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LOCALIZATION OF V-CATEGORIES

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ABSTRACT. Let V be a symmetric monoidal closed category with a suitably compatible simplicial model category structure. We show how to extend Dwyer and Kan's notion of simplicial localization to V-categories. This may for instance be applied to the case where our categories are enriched in suitable models for spectra.

If M is a monoid, we may "group complete", or "localize", it by adding formal inverses. If M is a simplicial monoid, we may still do this in every degree, but this is a very crude completion, destroying much valuable homotopy information in M. Therefore, when homotopy theorists talk about group completion, they mean some "total derived functor" of the naïve group completion, for instance loops on the classifying space.

Dwyer and Kan [DK] explained how such homotopy acceptable localizations ought to be interpreted when "there are many objects"; i.e. what happens if you want to invert some of the maps in a category? Naïvely, you may simply invert them: in particular if W is a small category, you can consider $W[W^{-1}]$ where you have left the objects undisturbed, but formally inverted all the morphisms. If you want to make a homotopy invariant version of this, you have to derive this construction. We will come back to a concrete description of such a construction in section 0 which also covers the special case of ordinary categories.

In the case the category is enriched, things become a bit more delicate. As an example which may serve as a motivation and which captures the essential point, one might consider a ring A. In particular, if A is commutative there is a nice notion of localization, but even then there are some restrictions. One considers a subset $M \subset A$, closed under multiplication, but most certainly, M will not be closed under addition (because then $0 \in M$, and only very rarely do we want to invert zero), and so M is blind to the "enrichment" of A in abelian groups. What we do is to consider the ring homomorphism $\mathbf{Z}[M] \to A$, and form the tensor product $\mathbf{Z}[M[M^{-1}]] \otimes_{\mathbf{Z}[M]} A$, which is exactly A localized at M. There is nothing hindering us from pursuing a similar approach for non commutative rings: if $M \subseteq A$ is a submonoid under multiplication, then we may define the localization of A at M to be the pushout of $\mathbf{Z}[M[M^{-1}]] \leftarrow \mathbf{Z}[M] \to A$. However,

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if we are concerned about homotopy properties, we should modify this in the standard fashion.

The analog of this for categories enriched in spectra is this. A category $\mathcal{C}$ enriched in spectra may be localized with respect to a "subcategory" $\mathcal{W}$ which is only enriched in spaces. As a first approximation, the localization for commutative rings is very illustrating, when one remembers that tensor product is really pushout. One considers the map $\Sigma[\mathcal{W}] \to \mathcal{C}$, where $\Sigma[\mathcal{W}]$ is closely analogous to the monoid ring above, and performs the pushout

$$\Sigma[\mathcal{W}[\mathcal{W}^{-1}]] \coprod_{\Sigma[\mathcal{W}]} \mathcal{C}$$

The aim is to give a modification of this construction with good homotopy properties. The construction presented below achieves this by associating to a category enriched in spectra with a choice of weak equivalences (C, w) another such pair $L(\mathcal{C}, w)$ whose properties among other things are: (see section 0 for details)

(1) The "weak equivalences" in $L(\mathcal{C}, w)$ are isomorphisms (see 0.3)
(2) L is a homotopy functor (see 0.4)
(3) There is a chain of natural transformations $(\mathcal{C}, w) \leftarrow B(\mathcal{C}, w) \to L(\mathcal{C}, w)$ where $(\mathcal{C}, w) \leftarrow B(\mathcal{C}, w)$ is a weak equivalence (see 0.3)
(4) If $(\mathcal{C}, w)$ is "groupoid-like" (see 0.5) then $B(\mathcal{C}, w) \to L(\mathcal{C}, w)$ is a weak equivalence (see 0.6)

There also are more technical results, such as closure properties, and an analysis of what happens if the weak equivalences already are isomorphisms.

The reason for writing this paper is that if one wants to apply functors to enriched categories with weak equivalences, one usually can not expect these functors to behave well with respect to natural transformations consisting of weak equivalences. This problem is resolved if we localize before applying our functors, and the results in this paper show that in many situations this gives a good theory. In particular, the author was interested in questions about algebraic K-theory of such categories, and localization turned out to be the natural answer to a technical problem relating to trace maps see [D].

Although the intended applications are to categories enriched in spectra, the construction and its general properties are fairly context independent, and are applicable in other situations. For this reason, and because of the many competing versions of closed categories of spectra, we work with "V-categories", where V is any sufficiently nice symmetric monoidal closed category with a compatible model structure. This will also prove to be useful when one wants to compare with more algebraic situations.

Plan. In section 0 we give the main construction postponing some technical points. This section also contains the main results about the localization functor. If you are unfamiliar with enriched categories, you may choose to take a peek at the beginning of section 1 before starting to read section 0. The constructions of section 0 logically depend on some results referred away to later sections, but we have chosen this ordering so as to present the main results immediately. This hopefully will serve as a motivation for the more tiresome work in the later sections. Since most readers are not machines, it is hoped that they can deal with this without a system failure.

In section 1 we discuss the categorical problems about enriched categories that pop up in the constructions, in particular colimits deserve some special care. In section 2 we discuss how much of the enrichment that survives to the category of enriched categories. The third section finally deals with homotopy issues. We do not claim any particular originality, and we have become aware that similar structures have been obtained by Shipley and Schwede, [SS].

Although it is true that $V\mathcal{O}$-categories are monoids in the monoidal category of $V\mathcal{O}$-graphs, we will not use this, and the difficulties pointed out by Shipley and Schwede relating to fact that the category of $V\mathcal{O}$-graphs is not symmetric is not relevant to this paper.

In addition to the usual homotopical structure, we need for our constructions in section 0 homotopy invariance of pushouts under certain flatness conditions on the underlying graphs. This is provided in section 3 using the description of the pushout given in section 1.

For certain applications, we must allow our functors to actually do things on objects too, and section 3 ends with few critical points related to this.

The last section makes more precise some notions pertaining to the category of pairs which was used in the localization functor.

Notation. This paper is written simplicially, meaning that the category of "spaces" $\mathcal{S}$ is the category of simplicial sets. If $\mathcal{C}$ is a V-category and c and d are two objects in $\mathcal{C}$, then we let $\mathcal{C}(c, d) \in obV$ denote the corresponding morphism object.

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0. The general procedure

We start off with the general construction, postponing the technical noise which is bound to follow with statements on this level of generality.

Let $(V, \Box, \mathbf{I})$ be a closed category (which is short for closed symmetric monoidal category, see [McL] or [B]). Then we may talk about V-categories or "categories enriched in V" (which instead of morphism sets have "morphism objects" in V, see next section for some more details).

Assume furthermore that V has a structure of a closed simplicial model category [Q] and that the two structures are suitably compatible and have certain reasonable properties (see section 3 where the terminology used in this section is explained, and where the "compatibility" and "reasonable properties" are made explicit). Such a V will be called a *monoidal simplicial model category*. The data are:

(1) a monoidal model category $(V, \Box, \mathbf{I})$ with $\mathbf{I}$ cofibrant

(2) a monoidal Quillen adjunction

$$(V, \square, \mathbf{I}) \underset{R}{\overset{\Sigma}{\leftrightarrows}} (\mathcal{S}, \times, *)$$

where $\mathcal{S}$ is the category of simplicial sets.

0.0 Examples. Examples of monoidal simplicial model categories include:

(1) simplicial sets,
(2) symmetric spectra [HSS],
(3) Γ-spaces and simplicial functors [L1], [L2] and [S]
(4) simplicial k-modules (for k a simplicial ring).

That

$$V \underset{R}{\overset{\Sigma}{\leftrightarrows}} \mathcal{S}$$

is a monoidal Quillen adjunction implies that the left adjoint Σ preserves weak equivalences and cofibrations, and that both functors are lax monoidal. This last point is only to ensure that they define functors between V-categories and $\mathcal{S}$-categories by applying Σ and R to the morphism spaces. In particular, if $\mathcal{W}$ is a $\mathcal{S}$-category $\Sigma\mathcal{W}$ is the V-category with the same objects as $\mathcal{W}$, but with morphism object $\Sigma(\mathcal{W}(c, c')) \in obV$ for $c, c' \in ob\mathcal{W}$.

Presumably this close connection to simplicial sets could be relaxed, and, indeed, part of what was interesting about Dwyer and Kan's construction was that it *provided* a simplicial structure. However, trying to avoid simplicial sets in the current construction seemed not to be worth the while. In most cases the input comes naturally with such a structure, and if not, one should be able to provide one.

The constructions we are going to present have the pleasant property that they leave the objects in our categories untouched.

0.1. Definition. *Let $\mathcal{O}$ be a set. A $V\mathcal{O}$-category is a V-category $\mathcal{C}$ with $ob\mathcal{C} = \mathcal{O}$, and a $V\mathcal{O}$-functor $\mathcal{C} \xrightarrow{F} \mathcal{D}$ between two $V\mathcal{O}$-categories is a V-functor which is the identity on objects. A $V\mathcal{O}$-functor $\mathcal{C} \xrightarrow{F} \mathcal{D}$ is a weak equivalence (resp. fibration) if $\mathcal{C}(c, c') \to \mathcal{D}(c, c')$ are weak equivalences (resp. fibrations) in V for all $c, c' \in \mathcal{O}$. We call a $V\mathcal{O}$-functor a cofibration if it has the left lifting property with respect to all $V\mathcal{O}$-functors that are both weak equivalences and cofibrations.*

In many examples the following is true (for definitions and the precise statement, see theorem 3.3. and the discussion preceding it). We list it as a pre-theorem since it guides the mind in the right direction, although the actual constructions use less structure.

0.2. Pre-theorem. *If V has a monoidal fibrant replacement functor, then the category of $V\mathcal{O}$-categories is a cofibrantly generated simplicial model category.*

However, apart from the existence of pushouts, the only consequence of this we actually need is that we can choose a functorial factorization such that any map f is factored as $p \circ i$ where i is a cofibration and p is both a weak equivalence and a fibration. Both these statements can be had with much less effort and fewer assumptions.

Let us list the properties we actually use:

0.3. Facts. *Let V be a monoidal simplicial model category. Then*

1. *the functor $\Sigma\colon \mathcal{S} \to V$ preserves weak equivalences and cofibrations. The unit* $\mathbf{I}$ *of the monoidal structure on V is cofibrant.*
2. *Corollary 1.2: The category of $V\mathcal{O}$-categories is small-cocomplete.*
3. *Lemma 3.4: The category of $V\mathcal{O}$-categories has functorial factorizations into cofibrations followed by maps that are both fibrations and weak equivalences.*
4. *Lemma 3.5: the gluing lemma (under certain flatness hypotheses pushouts $V\mathcal{O}$-categories of preserve weak equivalences).*
5. *Lemma 3.6: under certain assumptions on the domain, a cofibration of $V\mathcal{O}$-categories induces cofibrations on morphism objects.*

0.4. The localization. Begin construction. The localization depends on the choice of weak equivalences, and so is really a functor of pairs $(\mathcal{C}, w)$ where $\mathcal{C}$ is a $V\mathcal{O}$-category, $\mathcal{C}$ an $\mathcal{SO}$-category and $w\colon \Sigma\mathcal{W} \to \mathcal{C}$ is a $V\mathcal{O}$-functor See section 4 for more details on these pairs.

For any small $\mathcal{O}$-category $\mathcal{W}$ you may formally invert all morphisms and get a groupoid $\mathcal{W}[\mathcal{W}^{-1}]$ which is universal with respect to all functors from $\mathcal{W}$ into groupoids. If $\mathcal{W}$ is a $\mathcal{SO}$-category we let $\mathcal{W}[\mathcal{W}^{-1}]$ be what you get by inverting all morphisms in every degree.

To construct the localization $L(\mathcal{C}, w)$ of the pair $(\mathcal{C}, w)$, we first let

$$QW \overset{q_{\mathcal{W}}}{\underset{\sim}{\twoheadrightarrow}} \mathcal{W}$$

be the functorial cofibrant replacement of $\mathcal{W}$ within the category of $\mathcal{SO}$-categories (which exists by lemma 3.4. since $\mathcal{S}$ satisfies all properties we require of V).

Then, let

$$\Sigma Q\mathcal{W} \overset{B_w w}{\rightarrowtail} B_w\mathcal{C} \overset{\sim}{\twoheadrightarrow} \mathcal{C}$$

be the functorial factorization (3.4) in $V\mathcal{O}$-categories of the composite $\Sigma Q\mathcal{W} \xrightarrow{\Sigma q_{\mathcal{W}}} \Sigma\mathcal{W} \xrightarrow{w} \mathcal{C}$. This defines the pair $B(\mathcal{C}, w) = (B_w\mathcal{C}, B_w w)$ and a natural map of pairs $B(\mathcal{C}, w) \to (\mathcal{C}, w)$ which is a weak equivalence of pairs (i.e. both $B_w\mathcal{C} \overset{\sim}{\twoheadrightarrow} \mathcal{C}$ and $Q\mathcal{W} \overset{\sim}{\twoheadrightarrow} \mathcal{W}$ are weak equivalences, see section 4).

Then the *localization*

$$L(\mathcal{C}, w) = (L_w\mathcal{C}, L_w w)$$

of $(\mathcal{C}, w)$ is defined by the pushout

$$\begin{tikzcd}
\Sigma Q\mathcal{W} \arrow[r, "B_w w"] \arrow[d] & B_w\mathcal{C} \arrow[d] \\
\Sigma(Q\mathcal{W}[Q\mathcal{W}^{-1}]) \arrow[r, "L_w w"] & L_w\mathcal{C}
\end{tikzcd}$$

in V-categories with set of objects $\mathcal{O}$. END CONSTRUCTION.

The construction is homotopy invariant:

0.5. PROPOSITION. *If $(\mathcal{C}, w) \to (\mathcal{C}', w')$ is a weak equivalence, then $L(\mathcal{C}, w) \to L(\mathcal{C}', w')$ is a weak equivalence.*

PROOF. The relevant diagram looks as follows:

$$\begin{tikzcd}
 & \Sigma\mathcal{W} \arrow[dl] \arrow[dd] & & \Sigma Q\mathcal{W} \arrow[ll] \arrow[rr] \arrow[dl] \arrow[dd] & & \Sigma Q\mathcal{W}[Q\mathcal{W}^{-1}] \arrow[dl] \arrow[dd] \\
\mathcal{C} \arrow[dd] & & B_w\mathcal{C} \arrow[ll] \arrow[rr] \arrow[dd] & & L_w\mathcal{C} \arrow[dd] & \\
 & \Sigma\mathcal{W}' \arrow[dl] & & \Sigma Q\mathcal{W}' \arrow[ll] \arrow[rr] \arrow[dl] & & \Sigma Q\mathcal{W}'[(Q\mathcal{W}')^{-1}] \arrow[dl] \\
\mathcal{C}' & & B_{w'}\mathcal{C}' \arrow[ll] \arrow[rr] & & L_{w'}\mathcal{C}' &
\end{tikzcd}$$

The four horizontal maps pointing leftward are all weak equivalences (Σ preserves weak equivalences). The fact that the two leftmost vertical maps are weak equivalences imply that the two vertical maps in the middle are weak equivalences.

Since $Q\mathcal{W} \to Q\mathcal{W}'$ is a weak equivalence of cofibrant $\mathcal{SO}$-categories, we get by [DK, 9.4] that $Q\mathcal{W}[Q\mathcal{W}^{-1}] \to Q\mathcal{W}'[(Q\mathcal{W}')^{-1}]$ is a weak equivalence and injective on morphism spaces. Hence, $\Sigma Q\mathcal{W}[Q\mathcal{W}^{-1}] \to \Sigma Q\mathcal{W}'[(Q\mathcal{W}')^{-1}]$ is a weak equivalence and a cofibration on all morphism objects (Σ preserves cofibrations and weak equivalences). By the definition of $L_w\mathcal{C}$ by means of the pushout the two rightmost vertical maps must then also be weak equivalences (using lemma 3.6, the conditions of lemma 3.5 are readily checked in the current application). ∎

The importance of this construction is that we have replaced $\mathcal{W}$ by a simplicial groupoid, or loosely: we have "inverted all the morphisms in $\mathcal{C}$ coming from $\mathcal{W}$". This is important in many applications. For instance, Dwyer and Kan show that if $V = \mathcal{S}$ and $\mathcal{C}$ is a simplicial model category with weak equivalences $\mathcal{W}$, then $\pi_0 L_w\mathcal{C}$ is nothing but the homotopy category of $\mathcal{C}$, and $L_w\mathcal{C}$ is equivalent to the full subcategory of $\mathcal{C}$ whose objects are both fibrant and cofibrant.

0.6. DEFINITION. *Let $\mathcal{B}$ be a $\mathcal{S}$-category. The category $\pi_0\mathcal{B}$ is the category with the same objects as $\mathcal{B}$, but with morphism sets from a to b the set of path components $\pi_0\mathcal{B}(a, b)$. We say that a $\mathcal{S}$-category $\mathcal{B}$ is groupoid-like if $\pi_0\mathcal{B}$ is a groupoid.*

0.7. THEOREM. *Let $(\mathcal{C}, w)$ be as above. If $\mathcal{W}$ is groupoid-like, then*

$$(\mathcal{C}, w) \leftarrow B(\mathcal{C}, w) \to L(\mathcal{C}, w)$$

are weak equivalences. In particular

$$\mathcal{C} \leftarrow B_w\mathcal{C} \rightarrow L_w\mathcal{C}$$

are weak equivalences.

PROOF. By construction $(\mathcal{C}, w) \leftarrow B(\mathcal{C}, w)$ is a weak equivalence (this is is always true). If $\mathcal{W}$ is groupoid-like, then $Q\mathcal{W}$ is groupoid-like and by [DK, 9.5] the functor $Q\mathcal{W} \rightarrow Q\mathcal{W}[Q\mathcal{W}^{-1}]$ is a weak equivalence, and an inclusion (i.e. cofibration) on every morphism space. Since $\Sigma\colon \mathcal{S} \rightarrow V$ preserves cofibrations and weak equivalences, and since by lemma 3.6 $\Sigma Q\mathcal{W} \rightarrow B_w\mathcal{C}$ is a cofibration on morphism objects, the gluing lemma 3.5 assures that $B_w\mathcal{C} \rightarrow L_w\mathcal{C}$ is a weak equivalence. ∎

It is important for various applications that if $\mathcal{W}$ already consists of isomorphisms, then the localization does not mess up things too badly. The theorem assures that localization does not change the homotopy type, and the following corollary says that there actually is a map between them (and not just in the homotopy category).

0.8. COROLLARY. *On the category of pairs $(\mathcal{C}, w\colon \Sigma\mathcal{W} \rightarrow \mathcal{C})$ such that $\mathcal{W}$ is a (simplicial) groupoid there is a natural weak equivalence*

$$L(\mathcal{C}, w) \xrightarrow{\sim} (\mathcal{C}, w)$$

such that

$$\begin{tikzcd} L(\mathcal{C}, w) \arrow[dr] & B(\mathcal{C}, w) \arrow[l] \arrow[d] \\ & (\mathcal{C}, w) \end{tikzcd}$$

commutes.

PROOF. Note that since all the morphisms in $\mathcal{W}$ are isomorphisms $Q\mathcal{W} \rightarrow \mathcal{W}$ factors naturally through $Q\mathcal{W} \rightarrow Q\mathcal{W}[Q\mathcal{W}^{-1}]$, and so by the universal property of the pushout

$$\begin{tikzcd} \Sigma Q\mathcal{W} \arrow[r, "B_w w"] \arrow[d] & B_w\mathcal{C} \arrow[d] \\ \Sigma(Q\mathcal{W}[Q\mathcal{W}^{-1}]) \arrow[r, "L_w w"] & L_w\mathcal{C} \end{tikzcd}$$

we get the desired map. The map is a weak equivalence by theorem 0.7. ∎

This construction also inherits the "closure properties" of [DK]. Recall that if $\mathcal{E}$ is a small category, a subcategory $\mathcal{F} \subseteq \mathcal{E}$ is said to be *closed* if $\mathcal{F}$ consists of exactly those morphisms in $\mathcal{E}$ that are mapped to isomorphisms under the functor $\mathcal{E} \rightarrow \mathcal{E}[\mathcal{F}^{-1}]$. The *closure* $\overline{\mathcal{F}}$ of a subcategory $\mathcal{F} \subseteq \mathcal{E}$ is the smallest closed subcategory of $\mathcal{E}$ containing $\mathcal{F}$.

0.9. Theorem. *Let $\mathcal{C}$ be a $V\mathcal{O}$-category, let $\mathcal{W}_1 \xrightarrow{f} \mathcal{W}_2 \xrightarrow{g} R\mathcal{C}$ be $\mathcal{SO}$-functors (R is the right adjoint of Σ), let $w_2: \Sigma\mathcal{W}_2 \to \mathcal{C}$ be the adjoint of g and let $w_1: \Sigma\mathcal{W}_1 \to \mathcal{C}$ be the adjoint of gf. Then f defines a map $(\mathcal{C}, w_1) \to (\mathcal{C}, w_2)$. The induced functor*

$$L_{w_1}\mathcal{C} \to L_{w_2}\mathcal{C}$$

is a weak equivalence if and only if

$$im\{\pi_0 g: \pi_0\mathcal{W}_2 \to \pi_0 R\mathcal{C}\} \subseteq \overline{im\{\pi_0 gf: \pi_0\mathcal{W}_1 \to \pi_0 R\mathcal{C}\}}$$

Proof. Let $Q\mathcal{W}_1 \rightarrowtail P \overset{\sim}{\twoheadrightarrow} Q\mathcal{W}_2$ be the functorial factorization of $Q\mathcal{W}_1 \xrightarrow{Qf} Q\mathcal{W}_2$ (where Q is the cofibrant replacement functor in $\mathcal{SO}$-categories). Then P is cofibrant and $P \overset{\sim}{\twoheadrightarrow} \mathcal{W}_2$ is a weak equivalence. Let $Q\mathcal{W}_i \rightarrowtail T_i \overset{\sim}{\twoheadrightarrow} R\mathcal{C}$ (resp. $P \rightarrowtail T \overset{\sim}{\twoheadrightarrow} R\mathcal{C}$) be the functorial factorization of $Q\mathcal{W}_i \overset{\sim}{\twoheadrightarrow} \mathcal{W}_i \to R\mathcal{C}$ (resp. $P \overset{\sim}{\twoheadrightarrow} Q\mathcal{W}_2 \overset{\sim}{\twoheadrightarrow} \mathcal{W}_2 \xrightarrow{g} R\mathcal{C}$). Then we have a sequence of pushouts

$$\begin{aligned} Q\mathcal{W}_1[Q\mathcal{W}_1^{-1}] \coprod_{Q\mathcal{W}_1} T_1 \to Q\mathcal{W}_1[Q\mathcal{W}_1^{-1}] \coprod_{Q\mathcal{W}_1} T \\ \to P[P^{-1}] \coprod_{P} T \to Q\mathcal{W}_2[Q\mathcal{W}_2^{-1}] \coprod_{Q\mathcal{W}_2} T_2 \end{aligned}$$

induced from $Q\mathcal{W}_1 \rightarrowtail P \overset{\sim}{\twoheadrightarrow} Q\mathcal{W}_2$ and $T_1 \overset{\sim}{\to} T \overset{\sim}{\to} T_2$ (from the functoriality of the factorizations). The outer maps are weak equivalences by the gluing lemma (which is true in $\mathcal{SO}$-categories) and the middle map is an equivalence by [DK, 10.5].

Let $\Sigma T_i \rightarrowtail B_i \overset{\sim}{\twoheadrightarrow} \mathcal{C}$ be the functorial factorization of $\Sigma T_i \overset{\sim}{\to} \Sigma R\mathcal{C} \to \mathcal{C}$. This induces a weak equivalence $B_{w_i}\mathcal{C} \overset{\sim}{\to} B_i$ under $\Sigma Q\mathcal{W}_i$. Define L_i by the pushouts

$$\begin{tikzcd} \Sigma Q\mathcal{W}_i \arrow[r] \arrow[d] & \Sigma T_i \arrow[r] \arrow[d] & B_i \arrow[d] \\ \Sigma(Q\mathcal{W}_i[Q\mathcal{W}_i^{-1}]) \arrow[r] & \Sigma L_i \arrow[r] & L_i \end{tikzcd}$$

(the leftmost square is a pushout since Σ is left adjoint). The induced map $L_{w_i}\mathcal{C} \to L_i$ is a weak equivalence by the gluing lemma. That $L_1 \to L_2$ is a weak equivalence now follows since $L_1^s \to L_2^s$ is a weak equivalence. ∎

This theorem implies that we may assume without loss of generality that the adjoint of w is an inclusion of a subcategory $\mathcal{W} \subseteq R\mathcal{C}$ such that $\pi_0\mathcal{W}$ is closed in $\pi_0 R\mathcal{C}$.

0.10. Further functoriality. The construction above may seem restrictive, due to the fact that we do not allow our set of objects to vary. This is superficial due to the following considerations.

Let $\mathfrak{P}$ be the category whose objects are pairs $(\mathcal{C}, w)$ where $\mathcal{C}$ is a small V-category and $w: \Sigma\mathcal{W} \to \mathcal{C}$ is a $V(ob\mathcal{C})$-functor. A morphism $(\mathcal{C}, w) \to (\mathcal{C}', w')$ is a $\mathcal{S}$-functor $WF: \mathcal{W} \to \mathcal{W}'$ and a V-functor $F: \mathcal{C} \to \mathcal{C}'$ such that $Fw = w'\Sigma WF$. See section 4 for further details.

0.11. Lemma. *The localization extends to functors*

$$B, L\colon \mathfrak{P} \to \mathfrak{P}$$

connected by natural transformations

$$id \leftarrow B \to L$$

On the subcategory of pairs $(\mathcal{C}, w\colon \Sigma\mathcal{W} \to \mathcal{C})$ *such that* $\mathcal{W}$ *consists of isomorphisms there is a natural weak equivalence* $L(\mathcal{C}, w) \to (\mathcal{C}, w)$ *such that*

$$\begin{tikzcd}
(\mathcal{C}, w) \arrow[dr] & B(\mathcal{C}, w) \arrow[l] \arrow[d] \\
 & (\mathcal{C}, w)
\end{tikzcd}$$

commutes.

Proof. This follows by the construction of the localization, plus lemma 1.5 and lemma 3.8. ∎

1. $V\mathcal{O}$-categories and graphs

To set the notation we briefly recall the relevant definitions of enriched categories, and make the relationship to graphs explicit.

Let V be a category. A V-graph $\mathcal{G}$ is a class of "objects" $ob\mathcal{G}$ together with a choice of "morphism spaces" $\mathcal{G}(c, d) \in obV$ for each ordered pair (c, d) of elements in $ob\mathcal{G}$. A map of V-graphs $F\colon \mathcal{G} \to \mathcal{H}$ is a function $F\colon ob\mathcal{G} \to ob\mathcal{H}$, together with maps $\mathcal{G}(c, d) \to \mathcal{H}(F(c), F(d))$ in V.

If V is furthermore equipped with a monoidal structure, $(V, \square, \mathbf{I})$, we may talk about V-categories. A V-category $\mathcal{C}$ is a V-graph together with units $\mathbf{I} \to \mathcal{C}(c, c)$ and composition $\mathcal{C}(c, d)\square\mathcal{C}(b, c) \to \mathcal{C}(b, d)$ satisfying the usual unital and associativity conditions. A *V-functor* F from $\mathcal{C}$ to $\mathcal{D}$ is an assignment $ob\mathcal{C} \to ob\mathcal{D}$ together with maps $\mathcal{C}(c, c') \to \mathcal{D}(F(c), F(c'))$ preserving unit and composition. A V-natural transformation between two V-functors $F, G\colon \mathcal{C} \to \mathcal{D}$ is a collection of maps $\eta_c\colon F(c) \to G(c) \in \mathcal{D}$ indexed by the objects of $\mathcal{C}$ such that the following diagram in V commutes

$$\begin{tikzcd}
\mathcal{C}(c, c') \arrow[r] \arrow[d] & \mathcal{D}(F(c), F(c')) \arrow[d, "(\eta_{c'})_*"] \\
\mathcal{D}(G(c), G(c')) \arrow[r, "(\eta_c)^*"] & \mathcal{D}(F(c), G(c'))
\end{tikzcd}$$

1.0. The free/forgetful adjoint pair. If V has finite coproducts (which we will call $\vee$), and the monoidal product (which we will call $\square$) is naturally distributive over the coproduct, then the forgetful functor

$$U\colon V\text{-categories} \to V\text{-graphs}$$

has a left adjoint, namely the free functor F which on an V-graph $\mathcal{G}$ is the V-category $F\mathcal{G}$ with the same objects as $\mathcal{G}$, but with morphism spaces given by

$$F\mathcal{G}(c,d) = \begin{cases} \bigvee_{q\geq 0}\bigvee_{c_1,\ldots,c_q\in ob\mathcal{G}} \mathcal{G}(c_1,d)\Box\mathcal{G}(c_2,c_1)\ldots\Box\mathcal{G}(c,c_q) & \text{if } c\neq d \\ \mathbf{I}\vee\bigvee_{q\geq 0}\bigvee_{c_1,\ldots,c_q\in ob\mathcal{G}} \mathcal{G}(c_1,c)\Box\mathcal{G}(c_2,c_1)\ldots\Box\mathcal{G}(c,c_q) & \text{if } c=d \end{cases}$$

The composition is given by concatenation of terms, with the $\mathbf{I}\subseteq F\mathcal{G}(d,d)$ acting as the unit via $\mathbf{I}\Box\mathcal{G}(c,d)\cong\mathcal{G}(c,d)$. The units of adjunction $\mathcal{G}\to UF\mathcal{G}$ and $FU\mathcal{C}\to\mathcal{C}$ are given by the obvious inclusion and composition. Note that the requirement that $\Box$ is distributive over $\vee$ is automatic if V is closed (symmetric monoidal), since $-\Box v$ is a left adjoint and so preserve all colimits, and likewise in the other factor because of symmetry.

Let $\mathcal{O}$ be a set. A $V\mathcal{O}$-graph $\mathcal{G}$ is simply a V-graph with $ob\mathcal{G}=\mathcal{O}$, but a map of $V\mathcal{O}$-graphs is a map of V-graphs which is the *identity* on objects. Otherwise said, a morphism $\mathcal{G}\to\mathcal{H}$ of $V\mathcal{O}$-graphs is a collection of morphisms $\mathcal{G}(c,d)\to\mathcal{H}(c,d)\in V$. This forms a category we will call $V\mathcal{O}$-graphs (which is isomorphic to $V^{\mathcal{O}\times\mathcal{O}}$).

Likewise, if V is monoidal, a $V\mathcal{O}$-category is a V-category $\mathcal{C}$ with $ob\mathcal{C}=\mathcal{O}$, and $V\mathcal{O}$-functors are demanded to be the identity on objects. This forms a category we will call $V\mathcal{O}$-categories. A natural transformation $\eta\colon F\to G$ (or rather, a $V\mathcal{O}$-natural transformation) between two $V\mathcal{O}$-functors is the same as a V-natural transformation (note however that η_c is always an endomorphism since $F(c)=G(c)=c$).

Note that the above free and forgetful functors restrict to an adjoint pair of functors between $V\mathcal{O}$-categories and $V\mathcal{O}$-graphs.

Limits of $V\mathcal{O}$-categories are limits in V-categories and are formed at each morphism object, i.e. if $X\colon J\to V\mathcal{O}$-categories is a functor, then $\varprojlim_{j\in J} X_j$ (if it exists) is the $V\mathcal{O}$-category with morphism objects

$$\left(\varprojlim_{j\in J} X_j\right)(c,c') = \varprojlim_{j\in J}(X_j(c,c'))$$

Hence the category of $V\mathcal{O}$-category is closed if V is closed.

1.1. Lemma. *If V is a closed category, then the forgetful functor*

$$U\colon V\mathcal{O}-\textit{categories}\to V\mathcal{O}-\textit{graphs}$$

preserves and creates filtered colimits.

Proof. Let J be filtered and X a functor from J to $V\mathcal{O}$-categories. Then we define the composition on the graph $\varinjlim_{j\in J} UX_j$ via

$$\begin{tikzcd} \left(\varinjlim_{j\in J} X_j(c,d)\right)\Box\left(\varinjlim_{j\in J} X_j(\overset{\cong}{\overline{b}},c)\right) & \varinjlim_{(i,j)\in J\times J}(X_i(c,d)\Box X_j(b,c)) \arrow[l] \\ \varinjlim_{j\in J} X_j(b,d) & \varinjlim_{j\in J}(X_j(c,d)\Box X_j(b,c)) \arrow[l] \arrow[u, "\cong"'] \end{tikzcd}$$

where the first isomorphism is due to the fact that V is closed, and the second is an isomorphism since J is filtered and hence the diagonal $J \to J \times J$ is final. The unit is induced by $I \to X_j(c,c) \to \lim_{\overrightarrow{j\in J}} X_j(c,c)$. ∎

1.2. Corollary. *If V is a cocomplete closed category, then the category of $V\mathcal{O}$-categories is cocomplete.*

Proof. This follows by [B, 4.3.6] and lemma 1.1, since the triple UF preserves filtered colimits and $V\mathcal{O}$-categories are the UF-algebras in the category of $V\mathcal{O}$-graphs. ∎

Another application of lemma 1.1 is the following observation

1.3. Lemma. *Let $F\colon \mathcal{C} \to \mathcal{D}$ be a functor from a category $\mathcal{C}$ to a category $\mathcal{D}$. Assume F has a right adjoint which preserves filtered colimits. Let $d \in obD$ be an object which is small relative to $\mathcal{D}$. Then $F(d)$ is small relative to $\mathcal{C}$.*

Proof. Let U be the right adjoint. Let $X\colon K \to \mathcal{C}$ be a functor and K filtering. Then

$$\mathcal{C}(Fd, \lim_{\overrightarrow{k\in K}} X_k) \cong \mathcal{D}(d, U \lim_{\overrightarrow{k\in K}} X_k) \cong \mathcal{D}(d, \lim_{\overrightarrow{k\in K}} UX_k)$$

Since d is small relative to $\mathcal{D}$, there is a regular cardinal κ such that for every regular cardinal $\lambda \geq \kappa$ and any functor $Y\colon \lambda \to \mathcal{D}$ the canonical map

$$\mathcal{D}(d, \lim_{\overrightarrow{\lambda}} Y) \xrightarrow{\cong} \lim_{\overrightarrow{\lambda}} \mathcal{D}(d, Y)$$

is an isomorphism. Hence for every functor $X\colon \lambda \to \mathcal{C}$ for $\lambda \geq \kappa$ the map

$$\mathcal{D}(d, \lim_{\overrightarrow{\lambda}} UX) \xrightarrow{\cong} \lim_{\overrightarrow{\lambda}} \mathcal{D}(d, UX) \cong \lim_{\overrightarrow{\lambda}} \mathcal{C}(Fd, X)$$

is an isomorphism, and we are done. ∎

1.4. Pushouts. This paper relies heavily on a close control over the pushouts in the category of $V\mathcal{O}$-categories. Before we start to describe these in detail, we note the following fact which is useful for extending the functoriality of the localization

1.5. Lemma. *The forgetful functor*

$$V\mathcal{O}\text{-categories} \to V\text{-categories}$$

preserves pushouts.

Proof. Let

$$\begin{array}{ccc} \mathcal{C}_0 & \xrightarrow{f_Y} & \mathcal{C}_Y \\ {\scriptstyle f_X}\downarrow & & \downarrow{\scriptstyle g_X} \\ \mathcal{C}_X & \xrightarrow{g_Y} & \mathcal{C}_{XY} \end{array} \qquad \in V\mathcal{O}\text{-categories}$$

be a pushout square in $V\mathcal{O}$-categories, and let

$$\begin{tikzcd} \mathcal{C}_0 \arrow[r, "f_Y"] \arrow[d, "f_X"'] & \mathcal{C}_Y \arrow[d, "h_X"] \\ \mathcal{C}_X \arrow[r, "h_Y"] & \mathcal{C} \end{tikzcd} \qquad \in V\text{-categories}$$

be a commutative diagram in V-categories. We must show that there is a unique $\mathcal{C}_{XY} \xrightarrow{h} \mathcal{C}$ with $hg_Y = h_Y$ and $hg_X = h_X$.

Let $\mathcal{D} \in V\mathcal{O}$-categories be given by $\mathcal{D}(c,d) = \mathcal{C}(h_X c, h_X d)$. This is clearly well defined, and we have a factorization

$$\mathcal{C}_X \xrightarrow{h'_X} \mathcal{D} \xrightarrow{i} \mathcal{C}$$

of h_X with i induced by $\mathcal{D}(c,d) = \mathcal{C}(h_X c, h_X d)$. Since h_X and h_Y are equal on objects (since the diagram is commutative, and f_X and f_Y are both the identity on objects) we also get a factorization

$$\mathcal{C}_Y \xrightarrow{h'_Y} \mathcal{D} \xrightarrow{i} \mathcal{C}$$

of h_Y such that

$$\begin{tikzcd} \mathcal{C}_0 \arrow[r, "f_Y"] \arrow[d, "f_X"'] & \mathcal{C}_Y \arrow[d, "h'_X"] \\ \mathcal{C}_X \arrow[r, "h'_Y"'] & \mathcal{D} \end{tikzcd} \qquad \in V\mathcal{O}\text{-categories}$$

commutes. Hence there is a unique map $\mathcal{C}_{XY} \xrightarrow{h'} \mathcal{D} \in V\mathcal{O}$-categories such that $h'g_X = h'_X$ and $h'g_Y = h'_Y$. The desired universal map is thus $\mathcal{C}_{XY} \xrightarrow{h'} \mathcal{D} \xrightarrow{i} \mathcal{C}$. This is universal since any map $\mathcal{C}_{XY} \to \mathcal{C}$ must factor through $\mathcal{D} \xrightarrow{i} \mathcal{C}$. ∎

We also need an explicit description of the pushout in $V\mathcal{O}$-categories. We do this by describing the morphism objects of a pushout by means of the colimit over a quite unruly diagram in V. The diagram is not directed, but it is a "Reedy" diagram which still makes it possible to get homotopical control over the pushout.

Remember what the "amalgamated product" looks like: if $A_X \leftarrow A_0 \rightarrow A_Y$ is a diagram of rings, we could try to model the pushout as the colimit of a diagram of abelian groups. It certainly should contain "words" in A_X and A_Y, where adjacent letters from the same ring should be multiplied, and elements from A_0 should be free to choose whether they wanted to be imaged in A_X or A_Y. Thus the pushout is going to be a colimit of a diagram of large tensor products of A_X, A_Y and A_0. Hence the need for an indexing category codifying multiplication, insertion of units and the maps

$A_X \leftarrow A_0 \rightarrow A_Y$. This is our next task, complicated by the fact that we must allow many objects (which is kind of nice since "insertion of units" are clearly marked by repetition of objects)

1.6. THE INDEXING CATEGORY. The rest of section 1 deals with the technical details of how best to describe the pushout. All of it is used in the proof of lemma 3.5 and 3.6 below, but is not needed elsewhere in this paper.

We use the usual notation with Δ the category of nonempty finite ordered sets. For every natural number q we let $[q] \in \Delta$ be the ordered set $\{0 < 1 < \cdots < q\}$, and for every $0 \leq i \leq q$ we let $d^i \in \Delta([q-1],[q])$ be the monomorphism that skips $i \in [q]$. We consider Δ as a full subcategory of the category of categories: a finite ordered set S is identified with the category whose objects are the elements of S, and which has a morphism $i \leftarrow j$ whenever $i < j$.

Let $\mathcal{O}$ be a set, $c, d \in \mathcal{O}$, and let $\mathcal{O}^{c,d}$ be the following category. The objects of $\mathcal{O}^{c,d}$ are triples (i, C, q) where q is a natural number and i and C are functions (the ordering is not used yet!)

$$i\colon [q] \rightarrow \{0, X, Y\}, \quad C\colon \Delta([q],[1]) \rightarrow \mathcal{O}$$

such that

$$\Delta([0],[1]) \rightarrow \Delta([q],[1]) \xrightarrow{C} \mathcal{O}$$

sends $d^0\colon [0] \rightarrow [1]$ to d and $d^1\colon [0] \rightarrow [1]$ to c. In other words

$$ob\mathcal{O}^{c,d} = \coprod_{q\geq 0} \{0, X, Y\}^{[q]} \times \mathcal{O}^{\Delta([q],[1])} \times_{\mathcal{O}^{\Delta([0],[1])}} \{c, d\}$$

We will often refer to the values $i_k = i(k)$ as the "indices" of $\alpha = (i, C, q)$ and the values of C as the "objects".

To each object $\alpha = (i, C, q)$ we attach a number

$$deg(\alpha) = \sum_{k=1}^{q} deg(i_k)$$

which we call the *degree* of α, where $deg(0) = 1$ and $deg(X) = deg(Y) = 2$. In fact, we consider $\{0, X, Y\}$ to be the objects of the category

$$X \leftarrow 0 \rightarrow Y$$

and for two functions $i, j\colon [q] \rightarrow \{0, X, Y\}$ we write $i \leq j$ if for all $k \in [q]$ there is a map from i_k to j_k in $X \leftarrow 0 \rightarrow Y$. Note that X and Y are not commensurable.

Let $\alpha = (i, C, q)$ and $\beta = (j, D, p)$ be objects of $\mathcal{O}^{c,d}$. Then the set of morphisms from α to β is the set of weakly monotone maps $\phi\colon [q] \rightarrow [p]$ having the property that D is the composite $\Delta([p],[1]) \xrightarrow{\phi^*} \Delta([q],[1]) \xrightarrow{C} \mathcal{O}$ and for all $k \in [q]$ we have that $i_k \leq j_{\phi(k)}$. For the record we display this as

$$\mathcal{O}^{c,d}(\alpha, \beta) = \{\phi \in \Delta([q],[p]) \mid D = C\phi^*, i \leq j\phi\}$$

Note that this last condition implies that for every $j \in [p]$ all the i_k's with $k \in \phi^{-1}(j)$ must be commensurable (X and Y do not both appear).

The category $\mathcal{O}^{c,d}$ enjoys the property that all morphisms can be uniquely factored into the composition of one map coming from a subcategory $\overleftarrow{\mathcal{O}}^{c,d}$ followed by a map in a subcategory $\overrightarrow{\mathcal{O}}^{c,d}$. We proceed to describe these subcategories.

Restricting our attention to the epimorphisms in Δ we get the *inverse category* $\overleftarrow{\mathcal{O}}^{c,d}$ if we do not allow any degrees to rise. More precisely, a map from $\alpha = (i, C, q)$ to $\beta = (j, D, p)$ in $\mathcal{O}^{c,d}$ is in the inverse category if

(1) the corresponding $\phi \in \Delta([q], [p])$ is an epimorphism and
(2) $i_k = j_{\phi(k)}$ for all $k \in [q]$

All non-identities in $\overleftarrow{\mathcal{O}}^{c,d}(\alpha, \beta)$ decrease the degree (since some i_k's are deleted). The inverse category is going to control the multiplications in the coproduct.

Likewise, restricting to the monomorphisms in Δ we get the *direct category* $\overrightarrow{\mathcal{O}}^{c,d}$ (here there are no further conditions on the behavior of the maps). It is worthwhile to notice that the maps that are identities when just considered as maps in Δ need not be so in $\overrightarrow{\mathcal{O}}^{c,d}$: source and target may still disagree and the map may just increase the degree of the indices i_k. These maps correspond to the maps $\mathcal{C}_X \leftarrow \mathcal{C}_0 \rightarrow \mathcal{C}_Y$ in the coproduct. The other maps in the direct category will involve insertion of units as well. All the non-identities in the direct category increase the degree.

Given a map $g: \alpha \rightarrow \beta \in \mathcal{O}^{c,d}$ we get a unique factorization $g = \overrightarrow{g} \circ \overleftarrow{g}$ where $\overrightarrow{g} \in \overrightarrow{\mathcal{O}}^{c,d}$ and $\overleftarrow{g} \in \overleftarrow{\mathcal{O}}^{c,d}$ (first delete all repetitions that are to be deleted to get from α to β, and then expand the length and raise the degrees).

For homotopical reasons the connectivity of under categories of the inverse category is important. The following obvious lemma will later be interpreted as "$\mathcal{O}^{c,d}$ has fibrant constants".

1.7. Lemma. *Let $\alpha \in ob\mathcal{O}^{c,d}$. Then the under category $\alpha/\overleftarrow{\mathcal{O}}^{c,d}$ has a final object given by deleting all possible repetitions of indices in α.*

1.8. The direct subcategory. We also need to understand the over categories $\overrightarrow{\mathcal{O}}^{c,d}/\alpha$. What objects can map to $\alpha = (i, C, q)$ via non-identity maps in the direct categories? For sure, they must have lower degree than α, and can leave a trace in α in the form of a repetition of c_k's or an index $i_k \neq 0$. In fact, for most practical purposes, the most important thing about $\overrightarrow{\mathcal{O}}^{c,d}/\alpha$ is a manageable subcategory which we call $\mathcal{F}^{c,d}_{\alpha}$. This can be described as follows:

Let
$$S(\alpha) = [q] - i^{-1}(0), \quad T(\alpha) = i^{-1}(0) \cap \{k \in [q] | C = C(d^k s^k)^*\}$$
and $U(\alpha) = S(\alpha) \cup T(\alpha) \subseteq [q]$. Given a subset $u \subseteq U(\alpha)$ let $u_S = u \cap S(\alpha)$ and $u_T = u \cap T(\alpha) = \{k_1 < \cdots < k_t\}$. Define p, $\psi_u : [q] \rightarrow [p]$ and $\phi_u : [p] \rightarrow [q]$ by

(1) $[p] \cong [q] - u_T$,

(2) $\psi_u = s^{k_1} \dots s^{k_t}$ and
(3) $\phi_u: [p] \cong [q] - u_T \subseteq [q]$.

Let

$$\beta_u = (i\psi_u, C(\psi_u)^*, p) \xrightarrow{f_u} \alpha \in \overrightarrow{\mathcal{O}}^{c,d}$$

where f_u corresponds to ϕ_u. By the definition of $T(\alpha)$ this is well defined since $\psi_u\phi_u = id_{[p]}$. The increases in indices is encoded in u_S.

Furthermore, if $v \subset u$, there is a unique map $f^u_v: \beta_v \to \beta_u \in \overrightarrow{\mathcal{O}}^{c,d}$ such that $f_v = f_u f^u_v$.

Now, let $\mathcal{F}^{c,d}_\alpha \subset \overrightarrow{\mathcal{O}}^{c,d}/\alpha$ be the subcategory with the f_u's as objects and the f^u_v's as maps. We see that $\mathcal{F}^{c,d}_\alpha$ is isomorphic to the opposite of the category of subsets of $U(\alpha)$.

If $U(\alpha) = \emptyset$ we see that $\mathcal{F}^{c,d}_\alpha = \overrightarrow{\mathcal{O}}^{c,d}/\alpha = id_\alpha$, but if $U(\alpha) \neq \emptyset$ then any non-identity map $\gamma \xrightarrow{g} \alpha \in \overrightarrow{\mathcal{O}}^{c,d}$ defines a proper subset $u = u_S \cup u_V \subseteq U(\alpha)$ where u_S are the indices that are increased (either from zero, or from not originally being there) and u_V are the objects that are repeated. Note that the indices corresponding to u_V may be nonzero. However, g factors as $\gamma \xrightarrow{g_1} \beta_u \xrightarrow{f_u} \alpha$, and this factorization is unique in the sense that for any other factorization $\gamma \xrightarrow{g_2} \beta_v \xrightarrow{f_v} \alpha$ with $\emptyset \neq v \subset U(\alpha)$ we have that $v \subset u$ and $g_1 = f^v_u g_2$:

$$\begin{tikzcd}
\gamma \arrow[r, "g_1"] \arrow[d, "g_2"'] & \beta_u \arrow[dl, "f^u_v"'] \arrow[d, "f^u"] \\
\beta_v \arrow[r, "f_v"] & \alpha
\end{tikzcd}$$

The important outcome of all this is

1.9. Lemma. *Let $\alpha \in \mathcal{O}^{c,d}$. Then*

$$\mathcal{F}^{c,d}_\alpha - id_\alpha \subseteq \overrightarrow{\mathcal{O}}^{c,d}/\alpha - id_\alpha$$

is a final subcategory [McL, p 213], *and $\mathcal{F}^{c,d}_\alpha$ isomorphic to a cube with id_α as the final object.*

1.10. An explicit description of the morphism objects in a pushout of $V\mathcal{O}$-categories. Given a diagram

$$\mathcal{C}_X \leftarrow \mathcal{C}_0 \rightarrow \mathcal{C}_Y$$

of $V\mathcal{O}$-categories, we now give an interpretation of the morphism objects from c to d of the pushout category $\mathcal{C}_X \coprod_{\mathcal{C}_0} \mathcal{C}_Y$ in terms of a functor $\mathcal{C}^{c,d}$ from $\mathcal{O}^{c,d}$ to V. Let $\mathbf{I}\mathcal{O}$ be the initial $V\mathcal{O}$-category with

$$\mathbf{I}\mathcal{O}(c,d) = \begin{cases} \mathbf{I}, & \text{if } c = d \\ \text{the initial object } * \in V, & \text{otherwise} \end{cases}$$

If $\alpha = (i, C, q) \in ob\mathcal{O}^{c,d}$, let $C_k = C(\phi_k)$ where $\phi_k\colon [q] \to [1]$ which takes the value 0 exactly k times. Then

$$\mathcal{C}^{c,d}(\alpha) = \Box_{k=0}^{q} \mathcal{C}_{i_k}(C_{k+1}, C_k)$$

with the maps in $\mathcal{O}^{c,d}$ being sent to maps induced by the insertion of the unit $\mathbf{I}$, the maps

$$\mathcal{C}_X \leftarrow \mathcal{C}_0 \rightarrow \mathcal{C}_Y$$

and composition (when made possible by repetitions of indices). Let

$$\mathcal{C}_{XY}(c, d) = \varinjlim_{\alpha \in \mathcal{O}^{c,d}} \mathcal{C}^{c,d}(\alpha) \in V$$

We must show that this defines a $V\mathcal{O}$-category $\mathcal{C}_{XY}$, and that $\mathcal{C}_{XY} \cong \mathcal{C}_X \coprod_{\mathcal{C}_0} \mathcal{C}_Y$. The identity of $\mathcal{C}_{XY}$ is given by the canonical map

$$\mathbf{I}\mathcal{O}(c, d) \to \mathcal{C}_0(c, d) = \mathcal{C}^{c,d}(0, (c, d), 0)$$

and the composition is given by concatenation

$$\mathcal{C}^{c,d}(\alpha) \Box \mathcal{C}^{b,c}(\beta) \stackrel{\cong}{\to} \mathcal{C}^{b,d}(\alpha \sqcup \beta)$$

where $(i, C, q) \sqcup (j, D, p) = (i \sqcup j, C \sqcup D, q + p + 1)$,

$$[q] \sqcup [p] \cong [q + p + 1]$$

is concatenation (each element of $[p]$ is greater than each element in $[q]$),

$$i \sqcup j\colon [q] \sqcup [p] \to \{0, X, Y\}$$

is i on $[q]$ and j on $[p]$ and

$$C \sqcup D\colon \Delta([q] \sqcup [p], [1]) \to \mathcal{O}$$

is induced by the map $\Delta([q] \sqcup [p], [1]) \to \Delta([q], [1]) \vee \Delta([p], [1])$ gotten by observing that any weakly monotone map $[q] \sqcup [p] \to [1]$ must be constantly equal to 0 on $[q]$ or be constantly equal to 1 on $[p]$ (so the $\vee$ is over two different constant maps) and the fact that C and D both equal to c on the critical spot.

The associativity and unital axioms for $\mathcal{C}_{XY}$ are be fairly obvious, and likewise that $\mathcal{C}_{XY}$ actually describes the pushout.

2. Inheritance of enrichment

In the following section we will establish some properties of the free/forgetful pair between graphs and categories which will assure that the category of $V\mathcal{O}$-categories is an $\mathcal{S}$-category. This is used only in theorem 3.3. and theorem 4.1, and so is not a prerequisite for the localization construction given in section 0.

The category of monoids in a monoidal category $(V, \square, \mathbf{I})$ will usually not form a V-category. For instance, the category of associative rings is not enriched over abelian groups. However, in the cartesian closed situation the enrichment survives. The obvious example is $(V, \square, \mathbf{I})$ sets with cartesian product (monoids *do* form a category), and this captures the essence in the statements below.

If $(V, \square, \mathbf{I})$ and $(V', \square', \mathbf{I}')$ are two monoidal categories, a *monoidal adjunction*

$$(V, \square, \mathbf{I}) \underset{R}{\overset{L}{\leftrightarrows}} (V', \square', \mathbf{I}')$$

is a pair of (lax) monoidal functors such that L is left V'-adjoint to R; that is, there is a natural isomorphism in V'

$$RV(LX, v) \cong V'(X, Rv)$$

For the rest of the section we fix a complete and cocomplete closed category $(V, \square, \mathbf{I})$, a cartesian closed category $(V', \times, *)$ and a monoidal adjunction

$$(V, \square, \mathbf{I}) \underset{R}{\overset{L}{\leftrightarrows}} (V', \times, *)$$

such that L is strong monoidal (that is, the natural transformations $\mathbf{I} \to L*$ and $La \square Lb \to L(a \times b)$ are natural isomorphisms).

The reason to mess up the theory with V' is the canonical V'-functor

$$RV \times RV \to R(V \square V)$$

Composing this with

$$R(V \square V) \xrightarrow{R\square} RV$$

we get a "square"

$$\square : RV \times RV \to RV$$

sending $(v_1, v_2) \in obV \times V$ to $v_1 \square v_2$. The square is essential for the enrichment on monoids and V-categories.

2.1. Lemma. *Let $\mathcal{O}$ be a set and R, V and V' as above. The free/forgetful triple on $V\mathcal{O}$-graphs lifts to a V'-functor.*

Proof. Let $\mathcal{G}$ and $\mathcal{H}$ be $V\mathcal{O}$ graphs. The V'-object of morphisms from $\mathcal{G}$ to $\mathcal{H}$ is given by

$$\prod_{c,d\in\mathcal{O}} RV(\mathcal{G}(c,d),\mathcal{H}(c,d))$$

We have to show that there is a map in V'

$$\prod_{c,d\in\mathcal{O}} RV(\mathcal{G}(c,d),\mathcal{H}(c,d)) \to \prod_{c,d\in\mathcal{O}} RV(UF\mathcal{G}(c,d),UF\mathcal{H}(c,d))$$

lifting the morphism we get when forgetting all the way down to sets, but this is clear using the same formula (which is messy when you write it out, explanations of the various terms follow):

$$\begin{array}{c}
\prod_{c,d\in\mathcal{O}} RV(\mathcal{G}(c,d),\mathcal{H}(c,d)) \\
\Big\downarrow{\scriptstyle \Delta} \\
\prod_{c,d\in\mathcal{O}}\prod_{q\geq 0}\prod_{c=c_0,c_1,\dots c_q=d\in\mathcal{O}}\prod_{1\leq i\leq q} RV(\mathcal{G}(c_i,c_{i-1}),\mathcal{H}(c_i,c_{i-1})) \\
\Big\downarrow{\scriptstyle \square} \\
\prod_{c,d\in\mathcal{O}}\prod_{q\geq 0}\prod_{c=c_0,c_1,\dots c_q=d\in\mathcal{O}} RV(\square_{1\leq i\leq q}\mathcal{G}(c_i,c_{i-1}),\square_{1\leq i\leq q}\mathcal{H}(c_i,c_{i-1})) \\
\Big\downarrow \\
\prod_{c,d\in\mathcal{O}}\prod_{q\geq 0}\prod_{c=c_0,c_1,\dots c_q=d\in\mathcal{O}} RV(\square_{1\leq i\leq q}\mathcal{G}(c_i,c_{i-1}),UF\mathcal{H}(c_i,c_{i-1})) \\
\Big\downarrow{\scriptstyle \cong} \\
\prod_{c,d\in\mathcal{O}} RV(UF\mathcal{G}(c,d),UF\mathcal{H}(c,d))
\end{array}$$

The first map is the obvious diagonal (the empty product is given by $*$), the second is the square (the empty square is given by $\mathbf{I}$, and the corresponding maps from the empty products are given by the structure map $* \to R\mathbf{I} \cong RV(\mathbf{I},\mathbf{I})$). The third map is induced by the maps $\square_{1\leq i\leq q}\mathcal{H}(c_i,c_{i-1}) \to UF\mathcal{H}(c,d)$ and the last map is the isomorphism given by the formula for $UF\mathcal{G}(c,d)$. ∎

2.2. Corollary. *The category of $V\mathcal{O}$-categories is naturally a tensored and cotensored V'-category.*

Proof. To simplify the notation, let the V-object of morphisms from a $V\mathcal{O}$-graph $\mathcal{G}$ to another $\mathcal{H}$ be written $\{\mathcal{G},\mathcal{H}\}$. The previous lemma says that the free/forgetful triple induces a map $R\{\mathcal{G},\mathcal{H}\} \to R\{UF\mathcal{G},UF\mathcal{H}\}$ in V'. If $\mathcal{C}$ and $\mathcal{D}$ are $V\mathcal{O}$-categories, then we define the V'-object of morphisms from $\mathcal{C}$ to $\mathcal{D}$ as the equalizer of

$$\begin{tikzcd} R\{U\mathcal{C},U\mathcal{D}\} \arrow[r] \arrow[dr] & R\{UFU\mathcal{C},UFU\mathcal{D}\} \arrow[d] \\ & R\{UFU\mathcal{C},U\mathcal{D}\} \end{tikzcd}$$

(using the structure of $\mathcal{C}$ and $\mathcal{D}$ as UF-algebras and lemma 2.1.).

If $X \in obV'$ then the cotensor $\mathcal{C}^X$ is the $V\mathcal{O}$-category with morphism objects

$$\mathcal{C}^X(c,c') = V(LX,\mathcal{C}(c,c'))$$

This commutes with the monoidal structure (due to the fact that the diagonal map $X \to X \times X$ is a V'-map: there is no guarantee that there is a diagonal in $(V,\Box,\mathbf{I})$ which explains why we can't expect an enrichment in V), and so defines a $V\mathcal{O}$-category, and the natural isomorphism

$$V'(X,RV(\mathcal{D}(c,c'),\mathcal{C}(c,c'))) \cong RV(\mathcal{D}(c,c'),V(LX,\mathcal{C}(c,c')))$$

gives that it actually is a cotensor.

The category of $V\mathcal{O}$-graphs is tensored over V' by letting $X \otimes \mathcal{G}$ be the $V\mathcal{O}$-graph with morphisms $(X \otimes \mathcal{G})(c,d) = LX\Box\mathcal{G}(c,d)$. The tensor in $V\mathcal{O}$-categories of X and $\mathcal{C}$ is defined as the coequalizer of

$$F(X \otimes UFU\mathcal{C}) \rightrightarrows F(X \otimes U\mathcal{C})$$

where the upper map is induced by $UFU \xrightarrow{U(unit\ of\ adjunction)} U$ and the lower map is induced by the identity through

$$\begin{aligned} R\{X\otimes U\mathcal{C},X\otimes U\mathcal{C}\} &\cong V'(X,R\{U\mathcal{C},X\otimes U\mathcal{C}\}) \to V'(X,R\{UFU\mathcal{C},UF(X\otimes U\mathcal{C})\}) \\ &\cong R\{X\otimes UFU\mathcal{C},UF(X\otimes U\mathcal{C})\} \cong [F(X\otimes UFU\mathcal{C}),F(X\otimes U\mathcal{C})] \end{aligned}$$

which makes sense since UF is a V'-functor. ∎

3. The model structure

We will be working with enriched model categories by which we mean the following:

Let $(V,\Box,\mathbf{I})$ be a closed category. Assume given a model structure on V. We say that V is a (discrete) *monoidal model category* if the following condition holds: if $i\colon A \rightarrowtail B$ is a cofibration in V and $p\colon X \twoheadrightarrow Y$ is a fibration in V then

$$V(B,X) \to V(B,Y) \prod_{V(A,Y)} V(B,Y) \in V$$

is a fibration in V which is a weak equivalence if either i or p is a weak equivalence.

By adjointness this is equivalent to the axiom which Schwede and Shipley call the pushout product axiom and which says that if $A \overset{i}{\rightarrowtail} B$ and $C \overset{j}{\rightarrowtail} D$ are cofibrations in V, then the canonical map

$$(A\Box D) \coprod_{A\Box C} (B\Box C) \xrightarrow{(i,j)} B\Box D$$

is a cofibration, and if in addition one of the maps i and j is a weak equivalence, then (i,j) is a weak equivalence.

By induction and the fact that $\Box$ commutes with colimits we immediately get the following lemma:

3.1. Lemma. *Given cofibrations $j_i\colon A_0^i \rightarrowtail A_1^i \in V$ for $i = 1,\dots,n$. Consider the n-cube obtained by considering all n-fold square products $A_{i_1}^1\Box\dots\Box A_{i_n}^n$. Then the canonical map $(j_1,\dots,j_n)$ from the colimit of the punctured cube to $A_1^1\Box\dots\Box A_1^n$ is a cofibration, and if one of the $A_0^i \rightarrowtail A_1^i$ is a weak equivalence then so is $(j_1,\dots,j_n)$.*

A V-model category is a tensored and cotensored V-category $\mathcal{C}$ with a model structure such that if $i\colon A \rightarrowtail B$ is a cofibration in $\mathcal{C}$ and $p\colon X \twoheadrightarrow Y$ is a fibration in $\mathcal{C}$ then

$$\mathcal{C}(B,X) \to \mathcal{C}(B,Y) \prod_{\mathcal{C}(A,Y)} \mathcal{C}(B,Y) \in V$$

is a fibration in V which is a weak equivalence if either i or p is a weak equivalence.

If $(V,\Box,\mathbf{I})$ and $(V',\Box',\mathbf{I}')$ are two monoidal model categories, a monoidal Quillen adjunction

$$(V,\Box,\mathbf{I}) \underset{R}{\overset{L}{\leftrightarrows}} (V',\Box',\mathbf{I}')$$

is a pair of (lax) monoidal functors such that L preserves cofibrations and trivial cofibrations and L is V'-adjoint to R, that is there is a natural isomorphism

$$RV(LX,v) \cong V'(X,Rv)$$

In our intended applications $(V',\Box',\mathbf{I}')$ will be simplicial sets with cartesian product $(\mathcal{S},\times,*)$, and $\mathbf{I}$ will equal $L*$ (and so $\mathbf{I}$ will be cofibrant). The importance of this relation to the simplicial world is that the simplicial structure will survive even when restricting to subcategories of monoids or $V\mathcal{O}$-categories due to the results in the previous section. The fact that $\mathbf{I}$ is cofibrant will simplify many arguments regarding homotopy invariance of certain colimit constructions.

Formalizing this, we say that a *monoidal simplicial model category* is

(1) a monoidal model category $(V,\Box,\mathbf{I})$ with $\mathbf{I}$ cofibrant
(2) a monoidal Quillen adjunction

$$(V,\Box,\mathbf{I}) \underset{R}{\overset{\Sigma}{\leftrightarrows}} (\mathcal{S},\times,*)$$

If $\mathcal{O}$ is a set, the category of $V\mathcal{O}$-categories is a tensored and cotensored $\mathcal{S}$-category (corollary 2.2.).

If V is a model category and $\mathcal{O}$ is a set, then the category of $V\mathcal{O}$-graphs has a model structure given by declaring that a map $\mathcal{G} \to \mathcal{H}$ of $V\mathcal{O}$-graphs is a weak equivalence (resp. (co)fibration)) if $\mathcal{G}(c, c') \to \mathcal{H}(c, c') \in V$ is a weak equivalence (resp. (co)fibration)) for every $c, c' \in \mathcal{O}$.

3.2. Definition. *Let $\mathcal{O}$ be a set and V a monoidal model category. The fibrations (resp. weak equivalences) in the category of $V\mathcal{O}$-categories are the maps that induce fibrations (resp. weak equivalences) of $V\mathcal{O}$-graphs (i.e. on all morphism objects). The cofibrations are the $V\mathcal{O}$-functors having the left lifting property with respect to $V\mathcal{O}$-functors that are both weak equivalences and fibrations.*

If $v \in obV$ and $c_0, c_1 \in \mathcal{O}$, then $\mathcal{O}_{v,c_0,c_1}$ is the $V\mathcal{O}$-graph that has morphism objects

$$\mathcal{O}_{v,c_0,c_1}(c', c'') = \begin{cases} v & \text{if } c_0 = c' \text{ and } c_1 = c'' \\ * & \text{otherwise} \end{cases}$$

where $*$ is the initial object in V. If $f: v \to w \in V$, then $\mathcal{O}_{f,c_0,c_1}: \mathcal{O}_{v,c_0,c_1} \to \mathcal{O}_{w,c_0,c_1}$ is the obvious map of $V\mathcal{O}$-graphs. If I is a set of maps in V, we let $\mathcal{O}_I$ be the corresponding set of maps in $V\mathcal{O}$-graphs, and $F\mathcal{O}_I$ be its image in $V\mathcal{O}$-categories under the free functor F (see section 1.0).

We say that a simplicial closed model category V has a *monoidal fibrant replacement functor* T_0 if there is a lax monoidal functor $T_0: V \to V$ with a natural transformation

$$v \to T_0(v)$$

consisting of weak equivalences with $T_0(v)$ fibrant.

3.3. Theorem. *Let $\mathcal{O}$ be a set and V be a locally presentable cofibrantly generated monoidal simplicial model category with a monoidal fibrant replacement functor.*

Then the category of $V\mathcal{O}$-categories is a cofibrantly generated simplicial model category where the weak equivalences, fibrations and cofibrations are specified in the definition above.

If I (resp. J) is a set of generating cofibrations (resp. cofibrations that are weak equivalences) in V, then $F\mathcal{O}_I$ (resp. $F\mathcal{O}_J$) is a set of generating cofibrations (resp. cofibrations that are weak equivalences) in $V\mathcal{O}$-categories.

Proof. By Beck's theorem we have that $V\mathcal{O}$-categories are UF-algebras, where UF is the triple deriving from the free/forgetful adjoint pair between $V\mathcal{O}$-categories and $V\mathcal{O}$-graphs. By section 2 the category of $V\mathcal{O}$-categories is a tensored and cotensored $\mathcal{S}$-category. Since UF is simplicial (lemma 2.1), and the monoidality of the fibrant replacement functor assures that it lifts to a fibrant replacement functor for $V\mathcal{O}$-categories, the theorem follows from the arguments of [S, corollary B2]. ∎

The condition that V is locally presentable is just there to assure that the small object argument will not run into any problems: all objects are small with respect to the entire category. This condition is in practice satisfied by models stemming from the simplicial world, but not by topological examples.

Since the arguments involving the factorization we actually use do not depend on the fibrant replacement functor, but only the small object argument, we get immediately

3.4. Lemma. *Let $\mathcal{O}$ be a set and V be a locally presentable cofibrantly generated monoidal model category.*

If I is a set of generating cofibrations in V, then there is a functorial replacement functor which to every $V\mathcal{O}$-functor $f: \mathcal{C} \to \mathcal{D}$ assigns a factorization $\mathcal{C} \rightarrowtail \mathcal{E} \overset{\sim}{\twoheadrightarrow} \mathcal{D}$ into a cellular FO_I-cofibration $i: \mathcal{C} \rightarrowtail \mathcal{E}$ followed by $p: \mathcal{E} \overset{\sim}{\twoheadrightarrow} \mathcal{D}$ which is both a weak equivalence and a fibration.

We have not assumed that V was left proper, so we can not expect left properness for $V\mathcal{O}$-categories. However, we **do** have the following gluing lemma.

3.5. Lemma. *Let V be a monoidal model category with $\mathbf{I}$ cofibrant. Let*

$$\begin{tikzcd}
\mathcal{C}_X \arrow[d, "\simeq"'] & \mathcal{C}_0 \arrow[l] \arrow[r] \arrow[d, "\simeq"'] & \mathcal{C}_Y \arrow[d, "\simeq"'] \\
\mathcal{D}_X & \mathcal{D}_0 \arrow[l] \arrow[r] & \mathcal{D}_Y
\end{tikzcd}$$

be a diagram in $V\mathcal{O}$-categories where the units $\mathbf{I}\mathcal{O} \to \mathcal{C}_0$ and $\mathbf{I}\mathcal{O} \to \mathcal{D}_0$ and all the horizontal maps are cofibrations when considered as morphisms of $V\mathcal{O}$-graphs and the vertical maps are weak equivalences. Then the map of pushouts

$$\mathcal{C}_X \coprod_{\mathcal{C}_0} \mathcal{C}_Y \to \mathcal{D}_X \coprod_{\mathcal{D}_0} \mathcal{D}_Y$$

is a weak equivalence of $V\mathcal{O}$-categories.

Proof. Consider the description of the morphism objects of the pushout categories given in section 1.10. The morphism objects from c to d in $\mathcal{C}_{XY}$ was described as the colimit over a certain category $\mathcal{O}^{c,d}$ of a functor $\mathcal{C}^{c,d}: \mathcal{O}^{c,d} \to V$ which on the object $\alpha = (i, C, q) \in ob\mathcal{O}^{c,d}$ has the value $\mathcal{C}^{c,d}(\alpha) = \Box_{k=0}^{q} \mathcal{C}_{i_k}(C_{k+1}, C_k)$. We define the morphism objects of $\mathcal{D}_{XY}$ in the exact same way, except that we call the functor $\mathcal{O}^{c,d} \to V$ induced in the same way with $\mathcal{D}$'s instead of $\mathcal{C}$'s $\mathcal{D}^{c,d}$ instead of $\mathcal{C}^{c,d}$. The map of pushouts $\mathcal{C}_{X,Y} \to \mathcal{D}_{XY}$ is then induced by the obvious natural transformation

$$\mathcal{C}^{c,d}(\alpha) = \Box_{k=1}^{q} \mathcal{C}_{i_k}(C_k, C_{k-1}) \to \Box_{k=1}^{q} \mathcal{D}_{i_k}(C_k, C_{k-1}) = \mathcal{D}c, d(\alpha)$$

The strategy is to prove that $\mathcal{C}^{c,d}$ is Reedy cofibrant (see [H, 17.3.2.2]), and so since lemma 1.7. claims that $\mathcal{O}^{c,d}$ has "fibrant constants", that

$$\underset{\overrightarrow{\mathcal{O}^{c,d}}}{\operatorname{holim}} \mathcal{C}^{c,d} \to \underset{\overrightarrow{\mathcal{O}^{c,d}}}{\lim} \mathcal{C}^{c,d} = \mathcal{C}_{XY}(c, d)$$

is a weak equivalence by [H, 20.7.2.1] (and likewise for $\mathcal{D}$) and using that in this case the homotopy colimit is a homotopy invariant [H, 20.4.3].

To prove that $\mathcal{C}^{c,d}$ is Reedy cofibrant, we have to consider the latching category

$$L_\alpha = \overrightarrow{\mathcal{O}}^{c,d}/\alpha - id_\alpha$$

for all objects $\alpha = (i, C, q)$ in $\mathcal{O}^{c,d}$ and show that the map

$$\varinjlim_{L_\alpha} \mathcal{C}^{c,d} \to \mathcal{C}^{c,d}_\alpha$$

is a cofibration in V. This can conceivably be proven directly, but our task is simplified greatly by restricting our attention to the final subcategory

$$\mathcal{F}^{c,d}_\alpha - id_\alpha = \ell_\alpha \subseteq L_\alpha$$

Note that $\mathcal{C}^{c,d}|\mathcal{F}^{c,d}_\alpha$ is a cube, and lemma 3.1. will guarantee that the map from the punctured cube to the final node is a cofibration. In its simplest form it may look like

$$\begin{tikzcd}
\mathcal{C}_0(c,d)\Box\mathcal{C}_0(b,c) \arrow[r] \arrow[d] & \mathcal{C}_0(c,d)\Box\mathcal{C}_Y(b,c) \arrow[d] \\
\mathcal{C}_0(d,d)\Box\mathcal{C}_0(c,d)\Box\mathcal{C}_0(b,c) \arrow[r] & \mathcal{C}_0(d,d)\Box\mathcal{C}_0(c,d)\Box\mathcal{C}_Y(b,c)
\end{tikzcd}$$

where the vertical maps are given by insertion of a unit $\mathbf{I} \to \mathcal{C}_0(d,d)$ and horizontal maps are induced by $0 \to Y$. Discarding the $\mathcal{C}_0(c,d)\Box$ which commutes with colimits anyhow, we see that we have an example of the square product axiom, since both $\mathbf{I} \to \mathcal{C}_0(d,d)$ and $\mathcal{C}_0(b,c) \to \mathcal{C}_Y(b,c)$ are cofibrations in V. The general case is similar: the cube $\mathcal{C}^{c,d}|\mathcal{F}^{c,d}_\alpha$ is a cube of successively more square product factors (maps induced by units) or increase in indices (maps in the pushout). Discarding the square product factors being constant throughout the cube and inserting $\mathbf{I}$'s where units are to be inserted later on, we are left with a cube of the form given in lemma 3.1, and so

$$\varinjlim_{\ell_\alpha} \mathcal{C}^{c,d} \to \mathcal{C}^{c,d}_\alpha$$

is a cofibration in V. Since $\ell_\alpha \subseteq L_\alpha$ is final, the map

$$\varinjlim_{L_\alpha} \mathcal{C}^{c,d} \to \varinjlim_{\ell_\alpha} \mathcal{C}^{c,d}$$

is an isomorphism and so $\mathcal{C}^{c,d}$ is Reedy cofibrant. ∎

3.6. Lemma. *Let V be a monoidal model category with $\mathbf{I}$ cofibrant and with generating set of cofibrations I. Then every cellular FO_I-cofibration $\mathcal{C} \rightarrowtail \mathcal{D}$ of $V\mathcal{O}$-categories where the morphism objects of $\mathcal{C}$ are cofibrant in V induces cofibrations $\mathcal{C}(c,d) \rightarrowtail \mathcal{D}(c,d)$ of morphism objects.*

Proof. It is enough to prove that if $c, d \in \mathcal{O}$ and $f\colon v \to w \in I$, then for a pushout in $V\mathcal{O}$-categories

$$\begin{tikzcd} F\mathcal{O}_{v,c,d} \arrow[r] \arrow[d] & F\mathcal{O}_{w,c,d} \arrow[d] \\ \mathcal{C}_1 \arrow[r] & \mathcal{C}_2 \end{tikzcd}$$

with $\mathcal{C}_1(x,y) \in V$ cofibrant for all $x, y \in \mathcal{O}$ we have that the induced maps $\mathcal{C}_1(x,y) \to \mathcal{C}_2(x,y)$ are cofibrations. But this follows by an analysis of the diagrams defining the pushout as in the previous lemma, since the freeness of two of the categories imply that the multiplication part can be ignored, and we are left with a colimit of cobase changes, each of which are cofibrations since cofibrations are preserved by $\Box$-ing with cofibrant objects. ∎

3.7. Moving the objects. In many applications, it is convenient and sufficient to do the constructions solely within $V\mathcal{O}$-categories if we are only concerned with just one category at the time. However, we will have the occasion to use a wider functoriality which is the consequence of the following lemma.

3.8. Lemma. *The functorial factorizations coming from the cofibrant generation of the category of $V\mathcal{O}$-categories is functorial in the category of V-functors. More precisely, let $\mathcal{O}^1$ and $\mathcal{O}^2$ be two sets and consider the commutative diagram in V-categories*

$$\begin{tikzcd} \mathcal{C}^1 \arrow[r, "f^1"] \arrow[d, "F"'] & \mathcal{D}^1 \arrow[d, "G"] \\ \mathcal{C}^2 \arrow[r, "f^2"] & \mathcal{D}^2 \end{tikzcd}$$

where $f^i \in V\mathcal{O}^i$-categories. Let

$$\mathcal{C}^i \rightarrowtail \mathcal{Z}_{f^i} \overset{\sim}{\twoheadrightarrow} \mathcal{D}^i$$

be the functorial factorization of lemma 3.4 in $V\mathcal{O}^i$-categories, coming from a choice of generation of the cofibrantly generated model category V. Then there is a map $\mathcal{Z}_{f^1} \xrightarrow{\mathcal{Z}_{(F,G)}} \mathcal{Z}_{f^2}$ such that

$$\begin{tikzcd} \mathcal{C}^1 \arrow[r] \arrow[d, "F"'] & \mathcal{Z}_{f^1} \arrow[r, "\sim"] \arrow[d, "\mathcal{Z}_{(F,G)}"'] & \mathcal{D}^1 \arrow[d, "G"'] \\ \mathcal{C}^2 \arrow[r] & \mathcal{Z}_{f^2} \arrow[r, "\sim"] & \mathcal{D}^2 \end{tikzcd}$$

commutes. Furthermore, if $\mathcal{O}^3$ is yet another set, and

$$\begin{tikzcd} \mathcal{C}^2 \arrow[r, "f^2"] \arrow[d, "D"'] & \mathcal{D}^2 \arrow[d, "E"'] \\ \mathcal{C}^3 \arrow[r, "f^3"] & \mathcal{D}^3 \end{tikzcd}$$

is a commutative diagram in V-categories with f^3 in $V\mathcal{O}^3$-categories, then

$$\mathcal{Z}_{(DF,EG)} = \mathcal{Z}_{(D,E)}\mathcal{Z}_{(F,G)}$$

Proof. Consider the way the functorial factorizations are gotten in cofibrantly generated model categories via the small object argument (see e.g. [H, 13.4.14]). Let K be a set of maps in V permitting the small object argument. Recall the set $\mathcal{O}^i_K$ of maps in $V\mathcal{O}^i$-graphs introduced just before theorem 3.3. Furthermore, let F^i be the free functor from $V\mathcal{O}^i$-graphs to $V\mathcal{O}^i$-categories. Note that a map $F^1(\mathcal{O}^1_{v,c,d}) \to \mathcal{C}^1$ is the same as a map $v \to \mathcal{C}^1(c,d)$ in V, which induces a map $v \to \mathcal{C}^1(c,d) \to \mathcal{C}^2(Fc,Fd)$ which is the same as a map $F^2(\mathcal{O}^2_{v,Fc,Fd}) \to \mathcal{C}^2$. If $\sigma\colon v \to w \in V$, then a diagram

$$\begin{tikzcd} F^1(\mathcal{O}^1_{v,c,d}) \arrow[r, "F^1((\mathcal{O}^1)_{\sigma,c,d})"] \arrow[d] & F^1(\mathcal{O}^1_{w,c,d}) \arrow[d] \\ \mathcal{C}^1 \arrow[r, "f^1"] & \mathcal{D}^1 \end{tikzcd}$$

induces a diagram

$$\begin{tikzcd} v \arrow[r, "\sigma"] \arrow[d] & w \arrow[d] \\ \mathcal{C}^1(c,d) \arrow[r, "f^1"] \arrow[d, "F"'] & \mathcal{D}^1(c,d) \arrow[d, "G"'] \\ \mathcal{C}^2(Fc,Fd) \arrow[r, "f^2"] & \mathcal{D}^2(Gc,Gd) \end{tikzcd}$$

since F equals G on objects, which again gives a diagram

$$\begin{tikzcd} F^2(\mathcal{O}^2_{v,Fc,Fd}) \arrow[r, "F^2(\mathcal{O}^2_{\sigma,Fc,Fd})"] \arrow[d] & F^2(\mathcal{O}^2_{w,Fc,Fd}) \arrow[d] \\ \mathcal{C}^2 \arrow[r, "f^2"] & \mathcal{D}^2 \end{tikzcd}$$

Now, since the free functor commutes with coproducts, and the coproduct in $V\mathcal{O}^i$-graphs is just the coproduct in V of each morphism object we get a map of squares from

$$\begin{tikzcd}
\coprod_{t\in X^1} F^1(\mathcal{O}^1_{\sigma(t),v(t),w(t)}) \arrow[r] \arrow[d] & \coprod_{t\in X^1} F^1(\mathcal{O}^1_{\tau(t),v(t),w(t)}) \arrow[d] \\
\mathcal{C}^1 \arrow[r] & \mathcal{D}^1
\end{tikzcd}$$

to

$$\begin{tikzcd}
\coprod_{t\in X^2} F^2(\mathcal{O}^2_{\sigma(t),v(t),w(t)}) \arrow[r] \arrow[d] & \coprod_{t\in X^2} F^2(\mathcal{O}^2_{\tau(t),v(t),w(t)}) \arrow[d] \\
\mathcal{C}^2 \arrow[r] & \mathcal{D}^2
\end{tikzcd}$$

By 1.5 this induces a map of the corresponding pushouts, and since filtered colimits are computed on each morphism object, the result is clear. ∎

4. The category $\mathfrak{P}$ of pairs

In this section we develop some machinery pertaining to the category of pairs used in the localization functor. Note that the representation of the category of pairs used in section 0 is only isomorphic to the one we use here.

Let V be a closed category, and let R be a lax monoidal functor $(V\Box, \mathbf{I}) \to (\mathcal{S}, \times, *)$. Let $\mathfrak{P}^{\text{free}}$ be the category whose objects are pairs

$$(\mathcal{C}, \mathcal{W} \xrightarrow{w} R\mathcal{C})$$

where $\mathcal{C}$ is a (small) V-category and $\mathcal{W} \to R\mathcal{C}$ a $\mathcal{S}$-functor of (small) $\mathcal{S}$-categories. In section 0 we used that in that context R had a left adjoint $\Sigma\colon \mathcal{S} \to V$, and represented the pair $(\mathcal{C}, \mathcal{W} \to R\mathcal{C})$ by means of the adjoint $(\mathcal{C}, \Sigma\mathcal{W} \to \mathcal{C})$. That was convenient at the time (less notational burden at the time of the definition of the localization), but here it is simpler to stick with our current definition.

A morphism

$$(\mathcal{C}, w) \to (\mathcal{C}', w')$$

in $\mathfrak{P}^{\text{free}}$ is a pair

$$F\colon \mathcal{C} \to \mathcal{C}', \qquad WF\colon \mathcal{W} \to \mathcal{W}'$$

where F is a V-functor such that

$$\begin{tikzcd}
\mathcal{W} \arrow[r, "w"] \arrow[d, "WF"'] & R\mathcal{C} \arrow[d, "RF"] \\
\mathcal{W}' \arrow[r, "w'"] & R\mathcal{C}'
\end{tikzcd}$$

commutes.

In other words: $\mathfrak{P}^{\text{free}}$ is defined by the pullback

$$\begin{tikzcd}
\mathfrak{P}^{free} \arrow[r] \arrow[d] & V\text{-categories} \arrow[d] \\
\mathcal{S}\text{-functors} \arrow[r, "target"] & \mathcal{S}\text{-categories}
\end{tikzcd}$$

4.0. The subcategories $\mathfrak{P}^{\text{fix}} \subseteq \mathfrak{P} \subseteq \mathfrak{P}^{\text{free}}$. The localization as we have defined it eventually turned out to live in the full subcategory $\mathfrak{P} \subseteq \mathfrak{P}^{\text{free}}$ whose objects $(\mathcal{C}, w)$ have the property that $w\colon \mathcal{W} \to R\mathcal{C}$ is the identity on objects (see lemma 0.11.).

However, at the outset it was not clear that one could allow all the morphisms: the localization construction was originally performed in the much smaller subcategory $\mathfrak{P}^{\text{fix}} \subseteq \mathfrak{P}$ whose morphisms are morphisms

$$(\mathcal{C}, w) \stackrel{F,WF}{\longrightarrow} (\mathcal{C}', w')$$

where F (and hence WF) is the identity on objects.

The inclusion functor $\mathfrak{P} \subseteq \mathfrak{P}^{\text{free}}$ has a right adjoint $\phi\colon \mathfrak{P}^{\text{free}} \to \mathfrak{P}$ which is important for many applications. If $(\mathcal{C}, w\colon \mathcal{W} \to R\mathcal{C}) \in ob\mathfrak{P}^{\text{free}}$, then $\phi(\mathcal{C}, w) = (\phi_w\mathcal{C}, \phi w)$ each of whose factors are defined below. The first factor $\phi_w\mathcal{C}$ is the V-category with objects $ob\mathcal{W}$ and with morphism object from $c \in ob\mathcal{W}$ to $d \in ob\mathcal{W}$ given by

$$\phi_w\mathcal{C}(c, d) = \mathcal{C}(wc, wd)$$

The second factor $\phi w\colon \mathcal{W} \to R\phi_w\mathcal{C}$ is induced by w

$$\mathcal{W}(c, d) \to R\mathcal{C}(wc, wd) = R\phi_w\mathcal{C}(c, d)$$

This is a functor, since if $(\mathcal{C}, w) \stackrel{F,WF}{\longrightarrow} (\mathcal{C}', w')$ is a map in $\mathfrak{P}^{\text{free}}$, then we get a map $\phi(\mathcal{C}, w) \stackrel{\phi(F,WF)}{\longrightarrow} \phi(\mathcal{C}', w')$ in $\mathfrak{P}$ with first factor

$$\begin{aligned}
&\phi_w\mathcal{C}(c, d) = \mathcal{C}(wc, wd) \\
&\qquad \stackrel{F}{\to} \mathcal{C}'(Fwc, Fwd) = \mathcal{C}'(w'WFc, w'WFd) = \phi_{w'}\mathcal{C}'(WFc, WFd)
\end{aligned}$$

and second factor WF as before.

Note that the composite $\mathfrak{P} \subseteq \mathfrak{P}^{\text{free}} \stackrel{\phi}{\to} \mathfrak{P}$ is the identity. When considered as an endofunctor on $\mathfrak{P}^{\text{free}}$ ϕ is idempotent ($\phi^2 = \phi$) and there is a natural transformation $\phi \to id_{\mathfrak{P}^{\text{free}}}$ given by the obvious map $\phi_w\mathcal{C} \to \mathcal{C}$ which is the identity on morphisms.

Using this we get that ϕ is right adjoint to the inclusion as promised.

We say that a morphism $(\mathcal{C}, w) \stackrel{F,WF}{\longrightarrow} (\mathcal{C}', w')$ in $\mathfrak{P}^{\text{fix}}$ is a *weak equivalence* (resp. *fibration*) if both F and WF are. The following easy lemma is not needed in the text above, but is good for the intuition.

4.1. Lemma. *Let V be a locally presentable cofibrantly generated monoidal simplicial model category with a monoidal fibrant replacement functor. Then the category of fixed pairs $\mathfrak{P}^{\text{fix}}$ is a disjoint union of simplicial model categories.*

Proof. Let $\mathcal{O}$ be a set, and consider the component of $\mathfrak{P}^{\text{fix}}$ of pairs with set of objects $\mathcal{O}$. The limit, two out of three, retract and half of the lifting axiom are obvious. A cofibration is a map with the left lifting property with respect to maps that are both fibrations and weak equivalences. Note that a map

$$(\mathcal{A}, w_a \colon \mathcal{W}_a \to R\mathcal{A}) \to (\mathcal{B}, w_b \colon \mathcal{W}_b \to R\mathcal{B})$$

is a cofibration if and only if both $\mathcal{W}_a \to \mathcal{W}_b$ and

$$\mathcal{A} \coprod_{\Sigma\mathcal{W}_a} \Sigma\mathcal{W}_b \to \mathcal{B}$$

are cofibrations.

Hence the map is a weak equivalence and a cofibration if and only if $\mathcal{W}_a \to \mathcal{W}_b$ and $\mathcal{A} \coprod_{\Sigma\mathcal{W}_a} \Sigma\mathcal{W}_b \to \mathcal{B}$ are cofibrations and weak equivalences (by the two out of three axiom since cofibrations that are weak equivalences are stable under pushout). Thus the other part of the lifting axiom is true too.

The factorization axiom is proven by first factoring

$$\mathcal{W}_a \rightarrowtail Z_{WF} \twoheadrightarrow \mathcal{W}_b$$

(where either the first or the latter is a weak equivalence), and then letting $P_F = \mathcal{A} \coprod_{\Sigma\mathcal{W}_a} \Sigma Z_{WF}$, and lastly factoring $P_F \to \mathcal{B}$ as $P_F \rightarrowtail Z_F \twoheadrightarrow \mathcal{B}$. Let $w_F \colon Z_{WF} \to RZ_F$ be the adjoint of $\Sigma Z_{WF} \to P_F \to Z_F$. By construction $(\mathcal{A}, w_a) \to (Z_F, w_F)$ is a cofibration and $(Z_F, w_F) \to (\mathcal{B}, w_b)$ a fibration, and by proper choices we can make one of them a weak equivalence as well.

The simplicial structure is given by the simplicial structure on $V\mathcal{O}$-categories and $\mathcal{S}\mathcal{O}$-categories. ∎

We see that in this structure, the map $B(\mathcal{C}, w) \to (\mathcal{C}, w)$ constructed for use in the localization is a weak equivalence and a fibration, whereas $B(\mathcal{C}, w) \to L(\mathcal{C}, w)$ is a weak equivalence but not a cofibration since $Q\mathcal{W} \to Q\mathcal{W}[Q\mathcal{W}^{-1}]$ is not a cofibration of $\mathcal{S}$-categories (only a cofibration of $\mathcal{S}$-graphs).

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