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PERFECT MAPS ARE EXPONENTIABLE - CATEGORICALLY

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ABSTRACT. A categorical proof of the statement given by the title is provided, in generalization of a result for topological spaces proved recently by Clementino, Hofmann and Tholen.

1. Introduction

Using convergence structures, recently the authors of [CHT] established the fibred version of the fact that compact Hausdorff spaces are exponentiable in the category **Top** of topological spaces, by proving:

THEOREM. *Every perfect map of topological spaces is exponentiable in* **Top**.

Here perfect means proper (=stably closed, [B]) and separated [J], while a map $f : X \to Y$ is exponentiable if it is an exponentiable object in the fibred category $\mathbf{Top}/Y$ of spaces over Y; equivalently, if the change-of-base functor

$$f^* : \mathbf{Top}/Y \to \mathbf{Top}/X$$

has a right adjoint [N]. The proof of the Theorem in [CHT] is based on a characterization of exponentiable maps in terms of ultrafilter convergence, which on the one hand gives an explicit description of the exponential structures involved (i.e., of the right adjoint of f^*), but which on the other hand makes extensive use of the Axiom of Choice.

In this note we give an entirely constructive and general proof of the Theorem, in the sense that no Choice is used and that the argumentation is purely categorical, based on techniques developed in [CGT] and [T]. However, the result proved is also somewhat weaker than that of [CHT], in the sense that we show only preservation of quotient maps by the change-of-base functor, which then implies the existence of a right adjoint via Freyd's Adjoint Functor Theorem. At the end of the paper we discuss applications of the Theorem to some categories other than **Top**.

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2. The categorical theorem

2.1. We work in a finitely-complete category $\mathcal{X}$ with a proper, pullback-stable factorization system $(\mathcal{E}, \mathcal{M})$, as in [DG1], [DT], [CGT]. Referring to (isomorphism classes of) $\mathcal{M}$-morphisms into an object X as *subobjects* of X, for every morphism $f : X \to Y$ we therefore have the *image/inverse-image adjunction*

$$f(-) \dashv f^{-1}(-) : \mathrm{sub}Y \to \mathrm{sub}X,$$

with $\mathrm{sub}X = (\mathcal{M}/X)/\cong$. Stability means that in any pullback diagram

$$\begin{array}{ccc} Y & \xrightarrow{q} & S \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle g} \\ X & \xrightarrow{p} & T \end{array} \tag{1}$$

with $g \in \mathcal{E}$ also its pullback h along p must lie in $\mathcal{E}$. Equivalently, the *Beck-Chevalley Property* (*BCP*) holds true:

$$g^{-1}(p(a)) = q(h^{-1}(a))$$

for all $a \in \mathrm{sub}X$.

2.2. In addition, *we fix a pullback-stable subclass*

$$\mathcal{F}_0 \subseteq \mathcal{M}$$

which contains all isomorphisms and is closed under composition. Subobjects in $\mathcal{F}_0$ are referred to as *closed subobjects.* A morphism $f : X \to Y$ is *closed* if $f(-)$ maps closed subobjects to closed subobjects. (Since $\mathcal{F}_0$ is closed under composition, closed subobjects are given by closed morphisms in $\mathcal{M}$.) If every pullback of f is closed, then f is a *proper* morphism. The morphism f is *separated* if the diagonal map

$$\delta_f = < 1_X, 1_X >: X \to X \times_Y X$$

into its kernel pair is closed (as a morphism, or as a subobject). A *perfect* morphism is by definition both proper and separated.

An object X is *compact* (*Hausdorff*) if the morphism $X \to 1$ to the terminal object of $\mathcal{X}$ is proper (separated, respectively); equivalently: if every projection $X \times Z \to Z$ (the diagonal map $\delta_X : X \to X \times X$, resp.) is closed.

A *quotient map* is a morphism $f : X \to Y$ in $\mathcal{E}$ with the property that a subobject b in Y is closed whenever $f^{-1}(b)$ is closed.

Note that *all notions defined in* 2.2 *depend on the parameter* $\mathcal{F}_0$. Whenever there is need to emphasize this dependency, we augment these notions by the prefix $\mathcal{F}_0$. We also note that, denoting by $\mathcal{F}$ the class of closed morphisms in $\mathcal{X}$, we have $\mathcal{F}_0 = \mathcal{F} \cap \mathcal{M}$, and the notions defined here coincide with the ones given in [T] (except that there for proper morphisms the object term "compact" was used as well).

2.3. The *fibres* of a morphism $f : X \to Y$ are given by the pullback diagrams

$$\begin{tikzcd} f^{-1}y \arrow[r] \arrow[d, "f^{-1}(y)"'] & 1 \arrow[d, "y"] \\ X \arrow[r, "f"] & Y \end{tikzcd} \tag{2}$$

Proper morphisms are, by definition, *closed and have compact fibres* (see [T], Cor.3.4(3)), while the converse proposition fails in general (see [CGT], Remark 5.14). We say that *in $\mathcal{X}$ properness is fibre-determined* if every closed morphism with compact fibres is proper. (Actually, we shall use this property only if the morphism in question is also in $\mathcal{E}$ and separated. Note that the fibres of a separated morphism are Hausdorff, since separatedness is, like properness, stable under pullback.)

Recall that the object 1 is said to be *$\mathcal{E}$-projective* if for every morphism $f : X \to Y$ in $\mathcal{E}$ every point $y : 1 \to Y$ factors as $f \cdot x = y$.

2.4. We list a few properties to be used in the proof of 2.5 below; proofs may be found in [T]:

(1) every monomorphism is separated, and separated morphisms are closed under composition and stable under pullback;

(2) if a composite $g \cdot f$ is separated, so is f;

(3) every morphism representing a closed subobject is proper, and proper morphisms are closed under composition and stable under pullback;

(4) if a composite $g \cdot f$ is proper with g separated, then also f is proper.

In the general setting of 2.1, 2.2 we are now able to prove:

2.5. Theorem. *Let the terminal object of $\mathcal{X}$ be $\mathcal{E}$-projective, and let properness in $\mathcal{X}$ be fibre-determined. Then the pullback of a quotient map along any perfect morphism is again a quotient map.*

Proof. We consider the pullback diagram (1) with p perfect and g a quotient map. In order to show that h is a quotient map as well, we consider $a \in \mathrm{sub}X$ with $h^{-1}(a) \in \mathrm{sub}Y$ closed and must show that a is closed.

Step 1: As a pullback of p the morphism q is closed, so that

$$q(h^{-1}(a)) = g^{-1}(p(a))$$

is closed, whence $p(a)$ is closed by hypothesis on g. Hence, in the commutative diagram

$$\begin{tikzcd}
& Y \arrow[rr, "q"] \arrow[dd, "h" near start] & & S \arrow[dd, "g"] \\
h^{-1}(A) \arrow[ur, "h^{-1}(a)"] \arrow[rr, crossing over, "q'" near end] \arrow[dd, "h'"'] & & g^{-1}p(A) \arrow[ur, "g^{-1}(p(a))"] & \\
& X \arrow[rr, "p" near start] & & T \\
A \arrow[ur, "a"'] \arrow[rr, "p'"'] & & p(A) \arrow[ur, "p(a)"'] \arrow[from=uu, crossing over, "g'" near end] &
\end{tikzcd} \tag{3}$$

all diagonal morphisms but a are known to be closed. By 2.4(3), (4) it now suffices to show that p' is proper, since then the properness of

$$(*) \qquad p(a) \cdot p' = p \cdot a$$

and the separatedness of p give closedness of a. In order to show properness of p' we use the hypothesis that this property is fibre-determined in $\mathcal{X}$. (We note that p' is in $\mathcal{E}$ and separated, which follows from $(*)$ and properties 2.4(1), (2).)

Step 2: In order to show that the morphism p' is closed, we consider any $b \in \mathrm{sub}A$ closed and form the commutative diagram

$$\begin{tikzcd}
& h^{-1}(A) \arrow[rr, "q'"] \arrow[dd, "h'" near start] & & g^{-1}(p(A)) \arrow[dd, "g'"] \\
h^{-1}(B) \arrow[ur, "(h')^{-1}(b)"] \arrow[rr, crossing over, "q''" near end] \arrow[dd, "h''"'] & & g^{-1}(p(B)) \arrow[ur, "(g')^{-1}(p'(b))"] & \\
& A \arrow[rr, "p'" near start] & & p(A) \\
B \arrow[ur, "b"'] \arrow[rr, "p''"'] & & p(B) \arrow[ur, "p'(b)"'] \arrow[from=uu, crossing over, "g''" near end] &
\end{tikzcd} \tag{4}$$

We note that q' is closed, in fact proper, since

$$q \cdot h^{-1}(a) = g^{-1}(p(a)) \cdot q'$$

has this property (by 2.4(3),(4)). Furthermore, g' is a quotient map, as a pullback of g along the closed subobject $p(a)$; indeed, for any $c \in \mathrm{sub}p(A)$ with $(g')^{-1}(c)$ closed also

$$g^{-1}(p(a)) \cdot (g')^{-1}(c) = g^{-1}(p(a) \cdot c)$$

is closed, which implies closedness of $p(a) \cdot c$ and then of c.

Using BCP again, we conclude that

$$q'((h')^{-1}(b)) = (g')^{-1}(p'(b))$$

is closed, whence also $p'(b)$ is closed.

Step 3: We must still show that p' has compact fibres. But any point $z : 1 \to p(A)$ factors as $z = g' \cdot w$, with $w : 1 \to f^{-1}(p(A))$, by $\mathcal{E}$-projectivity of 1, and we can form the commutative diagram

$$\begin{tikzcd}
& h^{-1}(A) \arrow[rr, "q'"] \arrow[dd, "h'"'] & & g^{-1}(p(A)) \arrow[dd, "g'"] \\
(q')^{-1}w \arrow[ur] \arrow[rr] \arrow[dd, "h^*"'] & & 1 \arrow[ur, "w"] \arrow[dd] & \\
& A \arrow[rr, "p'"] & & p(A) \\
(p')^{-1}z \arrow[ur] \arrow[rr] & & 1 \arrow[ur, "z"'] &
\end{tikzcd} \tag{5}$$

Its front face is a pullback diagram since the back-, top- and bottom-faces are pullback diagrams. Hence h^* is an isomorphism, so that with the compact fibre $(q')^{-1}w$ of the proper morphism q' also $(p')^{-1}z$ is compact. ∎

2.6. If, in the setting of 2.1, we are given a closure operator c (in the sense of [DG1], [DT], [CGT]), then we may choose for $\mathcal{F}_0$ the class of c-closed subobjects, which is pullback-stable, and also closed under composition if c is weakly hereditary. Furthermore, if c is idempotent, "c-preserving" of [CGT] becomes "closed" in this paper's terminology, while "proper", "separated", "perfect" read as "c-compact", "c-Hausdorff", "c-perfect" in [CGT], respectively. Writing "c-quotient" instead of "($\mathcal{F}_0$-)quotient", we deduce from 2.5:

2.7. Corollary. *For an idempotent and weakly hereditary closure operator c, let c-properness be fibre-determined in $\mathcal{X}$, and assume the terminal object to be $\mathcal{E}$-projective. Then the pullback of a c-quotient map along any c-perfect morphism is again a c-quotient map.* ∎

3. Examples

3.1. (Topological spaces). An application of 2.5 to the category **Top** of topological spaces with its (surjective, embedding)-factorization structure and the usual notion of closure gives a choice-free proof of:

Pullbacks of quotient maps along perfect maps are quotient maps.

From this fact one derives with Freyd's Adjoint Functor Theorem (using the Axiom of Choice) Theorem A of [CHT]:

Perfect maps are exponentiable in **Top**.

If instead of $\mathcal{F}_0 = \{\text{closed embeddings}\}$ we choose $\mathcal{F}_0 = \{\text{open embeddings}\}$, 2.5 still yields an interesting but much easier and well-known statement:

Local homeomorphisms are exponentiable in **Top**.

3.2. (Partially ordered sets). Call a subset A of a poset X (up-)closed if $x \geq a \in A$ always implies $x \in A$. Then, in the category **PoSet** of posets and monotone maps with its (surjective,embedding)-factorization structure, closed maps $f : X \to Y$ are characterized by the property that for all $y_2 \geq f(x_1)$ one finds $x_2 \geq x_1$ with $f(x_2) = y_2$. Such maps are already proper, so that every poset is compact, and trivially properness is fibre-determined. Perfect maps have the additional property that $f(x_2) = f(x_2')$ with $x_2, x_2' \geq x_1$ always implies $x_2 = x_2'$. Theorem 2.5 confirms that *pullbacks of quotient maps along perfect maps are quotient maps.* It is worth noting that in **PoSet**, like in **Top**, our notion of quotient map assumes the usual meaning as a regular epimorphism of the category. Indeed, for $f : X \to Y$ and $y, y' \in Y$, let us write

$$y \leq_f y' :\Longleftrightarrow \exists x, x' \in X : x \leq x', f(x) = y, f(x') = y';$$

then f is a regular epimorphism precisely if the order of Y is the transitive hull of $\leq_f$. Such maps clearly have the property that $B \subseteq Y$ is closed whenever $f^{-1}(B) \subseteq X$ is; conversely, for $y \leq y'$ in Y, one lets B be the up-closure of $\{y\}$ with respect to the transitive hull of $\leq_f$ and obtains that B must be closed.
We note that generally quotient maps are not stable under pullback along arbitrary morphisms. Those $f : X \to Y$ that are stable have the property that $\leq_f$ is the order of Y; they are also known as biquotient maps (see [CH]). We note furthermore that we may also order-dualize the result obtained from 2.5: *pullbacks of quotient maps along local homeomorphisms are quotient maps*; as in [CH], here we call $f : X \to Y$ a local homeomorphism if for all $y_1 \leq f(x_2)$ there is exactly one $x_1 \leq x_2$ with $f(x_1) = y_1$.

In essence, for both topological spaces and partially ordered sets, the chosen class $\mathcal{F}_0$ is able to identify important classes of exponentiable maps since it "contains" essential information about the structures themselves, as is demonstrated negatively also by the following example.

3.3. (Pretopological spaces). A pretopological space may, like a topological space, be defined as a set with a closure operation, except that this operation is not required to be idempotent. Like in **Top**, there are therefore natural notions of closure and convergence in the category **PrTop**. In contrast to **Top** though, a pretopological space in which all ultrafilters converge is ($\mathcal{F}_0$-) compact (with $\mathcal{F}_0 = \{\text{closed embeddings}\}$), but not conversely: see [DG2]. Exponentiable objects in **PrTop** were completely characterized in [LS] as the finitely-generated pretopological spaces, a result which was generalized to all quotient-reflective subcategories of **PrTop** in [R]. In particular, we see that compact and separated objects in these categories are not exponentiable, in general.

We finally give an example showing that the crucial hypothesis of Theorem 2.5 (that properness be fibre-determined) fails to be a necessary condition.

3.4. (Continuous maps). We consider the category $\mathbf{Top^2}$ (with $\mathbf{2}$ the two-element chain), whose objects are continuous maps of topological spaces, and whose morphisms $(p, q) : h \to g$ are given by commutative squares as in diagram (1) in **Top**. It inherits the "pointwise" factorization structure from **Top**, and we may declare $\mathcal{F}_0$ to consist of the pointwise closed embeddings, i.e., of those morphisms (p, q) for which both p and q are closed embeddings. Then an arbitrary morphism (p, q) is closed (proper, separated, a quotient map) if both p and q have the respective property in **Top**. Consequently, pullbacks of quotient maps along perfect maps in $\mathbf{Top^2}$ are quotient maps. However, properness is not fibre-determined in $\mathbf{Top^2}$, simply because the points of an object h in $\mathbf{Top^2}$, given by the map $h : Y \to X$ in **Top**, are completely determined by the points of Y. Hence, they will not describe all points of X, unless h is surjective. For example, for every non-empty space X, the $\mathbf{Top^2}$-map from $(\emptyset \to X)$ into the terminal object is closed and has compact fibres (since there are none), but fails to be proper.

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