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n-PERMUTABLE LOCALLY FINITELY PRESENTABLE CATEGORIES

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ABSTRACT. We characterize n-permutable locally finitely presentable categories $Lex[\mathcal{C}^{op}, Set]$ by a condition on the dual of the essentially algebraic theory $\mathcal{C}^{op}$. We apply these results to exact Maltsev categories as well as to n-permutable quasivarieties and varieties.

Introduction

In recent years there has been a considerable interest in expressing properties of an essentially algebraic category (i.e. a locally finitely presentable category) in terms of properties of the corresponding essentially algebraic theory. From this point of view, complete answers have been given with respect to basic properties such as regularity, exactness, extensivity, cartesian closedness and so on (see [8], [9], [4] and [7]). In this note we analyse the condition of n-permutability of the composition of equivalence relations for a regular locally finitely presentable category $K = Lex[\mathcal{C}^{op}, Set]$.

It is known [5] that n-permutability can be equivalently stated by saying that, for any reflexive relation R, the corresponding generated equivalence relation $\overline{R}$ is given by a finite construction, more precisely $\overline{R} = R \circ R^o \circ R \circ R^o \circ \dots$ $(n-1)$-times. So, if we want to characterize n-permutability of K in terms of a corresponding condition on the essentially algebraic theory $\mathcal{C}^{op}$, or more simply on its dual $\mathcal{C}$ (where $\mathcal{C}$ is considered as a dense subcategory of K), we must interpret such a finiteness condition in $\mathcal{C}$. The answer to this problem is given in theorem 2.5, where we show that K is $(n+1)$-permutable $(n \geq 1)$ if and only if $\mathcal{C}$ is weakly regular and certain relations $R_n^{X,C}$, defined for any reflexive graph X in $\mathcal{C}$ and $C \in \mathcal{C}$, are transitive. The formal definition of $R_n^{X,C}$ simply corresponds to the syntactic interpretation in $\mathcal{C}$ of the following relation: if $X\colon\ X_1 \rightrightarrows X_0$ denotes a reflexive graph in $\mathcal{C}$ and $I\colon\ I \rightrightarrows X_0$ is its regular image in $Lex[\mathcal{C}^{op}, Set]$, two parallel arrows $C \underset{f_1}{\overset{f_0}{\rightrightarrows}} X_0$ are $R_n^{X,C}$-related if and only if (f_0, f_1) factorizes through the n-iterated composite $I \circ I^o \circ I \circ I^o \circ \dots$ (n-times). Notice that in $\mathcal{C}$ we must now consider reflexive graphs instead of reflexive relations; in fact conditions on reflexive relations do not suffice to force permutability in K. This theorem admits interesting applications in the case

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of an exact K and mainly in the case of quasivarieties and varieties. In these two last contexts the existence of a projective cover of $\mathcal{C}$ makes the syntactic conditions on $\mathcal{C}$ much simpler: indeed, for any P regular projective, the functor $Hom(P,-)\colon \mathcal{C} \to Set$ will preserve images and generated equivalences.

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1. Preliminaries on regular n-permutable categories

In this section we fix the notations and recall some properties of regular categories.
A category $\mathcal{A}$ is regular [3] if it is finitely complete, every kernel pair has a coequalizer and regular epimorphisms are stable under pullbacks. If $\mathcal{A}$ is regular, any arrow $f\colon A \to B$ can be factored as $f = i \circ p$ with p a regular epimorphism and i a monomorphism. A regular category $\mathcal{A}$ is exact when any equivalence relation is effective (a kernel pair).

A relation R from A to B will be denoted by $(r_0, r_1)\colon R \rightarrowtail A \times B$; for a relation R on an object A we shall also write $R \underset{r_1}{\overset{r_0}{\rightrightarrows}} A$. The set of equivalence relations on an object $A \in \mathcal{A}$ is denoted by $Eq(A)$. For any relation $(r_0, r_1)\colon R \rightarrowtail A \times B$ we can consider the opposite relation R^o given by $(r_1, r_0)\colon R \rightarrowtail B \times A$. Given two relations $(r_0, r_1)\colon R \rightarrowtail A \times B$ and $(s_0, s_1)\colon S \rightarrowtail B \times C$ in a regular category $\mathcal{A}$, the composite $S \circ R$ is defined as the image of the morphism $(r_0 \circ u_0, s_1 \circ u_1)\colon R \underset{B}{\times} S \to A \times C$:

$$
\begin{tikzcd}
 & & R \underset{B}{\times} S \arrow[dl, "u_0"'] \arrow[dr, "u_1"] & & \\
 & R \arrow[dl, "r_0"'] \arrow[dr, "r_1"] & & S \arrow[dl, "s_0"'] \arrow[dr, "s_1"] & \\
A & & B & & C
\end{tikzcd}
$$

In this section we always assume that the category $\mathcal{A}$ is regular: this assumption will assure that the composition of relations is associative. If R and S are equivalence relations on an object A, we have the increasing sequence

$$R \subseteq R \circ S \subseteq R \circ S \circ R \subseteq R \circ S \circ R \circ S \subseteq \dots$$

which we denote by

$$(R,S)_1 \subseteq (R,S)_2 \subseteq (R,S)_3 \subseteq (R,S)_4 \subseteq \dots$$

The smallest equivalence relation containing both R and S, denoted by $R \vee S$ (when it exists), contains all the terms of this sequence, and these terms are all different in general. If there is an $n \geq 2$ for which the relation $(R,S)_n$ is an equivalence relation, then $R \vee S = (R,S)_n$.

1.1. Definition. *A regular category $\mathcal{A}$ is n-permutable ($n \geq 2$) if for any $R, S \in Eq(A)$ and $A \in \mathcal{A}$, it holds $(R, S)_n = (S, R)_n$.*

Of course, if $n = 2$ we get the notion of Maltsev category: this notion was introduced in [6] as a weakening of the notion of abelian category. Since the Maltsev property can be expressed without the assumption of regularity of the category $\mathcal{A}$, we shall adopt the simpler and classical

1.2. Definition. *A category $\mathcal{A}$ is Maltsev if, for any $A \in \mathcal{A}$, any reflexive relation $R \rightarrowtail A \times A$ on A is an equivalence relation.*

The equivalence between the Maltsev axiom, the 2-permutability of the composition of equivalence relations and two other nice properties is recalled in the following

1.3. Theorem. *[6] Let $\mathcal{A}$ be a regular category. The following statements are equivalent:*

1. *$\mathcal{A}$ is a Maltsev category*
2. *$\mathcal{A}$ is 2-permutable*
3. *for any $A \in \mathcal{A}$ any reflexive relation R on A is transitive*
4. *for any $A \in \mathcal{A}$ and for any R, $S \in Eq(A)$, we have $R \circ S = R \vee S$*

1.4. Examples. A classical result of Maltsev [12] asserts that a finitary variety is Maltsev precisely when its theory contains a ternary operation $p(x, y, z)$ satisfying the axioms $p(x, y, y) = x$, $p(x, x, y) = y$; for instance in the variety of groups such a term $p(x, y, z)$ is given by $xy^{-1}z$. Among Maltsev varieties are then those of groups, abelian groups, modules over a fixed ring, rings, commutative rings, associative algebras and Lie algebras. The variety of quasi-groups is also Maltsev, as is the variety of Heyting algebras. There are non-varietal examples of exact Maltsev categories: any abelian category is exact Maltsev, as is the dual of the category of sets and, more generally, the dual of any topos. Finally, the category of topological groups is regular Maltsev [5].

It is interesting to know that the properties in theorem 1.3 remain equivalent in the n-permutable case ($n \geq 2$): indeed, we have the following

1.5. Theorem. *[5] Let $\mathcal{A}$ be a regular category. The following statements are equivalent:*

1. *for any reflexive relation R on an object $A \in \mathcal{A}$ the relation $(R, R^o)_{n-1}$ is an equivalence relation*
2. *$\mathcal{A}$ is n-permutable*
3. *for any reflexive relation R on an object $A \in \mathcal{A}$ the relation $(R, R^o)_{n-1}$ is transitive*
4. *for any $A \in \mathcal{A}$ and for any R, $S \in Eq(A)$, we have $(R, S)_n = R \vee S$*

1.6. Examples. Hagemann and Mitschke [11] proved that a finitary variety is n-permutable if and only if there exist $n+1$ ternary terms $p_0(x,y,z), p_1(x,y,z), \ldots, p_n(x,y,z)$ satisfying

$$\begin{aligned} p_0(x,y,z) &= x \\ p_i(x,x,y) &= p_{i+1}(x,y,y) \qquad \text{for} \quad 0 \le i \le n-1 \\ p_n(x,y,z) &= z. \end{aligned}$$

This result clearly includes Maltsev theorem, this latter being the special case where $n = 2$. The property of $(n+1)$-permutability can be shown to be strictly weaker than the one of n-permutability for each $n \geq 2$. In particular there are examples of 3-permutable varieties that fail to be Maltsev as, for instance, the variety of generalized right complemented semigroups: these algebras have two binary operations $\cdot$ and $*$ satisfying

$$\begin{aligned} x \cdot (x * y) &= y \cdot (y * x) \\ x \cdot (y * y) &= x. \end{aligned}$$

In this case the theorem of Hagemann and Mitschke can be applied by choosing $p_1(x,y,z)$ $= x \cdot (y * z)$ and $p_2(x,y,z) = z \cdot (y * x)$.

We recall that 3-permutable categories are called Goursat categories [5]. The property of 3-permutability, unlike 4-permutability, is strong enough to force the modularity of the lattice $Eq(A)$ of equivalence relations on any object A of the category.

2. Regular locally finitely presentable categories

A locally finitely presentable category $\mathcal{K}$ (see [10] or [2]) is a cocomplete category which admits a small set $\mathcal{S}$ of finitely presentable objects such that any object $K \in \mathcal{K}$ is a filtered colimit of objects of $\mathcal{S}$. Any such category is equivalent to a category $Lex[\mathcal{C}^{op}, Set]$ of finite limit preserving functors from a small finitely complete category $\mathcal{C}^{op}$ to the category of sets. Via the Yoneda embedding sending an object $C \in \mathcal{C}$ to the functor $Hom(-, C)$ the category $\mathcal{C}$ is a full subcategory of $Lex[\mathcal{C}^{op}, Set]$ and the objects of $\mathcal{C}$ form a family of dense generators. The dual category of $\mathcal{C}$ is called the essentially algebraic theory, while $Lex[\mathcal{C}^{op}, Set]$ is the category of models of the theory. Many properties of a locally finitely presentable category can be expressed just in terms of its essentially algebraic theory $\mathcal{C}$ and various results in this direction can be found in the literature, for instance in [7], [8],[9] and [4]. In this paper we are interested in the property of n-permutability of the composition of the equivalence relations; the regularity of $Lex[\mathcal{C}^{op}, Set]$ will be always required in order to express this kind of property.

Regular locally finitely presentable categories have been characterized in [7] as categories of finite limit preserving functors from the dual of a finitely cocomplete "weakly regular" category to the category of sets. We recall the definition:

2.1. Definition. *[7] A category $\mathcal{C}$ is weakly regular if every commutative diagram*

$$\begin{tikzcd} A \arrow[r, "f'"'] \arrow[d, "g'"] & B \arrow[d, "g"'] \\ C \arrow[r, "f"] & D \end{tikzcd}$$

in $\mathcal{C}$ in which f is a regular epi factors through a commutative diagram

$$\begin{tikzcd} \overline{A} \arrow[r, "\overline{f}"'] \arrow[d, "\overline{g}"] & B \arrow[d, "g"'] \\ C \arrow[r, "f"] & D \end{tikzcd}$$

where $\overline{f}$ is a regular epi.

Observe that any regular category is weakly regular: moreover one has the following

2.2. Theorem. *[7] Let $\mathcal{C}$ be a category with finite colimits. The following conditions are equivalent:*

1. *$Lex[\mathcal{C}^{op}, Set]$ is regular*
2. *$\mathcal{C}$ is weakly regular*

In order to express our main results we now introduce two important notions. The first one is the notion of n-iterated graph:

2.3. Definition. *Let X*

$$\begin{tikzcd} X_1 \arrow[r, shift left=3, "x_0"] \arrow[r, shift right=3, "x_1"'] & X_0 \arrow[l, "e"'] \end{tikzcd}$$

be a reflexive graph, $x_0 \circ e = 1_{X_0} = x_1 \circ e$. A graph K

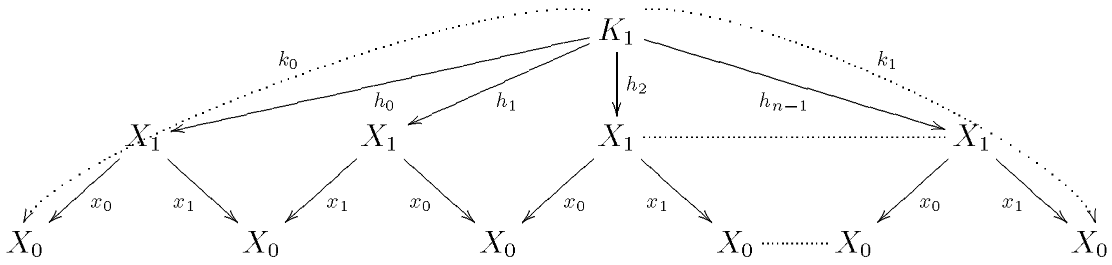
$$\begin{tikzcd} K_1 \arrow[r, shift left=1.5, "k_0"] \arrow[r, shift right=1.5, "k_1"'] & X_0 \end{tikzcd}$$

is a n-iterated of X (where $n \geq 1$) if there exist n arrows $h_0, h_1, \ldots, h_{n-1}$ from K_1 to X_1 such that

$$x_0 \circ h_0 = k_0,\ x_1 \circ h_0 = x_1 \circ h_1,\ x_0 \circ h_1 = x_0 \circ h_2, \ldots,\ x_i \circ h_{n-2} = x_i \circ h_{n-1},\ x_j \circ h_{n-1} = k_1$$

with $i = 1$ and $j = 0$ if n is even, while $i = 0$ and $j = 1$ if n is odd.

The conditions above can be expressed by the commutativity of the diagram below (in which we assume that n is odd)



By means of the notion of n-iterated graph, we now define a relation on the set $Hom(C, X_0)$:

2.4. Definition. *Let X*

$$\begin{tikzcd} X_1 \arrow[r, shift left=2, "x_0"] \arrow[r, shift right=2, "x_1"'] & X_0 \arrow[l, "e"'] \end{tikzcd}$$

be a reflexive graph and $n \geq 1$. Two arrows $C \overset{f_0}{\underset{f_1}{\rightrightarrows}} X_0$ are defined to be in the relation $R_n^{X,C}$, and we write $f_0 R_n^{X,C} f_1$, if there exists a factorisation of the graph $C \overset{f_0}{\underset{f_1}{\rightrightarrows}} X_0$ through a regular quotient $B \overset{b_0}{\underset{b_1}{\rightrightarrows}} X_0$ of a n-iterated graph $K \overset{k_0}{\underset{k_1}{\rightrightarrows}} X_0$ of X:

$$\begin{tikzcd} K \arrow[rr, shift left, "k_0"] \arrow[rr, shift right, "k_1"'] \arrow[dd, "\beta"'] & & X_0 \\ & & \\ B \arrow[uurr, shift left, "b_0"] \arrow[uurr, shift right, "b_1"'] & & C \arrow[uu, shift left, "f_0"] \arrow[uu, shift right, "f_1"'] \arrow[ll, "\gamma"] \end{tikzcd}$$

where

$$b_0 \circ \gamma = f_0, \quad b_1 \circ \gamma = f_1, \quad b_0 \circ \beta = k_0, \quad b_1 \circ \beta = k_1,$$

and β is a regular epi.

Remark that the relation $R_n^{X,C}$ is always reflexive: indeed, for any arrow $f_0 \colon C \to X_0$ the arrow $e \circ f_0 \colon C \to X_1$ shows that $f_0 R_n^{X,C} f_0$.

2.5. Theorem. *Let $\mathcal{C}$ be a category with finite colimits. The following conditions are equivalent ($n \geq 1$):*

1. *$Lex[\mathcal{C}^{op}, Set]$ is regular $(n+1)$-permutable*
2. *$\mathcal{C}$ is weakly regular and the relation $R_n^{X,C}$ is transitive for any reflexive graph X in $\mathcal{C}$ and $C \in \mathcal{C}$.*

PROOF. We first prove that $R_n^{X,C} = Hom(C, (I, I^o)_n)$ for any $C \in \mathcal{C}$ and any reflexive graph $X_1 \underset{x_1}{\overset{x_0}{\rightrightarrows}} X_0$ in $\mathcal{C}$ with regular image in $Lex[\mathcal{C}^{op}, Set]$ given by $I \underset{\overline{x}_1}{\overset{\overline{x}_0}{\rightrightarrows}} X_0$.

For this, we consider (f_0, f_1) in $R_n^{X,C}$ and we're going to show that (f_0, f_1) belongs to $Hom(C, (I, I^o)_n)$. By assumption there is a factorisation as in the diagram

$$
\begin{tikzcd}
K \arrow[r, shift left, "k_0"] \arrow[r, shift right, "k_1"'] \arrow[d, "\beta"'] & X_0 \\
B \arrow[ur, shift left, "b_0"] \arrow[ur, shift right, "b_1"'] & C \arrow[u, shift right, "f_0"] \arrow[u, shift left, "f_1"'] \arrow[l, "\gamma"]
\end{tikzcd}
$$

where $K \underset{k_1}{\overset{k_0}{\rightrightarrows}} X_0$ is a n-iterated of the reflexive graph $X_1 \underset{x_1}{\overset{x_0}{\rightrightarrows}} X_0$. Remark then that any n-iterated $K \underset{k_1}{\overset{k_0}{\rightrightarrows}} X_0$ factors through the limit $(L;\, l_0, l_1, \dots, l_{n-1})$

$$
\begin{tikzcd}
 & & & & & L \arrow[dllll, "l_0"'] \arrow[dll, "l_1"'] \arrow[d, "l_2"] \arrow[drrr, "l_{n-1}"] & & & & \\
 & X_1 \arrow[dl, "x_0"'] \arrow[dr, "x_1"] & & X_1 \arrow[dl, "x_1"'] \arrow[dr, "x_0"] & & X_1 \arrow[dl, "x_0"'] \arrow[dr, "x_1"] \arrow[rrr, dotted, no head] & & & X_1 \arrow[dl, "x_0"'] \arrow[dr, "x_1"] & \\
X_0 & & X_0 & & X_0 & & X_0 \arrow[r, dotted, no head] & X_0 & & X_0
\end{tikzcd}
$$

over the diagram

$$
\begin{tikzcd}
 & X_1 \arrow[dl, "x_0"'] \arrow[dr, "x_1"] & & X_1 \arrow[dl, "x_1"'] \arrow[dr, "x_0"] & & X_1 \arrow[dl, "x_0"'] \arrow[dr, "x_1"] \arrow[rrr, dotted, no head] & & & X_1 \arrow[dl, "x_0"'] \arrow[dr, "x_1"] & \\
X_0 & & X_0 & & X_0 & & X_0 \arrow[r, dotted, no head] & X_0 & & X_0
\end{tikzcd}
$$

(in the diagrams above we have assumed that n is odd). This limit can be clearly obtained by iterated pullbacks in $Lex[\mathcal{C}^{op}, Set]$. There is then an arrow $\eta\colon K \to L$ verifying, in particular, $x_0 \circ l_0 \circ \eta = k_0$ and $x_1 \circ l_{n-1} \circ \eta = k_1$.

The regularity of $Lex[\mathcal{C}^{op}, Set]$ implies that the induced arrow ϵ from $(L;\, l_0, l_1, \dots, l_{n-1})$ to the limit $(M;\, m_0, m_1, \dots, m_{n-1})$

$$\begin{tikzcd}[column sep=small]
 & & & & & M \arrow[dllll, "m_0"] \arrow[dll, "m_1"] \arrow[d, "m_2"] \arrow[drrr, "m_{n-1}"] & & & & \\
 & I \arrow[dl, "\overline{x}_0"] \arrow[dr, "\overline{x}_1"] & & I \arrow[dl, "\overline{x}_1"] \arrow[dr, "\overline{x}_0"] & & I \arrow[dl, "\overline{x}_0"] \arrow[dr, "\overline{x}_1"] \arrow[rrr, dotted, no head] & & & I \arrow[dl, "\overline{x}_0"] \arrow[dr, "\overline{x}_1"] & \\
X_0 & & X_0 & & X_0 & & X_0 \arrow[r, dotted, no head] & X_0 & & X_0
\end{tikzcd}$$

over the diagram

$$\begin{tikzcd}[column sep=small]
 & I \arrow[dl, "\overline{x}_0"] \arrow[dr, "\overline{x}_1"] & & I \arrow[dl, "\overline{x}_1"] \arrow[dr, "\overline{x}_0"] & & I \arrow[dl, "\overline{x}_0"] \arrow[dr, "\overline{x}_1"] \arrow[rrr, dotted, no head] & & & I \arrow[dl, "\overline{x}_0"] \arrow[dr, "\overline{x}_1"] & \\
X_0 & & X_0 & & X_0 & & X_0 \arrow[r, dotted, no head] & X_0 & & X_0
\end{tikzcd}$$

is a regular epi. By recalling the definition of the composite of relations in a regular category, one then has a regular epi $q \circ \epsilon \colon L \twoheadrightarrow (I, I^o)_n$ as in the diagram

$$\begin{tikzcd}
L \arrow[r, two heads, "\epsilon"] & M \arrow[rr, "(\overline{x}_0 \circ m_0, \overline{x}_1 \circ m_{n-1})"] \arrow[dr, two heads, "q"'] & & X_0 \times X_0 \\
 & & (I, I^o)_n \arrow[ur, tail, "(i_0, i_1)"'] &
\end{tikzcd}$$

The arrow $q \circ \epsilon \circ \eta \colon K \to (I, I^o)_n$ is such that

$$i_0 \circ q \circ \epsilon \circ \eta = \overline{x}_0 \circ m_0 \circ \epsilon \circ \eta = x_0 \circ l_0 \circ \eta = k_0 = b_0 \circ \beta$$

and similarly $i_1 \circ q \circ \epsilon \circ \eta = b_1 \circ \beta$. The commutativity of the diagram

$$\begin{tikzcd}
K \arrow[r, two heads, "\beta"'] \arrow[d, "q \circ \epsilon \circ \eta"] & B \arrow[d, "(b_0, b_1)"'] \\
(I, I^o)_n \arrow[r, tail, "(i_0, i_1)"] & X_0 \times X_0
\end{tikzcd}$$

gives a unique $\sigma \colon B \to (I, I^o)_n$ with $\sigma \circ \beta = q \circ \epsilon \circ \eta$ and $(i_0, i_1) \circ \sigma = (b_0, b_1)$. It follows then that $i_0 \circ \sigma \circ \gamma = b_0 \circ \gamma = f_0$ and $i_1 \circ \sigma \circ \gamma = b_1 \circ \gamma = f_1$. This shows that $(f_0, f_1) \in Hom(C, (I, I^o)_n)$.

Let us then prove that $(f_0, f_1) \in Hom(C, (I, I^o)_n)$ implies $(f_0, f_1) \in R_n^{X,C}$. Indeed, let $\alpha\colon C \to (I, I^o)_n$ be an arrow with $i_0 \circ \alpha = f_0$ and $i_1 \circ \alpha = f_1$. Keeping in mind that the regular epi $q \circ \epsilon\colon L \to (I, I^o)_n$ is a directed colimit of regular epis $(q \circ \epsilon)_j\colon L_j \to [(I, I^o)_n]_j$ in $\mathcal{C}$ (see for instance [1]) and that C is finitely presentable, we obtain an arrow α_j as in the diagram

$$\begin{tikzcd}
L \arrow[r, two heads, "q\circ\epsilon"] & (I,I^o)_n \arrow[r, shift left, "i_0"] \arrow[r, shift right, "i_1"'] & X_0 \\
L_j \arrow[u, "n_j"] \arrow[r, two heads, "(q\circ\epsilon)_j"'] & {[(I,I^o)_n]_j} \arrow[u, "m_j"'] & C \arrow[l, "\alpha_j"] \arrow[ul, "\alpha"'] \arrow[u, shift right, "f_0"'] \arrow[u, shift left, "f_1"]
\end{tikzcd}$$

with $\alpha = m_j \circ \alpha_j$. Since the graph $L_j \underset{i_1 \circ q \circ \epsilon \circ n_j}{\overset{i_0 \circ q \circ \epsilon \circ n_j}{\rightrightarrows}} X_0$ is a n-iterated of $X_1 \underset{x_1}{\overset{x_0}{\rightrightarrows}} X_0$, it follows that $f_0 R_n^{X,C} f_1$.

1. $\Rightarrow$ 2. By theorem 2.2 we know that $\mathcal{C}$ is weakly regular. The relation $(I, I^o)_n$ is transitive in $Lex[\mathcal{C}^{op}, Set]$ by assumption (and by theorem 1.5): this implies that $Hom(C, (I, I^o)_n) = R_n^{X,C}$ is a transitive relation in the category of sets.
2. $\Rightarrow$ 1. By theorem 2.2, the category $Lex[\mathcal{C}^{op}, Set]$ is regular. By theorem 1.5 we just need to show that any reflexive relation $A \underset{c}{\overset{d}{\rightrightarrows}} A_0$ is such that $(A, A^o)_n$ is transitive. Since any reflexive relation in $Lex[\mathcal{C}^{op}, Set]$ can be written as a filtered colimit of reflexive graphs in $\mathcal{C}$ and a filtered colimit of transitive relations is a transitive relation, then it suffices to check the property for a reflexive graph in $\mathcal{C}$. Let $X_1 \underset{x_1}{\overset{x_0}{\rightrightarrows}} X_0$ be a reflexive graph and let $I \underset{\overline{x}_1}{\overset{\overline{x}_0}{\rightrightarrows}} X_0$ denote its regular image in $Lex[\mathcal{C}^{op}, Set]$. We must prove that $(I, I^o)_n \underset{i_1}{\overset{i_0}{\rightrightarrows}} X_0$ is transitive. We recall that the category $\mathcal{C}$ is a dense generator in $K = Lex[\mathcal{C}^{op}, Set]$: this means that the inclusion of K into $Set^{\mathcal{C}^{op}}$ (via the restriction of the Yoneda embedding) is fully faithful. From this it follows that the relation $(I, I^o)_n \rightarrowtail X_0 \times X_0$ is transitive in $\mathcal{K}$ if and only if the relation $Hom(C, (I, I^o)_n) \rightarrowtail Hom(C, X_0) \times Hom(C, X_0)$ is transitive in Set for any $C \in \mathcal{C}$. Since $R_n^{X,C}$ is exactly $Hom(C, (I, I^o)_n)$, the result follows from $R_n^{X,C}$ transitive. ■

2.6. Corollary. *Let $\mathcal{C}$ be a category with finite colimits. The following conditions are equivalent:*

1. *$Lex[\mathcal{C}^{op}, Set]$ is regular Maltsev*
2. *$\mathcal{C}$ is weakly regular and the relation $R_1^{X,C}$ is transitive for any reflexive graph X in $\mathcal{C}$.*

3. Exact locally finitely presentable categories

We now turn to the problem of characterizing the locally finitely presentable categories that are exact Maltsev. From this point of view the property described in the following definition will have an essential role:

3.1. Definition. *A category $\mathcal{C}$ with coequalizers is pro-maltsev if for any reflexive graph X*

$$\begin{tikzcd} X_1 \arrow[r, shift left=2, "x_0"] \arrow[r, shift right=2, "x_1"'] & X_0 \arrow[l, "e"' description] \end{tikzcd}$$

and for any pair of arrows $f_0, f_1 \colon C \to X_0$ with $q \circ f_0 = q \circ f_1$, where q is the coequalizer of x_0 and x_1, (f_0, f_1) is in the relation $R_1^{X,C}$, i.e. there exists a factorization of C through a quotient of a 1-iterated $\begin{tikzcd} K \arrow[r, shift left, "k_0"] \arrow[r, shift right, "k_1"'] & X_0 \end{tikzcd}$ *of X:*

$$\begin{tikzcd} K \arrow[r, shift left, "k_0"] \arrow[r, shift right, "k_1"'] \arrow[d, "\beta"'] & X_0 \\ B \arrow[ur, shift left, "b_0"] \arrow[ur, shift right, "b_1"'] & C \arrow[l, "\gamma"] \arrow[u, shift left, "f_0"] \arrow[u, shift right, "f_1"'] \end{tikzcd}$$

We first remark that this property is stronger than the one expressed by the transitivity of the relation $R_1^{X,C}$:

3.2. Lemma. *If $\mathcal{C}$ is a pro-maltsev category, then the relation $R_1^{X,C}$ is transitive, for any reflexive graph X in $\mathcal{C}$ and any $C \in \mathcal{C}$.*

Proof. Let X

$$\begin{tikzcd} X_1 \arrow[r, shift left=2, "x_0"] \arrow[r, shift right=2, "x_1"'] & X_0 \arrow[l, "e"' description] \end{tikzcd}$$

be a reflexive graph and let $(f_0, f_1) \in R_1^{X,C}$ and $(f_1, f_2) \in R_1^{X,C}$; this means that there exist two configurations in $\mathcal{C}$ as in the diagrams

$$\begin{tikzcd} K \arrow[r, shift left, "k_0"] \arrow[r, shift right, "k_1"'] \arrow[d, "\beta"'] & X_0 \\ B \arrow[ur, shift left, "b_0"] \arrow[ur, shift right, "b_1"'] & C \arrow[l, "\gamma"] \arrow[u, shift left, "f_0"] \arrow[u, shift right, "f_1"'] \end{tikzcd} \qquad \begin{tikzcd} K' \arrow[r, shift left, "k'_0"] \arrow[r, shift right, "k'_1"'] \arrow[d, "\beta'"'] & X_0 \\ B' \arrow[ur, shift left, "b'_0"] \arrow[ur, shift right, "b'_1"'] & C \arrow[l, "\gamma'"] \arrow[u, shift left, "f_1"] \arrow[u, shift right, "f_2"'] \end{tikzcd}$$

where both $K \overset{k_0}{\underset{k_1}{\rightrightarrows}} X_0$ and $K' \overset{k'_0}{\underset{k'_1}{\rightrightarrows}} X_0$ are 1-iterated of the reflexive graph $X_1 \overset{x_0}{\underset{x_1}{\rightrightarrows}} X_0$.
If $q\colon X_0 \twoheadrightarrow Q$ denotes the coequalizer of x_0 and x_1, then

$$q \circ b_0 \circ \beta = q \circ b_1 \circ \beta,$$

hence, since β is a regular epi,

$$q \circ b_0 \circ \gamma = q \circ b_1 \circ \gamma,$$

so that $q \circ f_0 = q \circ f_1$. Similarly $q \circ f_1 = q \circ f_2$ and then $q \circ f_0 = q \circ f_2$: by the pro-maltsev assumption $f_0 R_1^{X,C} f_2$, proving that the relation $R_1^{X,C}$ is transitive. ■

3.3. Lemma. *Let $Lex[\mathcal{C}^{op}, Set]$ be an exact category. If the relation $R_1^{X,C}$ is transitive for any reflexive graph X in $\mathcal{C}$, then $\mathcal{C}$ is pro-maltsev.*

Proof. By corollary 2.6 the category $Lex[\mathcal{C}^{op}, Set]$ is exact Maltsev. With the same notations as in the lemma 3.2, if there are two arrows $f_0, f_1\colon C \to X_0$ such that $q \circ f_0 = q \circ f_1$, these must factorize through the regular image $I \overset{\overline{x}_0}{\underset{\overline{x}_1}{\rightrightarrows}} X_0$ in $Lex[\mathcal{C}^{op}, Set]$ of the reflexive graph $X_1 \overset{x_0}{\underset{x_1}{\rightrightarrows}} X_0$, since this relation is necessarily the kernel pair of q. Hence (f_0, f_1) is in $R_1^{X,C} = Hom(C, I)$. ■

We then get the following

3.4. Proposition. *Let $\mathcal{C}$ be a category with finite colimits. The following conditions are equivalent:*

1. *$Lex[\mathcal{C}^{op}, Set]$ is exact Maltsev*
2. *$\mathcal{C}$ is weakly regular and pro-maltsev*

Proof. 1. $\Rightarrow$ 2.
It follows by corollary 2.6 and lemma 3.3.
2. $\Rightarrow$ 1.
By corollary 2.6 and lemma 3.2 one knows that $Lex[\mathcal{C}^{op}, Set]$ is regular Maltsev. By using the pro-maltsev property it is possible to show that, given a reflexive graph $X_1 \overset{x_0}{\underset{x_1}{\rightrightarrows}} X_0$ in $\mathcal{C}$, its image factorisation $I \overset{\overline{x}_0}{\underset{\overline{x}_1}{\rightrightarrows}} X_0$ in $Lex[\mathcal{C}^{op}, Set]$ is an effective equivalence relation. Indeed, one can easily prove that the pro-maltsev property implies that the kernel pair in $Lex[\mathcal{C}^{op}, Set]$ of the coequalizer of the arrows x_0 and x_1 factorizes through $I \overset{\overline{x}_0}{\underset{\overline{x}_1}{\rightrightarrows}} X_0$: this certainly suffices to conclude that I is a kernel pair in $Lex[\mathcal{C}^{op}, Set]$.

Now, for any equivalence relation $A \underset{c}{\overset{d}{\rightrightarrows}} A_0$ in $Lex[\mathcal{C}^{op}, Set]$, write it as a filtered colimit of reflexive graphs in $\mathcal{C}$: since the regular image of any of these reflexive graphs is an effective equivalence relation, we get that $A \underset{c}{\overset{d}{\rightrightarrows}} A_0$ is an effective equivalence relation. ■

More generally, one can define the notion of pro-n-permutable category:

3.5. Definition. *A category $\mathcal{C}$ is pro-n-permutable ($n \geq 2$) if for any reflexive graph X*

$$\begin{tikzcd} X_1 \arrow[r, shift left=2, "x_0"] \arrow[r, shift right=2, "x_1"'] & X_0 \arrow[l, "e"' description] \end{tikzcd}$$

and for any pair of arrows $f_0, f_1 \colon C \to X_0$ with $q \circ f_0 = q \circ f_1$, where q is the coequalizer of x_0 and x_1, (f_0, f_1) is in the relation $R^{X,C}_{n-1}$.

By adopting the same technique as above, theorem 2.5 allows to generalize these results to the n-permutable case ($n \geq 2$):

3.6. Proposition. *Let $\mathcal{C}$ be a category with finite colimits. The following conditions are equivalent:*

1. *$Lex[\mathcal{C}^{op}, Set]$ is exact n-permutable*
2. *$\mathcal{C}$ is weakly regular and pro-n-permutable*

3.7. Remark. The notion of weakly regular pro-n-permutable category is clearly stronger than the one of pro-exact category in the sense of [7]: to see it, one just needs to remark that any n-iterated graph (in our sense) of a reflexive and symmetric graph X is an "iteration" of X as defined in that paper. Observe that pro-exactness corresponds, with the same notations as in definition 3.5, to

$$q \circ f_0 = q \circ f_1 \qquad \Rightarrow \qquad \exists n \geq 1, \quad (f_0, f_1) \in R^{X,C}_n.$$

The pro-n-permutability of a weakly regular category $\mathcal{C}$ can be accordingly thought as a synthetic way to express at the same time the "exactness" of the category $Lex[\mathcal{C}^{op}, Set]$ and the fact that the join of two equivalence relations in $Lex[\mathcal{C}^{op}, Set]$ can be obtained in a "finite number of steps". This last condition precisely expresses the n-permutability of the composition of equivalence relations.

4. Quasivarieties and varieties

By a quasivariety is meant a class of many-sorted finitary algebras that can be defined by implications of the form

$$\alpha_1 \wedge \alpha_2 \wedge \ldots \wedge \alpha_n \to \beta,$$

where $n \in \omega$ and α_i and β are equations (with both sides of the same sort). Any quasivariety is a locally finitely presentable category; in [1] Adámek and Porst characterized (the dual of) those essentially algebraic theories whose models form a quasivariety as being the finitely cocomplete categories which have enough regular projectives:

4.1. THEOREM. *[1] Let $\mathcal{C}$ be a category with finite colimits. The following conditions are equivalent:*

1. *$Lex[\mathcal{C}^{op}, Set]$ is a quasivariety*
2. *$\mathcal{C}$ has enough regular projectives*

Any quasivariety is a regular category: it seems then natural to investigate the condition of n-permutability for quasivarieties. For this, we begin with the following

4.2. DEFINITION. *Let $\mathcal{A}$ be a regular category and $n \geq 1$. A n-pseudo transitive relation is a reflexive graph $X_1 \underset{x_1}{\overset{x_0}{\rightrightarrows}} X_0$ such that its image factorisation $I \underset{\overline{x}_1}{\overset{\overline{x}_0}{\rightrightarrows}} X_0$ has the property that $(I, I^o)_n$ is a transitive relation.*

Of course, if $\mathcal{A}$ is regular Maltsev, any reflexive graph is a 1-pseudo transitive relation, since $I = (I, I^o)_1$ is an equivalence relation. We then define a notion of n-permutable object:

4.3. DEFINITION. *An object P in a category $\mathcal{C}$ is n-permutable ($n \geq 2$) if the functor $Hom(P, -)$ sends reflexive graphs to $(n-1)$-pseudo transitive relations (in Set).*

We can then give our characterization of n-permutable quasivarieties:

4.4. PROPOSITION. *Let $\mathcal{C}$ be a category with finite colimits and $n \geq 2$. The following conditions are equivalent:*

1. *$Lex[\mathcal{C}^{op}, Set]$ is a n-permutable quasivariety*
2. *$\mathcal{C}$ has enough n-permutable regular projectives*

PROOF. 1. $\Rightarrow$ 2. By theorem 4.1 the category $\mathcal{C}$ has enough regular projectives. It suffices to prove that, for any regular projective P and for any reflexive graph $X : X_1 \underset{x_1}{\overset{x_0}{\rightrightarrows}} X_0$, the relation

$$(J, J^o)_{n-1} \rightarrowtail Hom(P, X_0) \times Hom(P, X_0)$$

is transitive (where $J \underset{j_1}{\overset{j_0}{\rightrightarrows}} Hom(P, X_0)$ is the image factorisation in Set of the reflexive graph $Hom(P, X_1) \rightrightarrows Hom(P, X_0)$). Since P is regular projective then $(J, J^o)_{n-1} = Hom(P, (I, I^o)_{n-1})$, where $I \underset{\overline{x}_1}{\overset{\overline{x}_0}{\rightrightarrows}} X_0$ is the image of X in $Lex[\mathcal{C}^{op}, Set]$. The result then follows from theorem 2.5 and $R_{n-1}^{X,P} = Hom(P, (I, I^o)_{n-1})$.

2. $\Rightarrow$ 1. By theorems 2.5 and 4.1 we just have to prove that the relation $R^{X,C}_{n-1}$ is transitive for any reflexive graph X in $\mathcal{C}$ and $C \in \mathcal{C}$. Since $\mathcal{C}$ has a regular projective cover by assumption, it suffices to check that $R^{X,P}_{n-1}$ is transitive for P regular projective. This follows by $R^{X,P}_{n-1} = Hom(P, (I, I^o)_{n-1}) = (J, J^o)_{n-1}$ and P is n-permutable. ∎

As a corollary of this result, we now give a characterization of n-permutable finitary varieties. For this, we first recall the notion of effective projective object, due to Pedicchio and Wood:

4.5. Definition. *An object P in a category $\mathcal{C}$ is an effective projective object if the functor $Hom(P, -)$ preserves coequalizers of reflexive graphs.*

This notion plays an essential role in a recent work [14] by Pedicchio and Wood. In this paper the authors characterized (the dual of) those essentially algebraic theories whose models form a finitary variety of algebras: these are precisely the finitely cocomplete categories which have enough effective regular projectives:

4.6. Theorem. *[14] Let $\mathcal{C}$ be a category with finite colimits. The following conditions are equivalent:*

1. *$Lex[\mathcal{C}^{op}, Set]$ is a variety*
2. *$\mathcal{C}$ has enough effective regular projectives*

This theorem, together with proposition 4.4, gives the following corollary:

4.7. Corollary. *Let $\mathcal{C}$ be a category with finite colimits. The following conditions are equivalent:*

1. *$Lex[\mathcal{C}^{op}, Set]$ is a n-permutable variety*
2. *$\mathcal{C}$ has enough n-permutable effective regular projectives*

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