\mathcal{A}$, called the *action of* $\mathcal{V}$ *on* $\mathcal{A}$, along with invertible natural transformations

$$\begin{tikzcd} \mathcal{V}\mathcal{V}\mathcal{A} \arrow[r, "\otimes 1"] \arrow[d, "1\otimes"'] \arrow[dr, phantom, "\swArrow a"] & \mathcal{V}\mathcal{A} \arrow[d, "\otimes"] \\ \mathcal{V}\mathcal{A} \arrow[r, "\otimes"'] & \mathcal{A} \end{tikzcd} \qquad \begin{tikzcd} \mathcal{A} \arrow[r, "I1"] \arrow[d, "1"'] \arrow[dr, phantom, "\swArrow l"] & \mathcal{V}\mathcal{A} \arrow[d, "\otimes"] \\ \mathcal{A} \arrow[r, "1"'] & \mathcal{A} \end{tikzcd}$$

called the *associativity* and *unit* isomorphisms respectively, satisfying two coherence axioms. An arrow in $\mathcal{V}$-Act is called a *morphism of* $\mathcal{V}$*-actegories*, and if $\mathcal{A}$ and $\mathcal{B}$ are $\mathcal{V}$-actegories, then a morphism $F=(F,f)\colon\mathcal{A}\to\mathcal{B}$ amounts to a functor $F\colon\mathcal{A}\to\mathcal{B}$ equipped with a natural isomorphism,

$$\begin{tikzcd} \mathcal{V}\mathcal{A} \arrow[r, "\otimes"] \arrow[d, "1F"'] \arrow[dr, phantom, "\swArrow f"] & \mathcal{A} \arrow[d, "F"] \\ \mathcal{V}\mathcal{B} \arrow[r, "\otimes"'] & \mathcal{B} \end{tikzcd}$$

called the *structure isomorphism of* F, satisfying two coherence axioms. A 2-cell in $\mathcal{V}$-Act is called a *transformation of* $\mathcal{V}$*-actegories* and it amounts to a natural transformation satisfying one axiom. For an explicit description of $\mathcal{V}$-actegories, their morphisms and 2-cells, along with a variety of examples, see [McC99b]. The 2-category of *right* $\mathcal{V}$*-actegories* is of course the 2-category $\hom((\Sigma\mathcal{V})^{\mathrm{op}},\mathrm{Cat})$.

For a (left) $\mathcal{V}$-actegory $\mathcal{A}$, define a right $\mathcal{V}$-actegory $\hat{\mathcal{A}}=(\hat{\mathcal{A}},\hat{\otimes},\hat{a},\hat{r})$ as follows. The underlying category of $\hat{\mathcal{A}}$ is the underlying category of $\mathcal{A}$, and the tensor product is the composite $\otimes\circ\sigma\colon\mathcal{A}\times\mathcal{V}\to\mathcal{V}\times\mathcal{A}\to\mathcal{A}$, where σ is the symmetry on Cat. The symmetry for $\mathcal{V}$ and the associativity isomorphism for $\mathcal{A}$ provide the associativity isomorphism for

$\hat{\mathcal{A}}$, and the unit isomorphism for $\mathcal{A}$ provides the unit isomorphism for $\hat{\mathcal{A}}$. These data are coherent, so $\hat{\mathcal{A}}$ is a right $\mathcal{V}$-actegory. Also, if $(F,f)\colon \mathcal{A} \to \mathcal{B}$ is a morphism of $\mathcal{V}$-actegories, then $(\hat{F},\hat{f}) = (F,f)\colon \hat{\mathcal{A}} \to \hat{\mathcal{B}}$ is a morphism of right $\mathcal{V}$-actegories, and if $\alpha\colon (F,f) \Rightarrow (G,g)$ is a transformation of $\mathcal{V}$-actegories then $\hat{\alpha} = \alpha$ is a transformation of right $\mathcal{V}$-actegories. This assignment defines a 2-functor from the 2-category of (left) $\mathcal{V}$-actegories to the 2-category of right $\mathcal{V}$-actegories, which is an isomorphism.

For $\mathcal{V}$-actegories $\mathcal{A}$ and $\mathcal{B}$, define a $\mathcal{V}$-actegory $\mathcal{AB}$ as follows. The underlying category of $\mathcal{AB}$ is the cartesian product of the underlying categories of $\mathcal{A}$ and $\mathcal{B}$. The action of $\mathcal{V}$ is $\otimes \times \mathcal{B}$, and the associativity and unit isomorphisms for $\mathcal{AB}$ are $a \times \mathcal{B}$ and $l \times \mathcal{B}$ respectively. The assignment $(\mathcal{A},\mathcal{B}) \mapsto \mathcal{AB}$ defines a 2-functor $\mathcal{V}\text{-Act} \times \mathcal{V}\text{-Act} \to \mathcal{V}\text{-Act}$ which is associative up to canonical isomorphism, but not unital. The associativity isomorphism for $\mathcal{A}$ and the symmetry for $\mathcal{V}$ provide an isomorphism that makes $\hat{\otimes}\colon \mathcal{AV} \to \mathcal{A}$ into a morphism of $\mathcal{V}$-actegories.

Let $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ be $\mathcal{V}$-actegories. A *descent diagram from* $(\mathcal{A},\mathcal{B})$ *to* $\mathcal{C}$ is a pair (L,λ) where L is a morphism of $\mathcal{V}$-actegories $L\colon \mathcal{AB} \to \mathcal{C}$ and λ is an invertible transformation of $\mathcal{V}$-actegories

$$\begin{tikzcd}
& \mathcal{AB} \arrow[dr, "L"] & \\
\mathcal{AVB} \arrow[ur, "\hat{\otimes}1"] \arrow[dr, "1\otimes"'] \arrow[rr, phantom, "\Downarrow\lambda"] & & \mathcal{C} \\
& \mathcal{AB} \arrow[ur, "L"'] &
\end{tikzcd}$$

satisfying the following two axioms.

$$\begin{tikzcd}
& \mathcal{AVB} \arrow[rr, "\hat{\otimes}1"] \arrow[dr, phantom, "\Downarrow\hat{a}1"] & & \mathcal{AB} \arrow[dr, "L"] & \\
\mathcal{AVVB} \arrow[ur, "\hat{\otimes}11"] \arrow[rr, "1\otimes 1"] \arrow[dr, "11\otimes"'] & & \mathcal{AVB} \arrow[ur, "\hat{\otimes}1"] \arrow[dr, "1\otimes"] \arrow[rr, phantom, "\Downarrow\lambda"] \arrow[dl, phantom, "\Downarrow a"] & & \mathcal{C} \\
& \mathcal{AVB} \arrow[rr, "1\otimes"'] & & \mathcal{AB} \arrow[ur, "L"'] &
\end{tikzcd}
=
\begin{tikzcd}
& \mathcal{AVVB} \arrow[rr, "\hat{\otimes}1"] \arrow[dr, "1\otimes"] & & \mathcal{AB} \arrow[dr, "L"] & \\
\mathcal{AVVB} \arrow[ur, "\hat{\otimes}11"] \arrow[dr, "11\otimes"'] & & \mathcal{AB} \arrow[rr, "L"'] \arrow[ur, phantom, "\Downarrow\lambda"] \arrow[dr, phantom, "\Downarrow\lambda"] & & \mathcal{C} \\
& \mathcal{AVB} \arrow[ur, "\hat{\otimes}1"'] \arrow[rr, "1\otimes"'] & & \mathcal{AB} \arrow[ur, "L"'] &
\end{tikzcd}$$

$$\begin{tikzcd}
& \mathcal{AVB} \arrow[rr, "\hat{\otimes}1"] & & \mathcal{AB} \arrow[dr, "L"] & \\
\mathcal{AB} \arrow[ur, "1I1"] \arrow[rr, "1I1"] \arrow[dr, "1"'] & & \mathcal{AVB} \arrow[ur, "\hat{\otimes}1"] \arrow[dr, "1\otimes"] \arrow[rr, phantom, "\Downarrow\lambda"] \arrow[dl, phantom, "\Downarrow l"] & & \mathcal{C} \\
& \mathcal{AB} \arrow[rr, "1"'] & & \mathcal{AB} \arrow[ur, "L"'] &
\end{tikzcd}
=
\begin{tikzcd}
& \mathcal{AVB} \arrow[rr, "\hat{\otimes}1"] \arrow[dr, "\hat{\otimes}1"] & & \mathcal{AB} \arrow[dr, "L"] & \\
\mathcal{AB} \arrow[ur, "1I1"] \arrow[dr, "1"'] \arrow[r, phantom, "\Downarrow r"] & & \mathcal{AB} \arrow[rr, "L"'] & & \mathcal{C} \\
& \mathcal{AB} \arrow[ur, "1"'] \arrow[rr, "1"'] & & \mathcal{AB} \arrow[ur, "L"'] &
\end{tikzcd}$$

If (L,λ) and (L',λ') are descent diagrams from $(\mathcal{A},\mathcal{B})$ to $\mathcal{C}$, then a *descent arrow from* (L,λ) *to* (L',λ') is a transformation of $\mathcal{V}$-actegories $\beta\colon L \Rightarrow L'$ satisfying the following axiom.

$$
\begin{tikzcd}
 & \mathcal{A}\mathcal{B} \arrow[dr, bend left, "L"] \arrow[dr, bend right, "L'"'] \arrow[dr, phantom, "\Downarrow\beta"] & \\
\mathcal{A}\mathcal{V}\mathcal{B} \arrow[ur, "\hat{\otimes}1"] \arrow[dr, bend right, "1\otimes"'] \arrow[rr, phantom, "\Downarrow\lambda'"] & & \mathcal{C} \\
 & \mathcal{A}\mathcal{B} \arrow[ur, "L'"'] &
\end{tikzcd}
=
\begin{tikzcd}
 & \mathcal{A}\mathcal{B} \arrow[dr, "L"] & \\
\mathcal{A}\mathcal{V}\mathcal{B} \arrow[ur, bend left, "\hat{\otimes}1"] \arrow[dr, "1\otimes"'] \arrow[rr, phantom, "\Downarrow\lambda"] & & \mathcal{C} \\
 & \mathcal{A}\mathcal{B} \arrow[ur, bend left, "L"] \arrow[ur, bend right, "L'"'] \arrow[ur, phantom, "\Downarrow\beta"] &
\end{tikzcd}
$$

With the evident compositions, there is a category $\mathrm{Desc}(\mathcal{A},\mathcal{B};\mathcal{C})$ whose objects are descent diagrams from $(\mathcal{A},\mathcal{B})$ to $\mathcal{C}$ and whose arrows are descent arrows. The objects and arrows of this category are related to the *$\mathcal{V}$-bifunctors* and *$\mathcal{V}$-bimorphisms* of [Par96] respectively. More precisely, a descent diagram (L,λ) from $(\mathcal{A},\mathcal{B})$ to $\mathcal{C}$ gives rise to a $\mathcal{V}$-bifunctor, and each descent morphism gives rise to a $\mathcal{V}$-bimorphism in a canonical manner. This assignment, however, is not an equivalence of categories. In particular, $\mathcal{V}$-bifunctors are not coherent with respect to the unit, and extra data and axioms are required of $\mathcal{V}$-bifunctors.

If $F\colon \mathcal{C} \to \mathcal{C}'$ is a morphism of $\mathcal{V}$-actegories, then composition with F defines a functor $\mathrm{Desc}(\mathcal{A},\mathcal{B};F)\colon \mathrm{Desc}(\mathcal{A},\mathcal{B};\mathcal{C}) \to \mathrm{Desc}(\mathcal{A},\mathcal{B};\mathcal{C}')$, and similarly if $\alpha\colon F \Rightarrow G$ is a transformation of $\mathcal{V}$-actegories then composition with α defines a natural transformation $\mathrm{Desc}(\mathcal{A},\mathcal{B};\alpha)\colon \mathrm{Desc}(\mathcal{A},\mathcal{B};F) \Rightarrow \mathrm{Desc}(\mathcal{A},\mathcal{B};G)$. This assignment preserves compositions and hence defines a 2-functor $\mathrm{Desc}(\mathcal{A},\mathcal{B};-)\colon \mathcal{V}\text{-Act} \to \mathrm{Cat}$.

7.1. Definition. *If $\mathcal{A}$ and $\mathcal{B}$ are $\mathcal{V}$-actegories, then a tensor product of $\mathcal{A}$ and $\mathcal{B}$ is a $\mathcal{V}$-actegory $\mathcal{A}\otimes\mathcal{B}$ equipped with a 2-natural isomorphism* $\mathrm{Desc}(\mathcal{A},\mathcal{B};-) \cong \mathcal{V}\text{-Act}(\mathcal{A}\otimes\mathcal{B},-)$.

7.2. Proposition. *For $\mathcal{V}$-actegories $\mathcal{A}$ and $\mathcal{B}$, the 2-functor* $\mathrm{Desc}(\mathcal{A},\mathcal{B};-)\colon \mathcal{V}\text{-Act} \to \mathrm{Cat}$ *is 2-representable.*

Proof. The 2-category $\mathcal{V}$-Act is pseudo-cocomplete and admits splittings of idempotents [McC99b, Proposition 3.6], and it is shown in [McC99a, Section 2.3] that the existence of a 2-representation of $\mathrm{Desc}(\mathcal{A},\mathcal{B};-)$ follows from these colimits. ∎

The universal descent diagram from $(\mathcal{A},\mathcal{B})$ to $\mathcal{A}\otimes\mathcal{B}$ will usually be denoted by (K,κ). We now describe explicitly the 2-functoriality of this tensor product. If $(F,f)\colon \mathcal{A} \to \mathcal{A}'$ and $(G,g)\colon \mathcal{B} \to \mathcal{B}'$ are morphisms of $\mathcal{V}$-actegories, then

$$
\begin{tikzcd}
 & \mathcal{A}\mathcal{B} \arrow[rr, "FG"] \arrow[dr, phantom, "\Downarrow f1"] & & \mathcal{A}\mathcal{B} \arrow[dr, "K"] & \\
\mathcal{A}\mathcal{V}\mathcal{B} \arrow[ur, "\hat{\otimes}1"] \arrow[rr, "F1G"] \arrow[dr, "1\otimes"'] & & \mathcal{A}'\mathcal{V}\mathcal{B}' \arrow[ur, "\hat{\otimes}1"] \arrow[dr, "1\otimes"'] & \Downarrow\kappa & \mathcal{A}'\otimes\mathcal{B}' \\
 & \mathcal{A}'\mathcal{B}' \arrow[rr, "FG"'] \arrow[ur, phantom, "\Downarrow 1g^{-1}"] & & \mathcal{A}'\mathcal{B}' \arrow[ur, "K"'] &
\end{tikzcd}
$$

is a descent diagram from $(\mathcal{A},\mathcal{B})$ to $\mathcal{A}'\otimes\mathcal{B}'$, and so there exists a unique arrow $F\otimes G\colon \mathcal{A}\otimes\mathcal{B} \to \mathcal{A}'\otimes\mathcal{B}'$ such that $\mathrm{Desc}(\mathcal{A},\mathcal{B};F\otimes G)(K,\kappa)$ is equal to the above diagram.

Similarly, if $\alpha\colon (F,f) \to (G,g)$ and $\alpha'\colon (F',f') \Rightarrow (G',g')$ are transformations of $\mathcal{V}$-actegories, then

$$\begin{tikzcd} \mathcal{A}\mathcal{B} \arrow[r, bend left, "FG"] \arrow[r, bend right, "F'G'"'] \arrow[r, phantom, "\Downarrow\alpha\alpha'"] & \mathcal{A}'\mathcal{B}' \arrow[r, "K"] & \mathcal{A}' \otimes \mathcal{B}' \end{tikzcd}$$

is a descent arrow, so that there exists a unique 2-cell $\alpha \otimes \alpha'\colon F \otimes G \Rightarrow F' \otimes G'$ such that $\mathrm{Desc}(\mathcal{A},\mathcal{B};\alpha \otimes \alpha)(K,\kappa)$ is equal to the above diagram. This assignment respects compositions, and so defines a 2-functor $\otimes\colon \mathcal{V}\text{-Act} \times \mathcal{V}\text{-Act} \to \mathcal{V}\text{-Act}$.

We now proceed to the unit equivalences. For any $\mathcal{V}$-actegory $\mathcal{A}$, define a descent diagram (P,ρ) from $(\mathcal{V},\mathcal{A})$ to $\mathcal{A}$ as follows. The underlying functor of $P\colon \mathcal{V}\mathcal{A} \to \mathcal{A}$ is $\otimes$, and the associativity isomorphism provides P with the structure of a morphism of $\mathcal{V}$-actegories, which is easily seen to be coherent. The isomorphism ρ is given by the following diagram.

$$\begin{tikzcd} & \mathcal{V}\mathcal{A} \arrow[rr, "1"] \arrow[dr, phantom, "\Downarrow c1"] & & \mathcal{V}\mathcal{A} \arrow[dr, "\otimes"] & \\ \mathcal{V}\mathcal{V}\mathcal{A} \arrow[ur, "\hat{\otimes}1"] \arrow[rr, "\otimes 1"] \arrow[dr, "1\otimes"'] & & \mathcal{V}\mathcal{A} \arrow[ur, "1"'] \arrow[dr, "\otimes"] \arrow[dl, phantom, "\Swarrow a"] & & \mathcal{A} \\ & \mathcal{V}\mathcal{A} \arrow[rr, "\otimes"'] & & \mathcal{A} \arrow[ur, "1"'] & \end{tikzcd}$$

It is straightforward to show that ρ is a transformation of $\mathcal{V}$-actegories and the two coherence conditions that make (P,ρ) into a descent diagram hold, so by the universal property of $\mathcal{V} \otimes \mathcal{A}$, there exists a unique morphism of $\mathcal{V}$-actegories $\mathsf{l}_{\mathcal{A}}\colon \mathcal{V} \otimes \mathcal{A} \to \mathcal{A}$ such that $\mathrm{Desc}(\mathcal{V},\mathcal{A};\mathsf{l})(K,\kappa)$ is equal to (P,ρ). Similarly, if $F\colon \mathcal{A} \to \mathcal{B}$ is a morphism of $\mathcal{V}$-actegories, then it is straightforward to construct a 2-cell

$$\begin{tikzcd} \mathcal{V} \otimes \mathcal{A} \arrow[r, "\mathsf{l}_{\mathcal{A}}"] \arrow[d, "1\otimes F"'] \arrow[dr, phantom, "\Swarrow \mathsf{l}_F"] & \mathcal{A} \arrow[d, "F"] \\ \mathcal{V} \otimes \mathcal{B} \arrow[r, "\mathsf{l}_{\mathcal{B}}"'] & \mathcal{B} \end{tikzcd}$$

using the universal property of $\mathcal{V}\otimes\mathcal{A}$, and verify that this data constitutes a pseudonatural transformation

$$\begin{tikzcd} & \mathcal{V}\text{-Act} \arrow[dl, "\mathcal{V}\times\mathcal{V}\text{-Act}"'] \arrow[dr, "1"] \arrow[d, phantom, "\overset{\Rightarrow}{\mathsf{l}}"] & \\ \mathcal{V}\text{-Act}^2 \arrow[rr, "\otimes"'] & & \mathcal{V}\text{-Act} \end{tikzcd}$$

where $\mathcal{V}\colon 1 \to \mathcal{V}\text{-Act}$ is the 2-functor whose value on the only object of 1 is $\mathcal{V}$.

7.3. Lemma. *The pseudonatural transformation* l *described above is an equivalence.*

Proof. First note that to show that l is an equivalence, it suffices to show that for each $\mathcal{V}$-actegory $\mathcal{A}$, the morphism of $\mathcal{V}$-actegories $\mathsf{l}_{\mathcal{A}}\colon \mathcal{V}\otimes\mathcal{A}\to\mathcal{A}$ is an equivalence. Next, to show show each $\mathsf{l}_{\mathcal{A}}$ is an equivalence it suffices to show that the functor $\mathcal{V}\text{-Act}(\mathcal{A},-)\to \mathrm{Desc}(\mathcal{V},\mathcal{A};-)$ given by composition with (P,ρ) is the component of a 2-natural equivalence. This may be done by explicitly constructing its pseudoinverse; see [McC99a, Lemma 2.4.2] for details. ∎

A similar construction provides the right unit equivalence which is a pseudonatural equivalence as in the following diagram.

$$\begin{tikzcd}
 & \mathcal{V}\text{-Act} \arrow[dl, "\mathcal{V}\text{-Act}\times\mathcal{V}"'] \arrow[dr, "1"] \arrow[d, phantom, "\overset{\Rightarrow}{r}"] & \\
\mathcal{V}\text{-Act}^2 \arrow[rr, "\otimes"'] & {} & \mathcal{V}\text{-Act}
\end{tikzcd}$$

Observe that the sense of this pseudonatural transformation is the opposite to that given in the usual definition of a monoidal bicategory. We shall return to this point later.

To construct the associativity equivalence, it is expedient to use a multiple tensor product. Suppose $n\geq 2$ is a natural number and $\mathcal{A}_1,\mathcal{A}_2,\dots,\mathcal{A}_n$ and $\mathcal{C}$ are $\mathcal{V}$-actegories. A *descent diagram from* $(\mathcal{A}_1,\mathcal{A}_2,\dots,\mathcal{A}_n)$ *to* $\mathcal{C}$ is a pair (L,λ) where L is a morphism of $\mathcal{V}$-actegories $L\colon \mathcal{A}_1\mathcal{A}_2\dots\mathcal{A}_n\to\mathcal{C}$ and λ is an invertible transformation of $\mathcal{V}$-actegories

$$\begin{tikzcd}
\mathcal{A}_1\mathcal{V}\mathcal{A}_2\mathcal{V}\dots\mathcal{V}\mathcal{A}_n \arrow[r, "\hat{\otimes}\dots\hat{\otimes}1"] \arrow[d, "1\otimes\cdots\otimes"'] \arrow[dr, phantom, "\overset{\Leftarrow}{\lambda}"] & \mathcal{A}_1\mathcal{A}_2\dots\mathcal{A}_n \arrow[d, "L"] \\
\mathcal{A}_1\mathcal{A}_2\dots\mathcal{A}_n \arrow[r, "L"'] & \mathcal{C}
\end{tikzcd}$$

satisfying the following two axioms.

$$\begin{tikzcd}
 & \mathcal{A}_1\mathcal{V}\dots\mathcal{V}\mathcal{A}_n \arrow[rr, "\hat{\otimes}\dots\hat{\otimes}1"] \arrow[d, phantom, "\Downarrow\hat{a}\dots\hat{a}1"] & & \mathcal{A}_1\dots\mathcal{A}_n \arrow[dr, "L"] \arrow[d, phantom, "\Downarrow\lambda"] & \\
\mathcal{A}_1\mathcal{V}\mathcal{V}\dots\mathcal{V}\mathcal{V}\mathcal{A}_n \arrow[ur, "\hat{\otimes}1\dots\hat{\otimes}11"] \arrow[dr, "11\otimes\dots1\otimes"'] \arrow[rr, "1\otimes1\dots1\otimes1"] & {} & \mathcal{A}_1\mathcal{V}\dots\mathcal{V}\mathcal{A}_n \arrow[ur, "\hat{\otimes}\dots\hat{\otimes}1"'] \arrow[dr, "1\otimes\cdots\otimes"] & {} & \mathcal{C} \\
 & \mathcal{A}_1\mathcal{V}\dots\mathcal{V}\mathcal{A}_n \arrow[rr, "1\otimes\cdots\otimes"'] \arrow[u, phantom, "\Leftarrow 1a\dots a"] & & \mathcal{A}_1\dots\mathcal{A}_n \arrow[ur, "L"'] &
\end{tikzcd}$$

$$\|$$

$$\begin{tikzcd}
 & \mathcal{A}_1\mathcal{V}\dots\mathcal{V}\mathcal{A}_n \arrow[rr, "\hat{\otimes}\dots\hat{\otimes}1"] \arrow[dr, "1\otimes\cdots\otimes"] & & \mathcal{A}_1\dots\mathcal{A}_n \arrow[dr, "L"] \arrow[d, phantom, "\Leftarrow\lambda"] & \\
\mathcal{A}_1\mathcal{V}\mathcal{V}\dots\mathcal{V}\mathcal{V}\mathcal{A}_n \arrow[ur, "\hat{\otimes}11\dots\hat{\otimes}11"] \arrow[dr, "11\otimes\dots11\otimes"'] & & \mathcal{A}_1\dots\mathcal{A}_n \arrow[rr, "L"'] & {} & \mathcal{C} \\
 & \mathcal{A}_1\mathcal{V}\dots\mathcal{V}\mathcal{A}_n \arrow[rr, "1\otimes\cdots\otimes"'] \arrow[ur, "\hat{\otimes}\dots\hat{\otimes}1"] & & \mathcal{A}_1\dots\mathcal{A}_n \arrow[ur, "L"'] \arrow[u, phantom, "\Downarrow\lambda"] &
\end{tikzcd}$$

$$\begin{tikzcd}
 & \mathcal{A}_1\mathcal{V}\ldots\mathcal{V}\mathcal{A}_n \arrow[rr, "\hat{\otimes}1\ldots\hat{\otimes}1"] & & \mathcal{A}_1\ldots\mathcal{A}_n \arrow[dr, "L"] & \\
\mathcal{A}_1\ldots\mathcal{A}_n \arrow[ur, "1I1\ldots1I1"] \arrow[rr, "1I1\ldots1I1"] \arrow[dr, "1"'] & & \mathcal{A}_1\mathcal{V}\ldots\mathcal{V}\mathcal{A}_n \arrow[ur, "\hat{\otimes}1\ldots\hat{\otimes}1"] \arrow[dr, "1\otimes\ldots1\otimes"] \arrow[d, phantom, "\swarrow 1l\ldots l"] \arrow[rr, phantom, "\Downarrow\lambda"] & & \mathcal{C} \\
 & \mathcal{A}_1\ldots\mathcal{A}_n \arrow[rr, "1"'] & {} & \mathcal{A}_1\ldots\mathcal{A}_n \arrow[ur, "L"'] &
\end{tikzcd}$$

$$\|$$

$$\begin{tikzcd}
 & \mathcal{A}_1\mathcal{V}\ldots\mathcal{V}\mathcal{A}_n \arrow[rr, "\hat{\otimes}1\ldots\hat{\otimes}1"] \arrow[dr, "\hat{\otimes}1\ldots\hat{\otimes}1"] & & \mathcal{A}_1\mathcal{V}\ldots\mathcal{V}\mathcal{A}_n \arrow[dr, "L"] & \\
\mathcal{A}_1\ldots\mathcal{A}_n \arrow[ur, "1I1\ldots1I1"] \arrow[dr, "1"'] \arrow[rr, phantom, "\Downarrow r\ldots r"] & & \mathcal{A}_1\ldots\mathcal{A}_n \arrow[rr, "L"'] & & \mathcal{C} \\
 & \mathcal{A}_1\ldots\mathcal{A}_n \arrow[ur, "1"] \arrow[rr, "1"'] & & \mathcal{A}\mathcal{B} \arrow[ur, "L"'] &
\end{tikzcd}$$

If (L,λ) and (L',λ') are descent diagrams from $(\mathcal{A}_1,\ldots,\mathcal{A}_n)$ to $\mathcal{C}$, then a *descent arrow* from (L,λ) to (L',λ') is a transformation of $\mathcal{V}$-actegories $\beta\colon L \Rightarrow L'$ satisfying the following axiom.

$$\begin{tikzcd}
 & \mathcal{A}_1\ldots\mathcal{A}_n \arrow[dr, bend left, "L"] \arrow[dr, bend right, "L'"'] \arrow[dr, phantom, "\swarrow\beta"] & & & & \mathcal{A}_1\ldots\mathcal{A}_n \arrow[dr, "L"] \arrow[d, phantom, "\Downarrow\lambda"] & \\
\mathcal{A}_1\mathcal{V}\ldots\mathcal{V}\mathcal{A}_n \arrow[ur, "\hat{\otimes}1\ldots\hat{\otimes}1"] \arrow[dr, bend right, "1\otimes\ldots1\otimes"'] \arrow[r, phantom, "\swarrow\lambda'"] & {} & \mathcal{C} & = & \mathcal{A}_1\mathcal{V}\ldots\mathcal{V}\mathcal{A}_n \arrow[ur, bend left, "\hat{\otimes}1\ldots\hat{\otimes}1"] \arrow[dr, "1\otimes\ldots1\otimes"'] & {} & \mathcal{B} \\
 & \mathcal{A}_1\ldots\mathcal{A}_n \arrow[ur, "L'"'] & & & & \mathcal{A}_1\ldots\mathcal{A}_n \arrow[ur, bend left, "L"] \arrow[ur, bend right, "L'"'] \arrow[ur, phantom, "\Downarrow\beta"] &
\end{tikzcd}$$

Much as in the case $n = 2$, there is a category $\mathrm{Desc}(\mathcal{A}_1,\ldots,\mathcal{A}_n;\mathcal{C})$ of descent diagrams from $(\mathcal{A}_1,\ldots,\mathcal{A}_n)$ to $\mathcal{C}$ and arrows between them, and this is the value at $\mathcal{C}$ of a 2-functor $\mathrm{Desc}(\mathcal{A}_1,\ldots,\mathcal{A}_n;-)\colon \mathcal{V}\text{-Act} \to \mathrm{Cat}$.

7.4. Definition. *An n-fold tensor product of $\mathcal{V}$-actegories $\mathcal{A}_1,\ldots,\mathcal{A}_n$ is a $\mathcal{V}$-actegory $\mathcal{A}_1\otimes\cdots\otimes\mathcal{A}_n$ equipped with a a 2-natural isomorphism* $\mathrm{Desc}(\mathcal{A}_1,\ldots,\mathcal{A}_n;-) \cong \mathcal{V}\text{-Act}(\mathcal{A}_1\otimes\cdots\otimes\mathcal{A}_n,-)$.

Clearly when $n = 2$ this agrees with the definition of the binary tensor product given above. As with the binary tensor product, the n-fold tensor product exists, and extends to a 2-functor $-\otimes-\cdots-\otimes-\colon \mathcal{V}\text{-Act}^n \to \mathcal{V}\text{-Act}$. Now suppose $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ are $\mathcal{V}$-actegories and (K_1,κ_1) and (K_2,κ_2) are universal descent diagrams from $(\mathcal{A},\mathcal{B})$ to $\mathcal{A}\otimes\mathcal{B}$ and from $(\mathcal{A}\otimes\mathcal{B},\mathcal{C})$ to $(\mathcal{A}\otimes\mathcal{B})\otimes\mathcal{C}$ respectively. Then define a descent diagram from

$(\mathcal{A},\mathcal{B},\mathcal{C})$ to $(\mathcal{A}\otimes\mathcal{B})\otimes\mathcal{C}$ to be

$$\begin{tikzcd}
 & & \mathcal{A}\mathcal{B}\mathcal{C} \arrow[dr, "K_1 1"] \arrow[d, phantom, "\Downarrow\kappa_1^{-1}1"] & & \\
 & \mathcal{A}\mathcal{B}\mathcal{V}\mathcal{C} \arrow[ur, "1\hat{\otimes}1"] \arrow[r, "\hat{\otimes}1"] \arrow[dr, "K_1 11"'] \arrow[dd, phantom, "\Downarrow\kappa_1 11"] & \mathcal{A}\mathcal{B}\mathcal{C} \arrow[r, "K_1 1"] \arrow[d, phantom, "\Downarrow k_1 1"] & (\mathcal{A}\otimes\mathcal{B})\mathcal{C} \arrow[dr, "K_2"] \arrow[dd, phantom, "\Downarrow\kappa_2"] & \\
\mathcal{A}\mathcal{V}\mathcal{B}\mathcal{V}\mathcal{C} \arrow[uurr, bend left, "\hat{\otimes}\hat{\otimes}1"] \arrow[ur, "\hat{\otimes}11"] \arrow[dr, "1\otimes 11"'] \arrow[ddrr, bend right, "1\otimes\otimes"'] & & (\mathcal{A}\otimes\mathcal{B})\mathcal{V}\mathcal{C} \arrow[ur, "\hat{\otimes}1"] \arrow[dr, "1\otimes"] & & (\mathcal{A}\otimes\mathcal{B})\otimes\mathcal{C}, \\
 & \mathcal{A}\mathcal{B}\mathcal{V}\mathcal{C} \arrow[ur, "K_1 11"] \arrow[dr, "1\otimes"'] & & (\mathcal{A}\otimes\mathcal{B})\mathcal{C} \arrow[ur, "K_2"'] & \\
 & & \mathcal{A}\mathcal{B}\mathcal{C} \arrow[ur, "K_1 1"'] & &
\end{tikzcd}$$

where k_1 is the structure isomorphism of K_1; denote this descent diagram by (K',κ'). The proof of the following proposition may be found in [McC99a, Section 2.5]

7.5. PROPOSITION. *For any $\mathcal{V}$-actegories $\mathcal{A},\mathcal{B}$ and $\mathcal{C}$, the above diagram exhibits $(\mathcal{A}\otimes\mathcal{B})\otimes\mathcal{C}$ as a triple tensor product.* ■

Next, suppose that (K_3,κ_3) and (K_4,κ_4) are are universal descent diagrams from $(\mathcal{B},\mathcal{C})$ to $\mathcal{B}\otimes\mathcal{C}$ and from $(\mathcal{A},\mathcal{B}\otimes\mathcal{C})$ to $\mathcal{A}\otimes(\mathcal{B}\otimes\mathcal{C})$ respectively. Define a descent diagram from $(\mathcal{A},\mathcal{B},\mathcal{C})$ to $(\mathcal{A}\otimes\mathcal{B})\otimes\mathcal{C}$ to be the following diagram.

$$\begin{tikzcd}
 & & \mathcal{A}\mathcal{B}\mathcal{C} \arrow[dr, "1K_3"] & & \\
 & \mathcal{A}\mathcal{V}\mathcal{B}\mathcal{C} \arrow[ur, "\hat{\otimes}11"] \arrow[dr, "11K_3"] \arrow[dd, phantom, "\Downarrow 11\kappa_3"] & & \mathcal{A}(\mathcal{B}\otimes\mathcal{C}) \arrow[dr, "K_4"] \arrow[dd, phantom, "\Downarrow\kappa_4"] & \\
\mathcal{A}\mathcal{V}\mathcal{B}\mathcal{V}\mathcal{C} \arrow[uurr, bend left, "\hat{\otimes}\hat{\otimes}1"] \arrow[ur, "11\hat{\otimes}1"] \arrow[dr, "111\otimes"'] \arrow[ddrr, bend right, "1\otimes\otimes"'] & & \mathcal{A}\mathcal{V}(\mathcal{B}\otimes\mathcal{C}) \arrow[ur, "\hat{\otimes}1"] \arrow[dr, "1\otimes"] \arrow[dd, phantom, "\Downarrow k_3^{-1}"] & & \mathcal{A}\otimes(\mathcal{B}\otimes\mathcal{C}) \\
 & \mathcal{A}\mathcal{V}\mathcal{B}\mathcal{C} \arrow[ur, "11K_3"] \arrow[dr, "1\otimes 1"'] & & \mathcal{A}(\mathcal{B}\otimes\mathcal{C}) \arrow[ur, "K_4"'] & \\
 & & \mathcal{A}\mathcal{B}\mathcal{C} \arrow[ur, "1K_3"'] & &
\end{tikzcd}$$

Here k is the structure isomorphism of K. By Proposition 7.5, there exists a unique morphism of $\mathcal{V}$-actegories $\mathsf{a}\colon(\mathcal{A}\otimes\mathcal{B})\otimes\mathcal{C}\to\mathcal{A}\otimes(\mathcal{B}\otimes\mathcal{C})$ such that $\mathrm{Desc}(\mathcal{A},\mathcal{B},\mathcal{C};\mathsf{a})(K'\kappa')$ is equal to the above descent diagram.

7.6. Proposition. *The arrow* $\mathsf{a}\colon (\mathcal{A}\otimes\mathcal{B})\otimes\mathcal{C}\to\mathcal{A}\otimes(\mathcal{B}\otimes\mathcal{C})$ *form the components of a 2-natural isomorphism as in the following diagram.*

$$\begin{tikzcd}
\mathcal{V}\text{-Act}^3 \arrow[r, "\otimes\times 1"] \arrow[d, "1\times\otimes"'] \arrow[dr, phantom, "\Swarrow\mathsf{a}"] & \mathcal{V}\text{-Act}^2 \arrow[d, "\otimes"] \\
\mathcal{V}\text{-Act}^2 \arrow[r, "\otimes"'] & \mathcal{V}\text{-Act}
\end{tikzcd}$$

Proof. In a manner similar to Proposition 7.5, $\mathcal{A}\otimes(\mathcal{B}\otimes\mathcal{C})$ is a triple tensor product, so that the arrow $\mathsf{a}\colon (\mathcal{A}\otimes\mathcal{B})\otimes\mathcal{C}\to\mathcal{A}\otimes(\mathcal{B}\otimes\mathcal{C})$ is an isomorphism. This isomorphism is 2-natural which completes the proof. ■

We now proceed to the invertible modification π of the definition of a monoidal bicategory.

7.7. Proposition. *For* $\mathcal{V}$*-actegories* $\mathcal{A},\mathcal{B},\mathcal{C}$ *and* $\mathcal{D}$*, the following diagram commutes.*

$$\begin{tikzcd}
& (\mathcal{A}\otimes(\mathcal{B}\otimes\mathcal{C}))\otimes\mathcal{D} \arrow[dr, "\mathsf{a}"] & \\
((\mathcal{A}\otimes\mathcal{B})\otimes\mathcal{C})\otimes\mathcal{D} \arrow[ur, "\mathsf{a}\otimes 1"] \arrow[d, "\mathsf{a}"'] & & \mathcal{A}\otimes((\mathcal{B}\otimes\mathcal{C})\otimes\mathcal{D}) \arrow[d, "1\otimes\mathsf{a}"] \\
(\mathcal{A}\otimes\mathcal{B})\otimes(\mathcal{C}\otimes\mathcal{D}) \arrow[rr, "\mathsf{a}"'] & & \mathcal{A}\otimes(\mathcal{B}\otimes(\mathcal{C}\otimes\mathcal{D}))
\end{tikzcd}$$

Proof. In a manner similar to the proof of Proposition 7.5, it can be shown that $((\mathcal{A}\otimes\mathcal{B})\otimes\mathcal{C})\otimes\mathcal{D}$ is a 4-fold tensor product. Thus, in order to show that the above diagram commutes, it suffices to show that it commutes after composing with the universal arrow $\mathcal{ABCD}\to((\mathcal{A}\otimes\mathcal{B})\otimes\mathcal{C})\otimes\mathcal{D}$, and this is a long but straightforward calculation. ■

It follows from Proposition 7.7 that the modification

$$\begin{tikzcd}
& \mathcal{V}\text{-Act}^3 \arrow[rr, "\otimes 1"] \arrow[dr, phantom, "\Searrow\mathsf{a}1"] & & \mathcal{V}\text{-Act}^2 \arrow[dr, "\otimes"] & & & & \mathcal{V}\text{-Act}^3 \arrow[rr, "\otimes 1"] \arrow[dr, "1\otimes"] & & \mathcal{V}\text{-Act}^2 \arrow[dr, "\otimes"] \arrow[d, phantom, "\Swarrow\mathsf{a}"] & \\
\mathcal{V}\text{-Act}^4 \arrow[ur, "\otimes 11"] \arrow[rr, "1\otimes 1"] \arrow[dr, "11\otimes"'] & \arrow[dr, phantom, "\Swarrow 1\mathsf{a}"] & \mathcal{V}\text{-Act}^3 \arrow[ur, "\otimes 1"'] \arrow[dr, "1\otimes"] & \Downarrow\mathsf{a} & \mathcal{V}\text{-Act} & \overset{\pi}{\Rrightarrow} & \mathcal{V}\text{-Act}^4 \arrow[ur, "\otimes 11"] \arrow[dr, "11\otimes"'] & & \mathcal{V}\text{-Act}^2 \arrow[rr, "\otimes"'] & \arrow[d, phantom, "\Searrow\mathsf{a}"] & \mathcal{V}\text{-Act} \\
& \mathcal{V}\text{-Act}^3 \arrow[rr, "1\otimes"'] & & \mathcal{V}\text{-Act}^2 \arrow[ur, "\otimes"'] & & & & \mathcal{V}\text{-Act}^3 \arrow[ur, "\otimes 1"] \arrow[rr, "1\otimes"'] & & \mathcal{V}\text{-Act}^2 \arrow[ur, "\otimes"'] &
\end{tikzcd}$$

in the definition of a monoidal bicategory may be taken to be the identity.

We now consider the modification ν in the definition of a monoidal bicategory. Let (K',κ') be the universal descent diagram from $(\mathcal{A},\mathcal{V},\mathcal{B})$ to $(\mathcal{A}\otimes\mathcal{V})\otimes\mathcal{B}$ as constructed immediately before Proposition 7.5. Then

$$\begin{tikzcd}
(\mathcal{A}\otimes\mathcal{V})\mathcal{B} \arrow[dd, "K"'] \arrow[dr, "\mathsf{r}1"] & & \mathcal{A}\mathcal{V}\mathcal{B} \arrow[ll, "K1"'] \arrow[rr, "K1"] \arrow[dl, "\hat{\otimes}1"'] \arrow[d, "\otimes"] \arrow[dr, "1K"] & & (\mathcal{A}\otimes\mathcal{V})\mathcal{B} \arrow[d, "K"] \\
& \mathcal{A}\mathcal{B} \arrow[d, "K"'] \arrow[r, phantom, "\overset{\kappa}{\Rrightarrow}"] & \mathcal{A}\mathcal{B} \arrow[dl, "K"] & \mathcal{A}(\mathcal{V}\otimes\mathcal{B}) \arrow[l, "1\mathsf{l}"] \arrow[dr, "K"'] & (\mathcal{A}\otimes\mathcal{V})\otimes\mathcal{B} \arrow[d, "\mathsf{a}"] \\
(\mathcal{A}\otimes\mathcal{V})\otimes\mathcal{B} \arrow[r, "\mathsf{r}\otimes 1"'] & \mathcal{A}\otimes\mathcal{B} & & & \mathcal{A}\otimes(\mathcal{V}\otimes\mathcal{B}) \arrow[lll, "1\otimes\mathsf{l}"]
\end{tikzcd}$$

is an isomorphism of descent diagrams $\mathrm{Desc}(\mathcal{A},\mathcal{V},\mathcal{B};\mathsf{r}\otimes 1)(K',\kappa') \to \mathrm{Desc}(\mathcal{A},\mathcal{V},\mathcal{B};(1\otimes \mathsf{l})\circ\mathsf{a})(K',\kappa')$, and so there exists a unique invertible 2-cell

$$\begin{tikzcd}
(\mathcal{A}\otimes\mathcal{V})\otimes\mathcal{B} \arrow[rr, "\mathsf{a}"] \arrow[dr, "\mathsf{r}\otimes 1"'] & \arrow[d, phantom, "\Downarrow\nu"] & \mathcal{A}\otimes(\mathcal{V}\otimes\mathcal{B}) \arrow[dl, "1\otimes\mathsf{l}"] \\
& \mathcal{A}\otimes\mathcal{B} &
\end{tikzcd}$$

such that $\mathrm{Desc}(\mathcal{A},\mathcal{V},\mathcal{B};\nu)(K',\kappa')$ is equal to this morphism of descent diagrams.

7.8. Proposition. *The 2-cell ν forms the component at $(\mathcal{A},\mathcal{B})$ of an invertible modification*

$$\begin{tikzcd}
& \mathcal{V}\text{-Act}^2 \arrow[dl, "1\mathcal{V}1"'] \arrow[dr, "1"] \arrow[d, phantom, "\Uparrow\mathsf{r}1"] & & & & \mathcal{V}\text{-Act}^2 \arrow[dl, "1\mathcal{V}1"'] \arrow[dr, "1"] \arrow[d, phantom, "\Uparrow 1\mathsf{l}"] & \\
\mathcal{V}\text{-Act}^3 \arrow[rr, "\otimes 1"'] \arrow[d, "\otimes 1"'] & {} & \mathcal{V}\text{-Act}^2 \arrow[d, "\otimes"] & \overset{\nu}{\Longrightarrow} & \mathcal{V}\text{-Act}^3 \arrow[rr, "1\otimes"'] \arrow[d, "\otimes 1"'] & {} \arrow[d, phantom, "\overset{\mathsf{a}}{\Rrightarrow}"] & \mathcal{V}\text{-Act}^2 \arrow[d, "\otimes"] \\
\mathcal{V}\text{-Act}^2 \arrow[rr, "\otimes"'] & & \mathcal{V}\text{-Act} & & \mathcal{V}\text{-Act}^2 \arrow[rr, "\otimes"'] & {} & \mathcal{V}\text{-Act}
\end{tikzcd}$$

in $\mathrm{hom}(\mathcal{V}\text{-Act}^2,\mathcal{V}\text{-Act})$. ■

Similar constructions provides invertible modifications

$$\begin{tikzcd}
& \mathcal{V}\text{-Act}^3 \arrow[dr, "\otimes 1"] \arrow[d, phantom, "\Downarrow\mathsf{l}1"] & & & & \mathcal{V}\text{-Act}^3 \arrow[dr, "\otimes 1"] \arrow[d, "1\otimes"] & \\
\mathcal{V}\text{-Act}^2 \arrow[ur, "\mathcal{V}11"] \arrow[rr, "1"'] \arrow[d, "\otimes"'] & {} & \mathcal{V}\text{-Act}^2 \arrow[d, "\otimes"] & \overset{\lambda}{\Longrightarrow} & \mathcal{V}\text{-Act}^2 \arrow[ur, "\mathcal{V}11"] \arrow[d, "\otimes"'] & \mathcal{V}\text{-Act}^2 \arrow[dr, "\otimes"] \arrow[r, phantom, "\overset{\Lleftarrow}{\mathsf{a}}"] & \mathcal{V}\text{-Act}^2 \arrow[d, "\otimes"] \\
\mathcal{V}\text{-Act} \arrow[rr, "1"'] & & \mathcal{V}\text{-Act} & & \mathcal{V}\text{-Act} \arrow[ur, "\mathcal{V}1"] \arrow[rr, "1"'] & & \mathcal{V}\text{-Act}
\end{tikzcd}$$

$$\begin{tikzcd}
 & \mathcal{V}\text{-Act}^3 \arrow[d, "\otimes 1"] \arrow[dr, "\otimes 1"] & & & & \mathcal{V}\text{-Act}^3 \arrow[d, "\otimes 1"] \arrow[dr, "\otimes 1"] & \\
\mathcal{V}\text{-Act}^2 \arrow[ur, "11\mathcal{V}"] \arrow[d, "\otimes"'] & \mathcal{V}\text{-Act}^2 \arrow[dr, "\otimes"] \arrow[d, phantom, "\Downarrow \mathsf{r}"] & \mathcal{V}\text{-Act}^2 \arrow[d, "\otimes"] & \overset{\rho}{\Longrightarrow} & \mathcal{V}\text{-Act}^2 \arrow[ur, "11\mathcal{V}"] \arrow[d, "\otimes"'] \arrow[r, phantom, "1\mathsf{r}\,\Swarrow"] & \mathcal{V}\text{-Act}^2 \arrow[dr, "\otimes"] \arrow[r, phantom, "\overset{\Leftarrow}{\mathsf{a}}"] & \mathcal{V}\text{-Act}^2 \arrow[d, "\otimes"] \\
\mathcal{V}\text{-Act} \arrow[ur, "1\mathcal{V}"] \arrow[rr, "1"'] & & \mathcal{V}\text{-Act} & & \mathcal{V}\text{-Act} \arrow[ur, "1"] \arrow[rr, "1"'] & & \mathcal{V}\text{-Act}
\end{tikzcd}$$

in $\hom(\mathcal{V}\text{-Act}^2, \mathcal{V}\text{-Act})$.

As remarked above, the sense of the pseudonatural equivalence r given above is contrary to the sense of the right unit equivalence as given in [GPS95, Section 2.2]. As a consequence, the invertible modifications domain and codomain of ν and ρ given above are mates of the domain and codomain of ν and ρ as in [GPS95, Section 2.2]. We shall now use the calculus of mates to transform the data given above into the form of [GPS95, Section 2.2]. Let r^* be the adjoint pseudoinverse of r in the 2-category $\hom(\mathcal{V}\text{-Act}, \mathcal{V}\text{-Act})$ with invertible unit and counit η and ε respectively. Now, for any pair of objects $\mathcal{A}$ and $\mathcal{B}$ of $\mathcal{V}$-Act, define ν^* to be the following 2-cell.

$$\begin{tikzcd}
 & (\mathcal{A} \otimes \mathcal{V}) \otimes \mathcal{B} \arrow[dd, "\mathsf{r} \otimes 1"] \arrow[dr, "\mathsf{a}"] & \\
\mathcal{A} \otimes \mathcal{B} \arrow[ur, "\mathsf{r}^* \otimes 1"] \arrow[dr, "1"'] \arrow[r, phantom, "\overset{\varepsilon \otimes 1}{\Leftarrow}"] & \arrow[r, phantom, "\overset{\nu^{-1}}{\Leftarrow}"] & \mathcal{A} \otimes (\mathcal{V} \otimes \mathcal{B}) \arrow[dl, "1 \otimes \mathsf{l}"] \\
 & \mathcal{A} \otimes \mathcal{B} &
\end{tikzcd}$$

Then ν^* forms the component at $(\mathcal{A}, \mathcal{B})$ of an invertible modification as in the following diagram.

$$\begin{tikzcd}
 & \mathcal{V}\text{-Act}^2 \arrow[rr, "1"] \arrow[dr, phantom, "\Searrow 1\mathsf{l}"] & & \mathcal{V}\text{-Act}^2 \arrow[dr, "\otimes"] & & & & \mathcal{V}\text{-Act}^1 \arrow[dr, "\otimes"] & \\
\mathcal{V}\text{-Act}^2 \arrow[ur, "1"] \arrow[rr, "1\mathcal{V}1"] \arrow[dr, "1"'] & & \mathcal{V}\text{-Act}^3 \arrow[ur, "\otimes 1"] \arrow[dr, "1\otimes"] \arrow[rr, phantom, "\Downarrow \mathsf{a}"] & & \mathcal{V}\text{-Act} & \overset{\nu^*}{\Longrightarrow} & \mathcal{V}\text{-Act}^2 \arrow[ur, "1"] \arrow[dr, "1"'] & & \mathcal{V}\text{-Act} \\
 & \mathcal{V}\text{-Act}^2 \arrow[rr, "1"'] \arrow[ur, phantom, "\Swarrow 1\mathsf{r}^*"] & & \mathcal{V}\text{-Act} \arrow[ur, "\otimes"'] & & & & \mathcal{V}\text{-Act}^2 \arrow[ur, "\otimes"'] &
\end{tikzcd}$$

Similarly, for each pair of objects $\mathcal{A}$ and $\mathcal{B}$ of $\mathcal{V}$-Act, define ρ^* to be the following 2-cell.

$$\begin{tikzcd}
 & (\mathcal{A} \otimes \mathcal{B}) \otimes \mathcal{V} \arrow[r, "\mathsf{a}"] \arrow[dd, "\mathsf{r}"] & \mathcal{A} \otimes (\mathcal{B} \otimes \mathcal{V}) \arrow[dd, "1\mathsf{r}"'] \arrow[dr, "1"] & \\
\mathcal{A} \otimes \mathcal{B} \arrow[ur, "\mathsf{r}^*"] \arrow[dr, "1"'] \arrow[r, phantom, "\overset{\varepsilon^{-1}}{\Rightarrow}"] & \arrow[r, phantom, "\underset{\beta}{\Rightarrow}"] & \arrow[r, phantom, "\underset{\eta^{-1}}{\Rightarrow}"] & \mathcal{A} \otimes (\mathcal{B} \otimes \mathcal{V}) \\
 & \mathcal{A} \otimes \mathcal{B} \arrow[r, "1"'] & \mathcal{A} \otimes \mathcal{B} \arrow[ur, "1\mathsf{r}^*"'] &
\end{tikzcd}$$

Then ρ^* forms the component of an invertible modification as in the following diagram.

$$
\begin{tikzcd}
\mathcal{V}\text{-Act} \arrow[rr, "1"] & & \mathcal{V}\text{-Act} & & \mathcal{V}\text{-Act} \arrow[rr, "1"] \arrow[dr, "1\mathcal{V}"] & \arrow[d, phantom, "\Downarrow r^*"] & \mathcal{V}\text{-Act}. \\
\mathcal{V}\text{-Act}^2 \arrow[u, "\otimes"'] \arrow[rr, "1"] \arrow[dr, "11\mathcal{V}"'] & \arrow[d, phantom, "\Downarrow 1r^*"] & \mathcal{V}\text{-Act}^2 \arrow[u, "\otimes"'] & \overset{\rho^*}{\Rrightarrow} & \mathcal{V}\text{-Act}^2 \arrow[u, "\otimes"] \arrow[dr, "11\mathcal{V}"'] & \mathcal{V}\text{-Act}^2 \arrow[ur, "\otimes"] \arrow[r, phantom, "\overset{\mathsf{a}}{\Rrightarrow}"] & \mathcal{V}\text{-Act}^2 \arrow[u, "\otimes"'] \\
& \mathcal{V}\text{-Act}^3 \arrow[ur, "1\otimes"'] & & & & \mathcal{V}\text{-Act}^3 \arrow[u, "\otimes 1"] \arrow[ur, "1\otimes"'] &
\end{tikzcd}
$$

Thus $(\mathcal{V}\text{-Act}, \otimes, \mathcal{V}, \mathsf{a}, \mathsf{l}, \mathsf{r}^*, \pi, \nu^*, \lambda, \rho^*)$ defines data for a monoidal 2-category.

7.9. Theorem. *With data as defined above* $(\mathcal{V}\text{-Act}, \otimes, \mathcal{V}, \mathsf{a}, \mathsf{l}, \mathsf{r}^*, \pi, \nu^*, \lambda, \rho^*)$ *is a monoidal 2-category.*

Proof. Since the associativity equivalence a is 2-natural, and the modification π is the identity, both sides of the non-abelian cocycle condition [GPS95, Section 2.2, (TA1)] equation are the identity, and so trivially satisfied.

Next, one may show using the calculus of mates that the left normalization axiom [GPS95, Section 2.2, (TA2)] holds if and only if the following equation holds for all $\mathcal{V}$-actegories $\mathcal{A}, \mathcal{B}$ and $\mathcal{C}$, where juxtaposition denotes tensor.

$$
\begin{tikzcd}
& & (\mathcal{A}\mathcal{B})\mathcal{C} \arrow[dr, "\mathsf{a}"] & \\
((\mathcal{A}\mathcal{V})\mathcal{B})\mathcal{C} \arrow[urr, "(\mathsf{r}1)1"] \arrow[dr, "\mathsf{a}1"] \arrow[ddd, "\mathsf{a}"'] & \arrow[u, phantom, "\Downarrow \nu 1"] & & \mathcal{A}(\mathcal{B}\mathcal{C}) \\
& (\mathcal{A}(\mathcal{V}\mathcal{B}))\mathcal{C} \arrow[uur, "(1\mathsf{l})1"'] \arrow[dr, "\mathsf{a}"] & & \\
& \arrow[d, phantom, "\Swarrow \pi"] & \mathcal{A}((\mathcal{V}\mathcal{B})\mathcal{C}) \arrow[uur, "1(\mathsf{l}1)"] \arrow[dr, "1\mathsf{a}"'] \arrow[r, phantom, "\overset{1\lambda}{\Rrightarrow}"] & \\
(\mathcal{A}\mathcal{V})(\mathcal{B}\mathcal{C}) \arrow[rrr, "\mathsf{a}"'] & & & \mathcal{A}(\mathcal{V}(\mathcal{B}\mathcal{C})) \arrow[uuu, "1\mathsf{l}"']
\end{tikzcd}
=
\begin{tikzcd}
& & (\mathcal{A}\mathcal{B})\mathcal{C} \arrow[dr, "\mathsf{a}"] & \\
((\mathcal{A}\mathcal{V})\mathcal{B})\mathcal{C} \arrow[urr, "(\mathsf{r}1)1"] \arrow[ddd, "\mathsf{a}"'] & & & \mathcal{A}(\mathcal{B}\mathcal{C}) \\
& & & \\
& & \arrow[r, phantom, "\Searrow \nu"] & \\
(\mathcal{A}\mathcal{V})(\mathcal{B}\mathcal{C}) \arrow[rrr, "\mathsf{a}"'] \arrow[uuurrr, "\mathsf{r}(11)"] & & & \mathcal{A}(\mathcal{V}(\mathcal{B}\mathcal{C})) \arrow[uuu, "1\mathsf{l}"']
\end{tikzcd}
$$

Since $((\mathcal{A} \otimes \mathcal{V}) \otimes \mathcal{B}) \otimes \mathcal{C}$ is a 4-fold tensor product, it suffices to show that the above equation holds after composing with the universal arrow $\mathcal{A}\mathcal{V}\mathcal{B}\mathcal{C} \to ((\mathcal{A} \otimes \mathcal{V}) \otimes \mathcal{B}) \otimes \mathcal{C}$ and this is a long, but straightforward calculation with pasting diagrams. Similarly, the right normalization axiom [GPS95, Section 2.2, (TA3)] holds if and only if for all $\mathcal{V}$-actegories

$\mathcal{A}, \mathcal{B}$, and $\mathcal{C}$ the following equation holds, where juxtaposition denotes tensor.

$$
\begin{tikzcd}[column sep=small]
 & & (\mathcal{AB})\mathcal{C} \arrow[drr, "\mathsf{a}"] & & \\
((\mathcal{AB})\mathcal{V})\mathcal{C} \arrow[urr, "\mathsf{r1}"] \arrow[dr, "\mathsf{a1}"] \arrow[ddd, "\mathsf{a}"'] \arrow[r, phantom, "\Downarrow\rho 1"] & & & & \mathcal{A}(\mathcal{BC}) \\
 & (\mathcal{A}(\mathcal{BV}))\mathcal{C} \arrow[uur, "\mathsf{(1r)1}"'] \arrow[dr, "\mathsf{a}"'] & & & \\
 & \arrow[r, phantom, "\Swarrow\pi"] & \mathcal{A}((\mathcal{BV})\mathcal{C}) \arrow[uurr, "\mathsf{1(r1)}"] \arrow[dr, "\mathsf{1a}"] \arrow[r, phantom, "\overset{1\mu}{\Rrightarrow}"] & & \\
(\mathcal{AB})(\mathcal{VC}) \arrow[rrr, "\mathsf{a}"'] & & & \mathcal{A}(\mathcal{B}(\mathcal{VC})) \arrow[uuur, "\mathsf{1(1l)}"'] &
\end{tikzcd}
=
\begin{tikzcd}[column sep=small]
 & & (\mathcal{AB})\mathcal{C} \arrow[drr, "\mathsf{a}"] & & \\
((\mathcal{AV})\mathcal{B})\mathcal{C} \arrow[urr, "\mathsf{(r1)1}"] \arrow[dd, "\mathsf{a}"'] \arrow[dr, phantom, "\Searrow\nu"] & & & & \mathcal{A}(\mathcal{BC}) \\
 & & & & \\
(\mathcal{AV})(\mathcal{BC}) \arrow[uuurr, "\mathsf{(11)l}"'] \arrow[rrrr, "\mathsf{a}"'] & & & & \mathcal{A}(\mathcal{B}(\mathcal{VC})) \arrow[uu, "\mathsf{1l}"']
\end{tikzcd}
$$

Again, it suffices to show that the above equation holds after composing with the universal arrow $\mathcal{ABVC} \to ((\mathcal{A} \otimes \mathcal{B}) \otimes \mathcal{V}) \otimes \mathcal{C}$ and this is a long, but straightforward calculation with pasting diagrams. ∎

We now proceed to the braiding on $\mathcal{V}$-Act. Suppose that $\mathcal{A}$ and $\mathcal{B}$ are $\mathcal{V}$-actegories and (K_5, κ_5) and (K_6, κ_6) are are universal descent diagrams from $(\mathcal{A}, \mathcal{B})$ to $\mathcal{A} \otimes \mathcal{B}$ and from $(\mathcal{B}, \mathcal{A})$ to $\mathcal{B} \otimes \mathcal{A}$ respectively. There is a morphism of $\mathcal{V}$-actegories $P\colon \mathcal{AB} \to \mathcal{B} \otimes \mathcal{A}$ whose underlying functor is the composite $\mathcal{K}_6 \circ \sigma\colon \mathcal{AB} \to \mathcal{BA} \to \mathcal{B} \otimes \mathcal{A}$, and whose structure isomorphism has component at an object (V, A, B) of $\mathcal{VAB}$ given by the composite $k_6 \circ \kappa_6^{-1}\colon K_6(B, V \otimes A) \to K_6(V \otimes B, A) \to V \otimes K_6(B, A)$, where k_6 is the structure isomorphism of K_6. For each pair of objects A and B of $\mathcal{A}$ and each object V of $\mathcal{V}$, there is an arrow $P(V \otimes A, B) \to P(A, V \otimes B)$ given by $\kappa_6^{-1}\colon K_6(B, V \otimes A) \to K_6(V \otimes B, A)$ which is the component at (A, V, B) of an invertible transformation of $\mathcal{V}$-actegories

$$
\begin{tikzcd}
 & \mathcal{AB} \arrow[dr, "P"] & \\
\mathcal{AVB} \arrow[ur, "\hat{\otimes}1"] \arrow[dr, "1\otimes"'] \arrow[rr, phantom, "\Downarrow p"] & & \mathcal{B} \otimes \mathcal{A}. \\
 & \mathcal{AB} \arrow[ur, "P"'] &
\end{tikzcd}
$$

and (P, p) is a descent diagram. By the universal property of $\mathcal{A} \otimes \mathcal{B}$, there exists a unique morphism of $\mathcal{V}$ actegories $\rho\colon \mathcal{A} \otimes \mathcal{B} \to \mathcal{B} \otimes \mathcal{A}$ such that $\mathrm{Desc}(\mathcal{A}, \mathcal{B}; \rho)(K_5, \kappa_5)$ is equal to (P, p). It is straightforward to show that the arrow $\rho\colon \mathcal{A} \otimes \mathcal{B} \to \mathcal{B} \otimes \mathcal{A}$ constitutes the component at $(\mathcal{A}, \mathcal{B})$ of a 2-natural transformation

$$
\begin{tikzcd}
\mathcal{V}\text{-Act}^2 \arrow[rr, "\otimes"] \arrow[dr, "\sigma"'] & \Downarrow\rho & \mathcal{V}\text{-Act} \\
 & \mathcal{V}\text{-Act}^2 \arrow[ur, "\otimes"'] &
\end{tikzcd}
$$

in $\hom(\mathcal{V}\text{-Act}^2, \mathcal{V}\text{-Act})$.

7.10. Lemma. *For all $\mathcal{V}$-actegories $\mathcal{A}$ and $\mathcal{B}$ the composite $\rho\circ\rho\colon \mathcal{A}\otimes\mathcal{B} \to \mathcal{B}\otimes\mathcal{A} \to \mathcal{A}\otimes\mathcal{B}$ is the identity.* ∎

The following proposition is an immediate consequence.

7.11. Proposition. *The arrow $\rho\colon \mathcal{A}\otimes\mathcal{B} \to \mathcal{B}\otimes\mathcal{A}$ constitutes the component at $(\mathcal{A},\mathcal{B})$ of a 2-natural isomorphism in* $\hom(\mathcal{V}\text{-Act}^2, \mathcal{V}\text{-Act})$. ∎

7.12. Lemma. *For all $\mathcal{V}$-actegories $\mathcal{A},\mathcal{B}$ and $\mathcal{C}$, the diagrams*

$$\begin{tikzcd}
 & \mathcal{A}\otimes(\mathcal{B}\otimes\mathcal{C}) \arrow[r, "\rho"] & (\mathcal{B}\otimes\mathcal{C})\otimes\mathcal{A} \arrow[dr, "\mathsf{a}"] & \\
(\mathcal{A}\otimes\mathcal{B})\otimes\mathcal{C} \arrow[ur, "\mathsf{a}"] \arrow[dr, "\rho\otimes 1"'] & & & \mathcal{B}\otimes(\mathcal{C}\otimes\mathcal{A}) \\
 & (\mathcal{B}\otimes\mathcal{A})\otimes\mathcal{C} \arrow[r, "\mathsf{a}"'] & \mathcal{B}\otimes(\mathcal{A}\otimes\mathcal{C}) \arrow[ur, "1\otimes\rho"'] &
\end{tikzcd}$$

$$\begin{tikzcd}
 & (\mathcal{A}\otimes\mathcal{B})\otimes\mathcal{C} \arrow[r, "\rho"] & \mathcal{C}\otimes(\mathcal{A}\otimes\mathcal{B}) \arrow[dr, "\mathsf{a}^{-1}"] & \\
\mathcal{A}\otimes(\mathcal{B}\otimes\mathcal{C}) \arrow[ur, "\mathsf{a}^{-1}"] \arrow[dr, "1\otimes\rho"'] & & & (\mathcal{C}\otimes\mathcal{A})\otimes\mathcal{B} \\
 & \mathcal{A}\otimes(\mathcal{C}\otimes\mathcal{B}) \arrow[r, "\mathsf{a}^{-1}"'] & (\mathcal{A}\otimes\mathcal{C})\otimes\mathcal{B} \arrow[ur, "\rho\otimes 1"'] &
\end{tikzcd}$$

commute. ∎

It follows from Lemma 7.12 that the modifications R and S of the definition of a braided monoidal 2-category may be taken to be the identity 2-cells.

7.13. Proposition. *When equipped with the 2-natural isomorphism ρ, and the modifications R and S the monoidal 2-category $\mathcal{V}$-Act is braided.*

Proof. The axioms (BA1), (BA2), (BA3) and (BA4) of Appendix A hold trivially because in each case both sides of the equation are the identity 2-cell. ∎

We now consider the syllepsis for $\mathcal{V}$-Act. By Lemma 7.10 the modification

$$\begin{tikzcd}
 & \mathcal{V}\text{-Act}^2 \arrow[dd, "\otimes"] \\
\mathcal{V}\text{-Act}^2 \arrow[ur, bend left, "1"] \arrow[dr, bend right, "\otimes"'] & \\
 & \mathcal{V}\text{-Act}
\end{tikzcd}
\overset{v}{\Rrightarrow}
\begin{tikzcd}
 & & \mathcal{V}\text{-Act}^2 \arrow[dd, "\otimes"] \\
\mathcal{V}\text{-Act}^2 \arrow[urr, bend left, "1"] \arrow[drr, bend right, "\otimes"'] \arrow[r, "\sigma"] & \mathcal{V}\text{-Act}^2 \arrow[ur, "\sigma"] \arrow[dr, "\otimes"] \arrow[r, phantom, "\Downarrow\rho"] \arrow[d, phantom, "\Downarrow\rho"] & \\
 & \, & \mathcal{V}\text{-Act}
\end{tikzcd}$$

in the definition of a sylleptic monoidal 2-category may be taken to be the identity.

7.14. PROPOSITION. *When equipped with the modification v, the braided monoidal bicategory* $\mathcal{V}$-Act *is symmetric.*

Proof. The axioms (SA1) and (SA2) of Appendix B hold trivially because in each case, both sides of the equation are the identity 2-cell, so that $\mathcal{V}$-Act is sylleptic. Also,the axiom of Section 4 holds trivially because both sides of the equation are the identity 2-cell. ∎

8. Gray-limits and symmetric monoidal 2-categories

In [McC99b], a 2-category $\mathcal{V}$-Act$//\mathcal{V}$ is constructed whose objects $(\mathcal{A}, X, \sigma)$ consist of a $\mathcal{V}$-actegory $\mathcal{A}$, a set X and an X-indexed family $\sigma_x \colon \mathcal{A} \to \mathcal{V}$ of morphisms of $\mathcal{V}$-actegories. For example, if C is a $\mathcal{V}^{\mathrm{op}}$-category, then the category Comod(C) of C-comodules is a $\mathcal{V}$-actegory, and there is a family $\omega_c \colon \mathrm{Comod}(C) \to \mathcal{V}$ of morphisms of $\mathcal{V}$-actegories indexed by the objects of C. Indeed this is the value at the object C of a 2-functor

$$\mathrm{Comod}\colon \mathcal{V}^{\mathrm{op}}\text{-}\mathrm{Cat}^{\mathrm{op}} \to \mathcal{V}\text{-}\mathrm{Act}//\mathcal{V}.$$

This 2-functor was studied in [McC99b]; it was shown to be bi-fully faithful and its image was characterized in elementary terms. In this section we shall equip the 2-category $\mathcal{V}$-Act$//\mathcal{V}$ with the structure of a symmetric monoidal 2-category, and in Section 9, we shall equip the 2-functor Comod with the structure of a symmetric weak monoidal 2-functor. We shall show that this weak monoidal 2-functor is monoidally bi-fully-faithful in Section 10, which allows us to conclude our main result on the reconstruction of balanced coalgebroids.

The 2-category $\mathcal{V}$-Act$//\mathcal{V}$ is constructed by Gray-categorical means [McC99b, Section 3]. Indeed, it is formed by taking a Gray-limit followed by a certain factorization. The objects and arrows involved in these constructions are in fact symmetric monoidal, and we shall show that these constructions "lift" to make $\mathcal{V}$-Act$//\mathcal{V}$ a symmetric monoidal 2-category. These techniques also allow make the 2-category Cat$/\mathcal{V}$, as described in Section 2, into a symmetric monoidal 2-category. We thank S. Lack for suggesting this approach. The following description of the construction of $\mathcal{V}$-Act$//\mathcal{V}$ is taken from [McC99b, Section 2] and [McC99a, Section 1.1].

We first recall the symmetric monoidal closed category Gray. Let Gray_0 denote the category of 2-categories and 2-functors, and let Gray denote the symmetric monoidal closed category whose underlying category is Gray_0 and whose tensor product is the Gray *tensor product*; see for example [DS97]. A Gray-*category* is a Gray-enriched category in the sense of [Kel82], and may be considered to be a *semi-strict tricategory* [GPS95]. In particular, every 3-category may be considered to be a Gray-category. There is a factorization system $(\mathcal{E}, \mathcal{M})$ on the category Gray_0. The class $\mathcal{E}$ consists of those 2-functors that are bijective on objects and bijective on arrows, and the class $\mathcal{M}$ consists of those 2-functors that are locally fully faithful.

Let D be the free category on the graph $1 \to 2 \leftarrow 3$. Every category may be considered as a 3-category with no non-identity 2-cells or 3-cells, and every 3-category may be considered as a Gray-category, and we now consider D as a Gray-category in this way. To give a Gray-functor $S\colon D \to \mathrm{Gray}$ amounts to giving a diagram $F\colon \mathcal{K} \to \mathcal{L} \leftarrow \mathcal{M}\colon G$ of 2-categories and 2-functors. Let 1 denote the terminal 2-category, and let $\mathbb{2}$ denote the locally discrete 2-category with two objects x and y, and with exactly one non-identity arrow $x \to y$. Let $J\colon D \to \mathrm{Gray}$ be the Gray-functor defined by $x\colon 1 \to \mathbb{2} \leftarrow 1\colon y$, where x and y are named by their value on the only object of 1. If $T\colon D \to \mathrm{Gray}$ is a Gray-functor, then we may consider the limit of T weighted by J in the usual sense of enriched category theory [Kel82, Chapter 3]; this is a 2-category $\{J, T\}$ equipped with a Gray-natural isomorphism $\mathrm{Gray}(\mathcal{K}, \{J, T\}) \cong [D, \mathrm{Gray}](J, \mathrm{Gray}(\mathcal{K}, T))$. The universal property of $\{J, T\}$ provides 2-functors $p\colon \{J, T\} \to \mathcal{K}$ and $q\colon \{J, T\} \to \mathcal{M}$.

The 2-category $\{J, T\}$ has the following description [McC99a, Section 1]. An object (x, σ, y) of $\{J, T\}$ consists of an object x of $\mathcal{K}$, an object y of $\mathcal{M}$, and an arrow $\sigma\colon Fx \to Gy$ of $\mathcal{L}$. An arrow $(r, \varphi, s)\colon (x, \sigma, y) \to (x', \sigma', y')$ consists of arrows $r\colon x \to x'$ of $\mathcal{K}$ and $s\colon y \to y'$ and an invertible 2-cell

$$\begin{tikzcd} Fx \arrow[r, "\sigma"] \arrow[d, "Fr"'] \arrow[dr, phantom, "\swarrow\varphi"] & Gy \arrow[d, "Gs"] \\ Fx' \arrow[r, "\sigma'"'] & Gy' \end{tikzcd}$$

in $\mathcal{L}$. A 2-cell (α_1, α_2) of $\{J, T\}$ is a pair of two cells $\alpha_1\colon r \Rightarrow r'$ and $\alpha_2\colon s \Rightarrow s'$ in $\mathcal{K}$ and $\mathcal{M}$ respectively, satisfying the following equation.

$$\begin{tikzcd} & Gy \arrow[dr, bend left, "Gs"] \arrow[d, phantom, "\swarrow\varphi"] & \\ Fx \arrow[ur, "\sigma"] \arrow[dr, bend left, "Fr"] \arrow[dr, bend right, "Fr'"'] \arrow[dr, phantom, "\swarrow\alpha_1"] & {} & Gy' \\ & Fx' \arrow[ur, "\sigma'"'] & \end{tikzcd} = \begin{tikzcd} & Gy \arrow[dr, bend left, "Gs"] \arrow[dr, bend right, "Gs'"'] \arrow[dr, phantom, "\swarrow\varphi'"] & \\ Fx \arrow[ur, "\sigma"] \arrow[dr, bend right, "Fr'"'] & \swarrow\alpha_2 & Gy' \\ & Fx' \arrow[ur, "\sigma'"'] & \end{tikzcd}$$

There are evident projection 2-functors $p\colon \{J, T\} \to \mathcal{K}$ and $q\colon \{J, T\} \to \mathcal{M}$.

We now define the 2-category $\mathcal{V}\text{-Act}//\mathcal{V}$. Let Set be the locally discrete 2-category whose underlying category is the category of sets and let $\mathbb{D}\colon \mathrm{Set} \to \mathrm{Cat}$ be the evident 2-functor whose value on a set X is the discrete category with X as its set of objects. Let $T\colon D \to \mathrm{Gray}$ be defined by the following diagram[1],

$$\mathcal{V}\text{-Act} \xrightarrow{\mathcal{V}\text{-Act}(-,\mathcal{V})} \mathrm{Cat}^{\mathrm{op}} \xleftarrow{\mathbb{D}^{\mathrm{op}}} \mathrm{Set}^{\mathrm{op}}$$

and let $p\colon \{J, T\} \to \mathcal{V}\text{-Act}$ be the induced 2-functor. Define a 2-category $\mathcal{V}\text{-Act}//\mathcal{V}$ to be

[1]Here we are taking the liberty of ignoring issues of size. This can be rectified by an appropriate change of universe.

the 2-category appearing in the factorization

$$\begin{tikzcd} \{J,T\} \arrow[rr, "p"] \arrow[dr, "E"'] & & \mathcal{V}\text{-Act} \\ & \mathcal{V}\text{-Act}//\mathcal{V} \arrow[ur, "M"'] & \end{tikzcd}$$

with E bijective on objects and bijective on arrows, and M locally fully faithful. Using the description of $\{J,T\}$ given above, we may describe $\mathcal{V}$-Act$//\mathcal{V}$ explicitly as follows. An object $(\mathcal{A}, X, \sigma)$ of $\mathcal{V}$-Act$//\mathcal{V}$ consists of a $\mathcal{V}$-actegory $\mathcal{A}$, a set X and an X-indexed family $\sigma_x \colon \mathcal{A} \to \mathcal{V}$ of morphism of $\mathcal{V}$-actegories. An arrow $(F, t, \varphi) \colon (\mathcal{A}, X, \sigma) \to (\mathcal{B}, Y, \sigma)$ consists of a morphism of $\mathcal{V}$-actegories $F \colon \mathcal{A} \to \mathcal{B}$, a function $t \colon Y \to X$ and a family $\varphi_y \colon \sigma_y \circ F \Rightarrow \sigma_{ty}$ of invertible transformations of $\mathcal{V}$-actegories. A 2-cell $(F, t, \varphi) \Rightarrow (G, s, \psi)$ is simply a transformation of $\mathcal{V}$-actegories from F to G.

The same construction serves to define the 2-category Cat$/\mathcal{V}$. Indeed, let $T' \colon \mathcal{D} \to$ Gray be defined by the following diagram,

$$\mathrm{Cat} \xrightarrow{\mathrm{Cat}(-,\mathcal{V})} \mathrm{Cat}^{\mathrm{op}} \xleftarrow{1} 1$$

where $1 \colon 1 \to \mathrm{Cat}^{\mathrm{op}}$ is the 2-functor whose value on the only object of 1 is the discrete category with one object. Let $p' \colon \{J, T'\} \to$ Cat be the induced 2-functor. One may show that the 2-category $\mathcal{K}$ appearing in the factorization

$$\begin{tikzcd} \{J,T'\} \arrow[rr, "p'"] \arrow[dr, "E'"'] & & \mathrm{Cat} \\ & \mathcal{K} \arrow[ur, "M'"'] & \end{tikzcd}$$

with E' bijective on objects and bijective on arrows, and M' locally fully faithful is isomorphic to Cat$/\mathcal{V}$.

We now show that taking the J-limit of a 2-functor $T \colon D \to$ Gray "lifts" to monoidal 2-categories. More precisely, suppose $\mathcal{K}$, $\mathcal{L}$ and $\mathcal{M}$ are monoidal 2-categories and $F \colon \mathcal{K} \to \mathcal{L}$ and $G \colon \mathcal{M} \to \mathcal{L}$ are opweak and weak monoidal 2-functors respectively. Let $T \colon D \to$ Gray be the Gray-functor defined by the diagram $F \colon \mathcal{K} \to \mathcal{L} \leftarrow \mathcal{M} \colon G$, by taking the underlying 2-categories and 2-functors, and let $\{J,T\}$ be the J-weighted limit of T. We shall show that $\{J,T\}$ inherits a monoidal structure such that the canonical projections $p \colon \{J,T\} \to \mathcal{K}$ and $q \colon \{J,T\} \to \mathcal{L}$ are strict monoidal 2-functors. We shall not, however, analyze the universal property of this construction. As the details of this construction are straightforward much will be omitted. We use the description of $\{J,T\}$ given above.

Suppose that (a, σ, b) and (a', σ', b') are objects of $\{J,T\}$. Define $(a, \sigma, b) \otimes (a', \sigma', b')$ to be $(a \otimes a', \sigma'', b \otimes b')$ where σ'' is

$$F(a \otimes a') \xrightarrow{\chi} Fa \otimes Fa' \xrightarrow{\sigma \otimes \sigma'} Fb \otimes Fb' \xrightarrow{\chi} F(b \otimes b')$$

and the arrows $\chi\colon F(a\otimes a')\to Fa\otimes Fa'$ and $\chi\colon Fb\otimes Fb'\to F(b\otimes b')$ are part of the structure of F and G being opweak and weak monoidal 2-functors respectively. Suppose $(r,\varphi,s)\colon(a,\sigma,b)\to(c,\sigma,d)$ and $(r',\varphi',s')\colon(a',\sigma',b')\to(c',\sigma',d')$ are arrows in $\{J,T\}$ Define $(r,\varphi,s)\otimes(r',\varphi',s')$ to be $(r\otimes r',\varphi'',s\otimes s')$, where φ'' is the 2-cell in the following diagram.

$$\begin{tikzcd}
F(a\otimes a') \arrow[r, "\chi"] \arrow[d, "F(r\otimes r')"'] \arrow[dr, phantom, "\swarrow"] & Fa\otimes Fa' \arrow[r, "\sigma\otimes\sigma"] \arrow[d, "Fr\otimes Fr'"] \arrow[dr, phantom, "\swarrow\varphi\otimes\varphi'"] & Gb\otimes Gb' \arrow[r, "\chi"] \arrow[d, "Gs\otimes Gs'"] \arrow[dr, phantom, "\swarrow"] & G(b\otimes b') \arrow[d, "G(s\otimes s')"] \\
F(c\otimes c') \arrow[r, "\chi"'] & Fc\otimes Fc' \arrow[r, "\sigma'\otimes\sigma'"'] & Gd\otimes Gd' \arrow[r, "\chi"'] & G(d\otimes d')
\end{tikzcd}$$

It is straight forward to provide the remaining data for a homomorphism $\otimes\colon\{J,T\}\times\{J,T\}\to\{J,T\}$, and verify that it satisfies the required coherence conditions. The unit object of $\{J,T\}$ is defined to be $(I,FI\to I\to GI,I)$. We now consider the associativity equivalence: if (a,σ,d), (b,σ,e) and (c,σ,f) are objects of $\{J,T\}$, then there is an invertible 2-cell φ in $\mathcal{L}$ given by the following diagram.

$$\begin{tikzcd}
 & F(ab)Fc \arrow[dr, "\chi Fc"] & & & F(de)Ff \arrow[dr, "\chi"] & \\
F((ab)c) \arrow[ur, "\chi"] \arrow[d, "F\mathsf{a}"'] \arrow[rr, phantom, shift right=6, "\swarrow\omega_F"] & & (FaFb)Fc \arrow[r, "(\sigma\sigma)\sigma"] \arrow[d, "\mathsf{a}"] \arrow[dr, phantom, "\swarrow"] & (GdGe)Gf \arrow[ur, "\chi Gf"] \arrow[d, "\mathsf{a}"] \arrow[rr, phantom, shift right=6, "\swarrow\omega_G"] & & G((de)f) \arrow[d, "G\mathsf{a}"] \\
F(a(bc)) \arrow[dr, "\chi"'] & & Fa(FbFc) \arrow[r, "\sigma(\sigma\sigma)"'] & Gd(GeGf) \arrow[dr, "Gd\chi"'] & & G(d(ef)) \\
 & Fa(FbFc) \arrow[ur, "Fa\chi"'] & & & GdG(ef) \arrow[ur, "\chi"'] &
\end{tikzcd}$$

Here, ω_F and ω_G are part of the structure of F and G being opweak and weak monoidal 2-functors respectively. Define an arrow $((a,\sigma,d)\otimes(b,\sigma,e))\otimes(c,\sigma,f)\to(a,\sigma,d)\otimes((b,\sigma,e)\otimes(c,\sigma,f))$ of $\{J,T\}$ to be $(\mathsf{a},\varphi,\mathsf{a})$. This arrow is the component at the object $((a,\sigma,d),(b,\sigma,e),(c,\sigma,f))$ of a pseudonatural equivalence as in the following diagram.

$$\begin{tikzcd}
\{J,T\}^3 \arrow[r, "\otimes 1"] \arrow[d, "1\otimes"'] \arrow[dr, phantom, "\swarrow\mathsf{a}"] & \{J,T\}^2 \arrow[d, "\otimes"] \\
\{J,T\}^2 \arrow[r, "\otimes"'] & \{J,T\}
\end{tikzcd}$$

It is straightforward to construct the remaining data in order to make $\{J,T\}$ into a monoidal 2-category, and verify that the axioms hold. Clearly the projections $p\colon\{J,T\}\to\mathcal{K}$ and $q\colon\{J,T\}\to\mathcal{M}$ are *strict* monoidal 2-functors. A similar argument proves the following.

8.1. Proposition. *Suppose that $\mathcal{K},\mathcal{L}$ and $\mathcal{M}$ are symmetric monoidal 2-categories and $F\colon\mathcal{K}\to\mathcal{L}$ and $G\colon\mathcal{M}\to\mathcal{L}$ are opweak and weak symmetric monoidal 2-functors respectively. Let $T\colon D\to$* Gray *be the* Gray*-functor defined by the diagram $F\colon\mathcal{K}\to\mathcal{M}\leftarrow\mathcal{L}\colon G$,* *by taking the underlying 2-categories and 2-functors. Then $\{J, T\}$ inherits the structure of a symmetric monoidal 2-category such that the canonical projections $p\colon \{J, T\} \to \mathcal{K}$ and $q\colon \{J, T\} \to \mathcal{L}$ are strict symmetric monoidal 2-functors.* ∎

We now consider factorizations of strict monoidal 2-functors. Suppose $\mathcal{K}$ and $\mathcal{L}$ are monoidal 2-categories and $F\colon \mathcal{K} \to \mathcal{L}$ is a strict monoidal 2-functor. Let $M \circ E\colon \mathcal{K} \to \mathcal{M} \to \mathcal{L}$ be a factorization of F in Gray_0, where E is bijective on objects and bijective on arrows and M is locally fully faithful. It is straightforward to make $\mathcal{M}$ into a monoidal 2-category such that the 2-functors $E\colon \mathcal{K} \to \mathcal{M}$ and $M\colon \mathcal{M} \to \mathcal{L}$ are *strict* monoidal 2-functors. Similarly one may prove the following proposition.

8.2. Proposition. *Suppose that $\mathcal{K}$ and $\mathcal{L}$ are symmetric monoidal 2-categories and $F\colon \mathcal{K} \to \mathcal{L}$ is a strict symmetric monoidal 2-functor. Let $M \circ E\colon \mathcal{K} \to \mathcal{M} \to \mathcal{L}$ be a factorization of F in* Gray_0 *where E is bijective on objects and bijective on arrows, and M is locally fully faithful. Then $\mathcal{M}$ inherits the structure of a symmetric monoidal 2-category such that E and M are strict symmetric monoidal 2-functors.* ∎

8.3. Proposition. *If $\mathcal{K}$ is a symmetric monoidal 2-category, and A is a symmetric pseudomonoid in $\mathcal{K}$, then the 2-functor $\mathcal{K}(-, A)\colon \mathcal{K} \to \mathrm{Cat}^{\mathrm{op}}$ is canonically equipped with the structure of a opweak symmetric monoidal 2-functor.*

Proof. If B and C are objects of $\mathcal{K}$, then define a functor $\chi\colon \mathcal{K}(B, A) \times \mathcal{K}(C, A) \to \mathcal{K}(B \otimes C, A)$ to be the following composite.

$$\mathcal{K}(B, A) \times \mathcal{K}(C, A) \xrightarrow{\otimes} \mathcal{K}(B \otimes C, A \otimes A) \xrightarrow{\mathcal{K}(B \otimes C, \mu)} \mathcal{K}(B \otimes C, A)$$

It is straightforward to show that this arrow is the component at the pair (B, C) of a pseudonatural transformation

$$\begin{tikzcd} \mathcal{K}^2 \arrow[r, "\mathcal{K}(-,A)^2"] \arrow[d, "\otimes"'] \arrow[dr, phantom, "\overset{\chi}{\Rightarrow}"] & \mathrm{Cat}^{\mathrm{op}2} \arrow[d, "\times"] \\ \mathcal{K} \arrow[r, "\mathcal{K}(-,A)"'] & \mathrm{Cat}^{\mathrm{op}} \end{tikzcd}$$

in $\hom(\mathcal{K}^2, \mathrm{Cat}^{\mathrm{op}})$. The unit $\eta\colon I \to A$ provides a functor $\iota\colon 1 \to \mathcal{K}(I, A)$ which is the only component of a pseudonatural transformation

$$\begin{tikzcd} & 1 \arrow[dl, "I"'] \arrow[dr, "1"] \arrow[d, phantom, "\underset{\iota}{\Rightarrow}"] & \\ \mathcal{K} \arrow[rr, "\mathcal{K}(-,A)"'] & {} & \mathrm{Cat}^{\mathrm{op}} \end{tikzcd}$$

in $\hom(1, \mathrm{Cat}^{\mathrm{op}})$. The remaining data for the symmetric pseudomonoid A provide the remaining data making $\mathcal{K}(-, A)$ a symmetric weak monoidal 2-functor and the axioms that state that A is a symmetric pseudomonoid imply that $\mathcal{K}(-, A)$ is indeed a symmetric weak monoidal 2-functor. ∎

We are now in a position to define the symmetric monoidal structure on the 2-category $\mathcal{V}$-Act$//\mathcal{V}$. By Example 4.3, the $\mathcal{V}$-actegory $\mathcal{V}$ is a symmetric pseudomonoid in $\mathcal{V}$-Act by virtue of it being the unit. Thus by Proposition 8.3, $\mathcal{V}\text{-Act}(-,\mathcal{V})\colon \mathcal{V}\text{-Act} \to \mathrm{Cat}^{\mathrm{op}}$ is canonically equipped with the structure of an opweak symmetric monoidal 2-functor. Clearly $\mathbb{D}^{\mathrm{op}}\colon \mathrm{Set}^{\mathrm{op}} \to \mathrm{Cat}^{\mathrm{op}}$ is a strict symmetric monoidal 2-functor, and so in particular, it is a weak symmetric monoidal 2-functor. By Proposition 8.1, $\{J,T\}$ inherits the structure of a symmetric monoidal 2-category such that the induced 2-functors $p\colon \{J,T\} \to \mathcal{V}$-Act and $q\colon \{J,T\} \to$ Set are *strict* symmetric monoidal 2-functors. Thus by Proposition 8.2, $\mathcal{V}$-Act$//\mathcal{V}$ inherits the structure of a symmetric monoidal 2-category such that 2-functors $E\colon \{J,T\} \to \mathcal{V}\text{-Act}//\mathcal{V}$ and $M\colon \mathcal{V}\text{-Act}//\mathcal{V} \to \mathcal{V}$-Act are strict symmetric monoidal 2-functors.

We now provide an explicit description of this structure. Let $(\mathcal{A},X,\sigma)$ and $(\mathcal{B},Y,\sigma)$ be objects of $\mathcal{V}$-Act$//\mathcal{V}$; thus $\mathcal{A}$ and $\mathcal{B}$ are $\mathcal{V}$-actegories, X and Y are sets and $\sigma_x\colon \mathcal{A} \to \mathcal{V}$ and $\sigma_y\colon \mathcal{B} \to \mathcal{V}$ are families of morphisms of $\mathcal{V}$-actegories indexed by X and Y respectively. Then $(\mathcal{A},X,\sigma)\otimes(\mathcal{B},Y,\sigma)$ is $(\mathcal{A}\otimes\mathcal{B}, X\times Y, \mathsf{I}\circ(\sigma_x\otimes\sigma_y))$. Let $(F,f,\varphi)\colon(\mathcal{A},X,\sigma) \to (\mathcal{C},Z,\sigma)$ and $(G,g,\psi)\colon(\mathcal{B},Y,\sigma) \to (\mathcal{D},W,\sigma)$ be arrows of $\mathcal{V}$-Act$//\mathcal{V}$; thus $F\colon \mathcal{A} \to \mathcal{C}$ and $G\colon \mathcal{B} \to \mathcal{D}$ are morphisms of $\mathcal{V}$-actegories, $f\colon Z \to X$ and $g\colon W \to Y$ are functions and φ and ψ are families of invertible 2-cells $\varphi_z\colon F\circ\sigma_z \Rightarrow \sigma_{fz}$ and $\psi_w\colon G\circ\sigma_w \Rightarrow \sigma_{gw}$ indexed by Z and W respectively. The arrow $(F,f,\varphi)\otimes(G,g,\psi)\colon(\mathcal{A}\otimes\mathcal{B}, X\times Y,\sigma) \to (\mathcal{C}\otimes\mathcal{D}, Z\times W,\sigma)$ is $(F\otimes G, f\times g, \mathsf{I}\circ(\varphi\otimes\psi))$. Finally, if $\alpha\colon(F,f) \Rightarrow (F',f')$ and $\beta\colon (G,g) \Rightarrow (G',g')$ are 2-cells in $\mathcal{V}$-Act$//\mathcal{V}$, then $\alpha\otimes\beta\colon (F,f)\otimes(G,g) \Rightarrow (F',f')\otimes(G',g')$ is $\alpha\otimes\beta$. These data constitute the 2-functor $\otimes\colon \mathcal{V}\text{-Act}//\mathcal{V} \times \mathcal{V}\text{-Act}//\mathcal{V} \to \mathcal{V}\text{-Act}//\mathcal{V}$. The unit object of $\mathcal{V}$-Act$//\mathcal{V}$ is $(\mathcal{V},1,\mathrm{id})$ where 1 is the terminal set. If $(\mathcal{A},X,\sigma)$, $(\mathcal{B},Y,\sigma)$ and $(\mathcal{C},Z,\sigma)$ are objects of $\mathcal{V}$-Act$//\mathcal{V}$, then the associativity isomorphism $\mathsf{a}\colon ((\mathcal{A},X,\sigma)\otimes(\mathcal{B},Y,\sigma))\otimes(\mathcal{C},Z,\sigma) \to (\mathcal{A},X,\sigma)\otimes((\mathcal{B},Y,\sigma)\otimes(\mathcal{C},Z,\sigma))$ is $(\mathsf{a}, a^{-1}, \varphi)$ where $\mathsf{a}\colon (\mathcal{A}\otimes\mathcal{B})\otimes\mathcal{C} \to \mathcal{A}\otimes(\mathcal{B}\otimes\mathcal{C})$ is the 2-natural associativity isomorphism for $\mathcal{V}$-Act, $a\colon (X\times Y)\times Z \to X\times(Y\times Z)$ is the associativity isomorphism for the monoidal category of sets, and for each element (x,y,z) of $X\times Y\times Z$, φ is the invertible 2-cell exhibited by the following diagram.

$$\begin{tikzcd}
(\mathcal{A}\otimes\mathcal{B})\otimes\mathcal{C} \arrow[rr, "\mathsf{a}"] \arrow[d, "(\sigma_x\otimes\sigma_y)\otimes\sigma_z"'] & & \mathcal{A}\otimes(\mathcal{B}\otimes\mathcal{C}) \arrow[d, "\sigma_x\otimes(\sigma_y\otimes\sigma_z)"] \\
(\mathcal{V}\otimes\mathcal{V})\otimes\mathcal{V} \arrow[rr, "\mathsf{a}"] \arrow[d, "\mathsf{I}\otimes\mathcal{V}"'] & \arrow[d, phantom, "\Leftarrow a"] & \mathcal{V}\otimes(\mathcal{V}\otimes\mathcal{V}) \arrow[d, "\mathcal{V}\otimes\mathsf{I}"] \\
\mathcal{V}\otimes\mathcal{V} \arrow[r, "\mathsf{I}"'] & \mathcal{V} & \mathcal{V}\otimes\mathcal{V} \arrow[l, "\mathsf{I}"]
\end{tikzcd}$$

Here a is the associativity isomorphism for the pseudomonoid $\mathcal{V}$. These data form the component of the 2-natural isomorphism for $\mathcal{V}$-Act$//\mathcal{V}$. The left unit equivalence $\mathsf{l}\colon (\mathcal{V},1,\mathrm{id})\otimes(\mathcal{A},X,\sigma) \to (\mathcal{A},X,\sigma)$ is $(\mathsf{l}, l^{-1}, \varphi)$ where $\mathsf{l}\colon \mathcal{V}\otimes\mathcal{A} \to \mathcal{A}$ is the left unit equivalence for $\mathcal{V}$-Act, $l\colon 1\times X \to X$ is the left unit isomorphism for the category of sets, and φ is the following invertible 2-cell.

$$
\begin{tikzcd}
\mathcal{V} \otimes \mathcal{A} \arrow[rr, "|"] \arrow[d, "1 \otimes \sigma_x"'] & \arrow[d, phantom, "\Swarrow"] & \mathcal{A} \arrow[d, "\sigma_x"] \\
\mathcal{V} \otimes \mathcal{V} \arrow[rr, "|"] \arrow[dr, "|"'] & & \mathcal{V} \arrow[dl, "1"] \\
& \mathcal{V} &
\end{tikzcd}
$$

This data defines the component at $(\mathcal{A}, X, \sigma)$ of the left unit equivalence for $\mathcal{V}$-Act$//\mathcal{V}$; the isomorphisms expressing pseudonaturality of this equivalence are those expressing the pseudonaturality of the left unit equivalence for $\mathcal{V}$-Act. The right unit equivalence is similar. The modification π for $\mathcal{V}$-Act$//\mathcal{V}$ is the identity modification. Finally the modifications ν, λ and ρ for $\mathcal{V}$-Act$//\mathcal{V}$ are the modifications ν, λ and ρ for $\mathcal{V}$-Act. This completes the explicit description of the monoidal 2-category $\mathcal{V}$-Act$//\mathcal{V}$.

We now describe the braiding on $\mathcal{V}$-Act$//\mathcal{V}$. Let $(\mathcal{A}, X, \sigma)$ and $(\mathcal{B}, Y, \sigma)$ be objects of $\mathcal{V}$-Act$//\mathcal{V}$. The braiding $\rho\colon (\mathcal{A}, X, \sigma) \otimes (\mathcal{B}, Y, \sigma) \to (\mathcal{B}, Y, \sigma) \otimes (\mathcal{A}, X, \sigma)$ is $(\rho, \sigma^{-1}, \varphi)$, where $\rho\colon \mathcal{A} \otimes \mathcal{B} \to \mathcal{B} \otimes \mathcal{A}$ is the braiding on $\mathcal{V}$-Act, $\sigma\colon X \times Y \to Y \times X$ is the symmetry on the category of sets, and for each element (x, y) of $X \times Y$, the 2-cell φ is given by the following diagram.

$$
\begin{tikzcd}
\mathcal{A} \otimes \mathcal{B} \arrow[rr, "\rho"] \arrow[d, "\sigma_x \otimes \sigma_y"'] & & \mathcal{B} \otimes \mathcal{A} \arrow[d, "\sigma_y \otimes \sigma_x"] \\
\mathcal{V} \otimes \mathcal{V} \arrow[rr, "\rho"] \arrow[dr, "|"'] & \arrow[d, phantom, "\overset{c}{\Leftarrow}"] & \mathcal{V} \otimes \mathcal{V} \arrow[dl, "|"] \\
& \mathcal{V} &
\end{tikzcd}
$$

Here, c is the braiding on the braided pseudomonoid $\mathcal{V}$. This arrow is the component of the 2-natural braiding isomorphism for $\mathcal{V}$-Act$//\mathcal{V}$. The modifications R and S in the definition of braided monoidal bicategory are the identity, as is the syllepsis, and so the explicit description of the symmetric monoidal 2-category $\mathcal{V}$-Act$//\mathcal{V}$ is complete.

Similarly, the 2-category Cat$/\mathcal{V}$ is a symmetric monoidal 2-category. By definition, $\mathcal{V}$ is a symmetric pseudomonoid in Cat. Thus by Proposition 8.3, the 2-functor Cat$(-, \mathcal{V})\colon$ Cat $\to$ Cat$^{\mathrm{op}}$ is canonically equipped with the structure of an opweak symmetric monoidal 2-functor. Clearly $1\colon 1 \to$ Cat$^{\mathrm{op}}$ is a strict symmetric monoidal 2-functor, and so in particular, it is a weak symmetric monoidal 2-functor. Thus by Proposition 8.1, $\{J, T'\}$ inherits the structure of a symmetric monoidal 2-category such that the induced 2-functors $p'\colon \{J, T'\} \to$ Cat and $q'\colon \{J, T'\} \to 1$ are *strict* symmetric monoidal 2-functors. Thus by Proposition 8.2, Cat$/\mathcal{V}$ inherits the structure of a symmetric monoidal 2-category such that 2-functors $E'\colon \{J, T'\} \to$ Cat$/\mathcal{V}$ and $M'\colon$ Cat$/\mathcal{V} \to$ Cat are strict symmetric monoidal 2-functors. This symmetric monoidal 2-category was first described in [Str89].

9. The weak monoidal 2-functor Comod

In this section, we describe the symmetric weak monoidal 2-functor Comod: $\mathcal{V}^{\mathrm{op}}\text{-}\mathrm{Cat}^{\mathrm{op}} \to \mathcal{V}\text{-}\mathrm{Act}//\mathcal{V}$. Its underlying 2-functor was defined and studied in [McC99b], to which we refer the reader for a complete description. In Section 10, we shall show that Comod is monoidally bi-fully faithful, which allows the reconstruction of balanced coalgebroids from their representations.

We first recall the 2-category $\mathcal{V}^{\mathrm{op}}$-Cat of $\mathcal{V}^{\mathrm{op}}$-categories. For a symmetric monoidal category $\mathcal{V}$, the opposite category $\mathcal{V}^{\mathrm{op}}$ of $\mathcal{V}$ is a symmetric monoidal category with the same tensor product as $\mathcal{V}$, but with associativity, unit isomorphisms and symmetry taken to be the inverse of those of $\mathcal{V}$. We may thus consider the symmetric monoidal 2-category $\mathcal{V}^{\mathrm{op}}$-Cat of categories enriched in $\mathcal{V}^{\mathrm{op}}$ [Kel82, Section 1.2]. Objects, arrows and 2-cells of this 2-category are called *$\mathcal{V}^{\mathrm{op}}$-categories*, *$\mathcal{V}^{\mathrm{op}}$-functors* and *$\mathcal{V}^{\mathrm{op}}$-natural transformations* respectively. A $\mathcal{V}^{\mathrm{op}}$-category with exactly one object is a comonoid in $\mathcal{V}$, and a $\mathcal{V}^{\mathrm{op}}$-functor between one object $\mathcal{V}^{\mathrm{op}}$-categories is a comonoid morphism in the opposite direction. We thus consider the 2-category $\mathcal{V}^{\mathrm{op}}\text{-}\mathrm{Cat}^{\mathrm{op}}$ to be the appropriate context for the study of *many object coalgebras*, or *coalgebroids*.

For a $\mathcal{V}^{\mathrm{op}}$-category C, a *C-comodule*, or a *representation of C*, is an $\mathrm{obj}(C)$-indexed family M_a of objects of $\mathcal{V}$ equipped with arrows $\delta\colon M_a \to M_b \otimes C(a,b)$ of $\mathcal{V}$, called *coactions*, which are coassociative and counital; we shall denote such a C-comodule by M. The object M_a is called the *component of M at a*. There is a category $\mathrm{Comod}(C)$ of C-comodules, and for each object a of C, there is a functor $\omega_a\colon \mathrm{Comod}(C) \to \mathcal{V}$, called *evaluation at a*, whose value on a C-comodule M is the component of M at a. Observe that when C is a comonoid, that is, a $\mathcal{V}^{\mathrm{op}}$-category with one object, then $\mathrm{Comod}(C)$ is the usual category of C-comodules, and the only evaluation functor $\omega\colon \mathrm{Comod}(C) \to \mathcal{V}$ is the usual forgetful functor. Now let M be a C-comodule and V be an object of $\mathcal{V}$. The family $V \otimes M_a$ is a C-comodule with coactions $V \otimes \delta\colon V \otimes M_a \to V \otimes M_b \otimes C(a,b)$, and this is the value of a functor $\mathcal{V} \times \mathrm{Comod}(C) \to \mathrm{Comod}(C)$ at the pair (V, M) which makes $\mathrm{Comod}(C)$ into a $\mathcal{V}$-actegory. Indeed the $\mathcal{V}$-actegory $\mathrm{Comod}(C)$ is the value at C of a 2-functor Comod: $\mathcal{V}^{\mathrm{op}}\text{-}\mathrm{Cat}^{\mathrm{op}} \longrightarrow \mathcal{V}\text{-}\mathrm{Act}$. As a space saving measure, let us denote this 2-functor by S.

We now equip the 2-functor S with the structure of a symmetric weak monoidal 2-functor. For $\mathcal{V}^{\mathrm{op}}$-actegories C and D, define a morphism of $\mathcal{V}$-actegories $L\colon S(C)S(D) \to S(C \otimes D)$ as follows. For C- and D-comodules M and N respectively, define $L(M,N)$ to be the $\mathrm{obj}(C \otimes D)$-indexed family $M_c \otimes N_d$, with coactions given by the following composite.

$$M_c \otimes N_d \xrightarrow{\delta\otimes\delta} M_{c'} \otimes C(c,c') \otimes N_{d'} \otimes D(d,d') \longrightarrow M_{c'} \otimes N_{d'} \otimes C(c,c') \otimes D(d,d')$$

If $f\colon M \to M'$ and $g\colon N \to N'$ are C- and D-comodule morphisms respectively, then define $L(f,g)$ to be the $\mathrm{obj}(C \otimes D)$-indexed family $f_c \otimes g_d$. Clearly L is a strict morphism of $\mathcal{V}$-actegories. The symmetry on $\mathcal{V}$ provides an invertible 2-cell

$$\begin{tikzcd}
 & S(C)S(D) \arrow[dr, "L"] & \\
S(C)\mathcal{V}S(D) \arrow[ur, "\hat{\otimes}1"] \arrow[dr, "1\otimes"'] \arrow[rr, phantom, "\Downarrow\lambda"] & & S(C\otimes D) \\
 & S(C)S(D) \arrow[ur, "L"'] &
\end{tikzcd}$$

and (L,λ) is a descent diagram from $(S(C),S(D))$ to $S(C\otimes D)$. The universal property of $S(C)\otimes S(D)$ provides a unique morphism of $\mathcal{V}$-actegories $\chi\colon S(C)\otimes S(D)\to S(C\otimes D)$ such that $\mathrm{Desc}(S(C),S(D);\chi)(K,\kappa)$ is equal to the above descent diagram.

9.1. Lemma. *The arrow $\chi\colon S(C)\otimes S(D)\to S(C\otimes D)$ is the component at (C,D) of a 2-natural transformation*

$$\begin{tikzcd}
(\mathcal{V}^{\mathrm{op}}\text{-}\mathrm{Cat}^{\mathrm{op}})^2 \arrow[r, "S^2"] \arrow[d, "\otimes"'] \arrow[dr, phantom, "\Leftarrow\chi"] & \mathcal{V}\text{-}\mathrm{Act}^2 \arrow[d, "\otimes"] \\
\mathcal{V}^{\mathrm{op}}\text{-}\mathrm{Cat}^{\mathrm{op}} \arrow[r, "S"'] & \mathcal{V}\text{-}\mathrm{Act}
\end{tikzcd}$$

in $\hom((\mathcal{V}^{\mathrm{op}}\text{-}\mathrm{Cat}^{\mathrm{op}})^2,\mathcal{V}\text{-}\mathrm{Act})$. ∎

Let I denote the unit object of $\mathcal{V}^{\mathrm{op}}$-Cat. Define a morphism of $\mathcal{V}$-actegories $\iota\colon\mathcal{V}\to S(I)$ as follows. For an object V of $\mathcal{V}$ define $\iota(V)$ to be (V,δ) where the coaction δ is the inverse of the right unit isomorphism $V\to V\otimes I$. The coherence theorem for monoidal categories shows that (V,δ) is an I-comodule. For an arrow $f\colon V\to W$ of $\mathcal{V}$, define $\iota(f)$ to be f. Naturality of the unit isomorphism for $\mathcal{V}$ shows that $\iota(f)$ is indeed an I-comodule morphism. Clearly $\iota\colon\mathcal{V}\to S(I)$ is a strict morphism of $\mathcal{V}$-actegories and provides the (only) component of a 2-natural transformation

$$\begin{tikzcd}
 & 1 \arrow[dl, "I"'] \arrow[dr, "\mathcal{V}"] \arrow[d, phantom, "\Leftarrow\iota"] & \\
\mathcal{V}^{\mathrm{op}}\text{-}\mathrm{Cat}^{\mathrm{op}} \arrow[rr, "S"'] & {} & \mathcal{V}\text{-}\mathrm{Act}
\end{tikzcd}$$

in $\hom(1,\mathcal{V}\text{-}\mathrm{Act})$. This 2-natural transformation is clearly an isomorphism.

9.2. Lemma. *For all $\mathcal{V}^{\mathrm{op}}$-categories B,C and D, the diagram*

$$\begin{tikzcd}
 & S(BC)S(D) \arrow[r, "\chi"] & S((BC)D) \arrow[dr, "S\mathsf{a}"] & \\
(S(B)S(C))S(D) \arrow[ur, "\chi S(D)"] \arrow[dr, "\mathsf{a}"'] & & & S(B(CD)) \\
 & S(B)(S(C)S(D)) \arrow[r, "S(B)\chi"'] & S(B)S(CD) \arrow[ur, "\chi"'] &
\end{tikzcd}$$

commutes. ■

It follows from Lemma 9.2 that the modification

$$\begin{tikzcd}[column sep=small]
& \mathcal{V}\text{-Act}^3 \arrow[rr, "\otimes 1"] \arrow[d, phantom, "\Downarrow\chi 1"] & & \mathcal{V}\text{-Act}^2 \arrow[dr, "\otimes"] & & & & \mathcal{V}\text{-Act}^3 \arrow[rr, "\otimes 1"] \arrow[dr, "1\otimes"] & & \mathcal{V}\text{-Act}^2 \arrow[dr, "\otimes"] \arrow[d, phantom, "\Downarrow\mathsf{a}"] & \\
(\mathcal{V}^{\mathrm{op}}\text{-Cat}^{\mathrm{op}})^3 \arrow[ur, "S^3"] \arrow[rr, "\otimes 1"] \arrow[dr, "1\otimes"'] & & (\mathcal{V}^{\mathrm{op}}\text{-Cat}^{\mathrm{op}})^2 \arrow[ur, "S^2"'] \arrow[dr, "\otimes"] \arrow[rr, phantom, "\Downarrow\chi"] & & \mathcal{V}\text{-Act} \arrow[rr, phantom, "\overset{\omega}{\Rrightarrow}"] & & (\mathcal{V}^{\mathrm{op}}\text{-Cat}^{\mathrm{op}})^3 \arrow[ur, "S^3"] \arrow[dr, "1\otimes"'] \arrow[rr, phantom, "\Downarrow 1\chi"] & & \mathcal{V}\text{-Act}^2 \arrow[rr, "\otimes"] & & \mathcal{V}\text{-Act} \\
& (\mathcal{V}^{\mathrm{op}}\text{-Cat}^{\mathrm{op}})^2 \arrow[rr, "\otimes"'] \arrow[u, phantom, "\Downarrow\mathsf{a}"] & & \mathcal{V}^{\mathrm{op}}\text{-Cat}^{\mathrm{op}} \arrow[ur, "S"'] & & & & (\mathcal{V}^{\mathrm{op}}\text{-Cat}^{\mathrm{op}})^2 \arrow[ur, "S^2"'] \arrow[rr, "\otimes"'] & & \mathcal{V}^{\mathrm{op}}\text{-Cat}^{\mathrm{op}} \arrow[ur, "S"'] \arrow[u, phantom, "\Downarrow\chi"] &
\end{tikzcd}$$

in the definition of a weak monoidal 2-functor may be taken to be the identity.

Now suppose C is a $\mathcal{V}^{\mathrm{op}}$-category, and V and M are objects of $\mathcal{V}$ and $S(C)$ respectively. There is a C-comodule $M \otimes V$ whose component at an object c of C is $M_c \otimes V$, with coactions given by the following.

$$M_c \otimes V \xrightarrow{\delta \otimes V} M_d \otimes C(c,d) \otimes V \longrightarrow M_d \otimes V \otimes C(c,d)$$

Clearly the symmetry $M_c \otimes V \to V \otimes M_c$ forms the component at c of a C-comodule isomorphism $M \otimes V \to V \otimes M$. This arrow is the component at (V, M) of an invertible descent morphism exhibited by the following diagram.

$$\begin{tikzcd}
& S(C)\mathcal{V} \arrow[r, "K"] \arrow[dl, "K"'] \arrow[d, "1\iota"] \arrow[dr, "\hat{\otimes}"] & S(C) \otimes \mathcal{V} \arrow[d, "\mathsf{r}"] \\
S(C) \otimes \mathcal{V} \arrow[d, "1\otimes\iota"'] & S(C)S(I) \arrow[dl, "K"'] \arrow[dr, "L"] \arrow[r, phantom, "\Leftarrow"] & S(C) \\
S(C) \otimes S(I) \arrow[rr, "\chi"'] & & S(C \otimes I) \arrow[u, "S\mathsf{r}"']
\end{tikzcd}$$

The universal property of $S(C) \otimes \mathcal{V}$ implies that there exists a unique 2-cell

$$\begin{tikzcd}
S(C) \otimes \mathcal{V} \arrow[r, "\mathsf{r}"] \arrow[d, "1\otimes\iota"'] \arrow[dr, phantom, "\Downarrow\delta"] & S(C) \\
S(C) \otimes S(I) \arrow[r, "\chi"'] & S(C \otimes I) \arrow[u, "S(\mathsf{r})"']
\end{tikzcd}$$

such that $\mathrm{Desc}(S(C), \mathcal{V}); \delta)(K, \kappa)$ is the above descent morphism.

9.3. Lemma. *The 2-cell δ is the component at C of an invertible modification*

$$
\begin{tikzcd}
& \mathcal{V}\text{-Act}^2 \arrow[dr, "\otimes"] \arrow[d, phantom, "\Downarrow\mathsf{r}"] & \\
\mathcal{V}\text{-Act} \arrow[ur, "1I"] \arrow[rr, "1"'] & {} & \mathcal{V}\text{-Act} \\
\mathcal{V}^{\mathrm{op}}\text{-Cat}^{\mathrm{op}} \arrow[u, "S"] \arrow[rr, "1"'] & & \mathcal{V}^{\mathrm{op}}\text{-Cat}^{\mathrm{op}} \arrow[u, "S"']
\end{tikzcd}
\overset{\delta}{\Longrightarrow}
\begin{tikzcd}
& \mathcal{V}\text{-Act} \arrow[dr, "\otimes"] & \\
\mathcal{V}\text{-Act} \arrow[ur, "1I"] \arrow[r, phantom, "\overset{1\iota}{\Rightarrow}"] & \mathcal{V}^{\mathrm{op}}\text{-Cat}^{\mathrm{op}2} \arrow[u, "S^2"] \arrow[dr, "\otimes"] \arrow[r, phantom, "\Swarrow\chi"] \arrow[d, phantom, "\Downarrow\mathsf{r}"] & \mathcal{V}\text{-Act} \\
\mathcal{V}^{\mathrm{op}}\text{-Cat}^{\mathrm{op}} \arrow[u, "S"] \arrow[ur, "1I"] \arrow[rr, "1"'] & {} & \mathcal{V}^{\mathrm{op}}\text{-Cat}^{\mathrm{op}} \arrow[u, "S"']
\end{tikzcd}
$$

in $\hom(\mathcal{V}^{\mathrm{op}}\text{-Cat}^{\mathrm{op}}, \mathcal{V}\text{-Act})$. ■

Observe that as the sense of the pseudonatural equivalence r is opposite to that given in the usual definition of a weak monoidal 2-functor, the domain and codomain of the above modification are mates of the domain and codomain of the modification δ in the usual definition of a weak monoidal 2-functor. We shall to return to this point shortly.

9.4. Lemma. *For all $\mathcal{V}^{\mathrm{op}}$-categories C, the diagram*

$$
\begin{tikzcd}
\mathcal{V} \otimes S(C) \arrow[r, "\mathsf{l}"] \arrow[d, "\iota\otimes 1"'] & S(C) \\
S(I) \otimes S(C) \arrow[r, "\chi"'] & S(I \otimes C) \arrow[u, "S(\mathsf{l})"']
\end{tikzcd}
$$

commutes. ■

It follows from Lemma 9.4 that the invertible modification

$$
\begin{tikzcd}
& \mathcal{V}\text{-Act} \arrow[dr, "\otimes"] & \\
\mathcal{V}\text{-Act} \arrow[ur, "I1"] \arrow[r, phantom, "\overset{\iota 1}{\Rightarrow}"] & \mathcal{V}^{\mathrm{op}}\text{-Cat}^{\mathrm{op}2} \arrow[u, "S^2"] \arrow[dr, "\otimes"] \arrow[r, phantom, "\Swarrow\chi"] \arrow[d, phantom, "\Downarrow\mathsf{l}"] & \mathcal{V}\text{-Act} \\
\mathcal{V}^{\mathrm{op}}\text{-Cat}^{\mathrm{op}} \arrow[u, "S"] \arrow[ur, "I1"] \arrow[rr, "1"'] & {} & \mathcal{V}^{\mathrm{op}}\text{-Cat}^{\mathrm{op}} \arrow[u, "S"']
\end{tikzcd}
\overset{\gamma}{\Longrightarrow}
\begin{tikzcd}
& \mathcal{V}\text{-Act}^2 \arrow[dr, "\otimes"] \arrow[d, phantom, "\Downarrow\mathsf{l}"] & \\
\mathcal{V}\text{-Act} \arrow[ur, "I1"] \arrow[rr, "1"'] & {} & \mathcal{V}\text{-Act} \\
\mathcal{V}^{\mathrm{op}}\text{-Cat}^{\mathrm{op}} \arrow[u, "S"] \arrow[rr, "1"'] & & \mathcal{V}^{\mathrm{op}}\text{-Cat}^{\mathrm{op}} \arrow[u, "S"']
\end{tikzcd}
$$

in the definition of a weak monoidal 2-functor may be taken to be the identity.

We now return to the fact that the sense of the pseudonatural equivalence r is opposite to that given in the usual definition of a monoidal 2-category. As in Section 7, let r^* be the adjoint pseudoinverse of r in the 2-category $\hom(\mathcal{V}\text{-Act}, \mathcal{V}\text{-Act})$ with invertible unit and counit η and ε respectively. Similarly, let r be the *inverse* of the right unit *isomorphism* r^* in the 2-category $\hom(\mathcal{V}^{\mathrm{op}}\text{-Cat}^{\mathrm{op}}, \mathcal{V}^{\mathrm{op}}\text{-Cat}^{\mathrm{op}})$. For any $\mathcal{V}^{\mathrm{op}}$-category C, define δ^* to be the following 2-cell.

$$
\begin{tikzcd}
S(C) \arrow[r, "1"] \arrow[d, "\mathsf{r}^*"'] \arrow[dr, phantom, "\varepsilon\Downarrow"] & S(C) \arrow[r, "S\mathsf{r}^*"] \arrow[d, phantom, "\Downarrow\delta"] & S(C \otimes I) \\
S(C) \otimes \mathcal{V} \arrow[r, "1\otimes\iota"'] \arrow[ur, "\mathsf{r}"] & S(C) \otimes SI \arrow[r, "\chi"'] & S(C \otimes I) \arrow[u, "1"'] \arrow[ul, "S\mathsf{r}"]
\end{tikzcd}
$$

Then δ^* forms the component at C of a modification

$$\begin{tikzcd}
\mathcal{V}\text{-Act} \arrow[rr, "1"] & & \mathcal{V}\text{-Act} & & \mathcal{V}\text{-Act} \arrow[rr, "1"] \arrow[dr, "1I"] \arrow[r, phantom, "\Downarrow r^*"] & & \mathcal{V}\text{-Act} \\
\mathcal{V}^{\mathrm{op}}\text{-Cat}^{\mathrm{op}} \arrow[u, "S"] \arrow[rr, "1"] \arrow[dr, "1I"'] & & \mathcal{V}^{\mathrm{op}}\text{-Cat}^{\mathrm{op}} \arrow[u, "S"'] & \overset{\delta^*}{\Rrightarrow} & \mathcal{V}^{\mathrm{op}}\text{-Cat}^{\mathrm{op}} \arrow[u, "S"] \arrow[dr, "1I"'] \arrow[r, phantom, "\Downarrow 1\iota"] & \mathcal{V}\text{-Act}^2 \arrow[ur, "\otimes"] \arrow[r, phantom, "\overset{\chi}{\Rightarrow}"] & \mathcal{V}^{\mathrm{op}}\text{-Cat}^{\mathrm{op}} \arrow[u, "S"'] \\
& (\mathcal{V}^{\mathrm{op}}\text{-Cat}^{\mathrm{op}})^2 \arrow[ur, "\otimes"'] \arrow[u, phantom, "\Downarrow r^*"] & & & & (\mathcal{V}^{\mathrm{op}}\text{-Cat}^{\mathrm{op}})^2 \arrow[u, "S^2"] \arrow[ur, "\otimes"'] &
\end{tikzcd}$$

and $(S, \chi, \iota, \omega, \gamma, \delta^*)$ constitutes the data for a weak monoidal 2-functor $S\colon \mathcal{V}^{\mathrm{op}}\text{-Cat}^{\mathrm{op}} \to \mathcal{V}\text{-Act}$.

9.5. Proposition. *With data as described above, $S\colon \mathcal{V}^{\mathrm{op}}\text{-Cat}^{\mathrm{op}} \to \mathcal{V}\text{-Act}$ is a weak monoidal 2-functor.*

Proof. The axiom (AHTA1) of [GPS95, Section 3.1] is trivially satisfied as both sides are the identity. Next, one may show using the calculus of mates that the axiom (AHTA2) of [GPS95, Section 3.1] is satisfied if and only if the following equation holds for each pair of $\mathcal{V}^{\mathrm{op}}$-categories C and D,

$$
\begin{tikzcd}
 & S(C)S(D) \arrow[rrr, "\chi"] & & & S(CD) \arrow[ddr, "1"] & \\
 & \arrow[r, phantom, "\Downarrow\delta 1"] & S(CI)S(D) \arrow[ul, "S(\mathsf{r})1"'] \arrow[r, "\chi"'] & S(CI)D \arrow[ur, "S(\mathsf{r}1)"] \arrow[dr, "S(\mathsf{a})"'] & \arrow[r, phantom, "\Downarrow S(\nu)"] & \\
(S(C)\mathcal{V})S(D) \arrow[uur, "\mathsf{r}1"] \arrow[r, "(1\iota)1"] \arrow[ddr, "\mathsf{a}"'] & (S(C)S(I))S(D) \arrow[ur, "\chi 1"'] \arrow[dr, "\mathsf{a}"] & \arrow[r, phantom, "\Downarrow\omega"] & & S(C(ID)) \arrow[r, "S(1\mathsf{l})"] & S(CD) \\
 & \arrow[r, phantom, "\Downarrow"] & S(C)(S(I)S(D)) \arrow[r, "1\chi"'] & S(C)S(ID) \arrow[ur, "\chi"'] \arrow[dr, "1\mathsf{l}"] & \arrow[r, phantom, "\Downarrow"] & \\
 & S(C)(\mathcal{V}S(D)) \arrow[ur, "1(\iota 1)"'] \arrow[rrr, "1\mathsf{l}"'] & \arrow[r, phantom, "\Downarrow S(\mathsf{a})\gamma"] & & S(C)S(D) \arrow[uur, "\chi"'] &
\end{tikzcd}
$$

$$\Vert$$

$$
\begin{tikzcd}
 & S(C)S(D) \arrow[rr, "\chi"] \arrow[ddddrr, "1"] & & S(CD) \arrow[ddr, "1"] & \\
 & & & & \\
(S(C)\mathcal{V})S(D) \arrow[uur, "\mathsf{r}1"] \arrow[ddr, "\mathsf{a}"'] \arrow[r, phantom, "\Downarrow\nu"] & {} & & & S(CD) \\
 & & & & \\
 & S(C)(\mathcal{V}S(D)) \arrow[rr, "1\mathsf{l}"'] & & S(C)S(D) \arrow[uur, "\chi"'] &
\end{tikzcd}
$$

and to show this, it suffices to show that it holds after composing with the universal $S(C)\mathcal{V}S(D) \to (S(C)\otimes\mathcal{V})\otimes S(D)$. This is a calculation with pasting diagrams. ∎

We now equip the weak monoidal 2-functor $S\colon \mathcal{V}^{\mathrm{op}}\text{-}\mathrm{Cat}^{\mathrm{op}} \to \mathcal{V}\text{-}\mathrm{Act}$ with the datum necessary to make it into a weak *symmetric* monoidal 2-functor. Suppose C and D are $\mathcal{V}^{\mathrm{op}}$-categories and M and N are C- and D-comodules respectively. There is $D\otimes C$-comodule whose component at an element (d,c) of $\mathrm{obj}(D\otimes C)$ is $N_d\otimes M_c$ and whose coaction is given by the following composite.

$$N_d\otimes M_c \xrightarrow{\delta\otimes\delta} N_{d'}\otimes D(d,d')\otimes M_{c'}\otimes C(c,c') \longrightarrow N_{d'}\otimes M_{c'}\otimes D(d,d')\otimes C(c,c')$$

There is another $D\otimes C$-comodule whose component at an element (d,c) of $\mathrm{obj}(D\otimes C)$

is $M_c \otimes N_d$ and whose coaction is given by the following composite.

$$M_c \otimes N_d \xrightarrow{\delta\otimes\delta} M_{c'} \otimes C(c,c') \otimes N_{d'} \otimes D(d,d') \longrightarrow M_{c'} \otimes N_{d'} \otimes D(d,d') \otimes C(c,c')$$

The symmetry $\sigma\colon M_c \otimes N_d \to N_d \otimes M_c$ constitutes the component at (d,c) of a $D \otimes C$-comodule morphism, and this comodule morphism is the component at (M,N) of an invertible descent morphism exhibited by the following diagram.

$$\begin{tikzcd}
 & S(C)S(D) \arrow[r, "K"] \arrow[dl, "K"'] \arrow[d, "\sigma"] \arrow[dr, "\otimes"] & S(C)\otimes S(D) \arrow[d, "\chi"] \\
S(C)\otimes S(D) \arrow[d, "\rho"'] & S(D)S(C) \arrow[dl, "K"'] \arrow[dr, "L"] \arrow[r, phantom, "\Leftarrow"] & S(C\otimes D) \arrow[d, "S(\rho)"] \\
S(D)\otimes S(C) \arrow[rr, "\chi"'] & & S(D\otimes C)
\end{tikzcd}$$

The universal property of $S(C)\otimes S(D)$ implies that there exists a unique 2-cell

$$\begin{tikzcd}
S(C)\otimes S(D) \arrow[r, "\chi"] \arrow[d, "\rho"'] \arrow[dr, phantom, "\Leftarrow u"] & S(C\otimes D) \arrow[d, "S(\rho)"] \\
S(D)\otimes S(C) \arrow[r, "\chi"'] & S(D\otimes C)
\end{tikzcd}$$

such that $\mathrm{Desc}(S(C), S(D); u)(K, \kappa)$ is the above descent morphism.

9.6. Lemma. *The 2-cell u is the the component at (C,D) of an invertible modification*

$$\begin{tikzcd}
 & \mathcal{V}\text{-Act}^2 \arrow[dr, "\sigma"] & \\
\mathcal{V}\text{-Act}^2 \arrow[ur, "\sigma"] & (\mathcal{V}^{\mathrm{op}}\text{-Cat}^{\mathrm{op}})^2 \arrow[u, "S^2"] \arrow[dr, "\otimes"] \arrow[d, phantom, "\Downarrow\rho"] \arrow[r, phantom, "\Searrow\chi"] & \mathcal{V}\text{-Act} \\
(\mathcal{V}^{\mathrm{op}}\text{-Cat}^{\mathrm{op}})^2 \arrow[u, "S^2"] \arrow[ur, "\sigma"] \arrow[rr, "\otimes"'] & {} & \mathcal{V}^{\mathrm{op}}\text{-Cat}^{\mathrm{op}} \arrow[u, "S"']
\end{tikzcd}
\overset{u}{\Longrightarrow}
\begin{tikzcd}
 & \mathcal{V}\text{-Act}^2 \arrow[dr, "\otimes"] \arrow[d, phantom, "\Downarrow\rho"] & \\
\mathcal{V}\text{-Act}^2 \arrow[ur, "\sigma"] \arrow[rr, "\otimes"] & {} \arrow[d, phantom, "\Downarrow\chi"] & \mathcal{V}\text{-Act} \\
(\mathcal{V}^{\mathrm{op}}\text{-Cat}^{\mathrm{op}})^2 \arrow[u, "S^2"] \arrow[rr, "\otimes"'] & {} & \mathcal{V}^{\mathrm{op}}\text{-Cat}^{\mathrm{op}} \arrow[u, "S"']
\end{tikzcd}$$

in $\hom(\mathcal{V}^{\mathrm{op}}\text{-Cat}^{\mathrm{op}2}, \mathcal{V}\text{-Act})$. ∎

9.7. Proposition. *With datum described above $S\colon \mathcal{V}^{\mathrm{op}}\text{-Cat}^{\mathrm{op}} \to \mathcal{V}\text{-Act}$ is a symmetric weak monoidal 2-functor.* ∎

The underlying functor of $S\colon \mathcal{V}^{\mathrm{op}}\text{-Cat}^{\mathrm{op}} \to \mathcal{V}\text{-Act}//\mathcal{V}$ factors through the locally fully-faithful 2-functor $\mathcal{V}\text{-Act}//\mathcal{V} \to \mathcal{V}\text{-Act}$; for each $\mathcal{V}^{\mathrm{op}}$-category C and each object a of C, the evaluation functor $\omega_a\colon \mathrm{Comod}(C) \to \mathcal{V}$ is a strict morphism of $\mathcal{V}$-actegories so that the triple $(\mathrm{Comod}(C), \mathrm{obj}(C), \omega)$ is an object of $\mathcal{V}\text{-Act}//\mathcal{V}$. This assignment may be extended to arrows and 2-cells, providing a 2-functor Comod: $\mathcal{V}^{\mathrm{op}}\text{-}\mathrm{Cat}^{\mathrm{op}} \longrightarrow \mathcal{V}\text{-}\mathrm{Act}//\mathcal{V}$ lifting $S\colon \mathcal{V}^{\mathrm{op}}\text{-}\mathrm{Cat}^{\mathrm{op}} \to \mathcal{V}\text{-}\mathrm{Act}$; see [McC99b, Section 3] Let us denote this 2-functor by T. We shall now equip the 2-functor $T\colon \mathcal{V}^{\mathrm{op}}\text{-}\mathrm{Cat}^{\mathrm{op}} \to \mathcal{V}\text{-}\mathrm{Act}//\mathcal{V}$ with the structure of a symmetric weak monoidal 2-functor. Suppose C and D are $\mathcal{V}^{\mathrm{op}}$-actegories. Composing with the universal $S(C)S(D) \to S(C) \otimes S(D)$ shows for all elements c of $\mathrm{obj}(C)$ and all elements d of $\mathrm{obj}(D)$ that the following diagram commutes.

$$\begin{tikzcd}
S(C) \otimes S(D) \arrow[r, "\chi"] \arrow[d, "\omega_c \otimes \omega_d"'] & S(C \otimes D) \arrow[d, "\omega_{(c,d)}"] \\
\mathcal{V} \otimes \mathcal{V} \arrow[r, "|"'] & \mathcal{V}
\end{tikzcd}$$

Thus $(\chi, \mathrm{id}\colon \mathrm{obj}(C \otimes D) \to \mathrm{obj}(C) \times \mathrm{obj}(D), \mathrm{id})\colon (S(C), \mathrm{obj}(C), \omega) \otimes (S(D), \mathrm{obj}(D), \omega) \to (S(C \otimes D), \mathrm{obj}(C \otimes D), \omega)$ is an arrow in $\mathcal{V}\text{-}\mathrm{Act}//\mathcal{V}$.

9.8. Lemma. *The arrow χ is the component at (C, D) of a 2-natural transformation*

$$\begin{tikzcd}
(\mathcal{V}^{\mathrm{op}}\text{-}\mathrm{Cat}^{\mathrm{op}})^2 \arrow[r, "T^2"] \arrow[d, "\otimes"'] \arrow[dr, phantom, "\Leftarrow_{\chi}"] & (\mathcal{V}\text{-}\mathrm{Act}//\mathcal{V})^2 \arrow[d, "\otimes"] \\
\mathcal{V}^{\mathrm{op}}\text{-}\mathrm{Cat}^{\mathrm{op}} \arrow[r, "T"'] & \mathcal{V}\text{-}\mathrm{Act}//\mathcal{V}
\end{tikzcd}$$

in $\mathrm{hom}((\mathcal{V}^{\mathrm{op}}\text{-}\mathrm{Cat}^{\mathrm{op}})^2, \mathcal{V}\text{-}\mathrm{Act}//\mathcal{V})$. ∎

Clearly

$$\begin{tikzcd}
\mathcal{V} \arrow[rr, "\iota"] \arrow[dr, "\mathrm{id}"'] & & S(I) \arrow[dl, "\omega_0"] \\
& \mathcal{V} &
\end{tikzcd}$$

commutes, so $(\iota, 1 \to 1, \mathrm{id})\colon (\mathcal{V}, 1, \mathrm{id}) \to (S(I), \mathrm{obj}(I), \omega)$ is an arrow in $\mathcal{V}\text{-}\mathrm{Act}//\mathcal{V}$, and so provides the (only) component of a 2-natural transformation

$$\begin{tikzcd}
& 1 \arrow[dl, "I"'] \arrow[dr, "\mathcal{V}"] \arrow[d, phantom, "\Leftarrow_{\iota}"] & \\
\mathcal{V}^{\mathrm{op}}\text{-}\mathrm{Cat}^{\mathrm{op}} \arrow[rr, "T"'] & {} & \mathcal{V}\text{-}\mathrm{Act}//\mathcal{V}
\end{tikzcd}$$

in $\mathrm{hom}(1, \mathcal{V}\text{-}\mathrm{Act}//\mathcal{V})$. Now since the forgetful 2-functor $\mathcal{V}\text{-}\mathrm{Act}//\mathcal{V} \to \mathcal{V}\text{-}\mathrm{Act}$ is locally-fully-faithful, one may take the modifications ω, γ, δ and u in the definition of a symmetric weak monoidal 2-functor to be those of $S\colon \mathcal{V}^{\mathrm{op}}\text{-}\mathrm{Cat}^{\mathrm{op}} \to \mathcal{V}\text{-}\mathrm{Act}$.

9.9. Proposition. *With data as described above, $T\colon \mathcal{V}^{\mathrm{op}}\text{-}\mathrm{Cat}^{\mathrm{op}} \to \mathcal{V}\text{-}\mathrm{Act}//\mathcal{V}$ is a symmetric weak monoidal 2-functor.*

Proof. The axioms hold because they hold in the 2-category $\mathcal{V}$-Act and the forgetful 2-functor $\mathcal{V}\text{-}\mathrm{Act}//\mathcal{V} \to \mathcal{V}\text{-}\mathrm{Act}$ is locally-fully-faithful. ∎

10. Reconstruction of balanced coalgebroids

In this section, the symmetric weak monoidal 2-functor Comod: $\mathcal{V}^{\mathrm{op}}$-Cat$^{\mathrm{op}} \to \mathcal{V}$-Act$//\mathcal{V}$ described in Section 9 is shown to be monoidally bi-fully-faithful. This allows the reconstruction of a balanced structure on a $\mathcal{V}^{\mathrm{op}}$-category from the corresponding structure on its representations. In Section 11, this will be used to construct a balanced coalgebra with the property that its category of representations is equivalent the symmetric monoidal category of chain complexes. This coalgebra was first described in [Par81]. We maintain the notation of Section 9.

Recall from [McC99b, Section 4] that an object $(\mathcal{A}, X, \sigma)$ of $\mathcal{V}$-Act$//\mathcal{V}$ is said to be *cotractable* if for each pair x, y of elements of X, the left extension L_{xy} of σ_y along σ_x in $\mathcal{V}$-Act exists. Recall also that for such an object, there is a $\mathcal{V}^{\mathrm{op}}$-category $E(\mathcal{A}, X, \sigma)$ with the X as a set of objects, and for each pair x, y of elements of X, the hom object from x to y is $L_{xy}I$. It is shown [McC99b, Section 4] that for any $\mathcal{V}^{\mathrm{op}}$-category C and any cotractable object $(\mathcal{A}, X, \sigma)$ of $\mathcal{V}$-Act$//\mathcal{V}$, there is an pseudonatural equivalence

$$(\mathcal{V}^{\mathrm{op}}\text{-Cat}^{\mathrm{op}})(E(\mathcal{A}, X, \sigma), C) \simeq (\mathcal{V}\text{-Act}//\mathcal{V})((\mathcal{A}, X, \sigma), T(C)).$$

For a $\mathcal{V}^{\mathrm{op}}$-category C and any object c of C, the morphism of $\mathcal{V}$-actegories $\omega_c\colon S(C) \to \mathcal{V}$ has a right adjoint, so it follows that the the object $(S(C), \mathrm{obj}(C), \omega)$ is cotractable.

10.1. Lemma. *For any two $\mathcal{V}^{\mathrm{op}}$-categories C and D, the object $T(C) \otimes T(D)$ of $\mathcal{V}$-Act$//\mathcal{V}$ is cotractable.*

Proof. For each pair (c, d) in $\mathrm{obj}(C) \times \mathrm{obj}(D)$, let Q_{cd} be the composite $\mathsf{I} \circ (\omega_c \otimes \omega_d)\colon S(C) \otimes S(D) \to \mathcal{V} \otimes \mathcal{V} \to \mathcal{V}$. Recall from Section 8 that the object $T(C) \otimes T(D)$ of $\mathcal{V}$-Act$//\mathcal{V}$ is defined to be $(S(C) \otimes S(D), \mathrm{obj}(C) \times \mathrm{obj}(D), \omega)$ where for each pair $(c, d) \in \mathrm{obj}(C) \times \mathrm{obj}(D)$, the arrow $\omega_{(c,d)}$ is Q_{cd}. Since $\otimes\colon \mathcal{V}\text{-Act} \times \mathcal{V}\text{-Act} \to \mathcal{V}\text{-Act}$ is a 2-functor, it preserves adjunctions, so that the morphism of $\mathcal{V}$-actegories $\omega_c \otimes \omega_d\colon S(C) \otimes S(D) \to \mathcal{V} \otimes \mathcal{V}$ has a right adjoint. Since $\mathsf{I}\colon \mathcal{V} \otimes \mathcal{V} \to \mathcal{V}$ is an equivalence, it has a right adjoint. Since the composite of arrows with right adjoints has a right adjoint, it follows that Q_{cd} has a right adjoint. The result thus follows since the left extension of any arrow along an arrow with a right adjoint exists. ∎

In fact, using the construction of left extensions by right adjoints, it is not difficult to show that

$$\begin{tikzcd}
& S(C) \otimes S(D) \arrow[d, "\chi"] \arrow[ddl, bend right, "Q_{cd}"'] \arrow[ddr, bend left, "Q_{c'd'}"] & \\
& S(C \otimes D) \arrow[dl, "\omega_{(c,d)}"'] \arrow[dr, "\omega_{(c',d')}"] \arrow[d, phantom, "\Leftarrow \delta"] & \\
\mathcal{V} \arrow[rr, "-\otimes C(c,c') \otimes D(d,d')"'] & & \mathcal{V}
\end{tikzcd}$$

exhibits $- \otimes C(c, c') \otimes D(d, d')$ as the left extension of $Q_{c'd'}$ along Q_{cd}. It follows from Lemma 10.1 that $E(T(C) \otimes T(D))$ is well defined. Now let $\psi\colon E(T(C) \otimes T(D)) \to C \otimes D$

be the transpose of the arrow $\chi\colon T(C)\otimes T(D)\to T(C\otimes D)$ under the equivalence

$$(\mathcal{V}^{\mathrm{op}}\text{-}\mathrm{Cat}^{\mathrm{op}})(E(T(C)\otimes T(D)),C\otimes D)\simeq(\mathcal{V}\text{-}\mathrm{Act}//\mathcal{V})(T(C)\otimes T(D),T(C\otimes D)).$$

10.2. Proposition. *The arrow $\psi\colon E(T(C)\otimes T(D))\to C\otimes D$ in $\mathcal{V}^{\mathrm{op}}\text{-}\mathrm{Cat}^{\mathrm{op}}$ is an isomorphism.*

Proof. Write E for $E(T(C)\otimes T(D))$. By the construction in [McC99b], the arrow ψ is the identity on objects, and for all (c,d) and (c',d') in $\mathrm{obj}(C)\times\mathrm{obj}(D)$ the arrow $\psi\colon E((c,d),(c',d'))\to(C\otimes D)((c,d),(c',d'))$ is the unique arrow such that the following equation holds.

$$\begin{tikzcd}[column sep=small]
 & S(C)\otimes S(D) \arrow[dl, bend right, "Q_{cd}"'] \arrow[dr, bend left, "Q_{c'd'}"] \arrow[d, phantom, "\Leftarrow"] & \\
\mathcal{V} \arrow[rr, bend left, "-\otimes E((c,d),(c',d'))"] \arrow[rr, bend right, "-\otimes C(c,c')\otimes C(d,d')"'] \arrow[rr, phantom, "\Downarrow -\otimes\psi"] & {} & \mathcal{V}
\end{tikzcd}
\quad=\quad
\begin{tikzcd}[column sep=small]
 & S(C)\otimes S(D) \arrow[ddl, bend right, "Q_{cd}"'] \arrow[ddr, bend left, "Q_{c'd'}"] \arrow[d, "\chi"] & \\
 & S(C\otimes D) \arrow[dl, "\omega_{(c,d)}"'] \arrow[dr, "\omega_{(c',d')}"] \arrow[d, phantom, "\Leftarrow_{\delta}"] & \\
\mathcal{V} \arrow[rr, bend right, "-\otimes C(c,c')\otimes D(d,d')"'] & {} & \mathcal{V}
\end{tikzcd}$$

But since the right diagram is a left extension, each arrow $\psi\colon E((c,d),(c',d'))\to(C\otimes D)((c,d),(c',d'))$ is an isomorphism, and this completes the proof. ∎

We may now state the main theorem of this article.

10.3. Theorem. *For any symmetric monoidal category $\mathcal{V}$, the symmetric monoidal 2-functor* Comod: $\mathcal{V}^{\mathrm{op}}\text{-}\mathrm{Cat}^{\mathrm{op}}\to\mathcal{V}\text{-}\mathrm{Act}//\mathcal{V}$ *is monoidally bi-fully-faithful.*

Proof. It was shown in [McC99b, Proposition 4.7] that it is bi-fully-faithful as a 2-functor. Since the arrow $\iota\colon\mathcal{V}\to T(\mathcal{I})$ is an isomorphism,

$$\iota^*\colon(\mathcal{V}\text{-}\mathrm{Act}//\mathcal{V})(T(\mathcal{I}),T(D))\to(\mathcal{V}\text{-}\mathrm{Act}//\mathcal{V})(\mathcal{V},T(D))$$

is an isomorphism for all $\mathcal{V}^{\mathrm{op}}$-categories D. Finally, for all $\mathcal{V}^{\mathrm{op}}$-categories B, C and D,

$$\chi^*\colon(\mathcal{V}\text{-}\mathrm{Act}//\mathcal{V})(T(B\otimes C),T(D))\to(\mathcal{V}\text{-}\mathrm{Act}//\mathcal{V})(T(B)\otimes T(C),T(D))$$

is canonically isomorphic to

$$\begin{aligned}(\mathcal{V}\text{-}\mathrm{Act}//\mathcal{V})(T(B\otimes C),T(D)) &\simeq (\mathcal{V}^{\mathrm{op}}\text{-}\mathrm{Cat}^{\mathrm{op}})(B\otimes C,D)\\ &\simeq (\mathcal{V}^{\mathrm{op}}\text{-}\mathrm{Cat}^{\mathrm{op}})(E(T(B)\otimes T(C)),D)\\ &\simeq (\mathcal{V}\text{-}\mathrm{Act}//\mathcal{V})(T(B)\otimes T(C),T(D))\end{aligned}$$

where the first, second and third equivalences are given by bi-fully-faithfullness of T, composition with the isomorphism ψ, and the bi-adjunction $E\dashv T$ respectively. Thus χ^* is an equivalence, and this completes the proof. ∎

10.4. COROLLARY. *The following diagrams are bi-pullbacks in the 2-category of 2-categories, 2-functors and 2-natural transformations.*

$$\begin{tikzcd}
\mathrm{PsMon}(\mathcal{V}^{\mathrm{op}}\text{-}\mathrm{Cat}^{\mathrm{op}}) \arrow[r, "\mathrm{PsMon}(\mathrm{Comod})"] \arrow[d, "U"'] & \mathrm{PsMon}(\mathcal{V}\text{-}\mathrm{Act}//\mathcal{V}) \arrow[d, "U"] \\
\mathcal{V}^{\mathrm{op}}\text{-}\mathrm{Cat}^{\mathrm{op}} \arrow[r, "\mathrm{Comod}"'] & \mathcal{V}\text{-}\mathrm{Act}//\mathcal{V}
\end{tikzcd}$$

$$\begin{tikzcd}
\mathrm{BrPsMon}(\mathcal{V}^{\mathrm{op}}\text{-}\mathrm{Cat}^{\mathrm{op}}) \arrow[r, "\mathrm{BrPsMon}(\mathrm{Comod})"] \arrow[d, "U"'] & \mathrm{BrPsMon}(\mathcal{V}\text{-}\mathrm{Act}//\mathcal{V}) \arrow[d, "U"] \\
\mathcal{V}^{\mathrm{op}}\text{-}\mathrm{Cat}^{\mathrm{op}} \arrow[r, "\mathrm{Comod}"'] & \mathcal{V}\text{-}\mathrm{Act}//\mathcal{V}
\end{tikzcd}$$

$$\begin{tikzcd}
\mathrm{SymPsMon}(\mathcal{V}^{\mathrm{op}}\text{-}\mathrm{Cat}^{\mathrm{op}}) \arrow[r, "\mathrm{SymPsMon}(\mathrm{Comod})"] \arrow[d, "U"'] & \mathrm{SymPsMon}(\mathcal{V}\text{-}\mathrm{Act}//\mathcal{V}) \arrow[d, "U"] \\
\mathcal{V}^{\mathrm{op}}\text{-}\mathrm{Cat}^{\mathrm{op}} \arrow[r, "\mathrm{Comod}"'] & \mathcal{V}\text{-}\mathrm{Act}//\mathcal{V}
\end{tikzcd}$$

$$\begin{tikzcd}
\mathrm{BalPsMon}(\mathcal{V}^{\mathrm{op}}\text{-}\mathrm{Cat}^{\mathrm{op}}) \arrow[r, "\mathrm{BalPsMon}(\mathrm{Comod})"] \arrow[d, "U"'] & \mathrm{BalPsMon}(\mathcal{V}\text{-}\mathrm{Act}//\mathcal{V}) \arrow[d, "U"] \\
\mathcal{V}^{\mathrm{op}}\text{-}\mathrm{Cat}^{\mathrm{op}} \arrow[r, "\mathrm{Comod}"'] & \mathcal{V}\text{-}\mathrm{Act}//\mathcal{V}
\end{tikzcd}$$

Proof. This follows from Theorem 10.3, and Propositions 6.1 and 6.2. ■

Corollary 10.4 is a precise way of stating that there is a bijection between pseudomonoidal, braided pseudomonoidal, symmetric pseudomonoidal and balanced pseudomonoidal structures on a $\mathcal{V}^{\mathrm{op}}$-category and the corresponding structure on its category of representations.

11. An example

In this section, Corollary 10.4 is used to construct a balanced comonoid with the property that its category of representations is equivalent to the symmetric monoidal category of chain complexes.

Let R be a commutative ring, and let $\mathcal{V}$ be the symmetric monoidal category of R-modules. Recall that $\mathrm{Comon}(\mathcal{V})$ is the full sub-2-category of $\mathcal{V}^{\mathrm{op}}$-$\mathrm{Cat}^{\mathrm{op}}$ consisting of those $\mathcal{V}^{\mathrm{op}}$-categories with one object, and that the symmetric monoidal structure on $\mathcal{V}^{\mathrm{op}}$-$\mathrm{Cat}^{\mathrm{op}}$ restricts to $\mathrm{Comon}(\mathcal{V})$.

Recall the $\mathcal{V}$-actegory $\mathcal{C}$ of chain complexes as described in [McC99b, Section 6]. An object of $\mathcal{C}$ shall be denoted by $(M, \partial) = (M_n, \partial_n \colon M_n \to M_{n+1})_{n\in\mathbb{Z}}$. The R-module M_n is called the *n-th component* of (M, ∂) and ∂_n is called the *n-th boundary map*. An arrow

$f\colon (M,\partial) \to (N,\partial)$ of $\mathcal{C}$ is called a *chain map* and it amounts to a $\mathbb{Z}$-indexed family $f_n\colon M_n \to N_n$ of arrows that commute with the boundary maps. The arrow f_n will be call the *n-th component of f*. If V is an R-module, and (M_n,∂_n) is a chain complex, then there is a chain complex $V \otimes (M,\partial)$ whose n-th component is $V \otimes M_n$ and whose n-th boundary operator is $V \otimes \partial_n$. The chain complex $V \otimes (M,\partial)$ is the value at the pair (V,M) of a functor $\otimes\colon \mathcal{V} \times \mathcal{C} \to \mathcal{C}$ making $\mathcal{C}$ into a $\mathcal{V}$-actegory.

There is a functor $L\colon \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ given by the usual tensor product of chain complexes; see [Rot79] for example. In detail, if (M,∂) and (N,∂) are complexes, then the n-th component of $L((M,\partial),(N,\partial))$ is given by $\sum_{i+j=n} M_i \otimes N_j$ and the n-th boundary map is given by $\sum_{i+j=n}(-1)^j \partial_i \otimes N_j + M_i \otimes \partial_j$. For any object V of $\mathcal{V}$, and any natural number n, the canonical isomorphism $\sum_{i+j=n} V \otimes M_i \otimes N_j \to V \otimes (\sum_{i+j=n} M_i \otimes N_j)$ provides the n-th component of an invertible chain map $L(V \otimes ((M,\partial),(N,\partial))) \to V \otimes L((M,\partial),(N,\partial))$ which makes L into a morphism of $\mathcal{V}$-actegories. The symmetry on $\mathcal{V}$ provides an invertible transformation of $\mathcal{V}$-actegories

$$\begin{tikzcd}
 & \mathcal{C}\mathcal{C} \arrow[dr, "L"] & \\
\mathcal{C}\mathcal{V}\mathcal{C} \arrow[ur, "\hat{\otimes}1"] \arrow[dr, "1\otimes"'] \arrow[rr, phantom, "\Downarrow\lambda"] & & \mathcal{C} \\
 & \mathcal{C}\mathcal{C} \arrow[ur, "L"'] &
\end{tikzcd}$$

and (L,λ) is a descent diagram. Thus there exists a unique morphism of $\mathcal{V}$-actegories $\mu\colon \mathcal{C} \otimes \mathcal{C} \to \mathcal{C}$ such that $\mathrm{Desc}(\mathcal{C},\mathcal{C};\mu)(K,\kappa)$ is equal to the above descent diagram. Next, let $\mathcal{I}$ be the chain complex whose n-th component is I if $n = 0$, and 0 otherwise. There is a morphism of $\mathcal{V}$-actegories $\eta\colon \mathcal{V} \to \mathcal{C}$ whose value on an object V of $\mathcal{V}$ is $V \otimes \mathcal{I}$. For each object V of $\mathcal{V}$, and each chain complex (M,∂), the canonical isomorphism $V \otimes M_n \to V \otimes I \otimes M_n \to \sum_{i+j=n} V \otimes \mathcal{I}_i \otimes M_j$ is the n-th component of an invertible chain map which is the component at (V,M) of an invertible descent morphism exhibited by the following diagram.

$$\begin{tikzcd}
 & \mathcal{V}\mathcal{C} \arrow[rr, "K"] \arrow[dl, "K"'] \arrow[dr, "\eta 1"] \arrow[dd, "\otimes"] & & \mathcal{V} \otimes \mathcal{C} \arrow[dr, "\eta\otimes 1"] & \\
\mathcal{V} \otimes \mathcal{C} \arrow[dr, "1"'] & & \mathcal{C}\mathcal{C} \arrow[rr, "K"] \arrow[dr, "L"] \arrow[dl, phantom, "\Swarrow"] & & \mathcal{C} \otimes \mathcal{C} \arrow[dl, "\mu"] \\
 & \mathcal{C} \arrow[rr, "1"'] & & \mathcal{C} &
\end{tikzcd}$$

Thus there exists a unique 2-cell

$$
\begin{tikzcd}
 & \mathcal{C}\otimes\mathcal{C} \arrow[dd, "\mu"] \\
\mathcal{V}\otimes\mathcal{C} \arrow[ur, "\eta\otimes\mathcal{C}"] \arrow[dr, "l"'] \arrow[r, phantom, "\Downarrow l"] & \\
 & \mathcal{C}
\end{tikzcd}
$$

such that Desc$(\mathcal{V},\mathcal{C};l)(K,\kappa)$ is equal to the above descent morphism. Similarly, one may construct the associativity isomorphism a and right unit isomorphism r, and $(\mathcal{C},\mu,\eta,a,r,l)$ is a pseudomonoid in $\mathcal{V}$-Act.

For all $n\in\mathbb{Z}$ and chain complexes (M,∂) and (N,∂), define $s\colon \sum_{i+j=n} M_i\otimes N_j \to \sum_{k+l=n} N_k\otimes M_l$ to be the unique arrow of $\mathcal{V}$ such that the following diagram commutes for all i,j,k,l.

$$
\begin{tikzcd}
\sum_{i+j=n} M_i\otimes N_j \arrow[r, "s"] & \sum_{k+l=n} N_k\otimes M_l \\
M_i\otimes N_j \arrow[u, "i_{jk}"] \arrow[r, "s_{ijkl}"'] & N_k\otimes M_l \arrow[u, "i_{kl}"']
\end{tikzcd}
$$

Here, s_{ijkl} is defined to be $m\otimes n\mapsto(-1)^{ij}n\otimes m$ if $j=k$ and $i=l$, and 0 otherwise; of course s is the usual symmetry on the category of chain complexes. The arrow s is the component at $((M,\partial),(N,\partial))$ of an invertible descent morphism

$$
\begin{tikzcd}
 & \mathcal{C}\otimes\mathcal{C} \arrow[rr, "\rho"] & & \mathcal{C}\otimes\mathcal{C} \arrow[dd, "\mu"] \\
\mathcal{C}\mathcal{C} \arrow[ur, "K"] \arrow[dr, "K"'] \arrow[rr, "\sigma"] & & \mathcal{C}\mathcal{C} \arrow[ur, "K"'] \arrow[dr, "L"] & \\
 & \mathcal{C}\otimes\mathcal{C} \arrow[rr, "\mu"'] \arrow[r, phantom, "\Nearrow"] & & \mathcal{C}
\end{tikzcd}
$$

and so there exists a unique invertible 2-cell

$$
\begin{tikzcd}
\mathcal{C}\otimes\mathcal{C} \arrow[rr, "\rho"] \arrow[dr, "\mu"'] \arrow[rr, phantom, shift right=3ex, "\overset{\gamma}{\Rightarrow}"] & & \mathcal{C}\otimes\mathcal{C} \arrow[dl, "\mu"] \\
 & \mathcal{C} &
\end{tikzcd}
$$

such that Desc$(\mathcal{C},\mathcal{C};\gamma)(K,\kappa)$ is equal to the above descent morphism. The 2-cell γ is a symmetry for the pseudomonoid $\mathcal{C}$, so in particular, Example 5.3 shows that $\mathcal{C}$ is a balanced pseudomonoid in $\mathcal{V}$-Act.

Recall [McC99b, Section 6] that $\mathcal{V}$-Act$/\mathcal{V}$ is the full sub-2-category of $\mathcal{V}$-Act$//\mathcal{V}$ consisting of those objects $(\mathcal{A},X,\sigma)$ where X is the terminal set. Denote such an object by $(\mathcal{A}, \sigma)$. The symmetric monoidal structure on $\mathcal{V}$-Act$//\mathcal{V}$ restricts to one on $\mathcal{V}$-Act$/\mathcal{V}$. We shall show that $\mathcal{C}$ may be equipped with the structure of a symmetric pseudomonoid in $\mathcal{V}$-Act$/\mathcal{V}$. Recall the morphism of $\mathcal{V}$-actegories $G\colon \mathcal{C} \to \mathcal{V}$ described in [McC99b, Section 6]. The underlying functor of G is the functor whose value on an object (M, ∂) of $\mathcal{C}$ is $\sum_n M_n$. For each chain complex (M, ∂) and each object V of $\mathcal{V}$ the canonical isomorphism $\sum_n V \otimes M_n \to V \otimes \sum_n M_n$ is the component at $(V, (M, \partial))$ of the structure isomorphism of G. Thus $(\mathcal{C}, G)$ is an object of $\mathcal{V}$-Act$/\mathcal{V}$.

For complexes (M, ∂) and (N, ∂) define an arrow $\sum_n \sum_{i+j=n} M_i \otimes N_j \to \sum_l M_l \otimes \sum_k N_k$ to be the unique arrow such that the following diagram commutes for all i, j, k, l.

$$\begin{tikzcd}
\sum_n \sum_{i+j=n} M_i \otimes N_j \arrow[r] & \sum_l M_l \otimes \sum_k N_k \\
M_i \otimes N_j \arrow[u, "i_{nij}"] \arrow[r, "\sigma_{ijkl}"'] & M_l \otimes N_k \arrow[u, "i_{kl}"']
\end{tikzcd}$$

Here, σ_{ijkl} is the identity if $i = k$ and $j = l$, and 0 otherwise. This arrow is the component at $((M, \partial), (N, \partial))$ of an invertible descent morphism

$$\begin{tikzcd}
& \mathcal{C}\mathcal{C} \arrow[rr, "K"] \arrow[dl, "K"'] \arrow[dd, "L"] \arrow[dr, "GG"] & & \mathcal{C} \otimes \mathcal{C} \arrow[dr, "G\otimes G"] & \\
\mathcal{C} \otimes \mathcal{C} \arrow[dr, "\mu"'] & & \mathcal{V}\mathcal{V} \arrow[rr, "K"] \arrow[dr, "\otimes"] & & \mathcal{V} \otimes \mathcal{V} \arrow[dl, "|"] \\
& \mathcal{C} \arrow[rr, "G"'] \arrow[ur, phantom, "\Nearrow"] & & \mathcal{V} &
\end{tikzcd}$$

and so there exists a unique invertible 2-cell

$$\begin{tikzcd}
\mathcal{C} \otimes \mathcal{C} \arrow[r, "G\otimes G"] \arrow[d, "\mu"'] \arrow[dr, phantom, "\overset{\varphi}{\Nearrow}"] & \mathcal{V} \otimes \mathcal{V} \arrow[d, "|"] \\
\mathcal{C} \arrow[r, "G"'] & \mathcal{V}
\end{tikzcd}$$

such that Desc$(\mathcal{C}, \mathcal{C}; \varphi)(K, \kappa)$ is equal to the above descent morphism. Thus define an arrow $\mu\colon (\mathcal{C}, G) \otimes (\mathcal{C}, G) \to (\mathcal{C}, G)$ in $\mathcal{V}$-Act$/\mathcal{V}$ to be (μ, φ). Next, observe that for all objects V of $\mathcal{V}$, the canonical arrow $V \to \sum_n V \otimes \mathcal{I}_n$ is an isomorphism, and it is the component of an invertible transformation of $\mathcal{V}$-actegories as in the following diagram.

$$\begin{tikzcd}
\mathcal{V} \arrow[rr, "\eta"] \arrow[dr, "1"'] & \arrow[d, phantom, "\overset{\varphi}{\Leftarrow}"] & \mathcal{C} \arrow[dl, "G"] \\
& \mathcal{V} &
\end{tikzcd}$$

Thus define an arrow $\eta\colon (\mathcal{V}, \mathrm{id}) \to (\mathcal{C}, G)$ in $\mathcal{V}$-Act$/\mathcal{V}$ to be (η, φ). Since the forgetful 2-functor $\mathcal{V}$-Act$/\mathcal{V} \to \mathcal{V}$-Act is locally-fully-faithful, one may take the invertible 2-cells

a, r, l and γ in the definition of a symmetric pseudomonoid to be those described above, and then $\mathcal{C} = ((\mathcal{C}, G), \mu, \eta, a, r, l, \gamma)$ is a symmetric pseudomonoid in $\mathcal{V}$-Act$/\mathcal{V}$.

Let C be the comonoid of [McC99b, Example 6.9]. It was shown in this example that there is a canonical equivalence $F\colon \mathcal{C} \to \mathrm{Comod}(C)$ in $\mathcal{V}$-Act$/\mathcal{V}$. It follows that Comod(C) may be equipped with the structure of a symmetric pseudomonoid in $\mathcal{V}$-Act$/\mathcal{V}$ such that $F\colon \mathcal{C} \to \mathrm{Comod}(C)$ is an equivalence of symmetric pseudomonoids. Thus there are 2-functors $C\colon 1 \to \mathrm{Comon}(\mathcal{V})$ and $\mathrm{Comod}(C)\colon 1 \to \mathrm{SymPsMon}(\mathcal{V}\text{-Act}/\mathcal{V})$ making the following diagram commute.

$$\begin{tikzcd} 1 \arrow[r, "\mathcal{C}"] \arrow[d, "C"'] & \mathrm{SymPsMon}(\mathcal{V}\text{-Act}//\mathcal{V}) \arrow[d, "U"] \\ \mathrm{Comon}(\mathcal{V}) \arrow[r, "\mathrm{Comod}"'] & \mathcal{V}\text{-Act}/\mathcal{V} \end{tikzcd}$$

By Corollary 10.4, there exists an object D of SymPsMon(Comon$(\mathcal{V})$) and isomorphisms $G\colon \mathrm{SymPsMon}(\mathrm{Comod})(D) \cong \mathrm{Comod}(C)$ and $f\colon D \cong C$ in SymPsMon($\mathcal{V}$-Act$/\mathcal{V}$) and Comon$(\mathcal{V})$ respectively, such that $\mathrm{Comod}(f) = G$. The proof of Proposition 6.1 shows that one may take f to be the identity. Thus C is equipped with the structure of a symmetric pseudomonoid in Comon$(\mathcal{V})$ from the corresponding structure on its representations.

Pareigis [Par81] explicitly calculates this structure. Recall from [McC99b, Example 6.9], that the underlying R-module of C is $R\langle x, y, y^{-1}\rangle/(xy + yx, x^2)$, where $R\langle a, b\rangle$ means the non-commutative R-algebra generated by a and b. Observe that C has a basis given by the set $\{y^i x^j | i \in \mathbb{Z}, j \in \mathbb{Z}_2\}$. The comultiplication and counit are algebra homomorphisms and defined by

$$\begin{array}{rclcrcl} \delta(x) & = & x \otimes 1 + y^{-1} \otimes x & \qquad & \varepsilon(x) & = & 0 \\ \delta(y) & = & y \otimes y & & \varepsilon(y) & = & 1 \end{array}$$

respectively. The algebra structure on $R\langle x, y, y^{-1}\rangle/(xy + yx, x^2)$ equips C with the structure of a strict pseudomonoid in $\mathcal{V}^{\mathrm{op}}$-Cat$^{\mathrm{op}}$. Finally, the symmetry on C is the unique R-linear map $\gamma\colon C \otimes C \to R$ such that $\gamma(y^i \otimes y^j) = (-1)^{ij}$ and $\gamma(y^i x \otimes y^j) = \gamma(y^i \otimes y^j x) = \gamma(y^i x \otimes y^j x) = 0$.

A. Braided monoidal bicategories

In this section we provide the definitions of a braided monoidal bicategory, a braided weak monoidal homomorphism, a braided monoidal pseudonatural transformation and a braided monoidal modification.

Let $\mathcal{K} = (\mathcal{K}, \otimes, I, \mathsf{a}, \mathsf{l}, \mathsf{r}, \pi, \nu, \lambda, \rho)$ be a monoidal bicategory. This notation is that of [GPS95, Section 2.2], except that we denote the invertible modification μ of [GPS95, Section 2.2] by ν. We shall often write $\otimes$ as juxtaposition without comment, leave unlabelled the 2-cells expressing the pseudonaturality of a, l and r, and sometimes write as if $\otimes$ were a 2-functor. Let a^* be the adjoint pseudoinverse of a in $\hom(\mathcal{K}^3, \mathcal{K})$, with invertible unit η and counit ε respectively.

A.1. Lemma. *There exist canonical invertible modifications $\pi^*, \pi_1, \pi_2, \pi_3,$ and π_4 whose components at A, B, C, D are exhibited in the following diagrams.*

$$\begin{tikzcd}
& A((BC)D) \arrow[dr, "\mathsf{a}^*"] & \\
A(B(CD)) \arrow[ur, "1\mathsf{a}^*"] \arrow[d, "\mathsf{a}^*"'] \arrow[rr, phantom, "\Swarrow \pi^*"] & & (A(BC))D \arrow[d, "\mathsf{a}^*1"] \\
(AB)(CD) \arrow[rr, "\mathsf{a}^*"'] & & ((AB)C)D
\end{tikzcd}
\qquad
\begin{tikzcd}
& (A(BC))D & \\
A((BC)D) \arrow[ur, "\mathsf{a}^*"] \arrow[d, "1\mathsf{a}"'] \arrow[rr, phantom, "\Searrow \pi_1"] & & ((AB)C)D \arrow[ul, "\mathsf{a}1"'] \\
A(B(CD)) \arrow[rr, "\mathsf{a}^*"'] & & (AB)(CD) \arrow[u, "\mathsf{a}^*"']
\end{tikzcd}$$

$$\begin{tikzcd}
& A(B(CD)) \arrow[dr, "1\mathsf{a}^*"] & \\
(AB)(CD) \arrow[ur, "\mathsf{a}"] \arrow[d, "\mathsf{a}^*"'] \arrow[rr, phantom, "\Downarrow \pi_2"] & & A((BC)D) \arrow[d, "\mathsf{a}^*"] \\
((AB)C)D \arrow[rr, "\mathsf{a}1"'] & & (A(BC))D
\end{tikzcd}
\qquad
\begin{tikzcd}
& ((AB)C)D \arrow[dr, "\mathsf{a}"] & \\
(A(BC))D \arrow[ur, "\mathsf{a}^*1"] \arrow[d, "\mathsf{a}"'] \arrow[rr, phantom, "\Downarrow \pi_3"] & & (AB)(CD) \\
A((BC)D) \arrow[rr, "1\mathsf{a}"'] & & A(B(CD)) \arrow[u, "\mathsf{a}^*"']
\end{tikzcd}$$

$$\begin{tikzcd}
& A((BC)D) & \\
(A(BC))D \arrow[ur, "\mathsf{a}"] \arrow[d, "\mathsf{a}^*1"'] \arrow[rr, phantom, "\Searrow \pi_4"] & & A(B(CD)) \arrow[ul, "1\mathsf{a}^*"'] \\
((AB)C)D \arrow[rr, "\mathsf{a}"'] & & (AB)(CD) \arrow[u, "\mathsf{a}"']
\end{tikzcd}$$

Proof. One uses the calculus of mates and π. ∎

A *braiding* for $\mathcal{K}$ consists of the data (BD1), (BD2) and (BD3) satisfying the axioms (BA1), (BA2), (BA3) and (BA4) that follow.

(BD1) A pseudonatural equivalence

$$\begin{tikzcd}
\mathcal{K}^2 \arrow[rr, "\otimes"] \arrow[dr, "\sigma"'] & \arrow[d, phantom, "\Downarrow \rho"] & \mathcal{K} \\
& \mathcal{K}^2 \arrow[ur, "\otimes"'] &
\end{tikzcd}$$

in $\hom(\mathcal{K}^2, \mathcal{K})$;

(BD2) an invertible R modification in $\hom(\mathcal{K}^3, \mathcal{K})$ whose component at an object (A, B, C) of $\mathcal{K}^3$ is exhibited in the following diagram;

$$\begin{tikzcd}
 & A(BC) \arrow[r, "\rho"] & (BC)A \arrow[dr, "\mathsf{a}"] & \\
(AB)C \arrow[ur, "\mathsf{a}"] \arrow[dr, "\rho C"'] & & & B(CA) \\
 & (BA)C \arrow[r, "\mathsf{a}"'] \arrow[uu, phantom, shift left=8ex, "\Downarrow R"] & B(AC) \arrow[ur, "B\rho"'] &
\end{tikzcd}$$

(BD3) an invertible modification S in $\hom(\mathcal{K}^3, \mathcal{K})$ whose component at an object (A, B, C) of $\mathcal{K}^3$ is exhibited in the following diagram;

$$\begin{tikzcd}
 & (AB)C \arrow[r, "\rho"] & C(AB) \arrow[dr, "\mathsf{a}^*"] & \\
A(BC) \arrow[ur, "\mathsf{a}^*"] \arrow[dr, "A\rho"'] & & & (CA)B \\
 & A(CB) \arrow[r, "\mathsf{a}^*"'] \arrow[uu, phantom, shift left=8ex, "\Downarrow S"] & (AC)B \arrow[ur, "\rho B"'] &
\end{tikzcd}$$

(BA1) for all objects A, B, C and D of $\mathcal{K}$ the following equation holds;

$$
\begin{tikzcd}[column sep=small]
 & & & (A(BC))D \arrow[drr, "\mathsf{a}"] \arrow[d, phantom, "\Downarrow\pi"] & & & \\
 & ((AB)C)D \arrow[urr, "\mathsf{a}1"] \arrow[dr, "\mathsf{a}"] \arrow[ddl, "(\rho 1)1"'] & & {} & & A((BC)D) \arrow[dl, "1\mathsf{a}"] \arrow[ddr, "\rho"] & \\
 & {} \arrow[d, phantom, "\Lleftarrow"] & (AB)(CD) \arrow[rr, "\mathsf{a}"] \arrow[d, "\rho(11)"'] & & A(B(CD)) \arrow[d, "\rho"] & {} \arrow[d, phantom, "\Lleftarrow"] & \\
((BA)C)D \arrow[rr, "\mathsf{a}"] \arrow[dd, "\mathsf{a}1"'] & {} & (BA)(CD) \arrow[d, "\mathsf{a}"] \arrow[r, phantom, "\Lleftarrow_R"] & {} & (B(CD))A \arrow[d, "\mathsf{a}"] & {} & ((BC)D)A \arrow[ll, "\mathsf{a}1"'] \arrow[dd, "\mathsf{a}"] \\
 & {} \arrow[d, phantom, "\Lleftarrow_{\pi^{-1}}"] & B(A(CD)) \arrow[rr, "1\rho"'] & & B((CD)A) \arrow[dd, "1\mathsf{a}"] & {} \arrow[d, phantom, "\Lleftarrow_{\pi}"] & \\
(B(AC))D \arrow[drr, "\mathsf{a}"'] & {} & & & & {} & (BC)(DA) \arrow[dll, "\mathsf{a}"] \\
 & & B((AC)D) \arrow[uu, "1\mathsf{a}"] \arrow[d, "1(\rho 1)"'] \arrow[rr, phantom, "\Lleftarrow_{1R}"] & & B(C(DA)) & & \\
 & & B((CA)D) \arrow[rr, "1\mathsf{a}"'] & & B(C(AD)) \arrow[u, "1(1\rho)"'] & &
\end{tikzcd}
$$

$$\|$$

$$
\begin{tikzcd}[column sep=small]
 & ((AB)C)D \arrow[dl, "(\rho 1)1"'] \arrow[dr, "\mathsf{a}1"] & & A((BC)D) \arrow[dr, "\rho"] & \\
((BA)C)D \arrow[dd, "\mathsf{a}1"'] & {} \arrow[dd, phantom, "\Lleftarrow_R"] & (A(BC))D \arrow[ur, "\mathsf{a}"] \arrow[dd, "\rho 1"'] & {} \arrow[dd, phantom, "\Lleftarrow_R"] & ((BC)D)A \arrow[dd, "\mathsf{a}"] \\
 & & & & \\
(B(AC))D \arrow[dr, "(1\rho)1"'] \arrow[dd, "\mathsf{a}"'] & {} & ((BC)A)D \arrow[dl, "\mathsf{a}1"] \arrow[dr, "\mathsf{a}"] & {} & (BC)(DA) \arrow[dd, "\mathsf{a}"] \\
{} \arrow[r, phantom, "\Lleftarrow"] & (B(CA))D \arrow[dd, "\mathsf{a}"] \arrow[rr, phantom, "\Lleftarrow_{\pi^{-1}}"] & & (BC)(AD) \arrow[ur, "(11)\rho"] \arrow[dd, "\mathsf{a}"] \arrow[r, phantom, "\Lleftarrow"] & {} \\
B((AC)D) \arrow[dr, "1(\rho 1)"'] & & & & B(C(DA)) \\
 & B((CA)D) \arrow[rr, "1\mathsf{a}"'] & & B(C(AD)) \arrow[ur, "1(1\rho)"'] &
\end{tikzcd}
$$

(BA2) for all objects A, B, C and D of $\mathcal{K}$ the following equation holds;

$$
\begin{tikzcd}[column sep=small]
 & & & & (A((BC)D)) \arrow[drr, "\mathsf{a}^*"] & & & \\
 & & A(B(CD)) \arrow[urr, "1\mathsf{a}^*"] \arrow[dr, "\mathsf{a}^*"] \arrow[ddl, "1(1\rho)"'] \arrow[rrrr, phantom, "\Downarrow\pi^*"] & & & & (A(BC))D \arrow[dl, "\mathsf{a}^*1"] \arrow[ddr, "\rho"] & \\
 & & & (AB)(CD) \arrow[rr, "\mathsf{a}^*"] \arrow[d, "(11)\rho"'] & & ((AB)C)D \arrow[d, "\rho"] \arrow[dr, phantom, "\Lleftarrow"] & & \\
 & A(B(DC)) \arrow[rr, "\mathsf{a}^*"] \arrow[dd, "1\mathsf{a}^*"'] \arrow[ur, phantom, "\Lleftarrow"] \arrow[dr, phantom, "\Lleftarrow_{\pi^{*-1}}"] & & (AB)(DC) \arrow[d, "\mathsf{a}^*"] \arrow[rr, phantom, "\Lleftarrow_S"] & & D((AB)C) \arrow[d, "\mathsf{a}^*"'] & & D(A(BC)) \arrow[ll, "1\mathsf{a}^*"'] \arrow[dd, "\mathsf{a}^*"] \arrow[dl, phantom, "\Lleftarrow_{\pi^*}"] \\
 & & & ((AB)D)C \arrow[rr, "\rho 1"'] & & (D(AB))C \arrow[dd, "\mathsf{a}^*1"] & & \\
 & A((BD)C) \arrow[drr, "\mathsf{a}^*"'] & & & & & & (DA)(BC) \arrow[dll, "\mathsf{a}^*"] \\
 & & & (A(BD))C \arrow[uu, "\mathsf{a}^*1"] \arrow[d, "(1\rho)1"'] \arrow[rr, phantom, "\Lleftarrow_{S1}"] & & ((DA)B)C & & \\
 & & & (A(DB))C \arrow[rr, "\mathsf{a}^*1"'] & & ((AD)B)C \arrow[u, "(\rho 1)1"'] & &
\end{tikzcd}
$$

$$\|$$

$$
\begin{tikzcd}[column sep=small]
 & A(B(CD)) \arrow[dl, "1(1\rho)"'] \arrow[dr, "1\mathsf{a}^*"] & & (A(BC))D \arrow[dr, "\rho"] & \\
A(B(DC)) \arrow[d, "1\mathsf{a}^*"'] \arrow[dr, phantom, "\Lleftarrow_S"] & & A((BC)D) \arrow[ur, "\mathsf{a}"] \arrow[d, "1\rho"'] \arrow[dr, phantom, "\Lleftarrow_S"] & & D(A(BC)) \arrow[d, "\mathsf{a}^*"] \\
A((BD)C) \arrow[dr, "1(\rho 1)"'] \arrow[dd, "\mathsf{a}^*"'] & & A(D(BC)) \arrow[dl, "1\mathsf{a}^*"] \arrow[dr, "\mathsf{a}^*"] & & (DA)(BC) \arrow[dd, "\mathsf{a}^*"] \\
 & A((DB)C) \arrow[dd, "\mathsf{a}^*"'] \arrow[rr, phantom, "\Lleftarrow_{\pi^{*-1}}"] & & (AD)(BC) \arrow[ur, "\rho(11)"] \arrow[dd, "\mathsf{a}^*"'] \arrow[dr, phantom, "\Lleftarrow"] & \\
(A(BD))C \arrow[dr, "1(\rho 1)"'] \arrow[ur, phantom, "\Lleftarrow"] & & & & ((DA)B)C \\
 & (A(DB))C \arrow[rr, "\mathsf{a}^*1"'] & & ((AD)B)C \arrow[ur, "(\rho 1)1"'] &
\end{tikzcd}
$$

(BA3) for all objects A, B, C and D of $\mathcal{K}$ the following equation holds;

$$
\begin{tikzcd}
 & & & (AB)(CD) \arrow[rr,"\rho"] & & (CD)(AB) \arrow[dr,"\mathsf{a}"] & & & \\
 & & ((AB)C)D \arrow[ur,"\mathsf{a}"] \arrow[dr,"\rho 1"] \arrow[rrrr, phantom, "\Downarrow R"] & & & & C(D(AB)) \arrow[dr,"1\mathsf{a}^*"] & & \\
 & (A(BC))D \arrow[ur,"\mathsf{a}^*1"] \arrow[dl,"\mathsf{a}"'] \arrow[dd,"(1\rho)1"] & & (C(AB))D \arrow[rr,"\mathsf{a}"] \arrow[dd,"\mathsf{a}^*1"] \arrow[ddrr, phantom, "\Downarrow\pi_4"] & & C((AB)D) \arrow[ur,"1\rho"] \arrow[ddrr, phantom, "\Downarrow 1S"] & & C((DA)B) \arrow[dr,"\mathsf{a}^*"] \arrow[ddr, phantom, "\Downarrow"] & \\
A((BC)D) \arrow[dd,"1(\rho 1)"'] \arrow[rrrr, phantom, "\Downarrow S1"] \arrow[dr, phantom, "\Nwarrow"] & & & & & & & & (C(DA))B \\
 & (A(CB))D \arrow[dl,"\mathsf{a}"] \arrow[dr,"\mathsf{a}^*1"] & & ((CA)B)D \arrow[dr,"\mathsf{a}"] & & C(A(BD)) \arrow[uu,"1\mathsf{a}^*"] \arrow[dr,"1(1\rho)"] & & C((AD)B) \arrow[uu,"1(\rho 1)"] \arrow[dr,"\mathsf{a}^*"] & \\
A((CB)D) \arrow[dr,"1\mathsf{a}"'] \arrow[rrr, phantom, "\Downarrow\pi_3"] & & ((AC)B)D \arrow[ur,"(\rho 1)1"] \arrow[dr,"\mathsf{a}"] \arrow[rr, phantom, "\Downarrow"] & & (CA)(BD) \arrow[ur,"\mathsf{a}"] \arrow[dr,"(11)\rho"] \arrow[rr, phantom, "\Downarrow"] & & C(A(DB)) \arrow[ur,"1\mathsf{a}^*"] \arrow[rr, phantom, "\Downarrow\pi_2"] & & ((CA)D)B \arrow[uu,"(1\rho)1"'] \\
 & A(C(BD)) \arrow[rr,"\mathsf{a}^*"'] & & (AC)(BD) \arrow[ur,"(\rho 1)1"] \arrow[dr,"(11)\rho"'] \arrow[rr, phantom, "\Downarrow"] & & (CA)(DB) \arrow[ur,"\mathsf{a}"] \arrow[rr,"\mathsf{a}^*"'] & & ((CA)D)B \arrow[ur,"\mathsf{a}1"'] & \\
 & & & & (AC)(DB) \arrow[ur,"\rho(11)"'] & & & &
\end{tikzcd}
$$

$$\|$$

$$
\begin{tikzcd}
 & & ((AB)C)D \arrow[drr,"\mathsf{a}"] & & & & & & C(D(AB)) \arrow[dr,"1\mathsf{a}^*"] & & \\
 & (A(BC))D \arrow[ur,"\mathsf{a}^*1"] \arrow[d,"\mathsf{a}"] \arrow[rrr, phantom, "\Downarrow\pi_3"] & & & (AB)(CD) \arrow[rr,"\rho"] & & (CD)(AB) \arrow[urr,"\mathsf{a}"] \arrow[dr,"\mathsf{a}^*"] \arrow[rrr, phantom, "\Swarrow\pi_2"] & & & C((DA)B) \arrow[d,"\mathsf{a}^*"] & \\
 & A((BC)D) \arrow[rr,"1\mathsf{a}"] \arrow[dl,"1(\rho 1)"'] & & A(B(CD)) \arrow[ur,"\mathsf{a}^*"] \arrow[dr,"1\rho"] \arrow[rrrr, phantom, "\Downarrow S"] & & & & ((CD)A)B \arrow[rr,"\mathsf{a}1"] & & (C(DA))B & \\
A((CB)D) \arrow[dr,"1\mathsf{a}"'] \arrow[rrrr, phantom, "\Leftarrow 1R"] & & & & A((CD)B) \arrow[rr,"\mathsf{a}^*"] \arrow[dl,"1\mathsf{a}"] & & (A(CD))B \arrow[ur,"\rho 1"] \arrow[rrrr, phantom, "\Searrow R1"] & & & & (C(AD))B \arrow[ul,"(1\rho)1"'] \\
 & A(C(BD)) \arrow[rr,"1(1\rho)"] \arrow[drr,"\mathsf{a}^*"'] & & A(C(DB)) \arrow[drr,"\mathsf{a}^*"] \arrow[d, phantom, "\Downarrow"] \arrow[rrrr, phantom, "\Downarrow\pi_1"] & & & & ((AC)D)B \arrow[ul,"\mathsf{a}1"] \arrow[rr,"(\rho 1)1"'] \arrow[d, phantom, "\Downarrow"] & & ((CA)D)B \arrow[ur,"\mathsf{a}1"'] & \\
 & & & (AC)(BD) \arrow[rr,"(11)\rho"'] & & (AC)(DB) \arrow[urr,"\mathsf{a}^*"] \arrow[rr,"\rho(11)"'] & & (CA)(DB) \arrow[urr,"\mathsf{a}^*"'] & & &
\end{tikzcd}
$$

(BA4) for all objects A, B and C of $\mathcal{K}$ the following equation holds.

$$
\begin{tikzcd}
 & & B(CA) \arrow[r, "\mathsf{a}^*"] & B(CA) \arrow[dr, "\mathsf{a}^*"] & & \\
 & B(AC) \arrow[ur, "1\rho"] \arrow[dr, "\mathsf{a}^*"] \arrow[r, phantom, "\Downarrow\varepsilon"] & \arrow[rr, phantom, "\Downarrow S^{*-1}"] & & (CB)A & \\
(BA)C \arrow[ur, "\mathsf{a}"] \arrow[rr, "1"'] & & (BA)C \arrow[r, "\rho"] & C(BA) \arrow[ur, "\mathsf{a}^*"] \arrow[rr, "1"] & & C(BA) \arrow[ul, "\mathsf{a}^*"'] \\
(AB)C \arrow[u, "\rho 1"] \arrow[rr, "1"'] \arrow[dr, "\mathsf{a}"'] \arrow[urr, phantom, "\Downarrow"] & & (AB)C \arrow[u, "\rho 1"] \arrow[r, "\rho"'] \arrow[ur, phantom, "\Downarrow"] & C(AB) \arrow[u, "1\rho"] \arrow[rr, "1"] \arrow[dr, "\mathsf{a}^*"] \arrow[urr, phantom, "\Downarrow"] & & C(AB) \arrow[u, "1\rho"'] \\
 & A(BC) \arrow[ur, "\mathsf{a}^*"'] \arrow[dr, "1\rho"] \arrow[r, phantom, "\Downarrow\varepsilon^{-1}"] & \arrow[rr, phantom, "\Downarrow S"] & & (CA)B \arrow[ur, "\mathsf{a}"'] \arrow[ul, phantom, "\Downarrow\eta"] & \\
 & & A(CB) \arrow[r, "\mathsf{a}^*"'] & (AC)B \arrow[ur, "\rho B"'] & &
\end{tikzcd}
$$

$$\Vert$$

$$
\begin{tikzcd}
 & & (BA)C \arrow[r, "\mathsf{a}"] & B(AC) \arrow[dr, "1\rho"] & & \\
 & (AB)C \arrow[ur, "\rho 1"] \arrow[dl, "\mathsf{a}"'] \arrow[dr, "\mathsf{a}"] \arrow[d, phantom, "\Downarrow"] & \arrow[rr, phantom, "\Downarrow R^{-1}"] & & B(CA) \arrow[dr, "\mathsf{a}^*"] \arrow[d, phantom, "\Downarrow\varepsilon"] & \\
A(BC) \arrow[rr, "1"] \arrow[d, "1\rho"'] \arrow[drr, phantom, "\Downarrow"] & & A(BC) \arrow[r, "\rho"'] \arrow[d, "1\rho"] \arrow[dr, phantom, "\Downarrow"] & (BC)A \arrow[ur, "\mathsf{a}"] \arrow[rr, "1"'] \arrow[d, "\rho 1"] \arrow[drr, phantom, "\Downarrow"] & & (BC)A \arrow[d, "\rho 1"] \\
A(CB) \arrow[rr, "1"'] \arrow[dr, "\mathsf{a}^*"'] & & A(CB) \arrow[r, "\rho"] & (CB)A \arrow[rr, "1"] \arrow[dr, "\mathsf{a}"'] & & (CB)A \\
 & (AC)B \arrow[ur, "\mathsf{a}"] \arrow[dr, "\rho 1"'] \arrow[u, phantom, "\Downarrow\eta"] & \arrow[rr, phantom, "\Downarrow R"] & & C(BA) \arrow[ur, "\mathsf{a}^*"'] \arrow[u, phantom, "\Downarrow\varepsilon^{-1}"] & \\
 & & (CA)B \arrow[r, "\mathsf{a}"'] & C(AB) \arrow[ur, "1\rho"'] & &
\end{tikzcd}
$$

A *braided monoidal bicategory* is a monoidal bicategory equipped with a braiding. We now proceed to the definition of a braided weak monoidal homomorphism.

A.2. LEMMA. *Suppose $\mathcal{K}$ and $\mathcal{L}$ are monoidal bicategories, and $T : \mathcal{K} \to \mathcal{L}$ is a weak homomorphism. Then there exists a canonical invertible modification ω^* whose component at an object (A, B, C) of $\mathcal{K}^3$ is exhibited by the following diagram.*

$$\begin{tikzcd}
& TAT(BC) \arrow[r, "\chi"] & T((A(BC)) \arrow[dr, "T\mathsf{a}^*"] & \\
TA(TBTC) \arrow[ur, "1\chi"] \arrow[dr, "\mathsf{a}^*"'] \arrow[rrr, phantom, "\Downarrow\omega^*"] & & & T((AB)C) \\
& (TATB)TC \arrow[r, "\chi 1"'] & T(AB)TC \arrow[ur, "\chi"'] &
\end{tikzcd}$$

∎

Suppose $T \colon \mathcal{K} \to \mathcal{L}$ is a weak homomorphism of monoidal bicategories. Then a *braiding* for T consists of the datum (BHD1) satisfying the axioms (BHA1) and (BHA2) that follow.

(BHD1) An invertible modification

$$\begin{tikzcd}
& \mathcal{L}^2 \arrow[dr, "\otimes"] \arrow[d, phantom, shift left=4, "\Downarrow\chi"] & \\
\mathcal{L}^2 \arrow[ur, "\sigma"] & \mathcal{K}^2 \arrow[u, "T^2"'] \arrow[dr, "\otimes"] \arrow[d, phantom, "\Downarrow\rho"] & \mathcal{L} \\
\mathcal{K}^2 \arrow[u, "T^2"] \arrow[ur, "\sigma"] \arrow[rr, "\otimes"'] & & \mathcal{K} \arrow[u, "T"']
\end{tikzcd}
\quad \overset{u}{\Rrightarrow} \quad
\begin{tikzcd}
& \mathcal{L}^2 \arrow[dr, "\otimes"] \arrow[d, phantom, "\Downarrow\rho"] & \\
\mathcal{L}^2 \arrow[ur, "\sigma"] \arrow[rr, "\otimes"'] & {} \arrow[d, phantom, "\Downarrow\chi"] & \mathcal{L} \\
\mathcal{K}^2 \arrow[u, "T^2"] \arrow[rr, "\otimes"'] & {} & \mathcal{K} \arrow[u, "T"']
\end{tikzcd}$$

in $\hom(\mathcal{K}^2, \mathcal{L})$;

(BHA1) for all objects A, B and C of $\mathcal{K}$ the following equation holds;

$$
\begin{tikzcd}[column sep=small]
 & TA(TBTC) \arrow[rrrr, bend left, "\rho"] \arrow[dr, "1\chi"] & & \arrow[r, phantom, "\Rrightarrow"] & & (TBTC)TA \arrow[dll, "\chi 1"'] \arrow[dr, "\mathsf{a}"] & \\
(TATB)TC \arrow[ur, "\mathsf{a}"] \arrow[dd, "\chi 1"'] \arrow[rr, phantom, "{}^{\omega}\!\Rrightarrow"] & & TAT(BC) \arrow[r, "\rho"] \arrow[dd, "\chi"] \arrow[dr, phantom, "\Rrightarrow"] & T(BC)TA \arrow[dd, "\chi"] \arrow[rrr, phantom, "\overset{\omega}{\Rrightarrow}"] & & & TB(TCTA \arrow[dd, "1\chi"] \\
 & & & & & & \\
T(AB)TC \arrow[dr, "\chi"'] & & T(A(BC)) \arrow[r, "T\rho"] & T((BC)A) \arrow[drr, "T\mathsf{a}"'] & & & TBT(CA) \arrow[dl, "1\chi"] \\
 & T((AB)C) \arrow[ur, "T\mathsf{a}"'] \arrow[dr, "T(\rho 1)"'] & & \arrow[r, phantom, "TR\Uparrow"] & & T(B(CA)) & \\
 & & T((BA)C) \arrow[r, "T\mathsf{a}"'] & T(B(AC)) \arrow[urr, "T(1\rho)"'] & & &
\end{tikzcd}
$$

$$\|$$

$$
\begin{tikzcd}[column sep=small]
 & & TA(TBTC) \arrow[r, "\rho"] & (TBTC)TA \arrow[dr, "\mathsf{a}"] & & \\
 & (TATB)TC \arrow[ur, "\mathsf{a}"] \arrow[ddl, "\chi 1"'] \arrow[dr, "\rho 1"] \arrow[rrr, phantom, "R\Uparrow"] & & & TB(TCTA) \arrow[ddr, "1\chi"] & \\
 & & (TBTA)TC \arrow[r, "\mathsf{a}"] \arrow[dl, "\chi 1"'] \arrow[ll, phantom, "\Rrightarrow"] & TB(TATC) \arrow[ur, "1\rho"] \arrow[dr, "1\chi"'] \arrow[r, phantom, "\Uparrow"] & & \\
T(AB)TC \arrow[dr, "T\rho 1"] \arrow[dd, "\chi"'] & & & & & TBT(CA) \arrow[dd, "\chi"] \\
 & T(BA)TC \arrow[ddr, "\chi"] \arrow[dl, phantom, "\Rrightarrow"] \arrow[rrr, phantom, "{}^{\omega}\!\Rrightarrow"] & & & TBT(AC) \arrow[ur, "1T\rho"] \arrow[ddl, "\chi"'] \arrow[dr, phantom, "\Rrightarrow"] & \\
T((AB)C) \arrow[drr, "T(\rho 1)"'] & & & & & T(B(CA)) \\
 & & T((BA)C) \arrow[r, "T\mathsf{a}"'] & T(B(AC)) \arrow[urr, "T(1\rho)"'] & &
\end{tikzcd}
$$

(BHA2) for all objects A, B and C of $\mathcal{K}$ the following equation holds;

$$
\begin{tikzcd}
 & (TATB)TC \arrow[rrr, bend left, "\rho"] \arrow[dr, "\chi 1"] \arrow[rrr, phantom, "\Rightarrow"] & & & TC(TATB) \arrow[dl, "1\chi"'] \arrow[dr, "\mathsf{a}^*"] & \\
TA(TBTC) \arrow[ur, "\mathsf{a}^*"] \arrow[d, "1\chi"'] \arrow[rr, phantom, "\omega^*\Nearrow"] & & T(AB)TC \arrow[r, "\rho"] \arrow[d, "\chi"] \arrow[dr, phantom, "\Nearrow"] & TCT(AB) \arrow[d, "\chi"] \arrow[rr, phantom, "\overset{\omega^*}{\Rightarrow}"] & & (TCTA)TB \arrow[d, "\chi 1"] \\
TAT(BC) \arrow[dr, "\chi"'] & & T((AB)C) \arrow[r, "T\rho"] & T(C(AB)) \arrow[dr, "T\mathsf{a}^*"'] & & T(CA)TB \arrow[dl, "\chi"] \\
 & T(A(BC)) \arrow[ur, "T\mathsf{a}^*"'] \arrow[dr, "T(1\rho)"'] \arrow[rrr, phantom, "TS^{-1}\Uparrow"] & & & T((CA)B) & \\
 & & T(A(CB)) \arrow[r, "T\mathsf{a}^*"'] & T((AC)B) \arrow[ur, "T(\rho 1)"'] & &
\end{tikzcd}
$$

$$\Vert$$

$$
\begin{tikzcd}
 & & (TATB)TC \arrow[r, "\rho"] & TC(TATB) \arrow[dr, "\mathsf{a}^*"] & & \\
 & TA(TBTC) \arrow[ur, "\mathsf{a}^*"] \arrow[dl, "1\chi"'] \arrow[dr, "1\rho"] \arrow[rrr, phantom, "S\Uparrow"] & & & (TCTA)TB \arrow[dr, "\chi 1"] & \\
 & & TA(TCTB) \arrow[r, "\mathsf{a}^*"] \arrow[dl, "1\chi"'] & (TATC)TB \arrow[ur, "\rho 1"] \arrow[dr, "\chi 1"'] \arrow[r, phantom, "\Uparrow"] & & \\
TAT(BC) \arrow[dr, "1T\rho"] \arrow[ddr, "\chi"'] \arrow[urr, phantom, "\Nearrow"] & & & & & T(CA)TB \arrow[ddl, "\chi"] \\
 & TAT(CB) \arrow[ddr, "\chi"] \arrow[rrr, phantom, "\omega^*\Nearrow"] & & & T(AC)TB \arrow[ur, "T\rho 1"] \arrow[ddl, "\chi"'] \arrow[d, phantom, "\Nearrow"] & \\
 & T(A(BC)) \arrow[dr, "T(1\rho)"'] \arrow[u, phantom, "\Nearrow"] & & & T((CA)B) & \\
 & & T(A(CB)) \arrow[r, "T\mathsf{a}^*"'] & T((AC)B)) \arrow[ur, "T(\rho 1)"'] & &
\end{tikzcd}
$$

A *braided weak monoidal homomorphism* is a weak monoidal homomorphism equipped with a braiding.

Suppose $T, S\colon \mathcal{K} \to \mathcal{L}$ are braided weak moniodal homorphisms, and $\theta\colon T \to S$ is a monoidal pseudonatural transformation. Then θ is called braided when the axiom (BTA1) that follows is satisfied.

(BTA1) The following equation holds for all objects A and B of $\mathcal{K}$.

$$\begin{tikzcd}
 & TBTA \arrow[rr, "\chi"] & & T(BA) \arrow[dr, "\theta"] & \\
TATB \arrow[ur, "\rho"] \arrow[rr, "\chi"] \arrow[dr, "\theta\theta"'] & \arrow[r, phantom, "u\Nwarrow"] & T(AB) \arrow[ur, "T(\rho)"] \arrow[dr, "\theta"] & \Uparrow & S(BA) \\
 & SASB \arrow[rr, "\chi"'] & \arrow[u, phantom, "\Nearrow"] & S(AB) \arrow[ur, "T\rho"'] &
\end{tikzcd}
=
\begin{tikzcd}
 & TBTA \arrow[rr, "\chi"] \arrow[dr, "\theta\theta"] & & T(BA) \arrow[dr, "\theta"] & \\
TATB \arrow[ur, "\rho"] \arrow[dr, "\theta\theta"'] & \rho\Uparrow & SBSA \arrow[rr, "\chi"'] & \Nearrow & S(BA) \\
 & SASB \arrow[ur, "\rho"] \arrow[rr, "\chi"'] & & S(AB) \arrow[ur, "S(\rho)"'] \arrow[u, phantom, "u\Nwarrow"] &
\end{tikzcd}$$

Suppose $\theta, \phi : T \to S$ are braided monoidal pseudonatural transformations. Then a braided monoidal modification $s : \theta \to \phi$ is simply a monoidal modification $s : \theta \to \phi$.

B. Sylleptic monoidal bicategories

In this section we provide the definitions of a sylleptic monoidal bicategory and a sylleptic weak monoidal homomorphism.

Suppose $\mathcal{K}$ is a braided monoidal bicategory. A *syllepsis* for $\mathcal{K}$ is the datum (SD1) satisfying the axioms (SA1) and (SA2) that follow.

(SD1) An invertible modification

$$\begin{tikzcd}
 & \mathcal{K}^2 \arrow[dd, "\otimes"] \\
\mathcal{K}^2 \arrow[ur, bend left, "1"] \arrow[dr, bend right, "\otimes"'] & \\
 & \mathcal{K}
\end{tikzcd}
\overset{v}{\Longrightarrow}
\begin{tikzcd}
 & & \mathcal{K}^2 \arrow[dd, "\otimes"] \\
\mathcal{K}^2 \arrow[urr, bend left, "1"] \arrow[r, "\sigma"] \arrow[drr, bend right, "\otimes"'] & \mathcal{K}^2 \arrow[ur, "\sigma"] \arrow[dr, "\otimes"] \arrow[d, phantom, "\Downarrow\rho"] & \Downarrow\rho \\
 & & \mathcal{K}
\end{tikzcd}$$

in $\hom(\mathcal{K}^2, \mathcal{K})$;

(SA1) for all objects A, B and C of $\mathcal{K}$ the following equation holds;

$$
\begin{tikzcd}
 & A(BC) \arrow[rr, "1"] \arrow[dl, "1\rho"'] \arrow[dr, "\mathsf{a}^*"] & \arrow[d, phantom, "\Downarrow\eta"] & A(BC) & \\
A(CB) \arrow[dd, "\mathsf{a}^*"'] & \arrow[d, phantom, "\Leftarrow_{S}"] & (AB)C \arrow[ur, "\mathsf{a}"] \arrow[drr, "1"] \arrow[dd, "\rho"] & \arrow[r, phantom, "\Rrightarrow"] & \\
 & \, & \arrow[r, phantom, "\Swarrow v"] & \, & (AB)C \arrow[uul, "\mathsf{a}"'] \\
(AC)B \arrow[dr, "\rho 1"'] & & C(AB) \arrow[urr, "\rho"'] \arrow[dr, "1"] \arrow[dl, "\mathsf{a}^*"] & \arrow[r, phantom, "\Downarrow"] & \\
 & (CA)B \arrow[rr, "\mathsf{a}"'] & \arrow[u, phantom, "\Leftarrow_{\eta}"] & C(AB) \arrow[uur, "\rho"'] &
\end{tikzcd}
$$

$$\|$$

$$
\begin{tikzcd}
 & A(BC) \arrow[rr, "1"] \arrow[dr, "1\rho"] \arrow[ddl, "1\rho"'] & \arrow[d, phantom, "\Downarrow v"] & A(BC) & \\
 & \arrow[l, phantom, "\Leftarrow"] & A(CB) \arrow[ur, "1\rho"'] & & (AB)C \arrow[ul, "\mathsf{a}"'] \\
A(CB) \arrow[urr, "1"] \arrow[drr, "\mathsf{a}^*"] \arrow[ddr, "\mathsf{a}^*"'] & & \arrow[l, phantom, "\Searrow\eta"] & \arrow[r, phantom, "\overset{R^{-1}}{\Rightarrow}"] & \, \\
 & \arrow[d, phantom, "\Downarrow"] & (AC)B \arrow[uu, "\mathsf{a}"] & & C(AB) \arrow[uu, "\rho"'] \\
 & (AC)B \arrow[ur, "1"] \arrow[rr, "\rho 1"'] & \arrow[u, phantom, "\overset{v}{\Rightarrow}"] & (CA)B \arrow[ul, "\rho"'] \arrow[ur, "\mathsf{a}"] &
\end{tikzcd}
$$

(SA2) for all objects A, B and C of $\mathcal{K}$ the following equation holds.

$$
\begin{tikzcd}
 & (AB)C \arrow[rr, "1"] \arrow[dl, "\mathsf{a}"'] \arrow[dr, "\rho 1"] \arrow[rr, phantom, shift right=4ex, "\Downarrow v"] & & (AB)C & \\
A(BC) \arrow[dd, "\rho"'] \arrow[ddrr, phantom, "\overset{\Leftarrow}{R^{-1}}"] & & (BA)C \arrow[ur, "\rho 1"] \arrow[drr, "1"] \arrow[dd, "\mathsf{a}"] \arrow[urr, phantom, "\Searrow"] & & \\
 & & & & (BA)C \arrow[uul, "\rho 1"'] \\
(BC)A \arrow[dr, "\mathsf{a}"'] & & B(AC) \arrow[urr, "\mathsf{a}^*"'] \arrow[dl, "1\rho"'] \arrow[dr, "1"] \arrow[ur, phantom, "\Swarrow \varepsilon^{-1}"] \arrow[drr, phantom, "\Downarrow"] & & \\
 & B(CA) \arrow[rr, "1\rho"'] \arrow[rr, phantom, shift left=3ex, "\overset{\Leftarrow}{Bv}"] & & B(AC) \arrow[uur, "\mathsf{a}^*"'] &
\end{tikzcd}
$$

$$\|$$

$$
\begin{tikzcd}
 & (AB)C \arrow[rr, "1"] \arrow[dr, "\mathsf{a}"] \arrow[ddl, "\mathsf{a}"'] \arrow[rr, phantom, shift right=4ex, "\Downarrow \varepsilon^{-1}"] & & (AB)C & \\
 & & A(BC) \arrow[ur, "\mathsf{a}^*"] & & (BA)C \arrow[ul, "\rho C"'] \\
A(BC) \arrow[urr, "1"] \arrow[drr, "\rho"] \arrow[ddr, "\rho"'] \arrow[ur, phantom, "\Swarrow"] \arrow[rr, phantom, "\Searrow v"] \arrow[dr, phantom, "\Swarrow"] & & & \overset{S}{\Rrightarrow} & \\
 & & (BC)A \arrow[uu, "\rho"] & & B(AC) \arrow[uu, "\mathsf{a}^*"'] \\
 & (BC)A \arrow[ur, "1"] \arrow[rr, "\mathsf{a}"'] \arrow[rr, phantom, shift left=3ex, "\overset{\varepsilon^{-1}}{\Rrightarrow}"] & & B(CA) \arrow[ul, "\mathsf{a}^*"'] \arrow[ur, "1\rho"'] &
\end{tikzcd}
$$

A *sylleptic monoidal bicategory* is a braided monoidal bicategory equipped with a syllepsis.

If $\mathcal{K}$ and $\mathcal{L}$ are sylleptic monoidal bicategories, then a *sylleptic weak monoidal homomorphism* $T\colon \mathcal{K} \to \mathcal{L}$ is a braided weak monoidal homomorphism $T\colon \mathcal{K} \to \mathcal{L}$ such that the axiom (SHA1) that follows holds.

(SHA1) For all objects A and B of $\mathcal{K}$ the following equation holds.

$$
\begin{tikzcd}
 & TBTA \arrow[dr, "\rho"] & \\
TATB \arrow[ur, "\rho"] \arrow[rr, "1"] \arrow[d, "\chi"'] \arrow[rr, phantom, shift left=3ex, "v\Uparrow"] \arrow[drr, phantom, "\Rrightarrow"] & & TATB \arrow[d, "\chi"] \\
T(AB) \arrow[rr, "1"] \arrow[rr, bend right, "T1"'] \arrow[rr, phantom, shift right=3ex, "\Uparrow"] & & T(AB)
\end{tikzcd}
=
\begin{tikzcd}
 & & TBTA \arrow[drr, "\rho"] \arrow[d, "\chi"] & & \\
TATB \arrow[urr, "\rho"] \arrow[d, "\chi"'] \arrow[r, phantom, "u\Rrightarrow"] & & T(BA) \arrow[drr, "T\rho"] \arrow[r, phantom, "\overset{u}{\Rrightarrow}"] & & TATB \arrow[d, "\chi"] \\
T(AB) \arrow[urr, "T\rho"] \arrow[rrrr, "T(\rho\rho)"] \arrow[rrrr, bend right, "T1"'] \arrow[rrrr, phantom, shift left=3ex, "\Uparrow"] \arrow[rrrr, phantom, shift right=3ex, "Tv\Uparrow"] & & & & T(AB)
\end{tikzcd}
$$

If T and S are parallel sylleptic weak monoidal homomorphisms, then a s*sylleptic monoidal pseudonatural transformation* from T to S is simply a braided monoidal pseudonatural transformation from T to S. Also, a *sylleptic modification* is simply a braided modification.

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