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ON THE MONADICITY OF CATEGORIES WITH CHOSEN COLIMITS

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ABSTRACT. There is a 2-category $\mathcal{J}$**-Colim** of small categories equipped with a choice of colimit for each diagram whose domain J lies in a given small class $\mathcal{J}$ of small categories, functors strictly preserving such colimits, and natural transformations. The evident forgetful 2-functor from $\mathcal{J}$**-Colim** to the 2-category **Cat** of small categories is known to be monadic. We extend this result by considering not just conical colimits, but general weighted colimits; not just ordinary categories but enriched ones; and not just small classes of colimits but large ones; in this last case we are forced to move from the 2-category $\mathcal{V}$**-Cat** of small $\mathcal{V}$-categories to $\mathcal{V}$-categories with object-set in some larger universe. In each case, the functors preserving the colimits in the usual "up-to-isomorphism" sense are recovered as the *pseudomorphisms* between algebras for the 2-monad in question.

1. Introduction

An important structure on a category A is that of admitting a colimit for each diagram $J \to A$ with domain J in some small class $\mathcal{J}$ of small categories. Consider the 2-category $\mathcal{J}$**-Colim**, an object of which is a small category A (necessarily a $\mathcal{J}$-cocomplete one) together with, for each $J \in \mathcal{J}$, a *choice* of a colimit for each diagram $J \to A$; with the arrows being functors which preserve the chosen colimits strictly, and the 2-cells being arbitrary natural transformations. It is well known that the resulting forgetful 2-functor $U : \mathcal{J}\textbf{-Colim} \to \textbf{Cat}$ is monadic, where **Cat** is the 2-category of small categories. A recent proof of this fact, in the dual case of limits rather than colimits, is found in [12]: the argument rests upon the fact that to give to a category A the structure of an object of $\mathcal{J}$**-Colim** is precisely to give, for each $J \in \mathcal{J}$, a left adjoint to the diagonal functor from A to the functor category $[J, A]$, and that the structure describing such an adjoint consists of everywhere-defined operations and equational axioms, and so is monadic. Note that, in our use here of functors that preserve the colimits strictly, we have not abandoned those that preserve the colimits only to within a canonical isomorphism: for these reappear as the *pseudomorphisms* between algebras for the 2-monad in question. In fact the 2-category of algebras, pseudomorphisms, and algebra 2-cells is equivalent to the 2-category $\mathcal{J}$**-Cocts** of small $\mathcal{J}$-cocomplete categories, $\mathcal{J}$-cocontinuous functors, and natural transformations.

It is of course well-known that $\mathcal{J}$**-Cocts** can be seen as the 2-category of algebras

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for a "pseudomonad" or "doctrine" on **Cat** of the Kock-Zöberlein kind. (The epithet "lax idempotent" was suggested in [11] as a replacement for "Kock-Zöberlein".) The point of the present paper, however, is precisely the generalization, to the most general "class of colimits" context, of the stronger result above about the (strict) 2-monadicity of $\mathcal{J}$-**Colim**, which carries $\mathcal{J}$-**Cocts** along with it when we turn from strict morphisms of algebras to "pseudo" ones. For it is this 2-monadicity — raised as an open question by Kock on page 43 of [14] — that exhibits the giving of chosen colimits of the given class as endowing the category with a purely algebraic structure, and opens the way to applying the results of [3] on 2-monads.

Even in the case of unenriched categories, it has long been clear that a more versatile theory of colimits is obtained by considering not just the classical "conical" colimits, but also the more general *weighted* colimits of [9]. A *weight* is simply a functor $\phi : D_\phi^{\mathrm{op}} \to \mathbf{Set}$ with small domain, and the ϕ-weighted colimit of a functor $s : D_\phi^{\mathrm{op}} \to A$ is a representation of $[D_\phi^{\mathrm{op}}, \mathbf{Set}](\phi, A(s,a)) : A^{\mathrm{op}} \to \mathbf{Set}$, as in

$$A(\phi * s, a) \cong [D_\phi^{\mathrm{op}}, \mathbf{Set}](\phi, A(s,a));$$

often one speaks loosely of the representing object $\phi * s$ as the "ϕ-weighted colimit of s", but strictly speaking the colimit includes the isomorphism above, or equivalently the *unit* $\phi \to \mathcal{K}(s, \phi * s)$. As an example of the importance of weighted colimits, (pointwise) left Kan extensions are directly expressible as such colimits: given $k : D \to B$ and $s : D \to A$ we have $(\mathrm{Lan}_k s)b = B(k,b) * s$. One recovers the conical colimits within this framework as those with weights $\Delta 1 : J^{\mathrm{op}} \to \mathbf{Set}$, where $\Delta 1$ is the functor constant at 1; accordingly one often writes $\mathrm{colim}\, s$ for $\Delta 1 * s$. In fact to give the weighted colimit $\phi * s$ is just to give the conical colimit of $sd_\phi^{\mathrm{op}} : el(\phi)^{\mathrm{op}} \to \mathcal{K}$, where $el(\phi)$ is the category of elements of ϕ and $d_\phi : el(\phi) \to D_\phi^{\mathrm{op}}$ is the projection; and so the existence of *particular* weighted colimits reduces to the existence of *particular* conical colimits. It is not however the case that the existence of all colimits weighted by some class Φ of weights reduces to the existence of all conical colimits of functors with domain in some class $\mathcal{J}$ of small categories. If Φ is a small class of weights we can define, as we did for $\mathcal{J}$-**Colim**, a 2-category Φ-**Colim** of small categories with chosen Φ-colimits, functors preserving these strictly, and natural transformations, together with a forgetful 2-functor $U : \Phi\text{-}\mathbf{Colim} \to \mathbf{Cat}$; but the monadicity of this U will not follow from the known results of the first paragraph. It is however a special case of the monadicity established in Theorem 6.1 below.

Of course there is an elegant and well-developed theory of colimits in the context of categories enriched in a symmetric monoidal closed category $\mathcal{V}$ whose underlying ordinary category $\mathcal{V}_0$ is complete and cocomplete. A weight is now a $\mathcal{V}$-functor $\phi : D_\phi^{\mathrm{op}} \to \mathcal{V}$ with small domain, and the ϕ-weighted colimit of a $\mathcal{V}$-functor $s : D_\phi \to A$ is now defined by a representation

$$A(\phi * s, a) \cong [D_\phi^{\mathrm{op}}, \mathcal{V}](\phi, A(s,a)),$$

this of course now being an isomorphism not in **Set** but in $\mathcal{V}$. We can again express left Kan extensions in terms of weighted colimits, but there is no longer a reduction of $\phi * s$ to an ordinary colimit like $\mathrm{colim}(sd_\phi^{\mathrm{op}})$. If Φ is a small collection of such weights, we have once again a forgetful 2-functor $U : \mathbf{\Phi\text{-}Colim} \to \mathcal{V}\text{-}\mathbf{Cat}$, which will be shown in Theorem 6.1 below to be monadic; here of course $\mathcal{V}\text{-}\mathbf{Cat}$ is the 2-category of small $\mathcal{V}$-categories.

In the various situations above we have always taken the class Φ of weights to be small, which prevents us from dealing with such important cases as that of *all* (small) weights, and so of dealing with cocomplete categories. In the case of a large class of colimits we can no longer restrict attention to small categories — it is well-known, for instance, that the only cocomplete categories which are small are preorders — and so we adopt the following approach. We suppose given once and for all an inaccessible cardinal ∞, whereupon a set is said to be *small* if its cardinality is less than ∞; similarly by a *small* $\mathcal{V}$-category we mean one whose object-set is small, and by a *cocomplete* $\mathcal{V}$-category we mean a $\mathcal{V}$-category admitting ϕ-weighted colimits for all *small* weights ϕ — that is, all $\phi : D_\phi^{\mathrm{op}} \to \mathcal{V}$ with D_ϕ small. In Section 7 we moreover suppose given another inaccessible cardinal ∞' such that the collection of isomorphism classes of small weights, seen as objects of $\mathcal{V}\text{-}\mathbf{Cat}/\mathcal{V}$, has cardinality less than ∞'. We say that a set is *big* if its cardinality is less than ∞', and that a $\mathcal{V}$-category is big if its set of objects is big; and we write $\mathcal{V}\text{-}\mathbf{CAT}$ for the 2-category of big $\mathcal{V}$-categories. Given a class Φ of small weights we now have the 2-category $\mathbf{\Phi\text{-}COLIM}$ of big $\mathcal{V}$-categories with chosen Φ-limits, and the forgetful 2-functor $U : \mathbf{\Phi\text{-}COLIM} \to \mathcal{V}\text{-}\mathbf{CAT}$. In Theorem 7.1 we prove that this forgetful 2-functor U is monadic. In particular, cocomplete ordinary categories (with chosen colimits) are monadic over the 2-category $\mathbf{CAT}$ ($= \mathbf{Set}\text{-}\mathbf{CAT}$) of big but locally-small categories.

We shall shortly outline the method of proof of the monadicity of $U : \mathbf{\Phi\text{-}Colim} \to \mathcal{V}\text{-}\mathbf{Cat}$ for a small class Φ of weights. First, however, we make a comment about this method. The reader may wonder why our proof is not more direct. Can we not describe an object of $\mathbf{\Phi\text{-}Colim}$ as an object of $\mathcal{V}\text{-}\mathbf{Cat}$ provided with a structure — the chosen colimits — given by a small family of operations and equations, each having some small "arity", as was done in the classical case of $\mathcal{J}\text{-}\mathbf{Colim}$? One would have to assert that the appropriate morphism $A(\phi * s, a) \to [D^{\mathrm{op}}, \mathcal{V}](\phi, A(s, a))$ is invertible in $\mathcal{V}$, or equivalently that the induced function $\mathcal{V}_0(G, A(\phi * s, a)) \to \mathcal{V}_0(G, [D^{\mathrm{op}}, \mathcal{V}](\phi, A(s, a)))$ is bijective for each element G of a strongly-generating family $\mathcal{G}$ of objects of the ordinary category $\mathcal{V}_0$ underlying $\mathcal{V}$. Only when $\mathcal{V}_0$ is locally presentable have we a *small* family $\mathcal{G}$ of this kind giving suitable "arities" (in that each $\mathcal{V}_0(G, -)$ preserves α-filtered colimits for some α), opening the way to a direct proof as above. The 2-monadicity of $U : \mathbf{\Phi\text{-}Colim} \to \mathcal{V}\text{-}\mathbf{Cat}$ does, however, hold for every symmetric monoidal closed category $\mathcal{V}$ with the usual properties of being locally small, complete, and cocomplete. Furthermore, even in the case where a direct proof is possible, to describe all the operations and equations, as is required in such a direct proof, is extremely complicated and rather tedious. Finally, the technique we develop to construct a left adjoint will clearly be applicable to other problems: one such application is sketched in Section 8 below. It is for all three of these reasons — the greater generality of the theorem, the reduced technical complexity of the proof, and the applicability of the new technique — that we have adopted the method of proof we now sketch.

The main step is to construct a left adjoint to U; the monadicity of U is then deduced using the well-known theorem of Beck. The construction of the left adjoint involves consideration, alongside $\mathbf{\Phi\text{-}Colim}$, of the 2-category $\mathbf{\Phi\text{-}Cocts}$ of Φ-cocomplete $\mathcal{V}$-categories (those that admit Φ-colimits), Φ-cocontinuous $\mathcal{V}$-functors (those that preserve Φ-colimits in the usual non-strict sense, which merely requires the canonical comparison morphism to be invertible), and $\mathcal{V}$-natural transformations. We write $U' : \mathbf{\Phi\text{-}Cocts} \to \mathcal{V}\text{-}\mathbf{Cat}$ for the forgetful 2-functor, and henceforth use $U_s : \mathbf{\Phi\text{-}Colim} \to \mathcal{V}\text{-}\mathbf{Cat}$ for the 2-functor previously called U, so as to free the letter U for another purpose — it will turn out that U_s denotes a "strict-case" analogue of U. Forgetting the choice of colimits gives, of course, a 2-functor $L : \mathbf{\Phi\text{-}Colim} \to \mathbf{\Phi\text{-}Cocts}$ with $U'L = U_s$. It is convenient to factorize L further into a bijective-on-objects 2-functor J and a fully-faithful one M, as in

$$\mathbf{\Phi\text{-}Colim} \xrightarrow{J} \mathbf{\Phi\text{-}Cocts}_c \xrightarrow{M} \mathbf{\Phi\text{-}Cocts};$$

here the objects of $\mathbf{\Phi\text{-}Cocts}_c$ are, like those of $\mathbf{\Phi\text{-}Colim}$, $\mathcal{V}$-categories with a *choice* of Φ-colimits (the subscript c standing for "choice"), while the morphisms are, as in $\mathbf{\Phi\text{-}Cocts}$, merely the Φ-cocontinuous $\mathcal{V}$-functors, with the 2-cells as before. The 2-functor M is an equivalence in the weaker sense of being fully faithful and essentially surjective on objects, since it is actually surjective on objects; it is therefore an equivalence in the stronger sense of satisfying $MN \cong 1$ and $NM \cong 1$ for some N if we suppose the axiom of choice to hold. In fact our argument does not need the axiom of choice; it suffices to suppose that a definite choice of Φ-colimits has been made in the cocomplete $\mathcal{V}$-category $\mathcal{V}$. We shall now write U for $U'M : \mathbf{\Phi\text{-}Cocts}_c \to \mathcal{V}\text{-}\mathbf{Cat}$, so that $UJ = U_s$; it is this last relation between U_s and U that is central to our argument. Note that the 2-functor J, besides being bijective on objects, is faithful and locally fully faithful.

Recall from [9, Section 5.7] that U' has a left biadjoint Φ', whose value $\Phi'C$ at a small $\mathcal{V}$-category C is the closure of the representables in $[C^{\mathrm{op}}, \mathcal{V}]$ under Φ-colimits, the unit $y : C \to \Phi'C$ being the Yoneda embedding seen as landing in $\Phi'C$. That is to say, composition with y gives an equivalence of categories $\mathbf{\Phi\text{-}Cocts}(\Phi'C, A) \to \mathcal{V}\text{-}\mathbf{Cat}(C, U'A)$. We shall use the existence and properties of $\Phi'C$ to construct a left adjoint to U_s. Of course the more usual name for $\Phi'C$ is ΦC; but here we want to use ΦC for the object of $\mathbf{\Phi\text{-}Cocts}_c$ consisting of the Φ-cocomplete $\mathcal{V}$-category $\Phi'C$ together with some definite choice of Φ-colimits. This is no problem: we are supposing definite Φ-colimits to be chosen in $\mathcal{V}$, and we now choose Φ-colimits in $[C^{\mathrm{op}}, \mathcal{V}]$ to be the pointwise ones; then $\Phi'C$ is, being replete by definition, closed under these in $[C^{\mathrm{op}}, \mathcal{V}]$. So now $y : C \to \Phi C$ is the unit for a left biadjoint to U, composition with y giving an equivalence of categories $\mathbf{\Phi\text{-}Cocts}_c(\Phi C, A) \to \mathcal{V}\text{-}\mathbf{Cat}(C, UA)$.

Given a 2-category $\mathcal{K}$, a (fixed) class $\mathcal{M}$ of arrows in $\mathcal{K}$, and an object A of $\mathcal{K}$, we write $\mathcal{K} \downarrow A$ for the 2-category whose objects are arrows $m : B \to A$ in $\mathcal{M}$ with codomain

A, whose arrows are commutative triangles, as in

$$\begin{tikzcd} B \arrow[rr, "f"] \arrow[dr, "m"'] & & B' \arrow[dl, "m'"] \\ & A & \end{tikzcd},$$

and whose 2-cells from $f : (B, m) \to (B', m')$ to $g : (B, m) \to (B', m')$ are the 2-cells $\alpha : f \to g$ in $\mathcal{K}$ satisfying $m'\alpha = id_m$; thus $\mathcal{K} \downarrow A$ is a full sub-2-category of the slice 2-category $\mathcal{K}/A$.

Now consider the 2-categories $\mathcal{V}$-**Cat** $\downarrow \Phi C$ and Φ-**Colim** $\downarrow \Phi C$ where in each case the class $\mathcal{M}$ of arrows consists of those which are fully faithful as $\mathcal{V}$-functors. We shall see that the evident forgetful 2-functor $U_{\Phi C} : \Phi\text{-}\mathbf{Colim} \downarrow \Phi C \to \mathcal{V}\text{-}\mathbf{Cat} \downarrow \Phi C$ has a left adjoint; the value at $(y : C \to \Phi C)$ of this left adjoint has the form $(w : FC \to \Phi C)$, where FC has chosen Φ-colimits, strictly preserved by w. It is this FC which turns out to be the value at C of the left adjoint F to $U_s : \Phi\text{-}\mathbf{Colim} \to \mathcal{V}\text{-}\mathbf{Cat}$.

In fact this technique clearly applies more generally than to the study of the 2-functor $U_s : \Phi\text{-}\mathbf{Colim} \to \mathcal{V}\text{-}\mathbf{Cat}$. We consider the general context given by a diagram

$$\begin{tikzcd} \mathbb{A}_s \arrow[rr, "J"] \arrow[dr, "U_s"'] & & \mathbb{A} \arrow[dl, "U"] \\ & \mathbb{C} & \end{tikzcd}$$

of 2-categories and 2-functors, with J bijective on objects, faithful, and locally fully faithful, and describe conditions under which a left biadjoint to U may be used to construct a left adjoint to U_s.

The outline of the paper is as follows. We start by recalling the basic facts about the free cocompletions ΦC — we shall now drop the notation $\Phi' C$ — and about the *pseudolimit of an arrow*. We then prove Lemma 4.1 reducing, under suitable hypotheses, the problem of finding a left adjoint to U_s in the abstract context above to that of finding a left adjoint to $U_{\Phi C}$. In Section 5, we show that these hypotheses are satisfied in our case of $\mathbb{A}_s = \Phi$-**Colim**, concluding in Theorem 5.1 that our U_s has a left adjoint. In Section 6 we prove that this U_s is actually monadic, and furthermore that the pseudomorphisms for the resulting 2-monad are precisely the Φ-cocontinuous $\mathcal{V}$-functors. In Section 7 we turn to the case of a large class Φ of weights; and in Section 8 we give some further applications of our main abstract result, Lemma 4.1.

2. Background material on free cocompletions

In this section we recall without proof the main facts about free cocompletions; all can be found in [9, Section 5.7], except the last two, which appear in [2, Section 4].

Let C be an arbitrary $\mathcal{V}$-category. If C is small, we can of course form the $\mathcal{V}$-category $[C^{\mathrm{op}}, \mathcal{V}]$ of $\mathcal{V}$-functors from C^{op} to $\mathcal{V}$; its hom-objects are formed as certain (small) limits in the underlying ordinary category $\mathcal{V}_0$ of $\mathcal{V}$. Specifically, for $\mathcal{V}$-functors f and g from C^{op} to $\mathcal{V}$ (henceforth called *presheaves on* C), we define

$$[C^{\mathrm{op}}, \mathcal{V}](f, g) = \int_{c \in C} [fc, gc]$$

where $[fa, ga]$ denotes the internal hom in $\mathcal{V}$.

If C is not small, the above limit will not exist in general, but it will exist provided that f is *small*, in the sense that it may be formed as the left Kan extension of its restriction to some small full subcategory of C^{op}; small functors have also been called *accessible* [9]. We may now, following Lindner [15], define the $\mathcal{V}$-category $\mathcal{P}C$, whose objects are the small presheaves on C, and whose hom-objects are defined by the limit above. Representable presheaves are easily seen to be small, and so we have a (fully faithful) Yoneda embedding $y_C : C \to \mathcal{P}C$; we often abbreviate y_C to y. Of course $\mathcal{P}C$ is just $[C^{\mathrm{op}}, \mathcal{V}]$ if C is small.

2.1. THEOREM. *$\mathcal{P}C$ is the free cocompletion of C; that is, $\mathcal{P}C$ is cocomplete, and for any cocomplete $\mathcal{V}$-category A, composition with $y : C \to \mathcal{P}C$ induces an equivalence of categories between $\mathcal{V}$-$\mathbf{Cat}(C, A)$ and the full subcategory of $\mathcal{V}$-$\mathbf{Cat}(\mathcal{P}C, A)$ consisting of the cocontinuous $\mathcal{V}$-functors. Furthermore, a $\mathcal{V}$-functor $f : C \to A$ corresponds under this equivalence to the left Kan extension $\mathrm{Lan}_y f$ of f along y.*

We can now use $\mathcal{P}C$ to form free cocompletions with respect to a general class of colimits Φ. Let Φ be a class of colimits; we do not assume Φ to be small, but we do as always assume that for each $\phi : D_\phi^{\mathrm{op}} \to \mathcal{V}$ in Φ, the domain D_ϕ is small.

For any $\mathcal{V}$-category C, by a free Φ-cocompletion of C we mean a Φ-cocomplete $\mathcal{V}$-category $\overline{C}$ and a $\mathcal{V}$-functor $y : C \to \overline{C}$ such that for any Φ-cocomplete $\mathcal{V}$-category A, composition with y induces an equivalence of categories between $\mathcal{V}$-$\mathbf{Cat}(C, A)$ and the full subcategory of $\mathcal{V}$-$\mathbf{Cat}(\overline{C}, A)$ consisting of the Φ-cocontinuous $\mathcal{V}$-functors.

2.2. THEOREM. *If ΦC is the closure in $\mathcal{P}C$ of the representables under Φ-colimits, then the restricted Yoneda embedding $y : C \to \Phi C$ exhibits ΦC as the free Φ-cocompletion of C; furthermore the Φ-cocontinuous $\mathcal{V}$-functor $\Phi C \to A$ corresponding to an arbitrary $\mathcal{V}$-functor $f : C \to A$ is the left Kan extension $\mathrm{Lan}_y f$ of f along y.*

We also need:

2.3. THEOREM. *If Φ is a small class of weights, then ΦC is small if C is so.*

Finally we record:

2.4. THEOREM. *If A admits Φ-colimits, then the essentially unique Φ-cocontinuous $\mathcal{V}$-functor $a : \Phi A \to A$ with $ay_A \cong 1$ is left adjoint to y_A, with the isomorphism $ay_A \cong 1$ as counit.*

2.5. THEOREM. *Consider a $\mathcal{V}$-functor $f : A \to B$ where A and B admit Φ-colimits. Write $a : \Phi A \to A$ and $b : \Phi B \to B$ for the essentially unique Φ-cocontinuous $\mathcal{V}$-functors with $ay_A \cong 1$ and $by_B \cong 1$, and write $g : \Phi A \to \Phi B$ for the essentially unique Φ-cocontinuous $\mathcal{V}$-functor satisfying $gy_A \cong y_B f$. Then f is Φ-cocontinuous if and only if there is a $\mathcal{V}$-natural isomorphism $fa \cong bg$.*

3. Background material on pseudolimits of an arrows

We recall the notion of the pseudolimit of an arrow [10]. Let $\mathcal{K}$ be a 2-category, and $f : A \to B$ an arrow in $\mathcal{K}$. A *pseudolimit of the arrow* f is an invertible 2-cell

$$\begin{tikzcd} & A \arrow[dd, "f"] \\ L \arrow[ur, "u"] \arrow[dr, "v"'] \arrow[r, phantom, "\Uparrow\lambda"] & \\ & B \end{tikzcd}$$

in $\mathcal{K}$ which is *universal*: that is, given arrows $u' : L' \to A$ and $v' : L' \to B$ and an invertible 2-cell $\lambda' : v' \to fu'$, there is a unique arrow $x : L' \to L$ satisfying $\lambda x = \lambda'$ (and so in particular $ux = u'$ and $vx = v'$); and furthermore given 2-cells $\beta : vx \to vy$ and $\alpha : ux \to uy$ satisfying

$$\begin{tikzcd} & L \arrow[rr, "u"] & & A \arrow[dd, "f"] \\ L' \arrow[ur, "y"] \arrow[rr, "y"] \arrow[dr, "x"'] & & L \arrow[ur, "u"] \arrow[dr, "v"] \arrow[r, phantom, "\Uparrow\lambda"] \arrow[d, phantom, "\Uparrow\beta"] & \\ & L \arrow[rr, "v"'] & & B \end{tikzcd} = \begin{tikzcd} & L \arrow[rr, "u"] \arrow[dr, phantom, "\Uparrow\alpha"] & & A \arrow[dd, "f"] \\ L' \arrow[ur, "y"] \arrow[rr, "x"] \arrow[dr, "x"'] & & L \arrow[ur, "u"] \arrow[dr, "v"] \arrow[r, phantom, "\Uparrow\lambda"] & \\ & L \arrow[rr, "v"'] & & B \end{tikzcd}$$

there is a unique 2-cell $\xi : x \to y$ satisfying $u\xi = \alpha$ and $v\xi = \beta$.

In the case $\mathcal{K} = \mathcal{V}$-**Cat**, the pseudolimit of an arrow $f : A \to B$ has a very simple description. An object of L consists of objects a and b of A and B, and an isomorphism $\varphi : b \to fa$ in B; while the $\mathcal{V}$-valued-hom $L((a, b, \varphi), (a', b', \varphi'))$ is given by $A(a, a')$, and composition is defined as in A. We leave to the reader the (obvious) definitions of u, v, and λ; as well as the verification of the universal property.

It is clear from this description that the projection u in the pseudolimit of f is an equivalence, at least in the case $\mathcal{K} = \mathcal{V}$-**Cat**, but in fact this is true in any 2-category $\mathcal{K}$. We could prove this by observing that it is true in **Cat**, and that the notions of equivalence and of the pseudolimit of an arrow are both defined representably, but we choose instead to give an abstract 2-categorical proof:

3.1. Lemma. *If*

$$\begin{tikzcd} & A \arrow[dd, "f"] \\ L \arrow[ur, "u"] \arrow[dr, "v"'] \arrow[r, phantom, "\Uparrow\lambda"] & \\ & B \end{tikzcd}$$

is the pseudolimit of f, then u is an equivalence.

Proof. The arrows $1_A : A \to A$ and $f : A \to B$ and the identity 2-cell from f to $f1_A$ induce a unique arrow $s : A \to L$ satisfying $us = 1$, $vs = f$, and $\lambda s = id_f$. On the other hand the 2-cells $id_u : u1 \to usu$ and $\lambda : v \to fu = vsu$ induce a unique 2-cell $\sigma : 1 \to su$ satisfying $u\sigma = 1$ and $v\sigma = \lambda$, which is invertible since id_u and λ are so. Thus $su \cong 1$ and $us = 1$, giving the desired equivalence. ∎

4. The main lemma

We suppose given 2-categories $\mathbb{A}_s$, $\mathbb{A}$, and $\mathbb{C}$, 2-functors $U : \mathbb{A} \to \mathbb{C}$ and $J : \mathbb{A}_s \to \mathbb{A}$ with J bijective on objects, faithful, and locally fully faithful, and a left biadjoint Φ to U with unit $y : 1 \to U\Phi$. Write $U_s : \mathbb{A}_s \to \mathbb{C}$ for UJ; of course at this level of generality there is no reason why U_s should have a left biadjoint, let alone a left adjoint, but we shall provide conditions under which U_s does indeed have a left adjoint.

A special case is that where $\mathbb{A}_s$ is the 2-category **Φ-Colim** for a small class $\Phi = \{\phi : D_\phi^{\mathrm{op}} \to \mathcal{V}\}$ of weights, $\mathbb{A}$ is **Φ-Cocts**$_c$, $J :$ **Φ-Colim** $\to$ **Φ-Cocts**$_c$ is the inclusion, and $U :$ **Φ-Cocts**$_c \to$ **$\mathcal{V}$-Cat** is the forgetful 2-functor; of course the left biadjoint to U takes A to ΦA. Henceforth this special case will be called the MAIN EXAMPLE.

We further suppose given a class $\mathcal{M}$ of arrows in $\mathbb{C}$, containing the equivalences and the components y_C of y, and satisfying the property that if $mf \cong 1$ and $m \in \mathcal{M}$ then $fm \cong 1$. Since $mf \cong 1$ implies $mfm \cong m1$, the latter condition will be satisfied if $mx \cong my$ implies that $x \cong y$; this in turn is clearly the case if each m is *representably fully faithful* in $\mathbb{C}$, in the sense that the functor $\mathbb{C}(C, m)$ is fully faithful for each C in $\mathbb{C}$. In the MAIN EXAMPLE, $\mathcal{M}$ will be the class of fully faithful $\mathcal{V}$-functors, which are indeed representably fully faithful.

We recall the notation $\mathbb{C} \downarrow U\Phi C$ defined in the Introduction, and we shall also consider $\mathbb{A}_s \downarrow \Phi C$, and the 2-functor $U_{\Phi C} : \mathbb{A}_s \downarrow \Phi C \to \mathbb{C} \downarrow U\Phi C$ induced by U_s; here the chosen arrows in $\mathbb{A}_s$ are those whose image under U_s lie in $\mathcal{M}$, and so we shall often write $f \in \mathcal{M}$ to mean $U_s f \in \mathcal{M}$.

4.1. Lemma. [Main Lemma] *Let $\mathbb{A}_s$, $\mathbb{A}$, $\mathbb{C}$, and $\mathcal{M}$ be as above, and suppose that the following conditions are satisfied:*

(Ax1) Any arrow f in $\mathbb{A}$ for which Uf is an equivalence is itself an equivalence;

(Ax2) $\mathbb{A}$ has, and U preserves, pseudolimits of arrows; furthermore if $u : L \to A$ and $v : L \to B$ are the projections for the pseudolimit of an arrow $f : A \to B$ in $\mathbb{A}$, then u and v lie in $\mathbb{A}_s$, and an arrow $x : C \to L$ in $\mathbb{A}$ lies in $\mathbb{A}_s$ if and only ux and vx do so;

(Ax3) $U_A : \mathbb{A}_s \downarrow A \to \mathbb{C} \downarrow A$ has a left adjoint F_A for every object A in $\mathbb{A}_s$.

Then U_s has a left adjoint F whose value at C is the object FC appearing in $F_{\Phi C}(y : C \to \Phi C) = (w : FC \to \Phi C)$; furthermore Jw is an equivalence.

Proof. We shall often identify objects, arrows, and 2-cells of $\mathbb{A}_s$ with their images under J. The left adjoint $F_{\Phi C}$ sends $y : C \to U_s\Phi C$ to $w : FC \to \Phi C$; writing z for the y-component of the unit for this adjunction, we have a commutative triangle

$$\begin{tikzcd} U_sFC \arrow[rr, "U_sw"] & & U_s\Phi C \\ & C \arrow[ul, "z"] \arrow[ur, "y"'] & \end{tikzcd}$$

in which w and y lie in $\mathcal{M}$. By the universal property of ΦC, there is an (essentially unique) arrow $w' : \Phi C \to FC$ in $\mathbb{A}$ with an isomorphism $\theta : Uw'.y \cong z$ in $\mathbb{C}$. Now $U(Jw.w').y \cong U_sw.z = y$, and so by the universal property of ΦC once again, there is an isomorphism $Jw.w' \cong 1$. Thus $UJw.Uw' \cong 1$ and $UJw \in \mathcal{M}$, and so $Uw'.UJw \cong 1$ by the hypotheses on $\mathcal{M}$; whence UJw is an equivalence in $\mathbb{C}$. Thus Jw is an equivalence in $\mathbb{A}$ by *(Ax1)*, which implies in particular that (JFC, z) "has the same universal property as $(\Phi C, y)$".

We shall now show that $z : C \to U_sFC$ exhibits FC as the free object on C with respect to U_s. Suppose then that an arrow $f : C \to U_sB$ is given. By the universal property of ΦC we can find an arrow $g : \Phi C \to B$ in $\mathbb{A}$ and an isomorphism $\zeta : f \cong Ug.y$. Now form the pseudolimit

$$\begin{tikzcd} & \Phi C \arrow[d, "g"] \\ L \arrow[ur, "u"] \arrow[dr, "v"'] \arrow[r, phantom, "\Uparrow\lambda"] & B \end{tikzcd}$$

in $\mathbb{A}$. Since U preserves this pseudolimit by *(Ax2)*, the isomorphism $\zeta : f \to Ug.y$ induces a unique arrow $h : C \to UL$ in $\mathbb{C}$ satisfying $Uu.h = y$, $Uv.h = f$, and $U\lambda.h = \zeta$.

Now u is an equivalence in $\mathbb{A}$ by Lemma 3.1; thus Uu is an equivalence and so Uu lies in $\mathcal{M}$. Since $y : C \to U\Phi C$ also lies in $\mathcal{M}$, we can see h as an arrow $h : (y : C \to U\Phi C) \to (Uu : UL \to U\Phi C)$ in $\mathbb{C} \downarrow \Phi C$. The adjunction $F_{\Phi C} \dashv U_{\Phi C}$ therefore gives a unique arrow $k : (w : FC \to \Phi C) \to (u : L \to \Phi C)$ in $\mathbb{A}_s \downarrow \Phi C$ satisfying $Uk.z = h$. Now $vk : FC \to B$ lies in $\mathbb{A}_s$ since v and k do so, and $U(vk).z = Uv.Uk.z = Uv.h = f$, giving the existence part of the one-dimensional aspect of the universal property making F left adjoint to U_s.

As for the uniqueness, suppose that $\overline{f} : FC \to B$ in $\mathbb{A}_s$ satisfies $U\overline{f}.z = f$. Since FC shares the universal property of ΦC, the isomorphism $\zeta : U\overline{f}.z = f \cong Ug.y = Ug.Uw.z = U(gw).z$ is of the form $U\overline{\zeta}.z$ for a unique isomorphism $\overline{\zeta} : \overline{f} \to gw$ in $\mathbb{A}$. Now by the definition of pseudolimit, there is a unique arrow $k' : FC \to L$ in $\mathbb{A}$ satisfying $uk' = w$, $vk' = \overline{f}$, and $\lambda k' = \overline{\zeta}$; furthermore uk' and vk' lie in $\mathbb{A}_s$, hence so too by *(Ax2)* does k'. Finally $U\lambda.Uk'.z = U\overline{\zeta}.z = \zeta = U\lambda.h$ and so $Uk'.z = h$, giving $k = k'$; whence $\overline{f} = vk' = vk$, which is the desired uniqueness.

This completes the proof of the one-dimensional aspect of the universal property of the left adjoint; and the two-dimensional aspect is immediate, since FC is already known to share the universal property of ΦC. ∎

5. Verification of the axioms in the MAIN EXAMPLE

In the MAIN EXAMPLE, *(Ax1)* is obviously satisfied, and it is also not hard to see that *(Ax2)* is satisfied: for given the pseudolimit L of a $\mathcal{V}$-functor $f : A \to B$ in the notation of Section 3, and given a $\mathcal{V}$-functor $s : D_\phi \to L$, we may choose the colimit $\phi * s$ in L to be the object $(\phi * us, \phi * vs, \beta : \phi * vs \to f(\phi * us))$ of L, where β is the composite of $\phi * \lambda s : \phi * vs \to \phi * fus$ and the canonical isomorphism $\phi * fus \cong f(\phi * us)$. The straightforward verifications are left to the reader.

The key step therefore involves *(Ax3)*. We begin by observing that the 2-categories $\mathbf{\Phi}$**-Colim** $\downarrow A$ and $\mathcal{V}$**-Cat** $\downarrow A$ are *locally chaotic*, by which is meant that there is a unique 2-cell between any parallel pair of arrows. It will therefore suffice to prove that the ordinary functor $(U_A)_0 : (\mathbf{\Phi}\textbf{-Colim} \downarrow A)_0 \to (\mathcal{V}\textbf{-Cat} \downarrow A)_0$ has a left adjoint, for each $\mathcal{V}$-category A with chosen Φ-colimits; here $(\mathbf{\Phi}\textbf{-Colim} \downarrow A)_0$ and $(\mathcal{V}\textbf{-Cat} \downarrow A)_0$ denote the ordinary categories underlying $\mathbf{\Phi}$**-Colim** $\downarrow A$ and $\mathcal{V}$**-Cat** $\downarrow A$. To do this, we shall construct an endofunctor E of $(\mathcal{V}\textbf{-Cat} \downarrow A)_0$ for which $(\mathbf{\Phi}\textbf{-Colim} \downarrow A)_0$ is the category of algebras, and then prove that free E-algebras exist.

Given an object $(B, m : B \to A)$ of $(\mathcal{V}\textbf{-Cat} \downarrow A)_0$, we write $(EB)_0$ for the set $\{(\phi, s) | \phi \in \Phi, s : D_\phi \to B\}$, seen as a discrete $\mathcal{V}$-category, and $\overline{m}_0 : (EB)_0 \to A$ for the $\mathcal{V}$-functor taking (ϕ, s) to $\phi * ms$. We now factorize $\overline{m}_0$ as

$$(EB)_0 \xrightarrow{e} EB \xhookrightarrow{\overline{m}} A$$

where e is bijective on objects and $\overline{m}$ is fully faithful; recall that the bijective-on-objects $\mathcal{V}$-functors and the fully faithful ones constitute a factorization system on $\mathcal{V}\textbf{-Cat}_0$, the arrows of which we decorate as in the preceding diagram.

Given an arrow $f : (B, m) \to (B', m')$ in $(\mathcal{V}\textbf{-Cat} \downarrow A)_0$ we now write $(Ef)_0 : (EB)_0 \to (EB')_0$ for the $\mathcal{V}$-functor taking (ϕ, s) to (ϕ, fs); since $(\overline{m'})_0(Ef)_0(\phi, s) = (\overline{m'})_0(\phi, fs) = \phi * m'fs = \phi * ms = (\overline{m})_0(\phi, s)$, we have $(\overline{m'})_0(Ef)_0 = \overline{m}_0$, so that there is a unique $\mathcal{V}$-functor Ef rendering commutative

$$\begin{tikzcd}
(EB)_0 \arrow[r, "e"] \arrow[d, "(Ef)_0"'] & EB \arrow[d, "Ef"'] \arrow[dr, hook, "\overline{m}"] & \\
& & A \\
(EB')_0 \arrow[r, "e'"'] & EB' \arrow[ur, hook, "\overline{m'}"'] &
\end{tikzcd}$$

We now define E to be the endofunctor of $(\mathcal{V}\textbf{-Cat} \downarrow A)_0$ taking (B, m) to $(EB, \overline{m})$ and $f : (B, m) \to (B', m')$ to Ef.

To give to an object (B, m) of $(\mathcal{V}\textbf{-Cat} \downarrow A)_0$ the structure of an E-algebra is to give a $\mathcal{V}$-functor $b : EB \to B$ satisfying $mb = \overline{m}$. This determines a $\mathcal{V}$-functor $b_0 = be$ satisfying $mb_0 = mbe = \overline{m}e$; but since e is bijective on objects and m is fully faithful, such a b_0 equally determines b, so that to give to (B, m) the structure of an E-algebra is just to give a $\mathcal{V}$-functor $b_0 : (EB)_0 \to B$ satisfying $mb_0 = (\overline{m})_0$. This, however, is just to give, for each $\phi \in \Phi$ and each $s : D_\phi \to B$, an object $\phi \diamond s$ of B satisfying $m(\phi \diamond s) = \phi * ms$. Finally m is fully faithful and so reflects colimits, whence $\phi \diamond s$ must be a ϕ-weighted colimit of s; thus we see that an E-algebra structure on (B, m) is precisely *a choice in B of Φ-colimits, strictly preserved by m.*

Given two such E-algebras (B, m) and (B', m') with structure maps $b : EB \to B$ and $b' : EB' \to B'$, an arrow $f : (B, m) \to (B', m')$ in $(\mathcal{V}\textbf{-Cat} \downarrow A)_0$ is a morphism of E-algebras just when $fb = b'.Ef$, which happens if and only if $fb_0 = b'_0.(Ef)_0$; that is, if $f(\phi * s) = \phi * fs$ for each ϕ and each s. Thus an arrow in $(\mathcal{V}\textbf{-Cat} \downarrow A)_0$ between E-algebras is a morphism of E-algebras if and only if it (strictly) preserves the chosen Φ-colimits. This now proves that $(\Phi\textbf{-Colim} \downarrow A)_0$ is precisely the category of algebras for E, and $(U_A)_0 : (\Phi\textbf{-Colim} \downarrow A)_0 \to (\mathcal{V}\textbf{-Cat} \downarrow A)_0$ is the forgetful functor.

Thus we have reduced the problem of finding a left adjoint to $U_A : \Phi\textbf{-Colim} \downarrow A \to \mathcal{V}\textbf{-Cat} \downarrow A$ to the problem of showing that free E-algebras exist. This in turn will be the case — see for example [8, Proposition 3.1] and the references contained there — if we can show that $(\mathcal{V}\textbf{-Cat} \downarrow A)_0$ is cocomplete and E preserves α-filtered colimits for some regular cardinal α. Recall that an object c of a cocomplete category $\mathcal{K}$ is said to be *α-presentable* if the representable functor $\mathcal{K}(c, -) : \mathcal{K} \to \textbf{Set}$ preserves α-filtered colimits.

The functor $\mathrm{ob} : \mathcal{V}\textbf{-Cat}_0 \to \textbf{Set}$ taking a $\mathcal{V}$-category to its set of objects induces a functor $\mathrm{ob}_A : (\mathcal{V}\textbf{-Cat} \downarrow A)_0 \to \textbf{Set}/\mathrm{ob}A$, and this latter functor is easily seen to be an equivalence. Furthermore the functor $\partial_0 : \textbf{Set}/\mathrm{ob}A \to \textbf{Set}$ taking a function with codomain $\mathrm{ob}A$ to its domain creates colimits; so too, therefore, does the composite $\mathrm{ob} : (\mathcal{V}\textbf{-Cat} \downarrow A)_0 \to \textbf{Set}$ of ob_A and ∂_0. Thus in particular $(\mathcal{V}\textbf{-Cat} \downarrow A)_0$ is cocomplete, and an object (B, m) is α-presentable if and only if B has fewer than α objects.

Each $\mathcal{V}$-category D_ϕ is small, as is the class Φ, and so we may choose a regular cardinal α in such a way that for every $\phi \in \Phi$, the $\mathcal{V}$-category D_ϕ has fewer than α objects. We shall now show that E preserves α-filtered colimits for this α.

Suppose then that J is an α-filtered category, and $H : J \to (\mathcal{V}\textbf{-Cat} \downarrow A)_0$ is a diagram with colimit

$$\begin{tikzcd} & B \arrow[dr, "m"] & \\ B_j \arrow[ur, "k_j"] \arrow[rr, "m_j"'] & & A, \end{tikzcd}$$

where we have written m_j for the value of H at the object j of J; observe that k_j is necessarily fully faithful. We shall show that

$$\begin{tikzcd} & EB \arrow[dr, "\overline{m}"] & \\ EB_j \arrow[ur, "Ek_j"] \arrow[rr, "\overline{m_j}"'] & & A \end{tikzcd}$$

exhibits $(EB, \overline{m})$ as the colimit in $(\mathcal{V}\textbf{-Cat} \downarrow A)_0$ of EH, using the fact that $\mathrm{ob} : (\mathcal{V}\textbf{-Cat} \downarrow A)_0 \to \textbf{Set}$ creates colimits.

If (ϕ, s) is an object of EB then we can factorize s as

$$D_\phi \xrightarrow{e} E_\phi \xhookrightarrow{n} B$$

with e bijective on objects and n fully faithful. Now E_ϕ has fewer than α objects since D_ϕ does so, thus $(E_\phi, mn : E_\phi \to A)$ is α-presentable in $(\mathcal{V}\text{-}\mathbf{Cat} \downarrow A)_0$. Since J is α-filtered, we may factorize $n : (E_\phi, mn) \to (B, m)$ as $n = k_j n_j$ for some $j \in J$. Thus $s = ne = k_j n_j e$, and so $(\phi, s) = (Ek_j)(\phi, n_j e)$.

On the other hand, if $(\phi_j, s_j) \in EB_j$ and $(\phi_i, s_i) \in EB_i$ satisfy $(Ek_j)(\phi_j, s_j) = (Ek_i)(\phi_i, s_i)$, then $(\phi_j, k_j s_j) = (\phi_i, k_i s_i)$, and so $\phi_j = \phi_i$ and $k_j s_j = k_i s_i$. Since k_j and k_i are fully faithful, we may factorize s_j and s_i as $s_j = n_j e$ and $s_i = n_i e$ with e bijective on objects and with n_i and n_j fully faithful. Writing n for $mk_j n_j$ $(=mk_i n_i)$ and E_ϕ for the domain of n, we now have an α-presentable object $(E_\phi, n : E_\phi \to A)$ of $(\mathcal{V}\text{-}\mathbf{Cat} \downarrow A)_0$ with arrows $n_j : (E_\phi, n) \to (B_j, mk_j)$ and $n_i : (E_\phi, n) \to (B_i, mk_i)$ satisfying $k_j n_j = k_i n_i$. It follows that there exist arrows $\xi : j \to h$ and $\zeta : i \to h$ in J with $(H\xi)n_j = (H\zeta)n_i$, whence finally $(EH\xi)(\phi_j, s_j) = (\phi_j, H\xi.s_j) = (\phi_j, H\xi.n_j e) = (\phi_i, H\zeta.n_i e) = (\phi_i, H\zeta.s_i) = (EH\zeta)(\phi_i, s_i)$, and so E preserves α-filtered colimits as claimed.

This completes the verification of the hypotheses of the main lemma, and we now apply it to obtain:

5.1. Theorem. *For a small class Φ of weights, the forgetful 2-functor $U_s : \Phi\text{-}\mathbf{Colim} \to \mathcal{V}\text{-}\mathbf{Cat}$ has a left adjoint F. Furthermore, if $z_C : C \to U_s FC$ is the unit at C of the adjunction, then z_C exhibits FC as the free Φ-cocompletion of C.*

6. The monadicity of $\mathcal{V}$-categories with chosen colimits

Now that the 2-functor $U_s : \Phi\text{-}\mathbf{Colim} \to \mathcal{V}\text{-}\mathbf{Cat}$ is known to have a left adjoint, we shall prove it to be monadic by using Beck's criterion, in the "strict" form of Mac Lane's account [16, Theorem VI.7.1]; recall from Dubuc's thesis [6, Theorem II.2.1] that this applies unchanged to enriched categories, provided that we then understand "coequalizer" in the enriched sense; so that it applies in particular to our 2-categorical case.

We therefore consider in $\mathcal{V}\text{-}\mathbf{Cat}$ a diagram

$$\begin{tikzcd} A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & B \arrow[l, bend left, "j"] \arrow[r, "q"] & C \arrow[l, bend left, "i"] \end{tikzcd}$$

satisfying the "split fork" conditions $qf = qg$, $qi = 1$, $fj = 1$, $iq = gj$; wherein A and B have chosen Φ-colimits strictly preserved by f and g. We are to prove that C admits a unique choice of Φ-colimits for which q strictly preserves Φ-colimits, and that q is the coequalizer of f and g not only in $\mathcal{V}\text{-}\mathbf{Cat}$ but also in $\Phi\text{-}\mathbf{Colim}$.

There is no difficulty about the uniqueness of the Φ-colimits in C for which q is strictly Φ-cocontinuous: if $\phi : D_\phi^{\mathrm{op}} \to \mathcal{V}$ is in Φ and $s : D_\phi \to C$, we are obliged to define the colimit $\phi * s$ in C by

$$\phi * s = q(\phi * is)$$

with the unit

$$\phi \xrightarrow{\eta} B(is, \phi * is) \xrightarrow{q_*} C(s, q(\phi * is)),$$

where η is the unit for the colimit $\phi * is$ and q_* is here short for $q_{is,\phi * is}$, the effect of q on hom-objects; here and elsewhere, we make use without comment of the fact that $qis = s$.

We must now show that $q(\phi * is)$, with the unit above, is indeed a colimit $\phi * s$ in C. That is, we are to prove the invertibility of the $\mathcal{V}$-natural transformation α whose composite α_c for $c \in C$ is the composite appearing below:

$$\begin{tikzcd}
C(q(\phi * is), c) \arrow[r, "C(s,-)"] & {[D_\phi^{\mathrm{op}}, \mathcal{V}](C(s, q(\phi * is)), C(s, c))} \arrow[d, "{[D_\phi^{\mathrm{op}}, \mathcal{V}](q_*, C(s,c))}"] \\
{[D_\phi^{\mathrm{op}}, \mathcal{V}](\phi, C(s, c))} & {[D_\phi^{\mathrm{op}}, \mathcal{V}](B(is, \phi * is), C(s, c))} \arrow[l, "{[D_\phi^{\mathrm{op}}, \mathcal{V}](\eta, C(s,c))}"]
\end{tikzcd}$$

We assert that α has the inverse β, whose component β_c is the following composite

$$\begin{tikzcd}
{[D_\phi^{\mathrm{op}}, \mathcal{V}](\phi, C(s, c))} \arrow[r, "{[D_\phi^{\mathrm{op}}, \mathcal{V}](\phi, i_*)}"] & {[D_\phi^{\mathrm{op}}, \mathcal{V}](\phi, B(is, ic))} \arrow[d, "\pi"] \\
C(q(\phi * is), c) & B(\phi * is, ic) \arrow[l, "q_*"]
\end{tikzcd}$$

wherein π denotes the natural isomorphism expressing the universal property of the colimit $\phi * is$ in B. To ease the burden of writing these long expressions, let us simplify somewhat by writing $[X, Y]$ for $[D_\phi^{\mathrm{op}}, \mathcal{V}](X, Y)$, and, for instance, writing $[q_*, 1]$ for $[D^{\mathrm{op}}{}_\phi, \mathcal{V}](q_*, C(s, c))$.

We first show that $\beta\alpha = 1$. Since the domain of $\beta\alpha$ is the representable $\mathcal{V}$-functor $C(q(\phi * is), -)$, it suffices by the Yoneda lemma to put $c = q(\phi * is)$ and to show that $\beta_c\alpha_c$ sends the identity $1 \in C_0(q(\phi * is), q(\phi * is))$ to itself, where C_0 is the ordinary category underlying the $\mathcal{V}$-category C. The composite $[1, i_*][\eta, 1][q_*, 1]C(s, -)$ sends $1_{q(\phi * is)}$ to the top leg of the diagram

$$\begin{tikzcd}
\phi \arrow[r, "\eta"] \arrow[d, "\eta"'] & B(is, \phi * is) \arrow[r, "q_*"] & C(s, q(\phi * is)) \arrow[d, "i_*"] \\
B(is, \phi * is) \arrow[rr, "{B(is, \tau)}"'] & & B(is, iq(\phi * is)),
\end{tikzcd}$$

which, since η is the unit for the colimit $\phi * is$, is of the form $B(is, \tau)\eta$ for a unique $\tau : \phi * is \to iq(\phi * is)$, as shown in the diagram. In fact this τ is just what we obtain by applying π to the top leg of the diagram, so that finally we have

$$\beta_c\alpha_c(1) = q\tau : q(\phi * is) \to qiq(\phi * is) = q(\phi * is);$$

and it remains to show that $q\tau = 1$.

In fact τ is precisely the canonical comparison morphism

$$\phi * is = \phi * iqis \xrightarrow{\tau} iq(\phi * is),$$

and since $iq = gj$, this is equally the canonical comparison morphism $\phi * gjis \to gj(\phi * is)$, which in turn is the composite

$$\phi * gjis \xrightarrow{\overline{g}} g(\phi * jis) \xrightarrow{g\overline{j}} gj(\phi * is)\ ,$$

where $\overline{g}$ is the canonical comparison morphism associated to g, and $\overline{j}$ that associated to j. But $\overline{g} = 1$, since g preserves Φ-colimits strictly; so that finally $\tau = g\overline{j}$, and $q\tau = qg\overline{j}$; which is equally $qf\overline{j}$. Now if $\overline{f}$ is the canonical comparison morphism associated to f, that associated to fj is the composite

$$\phi * fjis \xrightarrow{\overline{f}} f(\phi * jis) \xrightarrow{f\overline{j}} fj(\phi * is),$$

which is the identity since $fj = 1$; while $\overline{f} = 1$ since f preserves ϕ-colimits strictly. It follows that $f\overline{j} = 1$, whence $qf\overline{j} = 1$; giving $q\tau = 1$, as desired, and so $\beta\alpha = 1$.

The proof that $\alpha\beta = 1$ must be more direct, since now the domain is no longer representable. A first simplification arises as follows: that the $\mathcal{V}$-functor q respects composition is expressed by the commutativity of

$$\begin{tikzcd}
B(\phi * is, ic) \otimes B(is, \phi * is) \arrow[r] \arrow[d, "q_* \otimes q_*"'] & B(is, ic) \arrow[d, "q_*"] \\
C(q(\phi * is), c) \otimes C(s, q(\phi * is)) \arrow[r] & C(s, c),
\end{tikzcd}$$

whose transpose under the tensor-hom adjunction is the commutative diagram

$$\begin{tikzcd}
B(\phi * is, ic) \arrow[r, "q_*"] \arrow[d, "B(is,-)"'] & C(q(\phi * is), c) \arrow[r, "C(s,-)"] & {[C(s, q(\phi * is)), C(s, c)]} \arrow[d, "{[q_*,1]}"] \\
{[B(is, \phi * is), B(is, ic)]} \arrow[rr, "{[1,q_*]}"'] & & {[B(is, \phi * is), C(s, c)].}
\end{tikzcd}$$

Accordingly, in the composite

$$\alpha_c\beta_c = [\eta, 1][q_*, 1]C(s, -)q_*\pi[1, i_*],$$

we can replace $[q_*, 1]C(s, -)q_*$ by $[1, q_*]B(is, -)$; and since we can then trivially replace $[\eta, 1][1, q_*]$ by $[1, q_*][\eta, 1]$, each being $[\eta, q_*]$, we get

$$\alpha_c\beta_c = [1, q_*][\eta, 1]B(is, -)\pi[1, i_*].$$

Now the composite $[\eta, 1]B(is, -)\pi$ here is just the identity of $[\phi, B(is, ic)]$, since η is by definition the unit of the representation π; moreover $[1, q_*][1, i_*] = 1$ since $qi = 1$; so that we do indeed have $\alpha_c\beta_c = 1$, or $\alpha\beta = 1$.

So $q(\phi * is)$ does provide a ϕ-colimit of s in C; let us write $\phi * s = q(\phi * is)$, it being understood here and below that such an equation asserts the equality not only of the objects but also of the respective units. We are next to show that q does indeed strictly preserve Φ-colimits; but for $r : D_\phi \to B$ we have

$$\begin{aligned} q(\phi * r) &= q(\phi * fjr) \\ &= qf(\phi * jr) \\ &= qg(\phi * jr) \\ &= q(\phi * gjr) \\ &= q(\phi * iqr) \\ &= \phi * qr. \end{aligned}$$

Finally we must show that q is the coequalizer of f and g in $\mathbf{\Phi\text{-}Colim}$. If E is an object of $\mathbf{\Phi\text{-}Colim}$ and $r : C \to E$ is such that rq preserves Φ-colimits strictly, then r too preserves them strictly; for if $s : D_\phi \to C$ we have

$$r(\phi * s) = rq(\phi * is) = \phi * rqis = \phi * rs.$$

Thus q is certainly the coequalizer of f and g in the underlying ordinary category $\mathbf{\Phi\text{-}Colim}_0$ of the 2-category $\mathbf{\Phi\text{-}Colim}$. To show that q is the coequalizer of f and g in $\mathbf{\Phi\text{-}Colim}$, we use the following argument, based on the existence in $\mathbf{\Phi\text{-}Colim}$ of cotensors with the arrow-category $\mathbf{2} = \{0 \to 1\}$; see [9, Section 3.8] for the general principle behind it.

To abbreviate, we temporarily introduce the notation $\mathcal{C}$ for $\mathbf{\Phi\text{-}Colim}$ and $\mathcal{C}_0$ for $\mathbf{\Phi\text{-}Colim}_0$. For an object E of $\mathcal{C}$, consider the functor category $[\mathbf{2}, E]$; this becomes an object of $\mathcal{C}$ when we give it the Φ-colimits formed pointwise from those in E, and then the evaluations $\partial_0, \partial_1 : [\mathbf{2}, E] \to E$ strictly preserve Φ-colimits. To give a functor $h : X \to [\mathbf{2}, E]$ is to give two functors $h_0, h_1 : X \to E$ and a natural transformation $\lambda : h_0 \to h_1$; and h is a morphism in $\mathcal{C}$ precisely when each of h_0 and h_1 is so. Accordingly we have a natural bijection

$$\mathcal{C}_0(X, [\mathbf{2}, E]) \cong \mathbf{Cat}_0(\mathbf{2}, \mathcal{C}(X, E)), \tag{$*$}$$

where $\mathbf{Cat}_0$ is the ordinary category underlying the 2-category $\mathbf{Cat}$.

Now since q is the coequalizer in $\mathcal{C}_0$, we have in $\mathbf{Set}$ the equalizer

$$\mathcal{C}_0(C, [\mathbf{2}, E]) \xrightarrow{\mathcal{C}_0(q,1)} \mathcal{C}_0(B, [\mathbf{2}, E]) \underset{\mathcal{C}_0(g,1)}{\overset{\mathcal{C}_0(f,1)}{\rightrightarrows}} \mathcal{C}_0(A, [\mathbf{2}, E]),$$

which by $(*)$ is also an equalizer

$$\mathbf{Cat}_0(\mathbb{2},\mathcal{C}(C,E)) \xrightarrow{\mathbf{Cat}_0(1,\mathcal{C}(q,1))} \mathbf{Cat}_0(\mathbb{2},\mathcal{C}(B,E)) \underset{\mathbf{Cat}_0(1,\mathcal{C}(g,1))}{\overset{\mathbf{Cat}_0(1,\mathcal{C}(f,1))}{\rightrightarrows}} \mathbf{Cat}_0(\mathbb{2},\mathcal{C}(A,E)).$$

But $\mathbf{Cat}_0(\mathbb{2},-) : \mathbf{Cat}_0 \to \mathbf{Set}$ preserves equalizers, and also reflects them, since it reflects isomorphisms; and so

$$\mathcal{C}(C,E) \xrightarrow{\mathcal{C}(q,1)} \mathcal{C}(B,E) \underset{\mathcal{C}(g,1)}{\overset{\mathcal{C}(f,1)}{\rightrightarrows}} \mathcal{C}(A,E)$$

is an equalizer in $\mathbf{Cat}_0$ for all C; which is what it means for q to be the coequalizer of f and g in $\mathcal{C}$. This completes the proof of:

6.1. Theorem. *If Φ is a small class of weights, then the 2-functor $U_s : \mathbf{\Phi\text{-}Colim} \to \mathcal{V}\text{-}\mathbf{Cat}$ is monadic.*

We have used $z : 1 \to U_s F$ for the unit of the (2-)adjunction $F \dashv U_s$; let us use $e : FU_s \to 1$ for the counit. This adjunction determines on $\mathcal{V}\text{-}\mathbf{Cat}$ the 2-monad $T = (T, z, m)$, where $T = U_s F$, where $z : 1 \to T$ is the $z : 1 \to U_s F$ above, and where $m : T^2 \to T$ is $U_s e F : U_s F U_s F \to U_s F$. Recall from the study [3] of 2-monads that it is convenient to use the name T-Alg$_s$ for the Eilenberg-Moore 2-category given by the T-algebras $(A, a : TA \to A)$, the strict T-algebra morphisms (or just strict T-morphisms) $f : (A,a) \to (B,b)$, which are those $f : A \to B$ with $fa = b.Tf$, and the T-transformations $\alpha : f \to g : (A,a) \to (B,b)$, which are those for which $\alpha a = b.T\alpha$; this leaves available the name T-Alg for the related 2-category whose arrows from (A,a) to (B,b), called *T-algebra morphisms* or just *T-morphisms*, are pairs $(f,\overline{f})$ where $f : A \to B$ is a $\mathcal{V}$-functor and $\overline{f}$ is an invertible $\mathcal{V}$-natural transformation $\overline{f} : b.Tf \to fa$ satisfying the usual two coherence conditions as in [3].

Theorem 6.1 asserts that the canonical comparison 2-functor $K : \mathbf{\Phi\text{-}Colim} \to T$-Alg$_s$ is invertible. This K sends the object A of $\mathbf{\Phi\text{-}Colim}$ to the T-algebra $(U_s A, U_s eA : U_s F U_s A \to U_s A)$. Here $U_s A$ is A itself, seen as merely a $\mathcal{V}$-category, and $U_s eA$ is then an action $a : TA \to A$. Since K is invertible, to give an action $a : TA \to A$ of T on A is equally to give a choice of Φ-colimits in A — which is of course possible only when A is Φ-cocomplete. Again, K sends a morphism $f : A \to B$ in $\mathbf{\Phi\text{-}Colim}$ to this same f seen as a strict T-morphism $f : (A,a) \to (B,b)$; thus, since K is invertible, a $\mathcal{V}$-functor $f : A \to B$ preserves the chosen Φ-colimits strictly precisely when it satisfies $fa = b.Tf$. As a final consequence of the invertibility of K, every $\mathcal{V}$-natural $\alpha : f \to g$ between morphisms in $\mathbf{\Phi\text{-}Colim}$ is automatically a T-transformation. It follows that we may actually, for convenience, *identify* $\mathbf{\Phi\text{-}Colim}$ and T-Alg$_s$ without harm; and to make the identification more useful, it is helpful to describe the action $a : TA \to A$ of the T-algebra $KA = (A,a)$ without explicit mention of the 2-monad T. In fact $a : TA \to A$, as we said above, is $U_s eA : U_s F U_s A \to U_s A$, so that it may be seen as the morphism $a = eA : TA \to A$ of $\mathbf{\Phi\text{-}Colim}$; and it satisfies the unit axiom $az_A = 1$. Accordingly, since $z_A : A \to TA$ is the unit $z_A : A \to U_s FA$ of the adjunction $F \dashv U_s$ of Theorem 5.1, we can characterize $a : TA \to A$ as the unique $\mathcal{V}$-functor strictly preserving the chosen Φ-colimits and satisfying $az_A = 1$.

Let us turn now to the 2-category $\mathbf{\Phi\text{-}Cocts}_c$, which has the same objects as $\mathbf{\Phi\text{-}Colim}$, and consider a morphism $f : A \to B$ therein — that is, a Φ-cocontinuous $\mathcal{V}$-functor. Write (A, a) and (B, b) as above for the associated T-algebras, and consider the two $\mathcal{V}$-functors $b.Tf$ and fa from TA to B. Here f preserves Φ-colimits, while a and b do so strictly, as we have just seen, and Tf too does so strictly, since as U_sFf it underlies the morphism Ff of $\mathbf{\Phi\text{-}Colim}$. Recalling from Theorem 5.1 that $z_A : A \to TA = U_sFA$ also exhibits FA (which is just TA seen as a Φ-cocomplete $\mathcal{V}$-category) as the free Φ-cocompletion of A in the sense of Section 2, we see that the equality $b.Tf.z_A = bz_Bf = f = faz_A$ implies the existence of a unique invertible 2-cell $\overline{f}$ of the form

$$\begin{tikzcd} TA \arrow[r, "a"] \arrow[d, "Tf"'] \arrow[dr, phantom, "\overset{\overline{f}}{\Rightarrow}"] & A \arrow[d, "f"] \\ TB \arrow[r, "b"'] & B \end{tikzcd}$$

whose composite with $z_A : A \to TA$ is the identity. Of course, by the last paragraph $\overline{f}$ is an identity if and only if f *strictly* preserves the chosen Φ-colimits. Recall from [3] that the pair $(f, \overline{f})$ is said to be a T-morphism if $\overline{f}$ also satisfies

$$\begin{tikzcd} T^2A \arrow[r, "m_A"] \arrow[d, "T^2f"'] & TA \arrow[r, "a"] \arrow[d, "Tf"'] \arrow[dr, phantom, "\overset{\overline{f}}{\Rightarrow}"] & A \arrow[d, "f"] \\ T^2B \arrow[r, "m_B"'] & TB \arrow[r, "b"'] & B \end{tikzcd} \quad = \quad \begin{tikzcd} T^2A \arrow[r, "Ta"] \arrow[d, "T^2f"'] \arrow[dr, phantom, "\overset{T\overline{f}}{\Rightarrow}"] & TA \arrow[r, "a"] \arrow[d, "Tf"'] \arrow[dr, phantom, "\overset{\overline{f}}{\Rightarrow}"] & A \arrow[d, "f"] \\ T^2B \arrow[r, "Tb"'] & TB \arrow[r, "b"'] & B. \end{tikzcd}$$

Since this equation expresses the equality of two 2-cells between Φ-cocontinuous $\mathcal{V}$-functors with codomain T^2A, and since $z_{TA} : TA \to T^2A$ expresses T^2A as the free Φ-cocompletion of TA, it holds if the two composites with z_{TA} coincide; however an easy calculation using naturality and $\overline{f}.z_A = id$ shows each composite with z_{TA} to be $\overline{f}$ itself.

Again, if $\alpha : f \to g : A \to B$ is a $\mathcal{V}$-natural transformation between Φ-cocontinuous $\mathcal{V}$-functors, we have the equality

$$\begin{tikzcd} TA \arrow[r, "a"] \arrow[d, bend right=50, "Tf"'] \arrow[d, bend left=50, "Tg"] \arrow[d, phantom, "\overset{T\alpha}{\Rightarrow}"] \arrow[dr, phantom, "\overset{\overline{g}}{\Rightarrow}"] & A \arrow[d, "g"] \\ TB \arrow[r, "b"'] & B \end{tikzcd} \quad = \quad \begin{tikzcd} TA \arrow[r, "a"] \arrow[d, "Tf"'] \arrow[dr, phantom, "\overset{\overline{f}}{\Rightarrow}"] & A \arrow[d, bend right=50, "f"'] \arrow[d, bend left=50, "g"] \arrow[d, phantom, "\overset{\alpha}{\Rightarrow}"] \\ TB \arrow[r, "b"'] & B, \end{tikzcd}$$

since each has the same composite with $z_A : A \to TA$; that is to say, α is a T-transformation $\alpha : (f, \overline{f}) \to (g, \overline{g}) : (A, a) \to (B, b)$ in the sense of [3]. Writing as there T-Alg for the 2-category of T-algebras, T-morphisms, and T-transformations, we have exhibited a 2-functor $K' : \mathbf{\Phi\text{-}Cocts}_c \to T\text{-Alg}$ which extends $K : \mathbf{\Phi\text{-}Colim} \to T\text{-Alg}_s$. In fact:

6.2. Theorem. *The above 2-functor $K' : \mathbf{\Phi\text{-}Cocts}_c \to T\text{-}\mathrm{Alg}$ extending the isomorphism $K : \mathbf{\Phi\text{-}Colim} \to T\text{-}\mathrm{Alg}_s$ is itself an isomorphism of 2-categories.*

Proof. It remains only to show that K' is bijective on morphisms. In fact, for objects A and B of $\mathbf{\Phi\text{-}Cocts}_c$, giving rise to the T-algebras (A, a) and (B, b), any $\mathcal{V}$-functor $f : A \to B$ having $fa \cong b.Tf$ is Φ-cocontinuous by Theorem 2.5; so that any T-morphism $f : (A, a) \to (B, b)$ is the image under K' of a unique morphism $A \to B$ in $\mathbf{\Phi\text{-}Cocts}_c$, namely f itself. ■

Recall from [11, Theorem 6.2] that a 2-monad $T = (T, z, m)$ on a 2-category $\mathcal{K}$ is *lax-idempotent* when for each T-algebra $(A, a : TA \to A)$ there is an adjunction $a \dashv z_A$ with identity counit; this is also called [14] the Kock-Zöberlein property. That this is the case here follows from Theorem 2.4, since $z_A : A \to TA$ is equivalent to $y : A \to \Phi A$, and since $a : TA \to A$ is Φ-cocontinuous with $az_A = 1$. Thus:

6.3. Theorem. *The 2-monad T whose Eilenberg-Moore objects is $U_s : \mathbf{\Phi\text{-}Colim} \to \mathcal{V}\text{-}\mathbf{Cat}$ is lax-idempotent.*

Before ending this section we make some observations on changing the class Φ, along with a kind of warning. Given two classes Φ and Ψ of weights with $\Phi \subseteq \Psi$, we have an evident forgetful 2-functor $P'' : \mathbf{\Psi\text{-}Cocts} \to \mathbf{\Phi\text{-}Cocts}$ over $\mathcal{V}\text{-}\mathbf{Cat}$ (the words "over $\mathcal{V}\text{-}\mathbf{Cat}$" meaning that P'' commutes with the forgetful 2-functors $\mathbf{\Phi\text{-}Cocts} \to \mathcal{V}\text{-}\mathbf{Cat}$ and $\mathbf{\Psi\text{-}Cocts} \to \mathcal{V}\text{-}\mathbf{Cat}$), which underlies a forgetful $P' : \mathbf{\Psi\text{-}Cocts}_c \to \mathbf{\Phi\text{-}Cocts}_c$, which in turn restricts to a forgetful $P : \mathbf{\Psi\text{-}Colim} \to \mathbf{\Phi\text{-}Colim}$, all over $\mathcal{V}\text{-}\mathbf{Cat}$. It may be the case that P'' is an *equality* of 2-categories: that is, every Φ-cocomplete $\mathcal{V}$-category is Ψ-cocomplete and every Φ-cocontinuous $\mathcal{V}$-functor is Ψ-cocontinuous. For a given Φ there is clearly a greatest Ψ with this property, namely the class Φ^* consisting of those weights $\psi : D_\psi^{\mathrm{op}} \to \mathcal{V}$ such that every Φ-cocomplete $\mathcal{V}$-category is ψ-cocomplete and every Φ-cocontinuous $\mathcal{V}$-functor is ψ-cocontinuous; and an explicit description of Φ^* was given by Albert and Kelly [2], who showed that $\psi \in \Phi^*$ if and only if the object ψ of $[D_\psi^{\mathrm{op}}, \mathcal{V}]$ lies in the closure $\Phi(D_\psi^{\mathrm{op}})$ of the representables under Φ-colimits.

We want to consider now the case where $\Phi \subseteq \Psi \subseteq \Phi^*$, so that $\mathbf{\Psi\text{-}Cocts} = \mathbf{\Phi\text{-}Cocts}$ and the forgetful $\mathbf{\Psi\text{-}Cocts}_c \to \mathbf{\Phi\text{-}Cocts}_c$ is an equivalence over $\mathcal{V}\text{-}\mathbf{Cat}$. For an example when $\mathcal{V} = \mathbf{Set}$, we may take Φ to consist of the weights for the initial object, binary coproducts, and coequalizers, while Ψ consists of the weights $\Delta 1 : J^{\mathrm{op}} \to \mathbf{Set}$ for all finite categories J; so that $\mathbf{\Phi\text{-}Colim}$ consists of categories with chosen initial object, binary coproducts, and coequalizers, while $\mathbf{\Psi\text{-}Colim}$ consists of categories with a choice of all finite colimits in the usual sense. (Here Φ^* consists of all weights $\psi : D_\psi^{\mathrm{op}} \to \mathbf{Set}$ which are finitely presentable in $[D_\psi^{\mathrm{op}}, \mathbf{Set}]$, which is much bigger than Ψ, and bigger even than the class of weights $\Delta 1 : J^{\mathrm{op}} \to \mathbf{Set}$ with J finitely presentable.) The point we want to emphasize is that the forgetful $P : \mathbf{\Psi\text{-}Colim} \to \mathbf{\Phi\text{-}Colim}$ is *not* in general an equivalence, even though the forgetful $P' : \mathbf{\Psi\text{-}Cocts}_c \to \mathbf{\Phi\text{-}Cocts}_c$ is clearly an equivalence, since it overlies the equality $P'' : \mathbf{\Psi\text{-}Cocts} = \mathbf{\Phi\text{-}Cocts}$. In fact, P here is not even fully faithful: there is no reason why a functor strictly preserving the *chosen* initial object, the *chosen* binary coproducts, and the *chosen* coequalizers should strictly preserve, say, the *chosen* ternary coproducts.

There is, however, a positive result. Using an argument by transfinite induction, Adámek and Kelly prove in the Appendix of the forthcoming article [1] that, when $\Phi \subseteq \Psi \subseteq \Phi^*$ as above, there is a 2-functor $Q : \mathbf{\Phi\text{-}Colim} \to \mathbf{\Psi\text{-}Colim}$ over $\mathcal{V}\text{-}\mathbf{Cat}$ satisfying $PQ = 1$; we may use their result as follows. If T and S are the 2-monads on $\mathcal{V}\text{-}\mathbf{Cat}$ corresponding respectively to $\mathbf{\Phi\text{-}Colim}$ and $\mathbf{\Psi\text{-}Colim}$, it follows from [13, Proposition 3.4] that P and Q, being 2-functors $P : S\text{-}\mathrm{Alg}_s \to T\text{-}\mathrm{Alg}_s$ and $Q : T\text{-}\mathrm{Alg}_s \to S\text{-}\mathrm{Alg}_s$ over $\mathcal{V}\text{-}\mathbf{Cat}$, are of the form ρ^* and σ^* for unique (strict) 2-monad morphisms $\rho : T \to S$ and $\sigma : S \to T$, which moreover satisfy $\sigma\rho = 1$ since $PQ = 1$. But by [13, (3.2)], ρ and σ also induce $\rho^\dagger : S\text{-}\mathrm{Alg} \to T\text{-}\mathrm{Alg}$ and $\sigma^\dagger : T\text{-}\mathrm{Alg} \to S\text{-}\mathrm{Alg}$, with of course $\rho^\dagger\sigma^\dagger = 1$. Here, however, $\rho^\dagger$ is the equivalence P', so that there is an isomorphism $\alpha : \sigma^\dagger\rho^\dagger \cong 1$. Now it follows from [13, Proposition 3.5] that there is a unique invertible modification $\theta : \rho\sigma \to 1$ of 2-monad morphisms having $\theta^\dagger = \alpha$, and we conclude that ρ and σ constitute an *equivalence* $T \simeq S$ *of 2-monads*, inducing the equivalence between T-Alg and S-Alg, while $T\text{-}\mathrm{Alg}_s$ and $S\text{-}\mathrm{Alg}_s$ remain inequivalent. Of course we expect $\mathbf{\Phi\text{-}Colim}$ to be a stranger creature than $\mathbf{\Phi\text{-}Cocts}$, but we need it to construct the 2-monad T for which $T\text{-}\mathrm{Alg} = \mathbf{\Phi\text{-}Cocts}_c$.

7. The case of a large class Φ

We now turn to the case where the class Φ of weights is not small. Our approach was sketched in the Introduction: we have been supposing the existence of an inaccessible cardinal ∞ and saying that a set is small if its cardinality is less than ∞, and we now suppose the existence of a further inaccessible cardinal ∞' such that the collection of isomorphism classes of small weights, seen as objects of $\mathcal{V}\text{-}\mathbf{Cat}/\mathcal{V}$, has cardinality less than ∞'. One easily sees that it suffices to suppose that the set of isomorphism classes of objects of $\mathbf{Set}$ and the set of isomorphism classes of objects of $\mathcal{V}$ both have cardinality less than ∞', and in the typical practical cases the latter cardinality is, like the former, equal to ∞, so that in these cases any inaccessible cardinal greater than ∞ will suffice. We now define a set to be big if its cardinality is less than ∞', a $\mathcal{V}$-category to be big if its set of objects is big, a $\mathcal{V}$-category to be big-complete if it has all big-limits, and big-cocomplete if it has all big-colimits.

Suppose now that Φ is a big class of small weights. Thus Φ might for example be the class of all small weights; but then, as mentioned in the Introduction, the only cocomplete small categories are preorders. There is little point then in restricting our attention to small $\mathcal{V}$-categories; rather, we seek a monadicity result involving big $\mathcal{V}$-categories, in the sense of the Introduction. Our previous results would carry over unchanged, replacing $\mathcal{V}\text{-}\mathbf{Cat}$ by the 2-category $\mathcal{V}\text{-}\mathbf{CAT}$ of big $\mathcal{V}$-categories, were it not for the fact that $\mathcal{V}$ was assumed only to be small-complete and small-cocomplete, not big-complete and big-cocomplete. To deal with this problem we adapt to our present needs the analysis of [9, Section 3.12] for dealing with enrichment in monoidal categories which fail to be complete or cocomplete.

Accordingly we write **SET** for the category of big sets; then the presheaf category $[\mathcal{V}_0^{\mathrm{op}}, \mathbf{SET}]$ has a symmetric monoidal closed structure given by convolution [5], and is big-complete and big-cocomplete, with the limits and colimits computed pointwise. The Yoneda embedding $Y : \mathcal{V}_0 \to [\mathcal{V}_0^{\mathrm{op}}, \mathbf{SET}]$ preserves the symmetric monoidal closed structure, as well as any limits which exist in $\mathcal{V}_0$, but does not preserve colimits. In order to overcome this, we write $\mathcal{V}'_0$ for the full subcategory of $[\mathcal{V}_0^{\mathrm{op}}, \mathbf{SET}]$ consisting of those functors which preserve small limits. This subcategory is reflective, by the results of [7], and so big-complete and big-cocomplete, and the restricted Yoneda embedding $Y : \mathcal{V}_0 \to \mathcal{V}'_0$ preserves small colimits and all existing limits. Furthermore $\mathcal{V}'_0$ has a symmetric monoidal closed structure, with the tensor product of two objects being the reflection into $\mathcal{V}'_0$ of their tensor product in $[\mathcal{V}_0^{\mathrm{op}}, \mathbf{SET}]$; we write $\mathcal{V}'$ for $\mathcal{V}'_0$ with this symmetric monoidal closed structure. The restricted Yoneda embedding $Y : \mathcal{V}_0 \to \mathcal{V}'_0$ also preserves the symmetric monoidal closed structure, and so induces a fully faithful 2-functor

$$Y_* : \mathcal{V}\text{-}\mathbf{CAT} \to \mathcal{V}'\text{-}\mathbf{CAT}$$

(which we henceforth treat as an inclusion) from $\mathcal{V}$-**CAT** to the 2-category $\mathcal{V}'$-**CAT** of big $\mathcal{V}'$-categories; and moreover this inclusion preserves small limits and small colimits.

A small weight, consisting of a $\mathcal{V}$-functor $\phi : D_\phi^{\mathrm{op}} \to \mathcal{V}$, which can also be seen as a $\mathcal{V}'$-functor, gives on composition with the $\mathcal{V}'$-functor $Y : \mathcal{V} \to \mathcal{V}'$ a "$\mathcal{V}'$-weight" $Y\phi : D_\phi^{\mathrm{op}} \to \mathcal{V}'$. Thus, for a $\mathcal{V}'$-functor $s : D_\phi^{\mathrm{op}} \to A$, we can speak of the colimit $(Y\phi) * s$. When, however, the $\mathcal{V}'$-category A is in fact a $\mathcal{V}$-category, the colimit $(Y\phi) * s$ coincides with the colimit $\phi * s$ (if either exists). The point is that $[D_\phi^{\mathrm{op}}, \mathcal{V}'](Y\phi, YA(s, a))$, which is the end in $\mathcal{V}'$ of $[\phi d, A(sd, a)]'$ (where $[-,-]'$ is the internal hom in $\mathcal{V}'$), coincides with $[D_\phi^{\mathrm{op}}, \mathcal{V}](\phi, A(s, a))$, since $[Ya, Yb]' \cong Y[a, b]$ and since $Y : \mathcal{V} \to \mathcal{V}'$ preserves limits. Thus it does no harm to speak of $(Y\phi) * s$ as a "ϕ-colimit".

It follows, if Φ is a big class of small weights, that we have in the category **2-Cat** of 2-categories and 2-functors a pullback diagram

$$\begin{tikzcd} \Phi\text{-}\mathbf{COLIM} \arrow[r] \arrow[d, "U_s"'] & \Phi\text{-}\mathbf{COLIM}' \arrow[d, "U'_s"] \\ \mathcal{V}\text{-}\mathbf{CAT} \arrow[r, "Y_*"'] & \mathcal{V}'\text{-}\mathbf{CAT}, \end{tikzcd}$$

where Φ-**COLIM**$'$ is the 2-category of big $\mathcal{V}'$-categories with chosen Φ-colimits, $\mathcal{V}'$-functors strictly preserving these, and $\mathcal{V}'$-natural transformations, while Φ-**COLIM**, as defined in the Introduction, is the 2-category of big $\mathcal{V}$-categories with chosen Φ-colimits, $\mathcal{V}$-functors strictly preserving these, and $\mathcal{V}$-natural transformations.

Now U'_s is monadic by Theorem 6.1, so that its pullback U_s along the fully faithful Y_* is monadic provided that it has a left adjoint. Write F' for the left adjoint to U'_s, and let A be a big $\mathcal{V}$-category. Although $[A^{\mathrm{op}}, \mathcal{V}]$ does not exist as a $\mathcal{V}$-category unless A is small, there is no problem forming $[A^{\mathrm{op}}, \mathcal{V}]$ as a $\mathcal{V}'$-category; it is small-cocomplete since $\mathcal{V}$ is so, with colimits being formed pointwise. Now $\mathcal{P}A$ is closed in $[A^{\mathrm{op}}, \mathcal{V}]$ under small colimits by [9, Proposition 5.34], and the $\mathcal{V}'$-functor $[A^{\mathrm{op}}, \mathcal{V}] \to [A^{\mathrm{op}}, \mathcal{V}']$ induced by

$Y : \mathcal{V} \to \mathcal{V}'$ preserves small colimits since Y does so; hence $\mathcal{P}A$ is closed in $[A^{\mathrm{op}}, \mathcal{V}']$ under small colimits and so in particular under Φ-colimits.

Now $F'A$ is equivalent to the closure $\Phi'A$ of the representables in $[A^{\mathrm{op}}, \mathcal{V}']$ under Φ-colimits; and $\mathcal{P}A$ is closed in $[A^{\mathrm{op}}, \mathcal{V}']$ under Φ-colimits, and contains the representables, and so contains $\Phi'A$. It follows that $\mathcal{P}A$ contains $\Phi'A$, and so that $\Phi'A$ is a $\mathcal{V}$-category since $\mathcal{P}A$ is one. Thus $F'A$ is equivalent to a $\mathcal{V}$-category; but it then follows easily that $F'A$ is isomorphic to a $\mathcal{V}$-category, and this last $\mathcal{V}$-category now gives the value at A of a left adjoint to U_s.

This completes the proof of:

7.1. Theorem. *The 2-functor* $U_s : \mathbf{\Phi\text{-}COLIM} \to \mathbf{\mathcal{V}\text{-}CAT}$ *is monadic, for any big class* Φ *of weights.*

7.2. Remark. Once again, it follows as in Theorem 6.2 and Theorem 6.3 that the 2-monad T on $\mathbf{\mathcal{V}\text{-}CAT}$ given by Theorem 7.1 is lax-idempotent, and that the isomorphism $K : \mathbf{\Phi\text{-}COLIM} \to T\text{-}\mathrm{Alg}_s$ extends to an isomorphism $K' : \mathbf{\Phi\text{-}COCTS}_c \to T\text{-}\mathrm{Alg}$.

8. Further applications

In this final section we briefly sketch another application of our results. Write $\mathbf{Reg}_c$ for the 2-category of small regular categories with chosen finite limits and chosen factorizations, regular functors, and natural transformations; and $U : \mathbf{Reg}_c \to \mathbf{Cat}$ for the forgetful 2-functor. Write $\mathbf{Reg}_s$ for the sub-2-category of $\mathbf{Reg}_c$ consisting of all the objects, those regular functors which strictly preserve the finite limits and the factorizations, and all the 2-cells between them; and write $J : \mathbf{Reg}_s \to \mathbf{Reg}_c$ for the inclusion and U_s for UJ. We seek a left adjoint to U_s.

U is well known to have a left biadjoint, whose value at a category A may be computed by a two-step process: first one forms the free finite-limit-completion of A, and then the free regular category [4] on that; moreover the unit y of the biadjunction is fully faithful. If we take $\mathcal{M}$ to be the class of fully faithful functors, then the standing hypotheses of Section 4 are satisfied. *(Ax1)* and *(Ax2)* are easily verified, and so once again the crucial step is in proving *(Ax3)*: that $U_R : \mathbf{Reg}_s \downarrow R \to \mathbf{Cat} \downarrow R$ has a left adjoint for each object R of $\mathbf{Reg}_s$.

The approach is similar to that taken in the case of the MAIN EXAMPLE. Once again $\mathbf{Reg}_s \downarrow R$ and $\mathbf{Cat} \downarrow R$ are locally chaotic, and so it suffices to show that $(U_R)_0 : (\mathbf{Reg}_s \downarrow R)_0 \to (\mathbf{Cat} \downarrow R)_0$ has a left adjoint, and we do this by showing that it is the forgetful functor from the category of algebras for an endofunctor E of $(\mathbf{Cat} \downarrow R)_0$, and that free E-algebras exist. The earlier argument proves once again the cocompleteness of $(\mathbf{Cat} \downarrow R)_0$, and so free E-algebras will exist if E preserves α-filtered colimits for some regular cardinal α.

Let $|\mathsf{Cat}_f|$ be a set of representatives of the isomorphism classes of finite categories. Given an object $(B, m : B \to R)$ of $(\mathbf{Cat} \downarrow R)_0$, let $E'B$ be the category

$$B^{\mathbf{2}} + \Sigma_{C \in |\mathsf{Cat}_f|} \mathbf{Cat}(C, B)$$

and let $m' : E'B \to R$ be the evident functor taking $k : a \to b$ in $B^{\mathbf{2}}$ to the object appearing in the chosen factorization in R of mk, and taking $x : C \to B$ to the chosen limit in R of $mx : C \to R$. Factorizing m' as

$$E'B \xrightarrow{e} EB \xhookrightarrow{\overline{m}} R$$

where e is bijective on objects and $\overline{m}$ is fully faithful, one can make $(EB, \overline{m})$ into the value at (B, m) of an endofunctor E of $(\mathbf{Cat} \downarrow R)_0$, much as in the case of the MAIN EXAMPLE.

The straightforward verifications that $(\mathbf{Reg}_s \downarrow R)_0$ is the category of algebras for E, and that E preserves filtered colimits, are left to the reader, giving:

8.1. THEOREM. *The 2-functor $U_s : \mathbf{Reg}_s \to \mathbf{Cat}$ admits a left adjoint.*

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