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HOW LARGE ARE LEFT EXACT FUNCTORS?

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ABSTRACT. For a broad collection of categories $\mathcal{K}$, including all presheaf categories, the following statement is proved to be consistent: every left exact (i.e. finite-limits preserving) functor from $\mathcal{K}$ to $\mathrm{S}et$ is small, that is, a small colimit of representables. In contrast, for the (presheaf) category $\mathcal{K} = \mathrm{A}lg(1,1)$ of unary algebras we construct a functor from $\mathrm{A}lg(1,1)$ to $\mathrm{S}et$ which preserves finite products and is not small. We also describe all left exact set-valued functors as directed unions of "reduced representables", generalizing reduced products.

1. Introduction

We study left exact (i.e. finite-limits preserving) set-valued functors on a category $\mathcal{K}$, and ask whether they all are small, i.e., small colimits of hom-functors. This depends of the category $\mathcal{K}$, of course, since even so well-behaved categories as $\mathrm{G}rp$, the category of groups, have easy counterexamples: recall the well-known example

$$F = \prod_{i \in \mathrm{Ord}} \mathrm{G}rp(A_i, -) : \mathrm{G}rp \longrightarrow \mathrm{S}et$$

of a functor preserving all limits and not having a left adjoint (thus, not being small), where A_i is a simple group of infinite cardinality $\aleph_i$.

In the present paper we are particularly interested in the case $\mathcal{K} = \mathrm{S}et^{\mathcal{A}}$, $\mathcal{A}$ small, since this corresponds to the question put by F. W. Lawvere, J. Rosický and the first author in [ALR1] of legitimacy of all $\mathcal{A}$-ary operations on the category $\mathrm{L}FP$ of locally finitely presentable categories. The main result of our paper is that the following statement

"all left exact functors from $\mathrm{S}et^{\mathcal{A}}$ to $\mathrm{S}et$, $\mathcal{A}$ small, are small"

is independent of set theory in the following sense: this is true if the set-theoretical axiom (R), introduced below, is assumed, and this is false if the negation of the following axiom

(M) there do not exist arbitrarily large measurable cardinals

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is assumed. For categories $\mathcal{K}$ with finite limits we are going to describe all left exact functors $F : \mathcal{K} \longrightarrow \mathrm{S}et$, generalizing the results for $\mathcal{K} = \mathrm{S}et$ by the third author [T]. She proved that left exact endofunctors of S*et* are precisely the (possibly large) directed unions of reduced-power functors $Q_{K,\mathcal{D}}$. Here $\mathcal{D}$ is a filter on a set K, and $Q_{K,\mathcal{D}}$ assigns to every set X its reduced power

$$\prod_{\mathcal{D}} X = \mathrm{colim}_{D\in\mathcal{D}} X^D,$$

or more precisely, $Q_{K,\mathcal{D}} = \mathrm{colim}_{D\in\mathcal{D}} \mathrm{S}et(D, -)$.

In the present paper we prove that this result extends to left exact set-valued functors on any category with finite limits: they are precisely the (possibly large) directed unions of "reduced hom-functors" defined analogously to $Q_{K,\mathcal{D}}$ above.

Returning to our question of smallness of left exact functors

$$F : \mathrm{S}et^{\mathcal{A}} \longrightarrow \mathrm{S}et \qquad (\mathcal{A} \quad \text{small}),$$

the negative answer has, for $\mathcal{A} = 1$, been already found by J. Reiterman [R]. The idea is simple: recall that $\neg$(M) is equivalent to the following statement:

$\neg$(M) For every ordinal i there exists a set K_i of power $\geq \aleph_i$ and a uniform ultrafilter $\mathcal{D}_i$ on K_i (i.e., an ultrafilter whose members have the same power as K_i) closed under intersections of less than $\aleph_i$ members.

The functor $F : \mathrm{S}et \longrightarrow \mathrm{S}et$ obtained by "transfinite composition" of the functors $Q_{K_i,\mathcal{D}_i}$ (i.e., $F = \mathrm{colim}_{i\in\mathrm{Ord}} F_i$ where $F_0 = Id$, $F_{i+1} = Q_{K_i,\mathcal{D}_i} \circ F_i$ and $F_j = \mathrm{colim}_{i<j} F_i$ for limit ordinals j) is left exact and large (i.e., not small).

Concerning the affirmative answer, it was A. Blass who showed in [B] that every left exact endofunctor of S*et* is small provided that the following set-theoretical axiom is assumed:

(R) Every uniform ultrafilter on an infinite set is regular

where an ultrafilter $\mathcal{D}$ on a set K of cardinality λ is called *regular* provided that $\mathcal{D}$ has λ members $D_i \in \mathcal{D}$ $(i \leq \lambda)$ such that every element of K lies in only finitely many of the sets D_i. An important property of regular ultrafilters $\mathcal{D}$ is that ultrapowers have the "full cardinality" of powers

$$\mathrm{card} \prod_{\mathcal{D}} X = \mathrm{card}\, X^\lambda \qquad \text{for all } X \text{ infinite}$$

see [CK]. The argument of A. Blass showing that (R) implies that every left exact functor $F : \mathrm{S}et \longrightarrow \mathrm{S}et$ is small probably does not request the full strength of (R); all we need is the following consequence of it:

(R′) There is a set Y such that for every cardinal μ there is a cardinal λ with the property that the ultrapowers of Y with respect to uniform ultrafilters on λ all have cardinality at least μ.

The argument goes as follows: suppose that, to the contrary, a large functor $F : \mathcal{S}et \longrightarrow \mathcal{S}et$ is left exact. For every cardinal μ choose λ as in (R′), then there exists, since F is large, a set X of cardinality $\operatorname{card} X \geq \lambda$ and an element $x \in FX$ such that the filter $\mathcal{D}(x)$ of all subsets Z of X with $x \in Fm(FZ)$ for the inclusion map $m : Z \longrightarrow X$ is uniform. (Since F is left exact, $\mathcal{D}(x)$ is indeed a filter.) Thus $\mathcal{D}(x)$ can be embedded into a uniform ultrafilter $\mathcal{D}^*(x)$. Then (R′) implies

$$\operatorname{card} Q_{X,\mathcal{D}(x)}Y \geq \operatorname{card} \prod_{\mathcal{D}^*(x)} Y \geq \mu.$$

This is in contradiction to $Q_{X,\mathcal{D}(x)}$ being a subfunctor of F: we cannot have $\operatorname{card} FY \geq \mu$ for all cardinals μ.

Now a decade after the paper of A. Blass it was proved by H.-D. Donder [D] that (R) is consistent with ZFC. We are going to extend Blass's argument to left exact set-valued functors on any category $\mathcal{K}$ which

(a) is finitely complete and well-powered

(b) admits a faithful left adjoint into $\mathcal{S}et$.

In particular, we conclude:

1.1. COROLLARY. *It is consistent with ZFC to state that all left exact functors $\mathcal{S}et^{\mathcal{A}} \longrightarrow \mathcal{S}et$, $\mathcal{A}$ small, are small.*

As mentioned above, this corollary answers the open problem put in [ALR1] whether it is consistent with set theory to assume that all small-ary operations on the category $\mathcal{L}FP$ are legitimate.

A completely different situation is shown to happen with operations on the category $\mathcal{V}AR$ of all finitary varieties studied in [ALR2]. The legitimacy of all small-ary operations on $\mathcal{V}AR$ would be equivalent to the statement that every functor

$$F : \mathcal{S}et^{\mathcal{A}} \longrightarrow \mathcal{S}et \qquad (\mathcal{A} \quad \text{small})$$

preserving finite products is small. But here is the answer dramatically different: for the free monoid $\mathcal{A}$ on two generators we prove that (in ZFC) there exists a large functor from $\mathcal{S}et^{\mathcal{A}}$ to $\mathcal{S}et$ preserving finite products.

2. Reduced Hom-functors

DEFINITION. By a *filter* on an object K of a finitely complete category we understand a non-empty set of subobjects of K closed under finite intersections and upwards-closed (i.e. given subobjects D_1, D_2 of K, if $D_1, D_2 \in \mathcal{D}$ then $D_1 \cap D_2 \in \mathcal{D}$ and if $D_1 \subseteq D_2$ then $D_1 \in \mathcal{D}$ implies $D_2 \in \mathcal{D}$).

Remark. (1) A filter always contains the largest subobject K.

(2) If $\mathcal{K} = \mathcal{S}et$, our concept coincides with the usual concept of a filter on a set K except that here we admit the trivial case of $\mathcal{D} =$ all subobjects of K.

Definition. Let $\mathcal{D}$ be a filter on an object K of $\mathcal{K}$, then the *reduced hom-functor* of K modulo $\mathcal{D}$ is the functor

$$Q_{K,\mathcal{D}} = \mathrm{colim}_{D\in\mathcal{D}}\mathcal{K}(D,-) \qquad \text{in } \mathcal{S}et^{\mathcal{K}}.$$

A functor in $\mathcal{S}et^{\mathcal{K}}$ is said to be *reduced representable* if it is naturally isomorphic to a reduced hom-functor for some filter on an object of $\mathcal{K}$.

Remark. Explicitly, $Q_{K,\mathcal{D}}$ is a colimit of the filtered diagram in $\mathcal{S}et^{\mathcal{K}}$ defined as follows: every element of $\mathcal{D}$ is represented by a monomorphism $m_D : D \longrightarrow K$; given $D' \subseteq D$ in $\mathcal{D}$ we have the unique monomorphism

$$m_{D',D} : D' \longrightarrow D \quad \text{with} \quad m_{D'} = m_D \circ m_{D',D}.$$

This leads to a diagram whose objects are the hom-functors

$$\mathcal{K}(D,-) \qquad (D \in \mathcal{D})$$

and whose morphisms are the natural transformations

$$(-) \circ m_{D',D} : \mathcal{K}(D,-) \longrightarrow \mathcal{K}(D',-) \qquad (D', D \in \mathcal{D},\ D' \subseteq D).$$

2.1. lemma. *Every reduced representable functor is left exact.*

Proof. A filtered colimit of left exact functors is always left exact because finite limits commute in presheaf categories with filtered colimits (including the large ones as far as they exist. ■

2.2. theorem. *Let $\mathcal{K}$ be a finitely complete, well-powered category. A set-valued functor on $\mathcal{K}$ is left exact if and only if it is a (possibly large) directed union of reduced representable functors.*

Remark. Directed unions are (possibly large) filtered colimits whose scheme is a directed partially ordered class and whose connecting morphisms are monomorphisms. In $\mathcal{S}et^{\mathcal{K}}$ each such diagram, provided that it has a colimit, has a colimit cocone formed by monomorphisms.

Proof. (i) Sufficiency follows from II.3 since directed unions of left exact functors are left exact.

(ii) To prove the necessity, let

$$F : \mathcal{K} \longrightarrow \mathcal{S}et$$

be a left exact functor. Let I be the class of all finite sets of elements of F, ordered by inclusion. (An element of F is a pair (K,k) where $K \in \mathrm{Obj}\,\mathcal{K}$ and $k \in FK$.) We use finite sets of elements, rather than just elements, in order to obtain F as a directed union rather than filtered colimit below.

For each element

$$i = \{(K_{i_1}, k_{i_1}), \ldots, (K_{i_n}, k_{i_n})\}$$

of I put

$$K_i = K_{i_1} \times \ldots \times K_{i_n}$$

and since F preserves this product, we can denote by

$$k_i \in FK_i$$

the unique element mapped by the t-th projection of $FK_i = FK_{i_1} \times \ldots \times FK_{i_n}$ to k_{i_t} $(t = 1, \ldots, n)$. Denote by $\mathcal{D}_i$ the filter on K_i of all subobjects

$$m_D : D \longmapsto K_i$$

such that k_i lies in the image of Fm_D. Since F preserves pullbacks, it is easy to see that $\mathcal{D}_i$ is indeed a filter. Moreover, F preserves monomorphisms, therefore Fm_D is a monomorphism, thus, there is a unique

$$k_i^D \in FD \qquad \text{with} \qquad Fm_D(k_i^D) = k_i.$$

(iii) We define a diagram

$$H : I \longrightarrow \mathrm{Set}^{\mathcal{K}}$$

on objects by

$$Hi = Q_{K_i, \mathcal{D}_i} \qquad (i \in I).$$

For $i \subseteq j$ in I we define the connecting morphism

$$h_{i,j} : H_i \longrightarrow H_j$$

by determining its composites with the colimit maps

$$c_i^D : \mathcal{K}(D, -) \longrightarrow H_i = Q_{K_i, \mathcal{D}_i} \qquad (D \in \mathcal{D}_i)$$

of H_i as follows. Given $m_D : D \longmapsto K_i$ in $\mathcal{D}_i$ we form a pullback of m_D and the first projection, π_1, of

$$K_j \cong K_i \times K_{i'}$$

(where i' denotes the complement of the set i in j), see Figure 1.

Since F preserves the pullback, from

$$Fm_D(k_i^D) = k_i = F\pi_1(k_i, k_{i'}) = F\pi_1(k_j^D)$$

we conclude that there exists $k_j^{D'} \in FD'$ with

$$Fm_{D'}(k_j^{D'}) = k_j \quad \text{and} \quad F\pi'(k_j^{D'}) = k_i^D. \tag{1}$$

$$\begin{tikzcd} D' \arrow[r, "\pi'"] \arrow[d, "m_{D'}"'] & D \arrow[d, "m_D"] \\ K_j \arrow[r, "\pi_1"] & K_i \end{tikzcd}$$

Figure 1.

Consequently, $m_{D'} : D' \longrightarrow K_j$ represents a member of $\mathcal{D}_j$. We compose $\mathcal{K}(\pi', -) : \mathcal{K}(D, -) \longrightarrow \mathcal{K}(D', -)$ with the colimit morphism $c_j^{D'} : \mathcal{K}(D', -) \longrightarrow H_j$ and obtain a morphism

$$c_j^{D'} \circ \mathcal{K}(\pi', -) : \mathcal{K}(D, -) \longrightarrow H_j.$$

Let us verify that these morphisms form a cocone, i.e., that given

$$D_0 \subseteq D \qquad \text{in } \mathcal{D}_i$$

(with the connecting morphism $m_{D_0,D}$), then

$$c_j^{D'} \circ \mathcal{K}(\pi', -) = \left[c_j^{D'_0} \circ \mathcal{K}(\pi'_0, -) \right] \circ \mathcal{K}(m_{D_0,D}, -) \tag{2}$$

where π'_0 is the morphism from the corresponding pullback for D_0, see Figure 2.

$$\begin{tikzcd} D'_0 \arrow[r, "\pi'_0"] \arrow[d, "m_{D'_0}"'] & D_0 \arrow[d, "m_{D_0}"] \\ K_j \arrow[r, "\pi_1"] & K_i \end{tikzcd}$$

Figure 2.

Use the universal property to define $m_{D'_0,D'}$, see Figure 3.
Since $c_j^{(-)}$ is a cocone, we have

$$c_j^{D'} = c_j^{D'_0} \circ \mathcal{K}(m_{D'_0,D'}, -)$$

therefore (2) holds:

$$c_j^{D'} \circ \mathcal{K}(\pi', -) \quad = \quad c_j^{D'_0} \circ \mathcal{K}(\pi' \circ m_{D'_0,D'}, -)$$

$$\begin{tikzcd}
D'_0 \arrow[rrr, "\pi'_0"] \arrow[dr, "m_{D'_0,D'}"] \arrow[ddr, swap, "m_{D'_0}"] & & & D_0 \arrow[dl, swap, "m_{D_0,D}"] \arrow[ddl, "m_{D_0}"] \\
& D' \arrow[r, "\pi'"] \arrow[d, "m_{D'}"] & D \arrow[d, swap, "m_D"] & \\
& K_j \arrow[r, "\pi_1"] & K_i &
\end{tikzcd}$$

Figure 3.

$$\begin{aligned}
&= c_j^{D'_0} \circ \mathcal{K}(m_{D_0,D} \circ \pi'_0, -) \\
&= c_j^{D'_0} \circ \mathcal{K}(\pi'_0, -) \circ \mathcal{K}(m_{D_0,D}, -).
\end{aligned}$$

Consequently, there is a unique morphism

$$h_{i,j} : H_i \longrightarrow H_j \qquad (i \subseteq j \text{ in } I)$$

factorizing the above cocone through the colimit cocone of H_i, i.e. with

$$h_{i,j} \circ c_i^D = c_j^{D'} \circ \mathcal{K}(\pi', -) \qquad \text{for all } D \in \mathcal{D}_i. \tag{3}$$

(iv) $h_{i,j}$ is a monomorphism in $\mathcal{S}et^{\mathcal{K}}$ (for any $i \subseteq j$). In fact, it is sufficient to prove that the right-hand side of (3) is a monomorphism for all $D \in \mathcal{D}_i$. That is given $X \in \mathrm{Obj}\,\mathcal{K}$ and $f, g : D \longrightarrow X$ with $F(f \circ \pi')(k_j^{D'}) = F(g \circ \pi')(k_j^{D'})$ then we are to show that $Ff(k_i^D) = Fg(k_i^D)$. This follows from (1).
(v) Our functor F is a colimit of the above directed diagram of all H_i $(i \in I)$ and $h_{i,j}$ $(i \leq j)$. In fact, define a natural transformation

$$h_i : H_i \longrightarrow F \qquad (i \in I)$$

by

$$h_i \circ c_i : \mathcal{K}(D, -) \longrightarrow F, \quad id_D \mapsto k_i^D \in FD$$

for all $D \in \mathcal{D}_i$. It is easy to see that $h_i \circ c_i^D$ is a cocone, i.e., if $D_0 \subseteq D$ then

$$h_i \circ c_i^D = h_i \circ c_i^{D_0} \circ \mathcal{K}(m_{D_0,D}, -)$$

because $Fm_{D_0,D}(k_i^{D_0}) = k_i^D$ (this follows from the fact that $Fm_{D_0}(k_i^{D_0}) = Fm_D(k_i^D)$).

Thus, h_i is well-defined.

The above cocone is collectively epic because for every element $k \in FK$ of K we have $i = \{(K, k)\}$ in I and then $h_i^K : Q_{K,\mathcal{D}_i} \longrightarrow FK$ maps the element $c_i^K(id_K)$ to k.

And each h_i is a monomorphism. In fact, assume that $h_i \circ c_i^D$ merges two elements $f, g : D \longrightarrow X$, then we prove that c_i^D merges them too. By assumption, $Ff(k_i^D) = Fg(k_i^D)$. Let $e : D_0 \longrightarrow D$ be an equalizer of f and g, then since Fe is an equalizer of Ff and Fg we have k_i^D in the image of Fe, thus $m_D \circ e : D_0 \longrightarrow K$ is a member of $\mathcal{D}_i$, and $e = m_{D_0,D}$. Since $c_i^D = c_i^{D_0} \circ \mathcal{K}(m_{D_0,D}, -)$, from $f \circ m_{D_0,D} = g \circ m_{D_0,D}$ we conclude $c_i^D(f) = c_i^D(g)$.

Consequently, $(H_i \xrightarrow{h_i} F)_{i \in I}$ is a filtered colimit. ■

3. All left exact set-valued functors are small

Definition. A category $\mathcal{K}$ is called *strongly left exact* provided that it is left exact (i.e., finitely complete) and well-powered and there exists a faithful left adjoint from $\mathcal{K}$ to $\mathcal{S}et$.

Examples. (1) Every category of presheaves

$$\mathcal{K} = \mathcal{S}et^{\mathcal{A}} \qquad (\mathcal{A} \text{ small})$$

is strongly left exact. The functor $V : \mathcal{K} \longrightarrow \mathcal{S}et$ defined on objects $H : \mathcal{A} \longrightarrow \mathcal{S}et$ by

$$V(H) = \coprod_{A \in \operatorname{Obj} \mathcal{A}} HA$$

and analogously on morphisms is obviously faithful. It preserves colimits, and since $\mathcal{K}$ has a set of generators (the hom-functors of objects of $\mathcal{A}$), it follows from the Special Adjoint Functor Theorem that V is a left adjoint.

(2) The category $\mathcal{K} = \mathcal{T}op$ of topological spaces is strongly left exact — just consider the usual forgetful functor and the indiscrete topology functor $\mathcal{S}et \longrightarrow \mathcal{T}op$.

The category $\mathcal{K} = \mathcal{R}el(n)$ of n-ary relations for any cardinal n and the category $\mathcal{K} = \mathcal{G}ra$ of undirected graphs are strongly left exact — just consider the usual forgetful functor. More generally, all topological categories (see [AHS]) over strongly left exact base-categories are strongly left exact.

(3) The category $\mathcal{G}rp$ of all groups is not strongly left exact. Under (R) this is a consequence of the following

3.1. theorem. *Assuming the axiom (R), every left exact functor $F : \mathcal{K} \longrightarrow \mathcal{S}et$ with $\mathcal{K}$ a strongly left exact category is small.*

Proof. Let $F : \mathcal{K} \longrightarrow \mathcal{S}et$ be a left exact functor, and let

$$V \dashv R : \mathcal{S}et \longrightarrow \mathcal{K}$$

be an adjoint situation with V faithful. Thus the unit $\varepsilon_K : K \longrightarrow RVK$ is formed by monomorphisms.

The functor FR is left exact, therefore, small, see Introduction above. Thus, there exists a cardinal λ such that for every set M and every element $x \in FRM$ there exists a function $f : M' \longrightarrow M$, $\operatorname{card} M' < \lambda$, with $x \in \operatorname{Im}(FRf)$. We are going to prove that for every object $K \in \mathcal{K}$ and every element $k \in FK$ there exists an object K' which is a subobject of RM' for some set M' with $\operatorname{card} M' < \lambda$ and there exists a morphism $g : K' \longrightarrow K$ with $k \in \operatorname{Im}(Fg)$. Since $\mathcal{K}$ is well-powered, all such objects K' have a small set of representatives with respect to isomorphism — therefore, F is small.

For the element $F\varepsilon_K(k) \in FRVK$ there exists a function $f : M' \longrightarrow VK$, $\operatorname{card} M' < \lambda$, and $y \in FRM'$ with

$$F\varepsilon_K(k) = FRf(y).$$

Let us form a pullback of ε_K and Rf, see Figure 4.

$$\begin{tikzcd} K' \arrow[r, "\varepsilon'_K"] \arrow[d, "g"'] & RM' \arrow[d, "Rf"] \\ K \arrow[r, "\varepsilon_K"] & RVK \end{tikzcd}$$

Figure 4.

Since F preserves this pullback, the above equality implies that there exists $z \in FK'$ with $Fg(z) = k$ and $F\varepsilon'_K(z) = y$. Thus $k \in \operatorname{Im}(Fg)$. Since ε_K is a monomorphism, so is ε'_K, i.e., K' is a subobject of RM'. ■

3.2. COROLLARY. *The statement "all left exact functors $Set^{\mathcal{A}} \longrightarrow Set$ ($\mathcal{A}$ any small category) are small" is consistent with set theory.*

In fact, we have remarked above that ZFC consistent implies ZFC+(R) consistent.

REMARK. Recall that P. Gabriel and F. Ulmer introduced in [GU] a category LFP of locally finitely presentable categories and all right adjoints preserving filtered colimits. In [ALR1] operations on LFP are studied whose arity is any small category $\mathcal{A}$: an $\mathcal{A}$-ary operation ω assigns to every object $\mathcal{K}$ of LFP an "operation map", i.e., a functor $\omega_{\mathcal{K}} : \mathcal{K}^{\mathcal{A}} \longrightarrow \mathcal{K}$, which all morphisms $H : \mathcal{K} \longrightarrow \mathcal{L}$ of LFP preserve in the expected sense: $H \circ \omega_{\mathcal{L}} \cong \omega_{\mathcal{K}} \circ H^{\mathcal{A}}$. It is proved in [ALR1] that $\mathcal{A}$-ary operations correspond bijectively to left exact functors on S$et^{\mathcal{A}}$, and operations called *legitimate* correspond precisely to small left exact functors. Thus:

3.3. COROLLARY. *It is consistent with ZFC to state that all operations of small arity on LFP are legitimate.*

4. Set functors preserving finite products

4.1. In [ALR2] the $\mathcal{A}$-ary operations on the category VAR of all varieties of algebras are studied. They correspond to set-valued functors on the category S$et^{\mathcal{A}}$ preserving finite products; and the so-called legitimate operations correspond to small functors preserving finite products. We will prove now that there is no analogy between the situation with $\mathcal{L}FP$ and $\mathcal{V}AR$, namely, the following holds in ZFC

- there is a large finite-products preserving functor $F : \mathcal{S}et^{\mathcal{A}} \longrightarrow \mathcal{S}et$ where $\mathcal{A}$ is the free monoid on two generators.

4.2. In fact, assuming the above axiom (M) (nonexistence of arbitrarily large measurable cardinals), we present large, finite-products preserving functors $F : \mathcal{K} \longrightarrow \mathcal{S}et$ for a very broad collection of categories $\mathcal{K}$. Namely, for all *algebraically universal* categories, i.e., categories $\mathcal{K}$ such that every variety of algebras has a full embedding into $\mathcal{K}$, see [PT]. The category

$$\mathcal{S}et^{\mathcal{A}} \cong \mathcal{A}lg(1,1)$$

for the above monoid $\mathcal{A}$ (equivalent to the category of unary algebras on two operations) is known to be algebraically universal, and so are the categories *Gra* (of graphs), *Sem* (of semigroups), and many others, see [PT].

The axiom (M) guarantees that every algebraically universal category $\mathcal{K}$ is *universal*, i.e., all concrete categories over $\mathcal{S}et$ have full embedding into $\mathcal{K}$. In particular, $\mathcal{K}$ has a large, full, discrete subcategory. We prove below that it also has a large strongly discrete subcategory where we introduce the following.

Definition. A full discrete subcategory $\mathcal{D}$ of a category $\mathcal{K}$ is called *strongly discrete* provided that given a finite product $D_1 \times D_2 \times \ldots \times D_n$ $(n \geq 1)$ of objects of $\mathcal{D}$ and a morphism $D_1 \times D_2 \times \ldots \times D_n \longrightarrow D$ with $D \in \mathcal{D}$, it follows that $D = D_i$ for some $i = 1, 2, \ldots, n$.

Example. of large functor $F : \mathcal{K} \longrightarrow \mathcal{S}et$ preserving finite products.

Let $\mathcal{D}$ be a large, strongly discrete subcategory of $\mathcal{K}$ and let $\mathcal{K}$ have finite products. Define F on objects X of $\mathcal{K}$ by

$$FX = \begin{cases} 1 & \text{if } \hom(D_1 \times \ldots \times D_n, X) \neq \emptyset \text{ for some } D_1, \ldots, D_n \in \mathcal{D}, \\ \emptyset & \text{else;} \end{cases}$$

the definition on morphisms is obvious. Then F clearly preserves finite products.

Suppose that F is small. Then there clearly exists a small collection $\mathcal{K}_0$ of objects $K \in \mathcal{K}$ with $FK = 1$ and such that for every object $X \in \mathcal{K}$ with $FX = 1$ we have $\hom(K, X) \neq \emptyset$ for some $K \in \mathcal{K}_0$. We derive a contradiction. For each $K \in \mathcal{K}_0$ since $FK = 1$, there exists a finite set $\mathcal{D}_K \subseteq \mathcal{D}$ such that we have a morphism from $\prod_{D \in \mathcal{D}_K} D$ into K.

Since $\mathcal{K}_0$ is small, also the union

$$\bar{\mathcal{D}} = \bigcup_{K \in \mathcal{K}_0} \mathcal{D}_K$$

is small. However, $\bar{\mathcal{D}} = \mathcal{D}$: for every object $D_0 \in \mathcal{D}$ we have $FD_0 = 1$, thus, by the choice of $\mathcal{K}_0$ there exists $K \in \mathcal{K}_0$ with $\hom(K, D_0) \neq \emptyset$. This implies $\hom(\prod_{D \in \mathcal{D}_K} D, D_0) \neq \emptyset$. By the definition of strong discreteness, we conclude that $D_0 \in \mathcal{D}_K \subseteq \bar{\mathcal{D}}$. This is a contradiction: $\bar{\mathcal{D}}$ is small but $\mathcal{D}$ is large.

4.3. THEOREM. *Assuming (M), every algebraically universal category has a large strongly discrete subcategory.*

PROOF. I. We construct a strongly discrete subcategory in the category $Rel(\Sigma)$ of relations of signature $\Sigma = \{R, T, S\}$ where R is binary, T unary and S ternary. This category is algebraically universal, and so is the category of directed graphs (one binary relation), see [PT]. Since (M) is assumed, it follows that there is a large, full, discrete subcategory $\mathcal{D}$ of the category of directed graphs. For every graph $G = (X, R_G)$ we denote by G^* the Σ-structure on the set $X \cup \{v\}$, $v \notin X$, where

$$R_{G^*} = R_G \cup \{(v, v)\},$$
$$T_{G^*} = X,$$
$$S_{G^*} = \{(x, y, z); \text{exactly one of } v = x,\ v = y,\ v = z \text{ holds}\} \cup \{(v, v, v)\}.$$

We then prove strong discreteness of $\mathcal{D}^* = \{D^*; D \in \mathcal{D}\}$ in $Rel(\Sigma)$. In fact, let

$$f : D_1^* \times D_2^* \times \ldots \times D_n^* \longrightarrow D^*$$

be a Σ-homomorphism for $D_i = (X_i, R_{D_i})$ and $D = (Y, R_D)$ in $\mathcal{D}$. Then we will prove that

$(*)$ there exists $i = 1, 2, \ldots, n$ with $f(\bar{x}) \in Y$ for all $x \in X_i$

where we put

$$\bar{x} = (v, v, \ldots, v, x, v, v, \ldots, v) \qquad x \text{ in position } i.$$

It follows that we obtain a graph homomorphism from D_i to D by $x \mapsto f(\bar{x})$ for $x \in X_i$: given $R(x_1, x_2)$ in D_i we have $R(\bar{x}_1, \bar{x}_2)$ in $D_1^* \times D_2^* \times \ldots \times D_n^*$, thus, $R(f(\bar{x}_1), f(\bar{x}_2))$ in D. This proves $D_i = D$ and, in case $n = 1$, $f = id$ – thus, $\mathcal{D}^*$ is strongly discrete.

Assuming that $(*)$ fails, we derive a contradiction by proving that f is the constant function with value v – this contradicts the preservation of T, of course. Given $z = (z_1, z_2, \ldots, z_n)$ in $D_1^* \times D_2^* \times \ldots \times D_n^*$ we prove $f(z) = v$ by induction on the number k of coordinates i with $z_i \neq v$. The case $k = 0$, i.e. $z = \bar{v}$, follows from the negation of $(*)$: choose $x_i \in X_i$ with $f(\bar{x}_i) = v$ $(i = 1, 2, \ldots, n)$. Then $S(\bar{x}_i, \bar{x}_i, \bar{v})$ holds in $D_1^* \times D_2^* \times \ldots \times D_n^*$, thus, $S(v, v, f(\bar{v}))$ holds in D^* and this proves $f(\bar{v}) = v$. Also the case $k = 1$, i.e. $z = \bar{y}_i$ for some $y_i \in X_i$, follows similarly: we have $S(\bar{x}_i, \bar{y}_i, \bar{v})$ in $D_1^* \times D_2^* \times \ldots \times D_n^*$, thus $S(v, f(\bar{y}_i), v)$ in D^*, i.e. $f(\bar{y}_i) = v$. In the induction step we have $k \geq 2$ and we choose a coordinate i with $z_i \in X_i$. Let z' denote the element obtained from z by changing the i-th coordinate only, with $z'_i = v$. Then $S(z, z', \bar{z}_i)$ in $D_1^* \times D_2^* \times \ldots \times D_n^*$. Since, by the induction hypothesis, $f(z') = f(\bar{z}_i) = v$, we conclude $S(f(z), v, v)$ in D^*, thus, $f(z) = v$.

II. For an arbitrary algebraically universal category $\mathcal{K}$ a full embedding $E : Rel(\Sigma) \longrightarrow \mathcal{K}$ exists, see [PT]. Then $E(\mathcal{D}^*)$ is strongly discrete in $\mathcal{K}$. In fact, given a finite product $\prod_{i=1}^n E(D_i)$ with $D_i \in \mathcal{D}^*$ and given $D \in \mathcal{D}^*$ for which a morphism $\prod_{i=1}^n E(D_i) \longrightarrow E(D)$ exists, then form a product $\prod_{i=1}^n D_i$ in $Rel(\Sigma)$ and observe that, since one always has a morphism $E(\prod_{i=1}^n D_i) \longrightarrow \prod_{i=1}^n E(D_i)$, there exists a morphism from $E(\prod_{i=1}^n D_i)$ to ED in $\mathcal{K}$. Since E is full, this yields a morphism from $\prod_{i=1}^n D_i$ to D, thus, $D = D_i$ for some $i = 1, 2, \ldots, n$. ■

4.4. COROLLARY. *For the free monoid on two generators, $\mathcal{A}$, there exist large functors from $\mathcal{S}et^{\mathcal{A}}$ to $\mathcal{S}et$ preserving finite products.*

In fact, if we assume (M), then this follows from IV.4 and IV.5 because $\mathcal{S}et^{\mathcal{A}} \cong \mathcal{A}lg(1,1)$ is algebraically universal. If $\neg$(M) is satisfied, we use the above functor $F : \mathcal{S}et \longrightarrow \mathcal{S}et$ of J. Reiterman presented in Section I, composed with the natural forgetful functor $U : \mathcal{A}lg(1,1) \longrightarrow \mathcal{S}et$. The composite FU preserves finite limits. And it is not small: if it were, it would preserve λ-directed colimits for some infinite cardinal λ. However, if one of the sets K_i in Reiterman's example (see Introduction) is chosen to have cardinality at least λ, we can present K_i as a λ-directed union of sets $L \subseteq K_i$ of cardinality less than λ, and for the free-algebra functor $V : \mathcal{S}et \longrightarrow \mathcal{A}lg(1,1)$ we obtain VK_i as a λ-directed colimit of the subalgebras VL. FU does not preserve this λ-directed colimit: in $FUV(K_i)$ consider the universal map $K_i \longrightarrow UVK_i$ as an element of $Q_{K_i,\mathcal{D}_i}UV(K_i)$. This element does not lie in the image of $FUV(m)$ for the embedding $m : L \longrightarrow K_i$ of any set $L \subseteq K_i$ of cardinality less than λ.

4.5. A DESCRIPTION OF FINITE-PRODUCTS PRESERVING FUNCTORS. Let $\mathcal{K}$ be a finitely complete category. A functor $\mathcal{K} \longrightarrow \mathcal{S}et$ preserves finite products iff it is a (possibly large) directed union of quotients of representables each of which preserves finite products.

This is a trivial consequence of the following observations: Let $F : \mathcal{K} \longrightarrow \mathcal{S}et$ preserve finite products, and let (K,k) be an element of F ($k \in FK$). Then the quotient of $\mathcal{K}(K,-)$ which is the image of the Yoneda transformation $\mathcal{K}(K,-) \longrightarrow F$ corresponding to k preserves finite products. The rest of the proof is analogous to that in II.4.

EXAMPLE. Let $\mathcal{D}$ be a filter on an object K. The quotient

$$R_{K,\mathcal{D}}$$

of $\mathcal{K}(K,-)$ given be the following congruence $\sim$

$$f \sim g \quad \text{iff} \quad f \circ m = g \circ m \quad \text{for some } m \in \mathcal{D} \qquad (f,g : K \longrightarrow X)$$

preserves finite products.

REMARK. In case $\mathcal{K} = \mathcal{S}et$, all finite-products preserving quotients of representables are naturally isomorphic to the functors $R_{K,\mathcal{D}}$. Moreover, whenever $\mathcal{D}$ is a "true filter", i.e., $\emptyset \notin \mathcal{D}$, then $R_{K,\mathcal{D}} \cong Q_{K,\mathcal{D}}$. However, given $K \neq \emptyset$ and $\mathcal{D} = \exp K$, then

$$Q_{K,\mathcal{D}} = \mathcal{S}et(\emptyset,-) \quad \text{—a constant functor with value } 1$$

whereas

$$R_{K,\mathcal{D}}\emptyset = \emptyset \quad \text{and} \quad R_{K,\mathcal{D}}X \cong 1 \quad \text{for all} \quad X \neq \emptyset.$$

This has been observed by the third author [T] who concluded that the only endofunctors of $\mathcal{S}et$ which are not left exact but preserve finite products are those naturally isomorphic to $R_{K,\exp K}$ ($K \neq \emptyset$). This result immediately extends to $\mathcal{K} =$ power of $\mathcal{S}et$, i.e., to the case $\mathcal{K} = \mathcal{S}et^{\mathcal{A}}$ for $\mathcal{A}$ small and discrete. Thus from III.4 we obtain

4.6. COROLLARY. *It is consistent with ZFC to state that all operations of small, discrete arity on VAR are legitimate.*

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