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A NOTE ON ACTIONS OF A MONOIDAL CATEGORY

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Abstract. An action $* : \mathcal{V} \times \mathcal{A} \longrightarrow \mathcal{A}$ of a monoidal category $\mathcal{V}$ on a category $\mathcal{A}$ corresponds to a strong monoidal functor $F : \mathcal{V} \longrightarrow [\mathcal{A}, \mathcal{A}]$ into the monoidal category of endofunctors of $\mathcal{A}$. In many practical cases, the ordinary functor $f : \mathcal{V} \longrightarrow [\mathcal{A}, \mathcal{A}]$ underlying the monoidal F has a right adjoint g; and when this is so, F itself has a right adjoint G as a monoidal functor—so that, passing to the categories of monoids (also called "algebras") in $\mathcal{V}$ and in $[\mathcal{A}, \mathcal{A}]$, we have an adjunction $\mathrm{Mon}\, F \dashv \mathrm{Mon}\, G$ between the category $\mathrm{Mon}\, \mathcal{V}$ of monoids in $\mathcal{V}$ and the category $\mathrm{Mon}[\mathcal{A}, \mathcal{A}] = \mathrm{Mnd}\, \mathcal{A}$ of monads on $\mathcal{A}$. We give sufficient conditions for the existence of the right adjoint g, which involve the existence of right adjoints for the functors $X * -$ and $- * A$, and make $\mathcal{A}$ (at least when $\mathcal{V}$ is symmetric and closed) into a tensored and cotensored $\mathcal{V}$-category $\mathbf{A}$. We give explicit formulae, as large ends, for the right adjoints g and $\mathrm{Mon}\, G$, and also for some related right adjoints, when they exist; as well as another explicit expression for $\mathrm{Mon}\, G$ as a large limit, which uses a new representation of any monad as a (large) limit of monads of two special kinds, and an analogous result for general endofunctors.

1. Introduction

Recall from [EK] that a *monoidal functor* $F : \mathcal{V} \longrightarrow \mathcal{V}'$ between monoidal categories $\mathcal{V} = (\mathcal{V}, \otimes, I)$ and $\mathcal{V}' = (\mathcal{V}', \otimes', I')$ consists of a triple $F = (f, \widetilde{f}, f^\circ)$ where $f : \mathcal{V} \longrightarrow \mathcal{V}'$ is an ordinary functor, $\widetilde{f}$ is a natural transformation with components $\widetilde{f}_{XY} : fX \otimes' fY \longrightarrow f(X \otimes Y)$, and $f^\circ : I' \longrightarrow fI$ is a morphism in $\mathcal{V}'$, these data being required to satisfy three *coherence conditions*, corresponding respectively to the associativities, the left identities, and the right identities of $\otimes$ and of $\otimes'$. Recall further that a *monoidal natural transformation* η from $F = (f, \widetilde{f}, f^\circ) : \mathcal{V} \longrightarrow \mathcal{V}'$ to $G = (g, \widetilde{g}, g^\circ) : \mathcal{V} \longrightarrow \mathcal{V}'$ is just an ordinary natural transformation $\eta : f \longrightarrow g : \mathcal{V} \longrightarrow \mathcal{V}'$ satisfying two coherence conditions—one for $\otimes$ and one for I; and that monoidal categories, monoidal functors, and monoidal natural transformations constitute a 2-category $\mathcal{MONCAT}$ with a forgetful 2-functor $U : \mathcal{MONCAT} \longrightarrow \mathcal{CAT}$ sending $(\mathcal{V}, \otimes, I)$ to the underlying category $\mathcal{V}$ and so on. Recall finally that a monoidal functor $F = (f, \widetilde{f}, f^\circ)$ is said to be *strong* when f° and all the $\widetilde{f}_{XY}$ are invertible.

Here we are using $\mathcal{MONCAT}$ and $\mathcal{CAT}$ to denote the appropriate "meta-2-categories" in which the structure of monoidal categories or of categories is codified, along with the corresponding morphisms and 2-cells, while size is absent from our thoughts: so that one cannot think of the objects of $\mathcal{CAT}$ as forming a set. Similarly we have the meta-category

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$\mathcal{SET}$ of sets. However we also have a notion of *small* set, determined by an inaccessible cardinal ∞, and hence notions of small category and of small monoidal category; we write **Set** for the category of small sets, which is the full subcategory of $\mathcal{SET}$ with the small sets as objects, and similarly **Cat** and **MonCat** for the 2-categories of small categories and of small monoidal categories. Of course any honest category (that is, one whose morphisms form a set) can be rendered small, by a suitable choice of ∞ : for we suppose every cardinal to be exceeded by some inaccessible one.

By an *action* of a monoidal category $\mathcal{V} = (\mathcal{V}, \otimes, I)$ on a category $\mathcal{A}$ we mean a strong monoidal functor $F = (f, \widetilde{f}, f^\circ) : \mathcal{V} \longrightarrow [\mathcal{A}, \mathcal{A}]$, where $[\mathcal{A}, \mathcal{A}]$ is the category of endofunctors of $\mathcal{A}$, provided with the (strict) monoidal structure $([\mathcal{A}, A], \circ, 1_{\mathcal{A}})$, wherein $\circ$ denotes composition and $1_{\mathcal{A}}$ is the identity endofunctor. Here, to give the functor $f : \mathcal{V} \longrightarrow [\mathcal{A}, \mathcal{A}]$ is equally to give a functor $* : \mathcal{V} \times \mathcal{A} \longrightarrow \mathcal{A}$ where $X * A = (fX)A$; to give the invertible and natural $\widetilde{f}_{XY} : (fX) \circ (fY) \longrightarrow f(X \otimes Y)$ (or rather their inverses) is equally to give a natural isomorphism with components $\alpha_{XYA} : (X \otimes Y) * A \longrightarrow X * (Y * A)$; to give the invertible $f^\circ : 1_{\mathcal{A}} \longrightarrow fI$ (or rather its inverse) is equally to give a natural isomorphism with components $\lambda_A : I * A \longrightarrow A$; and the coherence conditions for F become the commutativity of the three diagrams

$$\begin{tikzcd}
((X \otimes Y) \otimes Z) * A \arrow[r, "\alpha"] \arrow[d, "a*1"'] & (X \otimes Y) * (Z * A) \arrow[r, "\alpha"] & X * (Y * (Z * A)) \\
(X \otimes (Y \otimes Z)) * A \arrow[rr, "\alpha"'] & & X * ((Y \otimes Z) * A) \arrow[u, "1*\alpha"']
\end{tikzcd} \tag{1.1}$$

$$\begin{tikzcd}
(I \otimes X) * A \arrow[rr, "\alpha"] \arrow[dr, "\ell*1"'] & & I * (X * A) \arrow[dl, "\lambda"] \\
& X * A &
\end{tikzcd} \tag{1.2}$$

$$\begin{tikzcd}
(X \otimes I) * A \arrow[rr, "\alpha"] \arrow[dr, "r*1"'] & & X * (I * A) \arrow[dl, "1*\lambda"] \\
& X * A &
\end{tikzcd} \tag{1.3}$$

wherein a, ℓ, r are the associativity, left identity, and right identity isomorphisms for the monoidal structure on $\mathcal{V}$. Thus an action of $\mathcal{V}$ on $\mathcal{A}$ can equally be described by giving the functor $* : \mathcal{V} \times \mathcal{A} \longrightarrow A$ along with the natural isomorphisms α and λ satisfying (1.1) – (1.3). {In fact, (1.2) is a consequence of (1.1) and (1.3) : the proof of the corresponding result for a monoidal category given by Kelly in [KM] extends unchanged to the bicategory case, while the data $(\mathcal{V}, \otimes, I, a, \ell, r, *, \alpha, \lambda)$ satisfying the monoidal-category axioms and (1.1) – (1.3) can be seen as describing a two-object bicategory.}

Of course any monoidal category $\mathcal{V} = (\mathcal{V}, \otimes, I)$ as above has a canonical action on (the underlying category $\mathcal{V}$ of) itself, given by $X * Y = X \otimes Y$, $\alpha = a$, and $\lambda = \ell$; then the corresponding $f : \mathcal{V} \longrightarrow [\mathcal{V}, \mathcal{V}]$ has $(fX)Y = X \otimes Y$. For another simple example of an action, take $\mathcal{V}$ to be the cartesian closed category **Set** and $\mathcal{A}$ to be any category admitting small *multiples* (also called *copowers*—for $X \in$ **Set** and $A \in \mathcal{A}$, the multiple $X \cdot A$ is the coproduct of X copies of A); clearly we have an action of **Set** on $\mathcal{A}$ for which $X * A = X \cdot A$.

Actions in the sense above occur widely in mathematics; and we have moreover noticed that, in many important cases, the functor $f : \mathcal{V} \longrightarrow [\mathcal{A}, \mathcal{A}]$ admits a right adjoint $g : [\mathcal{A}, \mathcal{A}] \longrightarrow \mathcal{V}$. When this is so, the situation is richer than one might at first think. To begin with, because the monoidal functor F is strong, it follows from Theorem 1.5 of Kelly's [KD] that this monoidal $F : \mathcal{V} \longrightarrow [\mathcal{A}, \mathcal{A}]$ has a right adjoint $G : [\mathcal{A}, \mathcal{A}] \longrightarrow \mathcal{V}$ in the 2-category $\mathcal{MONCAT}$ exactly when the ordinary functor $f : \mathcal{V} \longrightarrow [\mathcal{A}, \mathcal{A}]$ has a right adjoint $g : [\mathcal{A}, \mathcal{A}] \longrightarrow \mathcal{V}$ in $\mathcal{CAT}$ (that is, a right adjoint in the classical sense). More precisely, given an adjunction $\eta, \epsilon : f \dashv g$, there are a unique $\widetilde{g}$ and a unique g° for which $(g, \widetilde{g}, g^\circ)$ is a monoidal functor $G : [\mathcal{A}, \mathcal{A}] \longrightarrow \mathcal{V}$ and for which η, ϵ constitute an adjunction $F \dashv G$ in $\mathcal{MONCAT}$. Next, there is a 2-functor Mon : $\mathcal{MONCAT} \longrightarrow \mathcal{CAT}$ taking a monoidal category $\mathcal{V}$ to the category $\mathrm{Mon}\,\mathcal{V}$ of *monoids* in $\mathcal{V}$ (also known by many writers as *algebras* in $\mathcal{V}$). Such a monoid is an object M of $\mathcal{V}$ together with morphisms $m : M \otimes M \longrightarrow M$ and $i : I \longrightarrow M$ satisfying the evident associativity and unit axioms—and is effectively the same thing as a monoidal functor $M = (M, m, i) : 1 \longrightarrow \mathcal{V}$, where 1 is the unit category with its unique monoidal structure. So Mon : $\mathcal{MONCAT} \longrightarrow \mathcal{CAT}$ may be seen as the representable 2-functor $\mathcal{MONCAT}(1, -)$; and like any 2-functor, it takes adjunctions to adjunctions. In particular, for an action $F : \mathcal{V} \longrightarrow [\mathcal{A}, \mathcal{A}]$ of $\mathcal{V}$ on $\mathcal{A}$, an adjunction $\eta, \epsilon : f \dashv g$ in $\mathcal{CAT}$, being in effect the same thing as an adjunction $\eta, \epsilon : F \dashv G$ in $\mathcal{MONCAT}$, gives us in $\mathcal{CAT}$ an adjunction

$$\mathrm{Mon}\,\eta,\ \ \mathrm{Mon}\,\epsilon\ :\ \ \mathrm{Mon}\,F\ \dashv\ \mathrm{Mon}\,G\ :\ \ \mathrm{Mon}\,\mathcal{V} \longrightarrow\ \mathrm{Mnd}\,\mathcal{A}, \tag{1.4}$$

where $\mathrm{Mnd}\,\mathcal{A}$ is the category of monads on $\mathcal{A}$ and monad-morphisms, which is another name for the category $\mathrm{Mon}[\mathcal{A}, \mathcal{A}]$ of monoids in $[\mathcal{A}, \mathcal{A}]$ and monoid-maps. Of course the underlying object of the monoid $(\mathrm{Mon}\,F)(M, m, i)$ is just fM and so on; speaking informally, one says that a monoidal functor takes monoids to monoids.

Because of these consequences, it is of interest to investigate conditions on an action $F : \mathcal{V} \longrightarrow [\mathcal{A}, \mathcal{A}]$ under which $f : \mathcal{V} \longrightarrow [\mathcal{A}, \mathcal{A}]$ has a right adjoint g; and we do this below. In Section 2 we show that a *necessary* condition for f to have a right adjoint is that each $- * A : \mathcal{V} \longrightarrow \mathcal{A}$ have a right adjoint; and we recall that, at least in the important case of a right-closed monoidal $\mathcal{V}$, to give a category $\mathcal{A}$ and an action of $\mathcal{V}$ on $\mathcal{A}$ with a right adjoint for each $- * A$ is equivalently to give a *tensored $\mathcal{V}$-category* $\mathbf{A}$ (of which $\mathcal{A}$ is then, to within isomorphism, the underlying ordinary category). In fact we defer to our appendix (Section 6) many of the details of the equivalence above, which has been discussed still more generally in [GP], so as not to interrupt too seriously the development of our central arguments. We turn in Section 3 to *sufficient* conditions for f to have a right adjoint: under reasonable hypotheses on $\mathcal{V}$ (which are satisfied by all the usual practical examples of a symmetric monoidal closed category), and under the hypotheses that $\mathcal{A}$ be locally small with a small generating set, we show that f will have a right adjoint when, to the the necessary condition above that each $-*A : \mathcal{V} \longrightarrow \mathcal{A}$ have a right adjoint, we add the further condition that each $X * - : \mathcal{A} \longrightarrow \mathcal{A}$ have a right adjoint. We go on to observe in addition that, at least in the simplest case where $\mathcal{V}$ is symmetric monoidal closed, to give an action of $\mathcal{V}$ on $\mathcal{A}$ for which both $-*A$ and $X*-$ have right adjoints is equivalently to give a $\mathcal{V}$-category $\mathbf{A}$ that is both tensored and cotensored. In fact, under these conditions, there is a right adjoint not only for $f : \mathcal{V} \longrightarrow [\mathcal{A}, \mathcal{A}]$ but also for the appropriate functor q from $\mathcal{V}$ to the category $(\mathbf{A}, \mathbf{A})$ of $\mathcal{V}$-functors $\mathbf{A} \longrightarrow \mathbf{A}$ and $\mathcal{V}$-natural transformations between these. We consider in particular (both here and later) the special cases given by $\mathcal{A} = \mathcal{V}$ and by $\mathcal{V} = \mathbf{Set}$. In Section 4 we treat the expressions for the right adjoint g of f and the right adjoint r of q as the ends $gT = \int_{A \in \mathcal{A}} \mathbf{A}(A, TA)$ and $rH = \int_{A \in \mathbf{A}} \mathbf{A}(A, HA)$; because these are *large* ends, we can use them only when we already know—for example, using the sufficient conditions of Section 3—that gT and rH exist. Finally, in Section 5, we give a different expression for gT as a large limit, using a new result expressing any monad as a canonical limit of monads of two special kinds, with a corresponding result for mere endofunctors. In addition, there are scattered throughout the sections a number of examples and of counter-examples.

2. An argument from the evaluation functors

Our concern being primarily with categories that occur in normal mathematical discourse, we have no reluctance about accepting hypotheses requiring $\mathcal{V}$ or $\mathcal{A}$ to be complete, cocomplete, or locally small; in many cases of interest, all of these will hold.

When $\mathcal{A}$ is locally small and admits (small) powers—and so certainly if $\mathcal{A}$ is locally small and complete—the evaluation functor $e(A) : [\mathcal{A}, \mathcal{A}] \longrightarrow \mathcal{A}$ sending $T : \mathcal{A} \longrightarrow \mathcal{A}$ to TA has a right adjoint sending $B \in \mathcal{A}$ to the power $B^{\mathcal{A}(-,A)}$. In that case a *necessary* condition for $f : \mathcal{V} \longrightarrow [\mathcal{A}, \mathcal{A}]$ to have a right adjoint is that each $e(A)f$ have a right adjoint. Since $e(A)f$ is the functor $-*A : \mathcal{V} \longrightarrow \mathcal{A}$, this is to require that each $-*A$ have a right adjoint—given, say, by a natural isomorphism

$$\kappa : \mathcal{A}(X * A, B) \;\cong\; \mathcal{V}(X, \mathbf{A}(A, B))\;. \tag{2.1}$$

For the rest of the paper, we shall suppose that we do have an adjunction (2.1), whether or not $\mathcal{A}$ is locally small and admits powers; we lose little of value by restricting ourselves to actions with this property. When $\mathcal{A}$ is $\mathcal{V}$ itself, with the canonical action given by $X * A = X \otimes A$, one often writes (2.1) in the form

$$\pi : \mathcal{V}(X \otimes A, B) \;\cong\; \mathcal{V}(X, [A, B])\;, \tag{2.2}$$

using $[A, B]$ rather than $\mathbf{V}(A, B)$; and then we say that the monoidal category $\mathcal{V}$ is *right closed*.

It is often-rediscovered folklore that, when we have an adjunction (2.1), the $\mathbf{A}(A, B)$ are the $\mathcal{V}$-valued homs of a $\mathcal{V}$-category $\mathbf{A}$ whose underlying ordinary category $\mathbf{A}_0$ is (canonically isomorphic to) $\mathcal{A}$. In fact the composition operation

$$M : \mathbf{A}(B,C) \otimes \mathbf{A}(A,B) \longrightarrow \mathbf{A}(A,C)$$

is the morphism corresponding under (an instance of) the adjunction (2.1) to the composite

$$(\mathbf{A}(B,C) \otimes \mathbf{A}(A,B)) * A \xrightarrow[\alpha]{} \mathbf{A}(B,C) * (\mathbf{A}(A,B) * A) \xrightarrow[1*\epsilon_{AB}]{} \mathbf{A}(B,C) * B \xrightarrow[\epsilon_{BC}]{} C, \quad (2.3)$$

where $\epsilon_{AB} : \mathbf{A}(A,B) * A \longrightarrow B$ is the B-component of the counit ϵ_A of the adjunction (2.1); the unit operation $j : I \longrightarrow \mathbf{A}(A,A)$ corresponds under (2.1) to $\lambda : I * A \longrightarrow A$; the associativity and unit axioms follow just as in the special case $\mathcal{A} = \mathcal{V}$, treated in Section 1.6 of [KB] or, in greater detail, in Sections II.3 and II.4 of [EK]; and the isomorphism between $\mathcal{A}$ and $\mathbf{A}_0$ is given by the isomorphisms

$$\mathcal{A}(A,B) \xrightarrow[\mathcal{A}(\lambda,1)]{} \mathcal{A}(I * A, B) \xrightarrow[\kappa]{} \mathcal{V}(I, \mathbf{A}(A,B)) = \mathbf{A}_0(A,B). \quad (2.4)$$

It is moreover straightforward to verify that, if $\mathcal{A}$ is identified with $\mathbf{A}_0$ by this isomorphism, the functor $\mathbf{A} : \mathcal{A}^{\mathrm{op}} \times \mathcal{A} \longrightarrow \mathcal{V}$ occurring in the adjunction (2.1) coincides with the functor $\hom_{\mathbf{A}} : \mathbf{A}_0^{\mathrm{op}} \times \mathbf{A}_0 \longrightarrow \mathcal{V}$ arising (as in Theorem I.8.2 of [EK]) from the $\mathcal{V}$-category $\mathbf{A}$ (and described more simply, when $\mathcal{V}$ is symmetric, in Section 1.6 of [KB]).

In particular, of course, when $\mathcal{V}$ is right closed, it is itself (isomorphic to) the underlying category $\mathbf{V}_0$ of a $\mathcal{V}$-category $\mathbf{V}$ having $\mathbf{V}(X,Y) = [X,Y]$ as its "internal hom".

When the $\mathcal{V}$ acting on $\mathcal{A}$ and admitting the adjunction (2.1) is in fact right closed, we can say more about the $\mathcal{V}$-category $\mathbf{A}$. By Yoneda, there is a unique natural transformation

$$k = k_{XAB} : \mathbf{A}(X * A, B) \longrightarrow [X, \mathbf{A}(A,B)] \quad (2.5)$$

making commutative the diagram

$$\begin{tikzcd}
\mathcal{A}(Y * (X * A), B) \arrow[rr, "\kappa"] \arrow[d, "\mathcal{A}(\alpha,B)"'] & & \mathcal{V}(Y, \mathbf{A}(X * A, B)) \arrow[d, "\mathcal{V}(Y,k)"] \\
\mathcal{A}((Y \otimes X) * A, B) \arrow[r, "\kappa"'] & \mathcal{V}(Y \otimes X, \mathbf{A}(A,B)) \arrow[r, "\pi"'] & \mathcal{V}(Y, [X, \mathbf{A}(A,B)])\ ;
\end{tikzcd} \quad (2.6)$$

and in fact k is invertible since, by the invertibility of α, κ, and π, each $\mathcal{V}(Y,k)$ is invertible. Recall now that we have the representable $\mathcal{V}$-functors $\mathbf{A}(X * A, -) : \mathbf{A} \longrightarrow \mathbf{V}$, $\mathbf{A}(A,-) : \mathbf{A} \longrightarrow \mathbf{V}$, and $[X,-] : \mathbf{V} \longrightarrow \mathbf{V}$, as well as the composite $\mathcal{V}$-functor $[X, \mathbf{A}(A,-)] : \mathbf{A} \longrightarrow \mathbf{V}$.

2.1. Lemma. *The k_{XAB} of (2.5) is equal to the composite*

$$\mathbf{A}(X * A, B) \xrightarrow[\mathbf{A}(A,-)_{X*A,B}]{} [\mathbf{A}(A, X * A), \mathbf{A}(A,B)] \xrightarrow[[\delta,1]]{} [X, \mathbf{A}(A,B)], \quad (2.7)$$

where $\delta = \delta_{AX} : X \longrightarrow \mathbf{A}(A, X * A)$ is the X-component of the unit δ_A of the adjunction (2.1). It follows from this that k_{XAB} is the B-component of a $\mathcal{V}$-natural isomorphism

$$k_{XA} \ :\ \mathbf{A}(X * A, -) \longrightarrow [X, \mathbf{A}(A,-)]. \quad (2.8)$$

Proof. Since $\mathbf{A}(A,-)_{X*A,B}$ corresponds under the adjunction (2.2) to the composition operation $M : \mathbf{A}(X*A,B)\otimes\mathbf{A}(A,X*A)\longrightarrow\mathbf{A}(A,B)$ we are equivalently to show that $\pi^{-1}(k)$ is the composite

$$\mathbf{A}(X*A,B)\otimes X\xrightarrow[1\otimes\delta]{}\mathbf{A}(X*A,B)\otimes\mathbf{A}(A,X*A)\xrightarrow[M]{}\mathbf{A}(A,B).$$

Again, since $\kappa^{-1}(u)$ for $u : X\longrightarrow\mathbf{A}(A,B)$ is $\epsilon(u*A)$, we are equivalently to show that $\kappa^{-1}\pi^{-1}(k)$ is the bottom leg of the diagram

$$\begin{tikzcd}
(\mathbf{A}(X*A,B)\otimes X)*A \arrow[rr, "\alpha"] \arrow[d, "(1\otimes\delta)*1"'] & & \mathbf{A}(X*A,B)*(X*A) \arrow[dl, "1*(\delta*1)"'] \arrow[dd, "1"] \\
(\mathbf{A}(X*A,B)\otimes\mathbf{A}(A,X*A))*A \arrow[r, "\alpha"] \arrow[ddd, "M*1"'] & \mathbf{A}(X*A,B)*(\mathbf{A}(A,X*A)*A) \arrow[dr, "1*\epsilon"] & \\
& & \mathbf{A}(X*A,B)*(X*A) \arrow[dd, "\epsilon"] \\
& & \\
\mathbf{A}(A,B)*A \arrow[rr, "\epsilon"'] & & B\ ,
\end{tikzcd}$$

wherein the top region commutes by naturality, the triangle by one of the triangular equations for the adjunction (2.1), and the bottom region by the definition of M; that is, we are to show that $\kappa^{-1}\pi^{-1}(k)=\epsilon\alpha$. This, however, is exactly what we find when we set $Y=\mathbf{A}(X*A,B)$ in (2.6) and evaluate both legs at $1\in\mathcal{V}(Y,\mathbf{A}(X*A,B))$. So k has the form (2.7); and now the $\mathcal{V}$-naturality in B of k_{XAB} follows from Proposition I.8.3 and I.8.4 of [EK]. (Alternatively, the k_{XA} of (2.8) is the unique $\mathcal{V}$-natural transformation corresponding, by the *enriched* Yoneda Lemma, to the morphism $\delta : X\longrightarrow\mathbf{A}(A,X*A)$ of $\mathcal{V}$.) ∎

The existence of the $\mathcal{V}$-enriched representation (2.8) of $[X,\mathbf{A}(A,-)]$ is expressed by saying that the $\mathcal{V}$-category $\mathcal{A}$ is *tensored*, with $X*A$ as the *tensor product* of X and A. (Although this notion of tensor product was originally introduced only for the case of a symmetric monoidal closed $\mathcal{V}$, it clearly makes sense for any right-closed monoidal $\mathcal{V}$; indeed Gordon and Power [GP] extend the definition to the case where $\mathcal{V}$ is replaced by a right-closed *bicategory*.)

In fact, for a right-closed monoidal $\mathcal{V}$, to give a category $\mathcal{A}$ and an action of $\mathcal{V}$ on $\mathcal{A}$ admitting the adjunction (2.1) is *essentially the same thing* as to give a tensored $\mathcal{V}$-category $\mathbf{A}$. To avoid interrupting our main argument further, we shall defer an account of the correspondence to our Appendix. Gordon and Power, in [GP], do much more : they not only establish this correspondence at the level of objects, but consider morphisms and

2-cells as well, to provide an equivalence of 2-categories; and they do so with a general right-closed bicategory in place of our monoidal $\mathcal{V}$. We have two reasons for giving, nevertheless, a brief account of our own : first, things are simpler in our more restricted situation, and with our more restricted goals; and secondly, the treatment in [GP] leaves rather a lot to the reader—such as the proof of our Lemma 2.1 above, covered only by their remark on page 181 (where our $\mathcal{V}$ is their $\mathcal{W}$) that "It is routine to verify that this structure does form a tensored $\mathcal{W}$-category".

Our Introduction contains two examples of such tensored $\mathcal{V}$-categories : first, the canonical action of $\mathcal{V}$ on itself corresponds, when $\mathcal{V}$ is right closed, to the tensored $\mathcal{V}$-category $\mathbf{V}$—and for later use, we shall denote the isomorphism k of (2.5) in this special case by

$$p : [X \otimes Y, Z] \longrightarrow [X, [Y, Z]]; \tag{2.9}$$

secondly, **Set** is certainly right closed, and a tensored **Set**-category is just an ordinary locally-small category admitting small multiples.

3. Use of the Special Adjoint Functor Theorem

We continue to suppose that we have an action $F : \mathcal{V} \longrightarrow [\mathcal{A}, \mathcal{A}]$, given either by $F = (f, \widetilde{f}, f^\circ)$ or, alternatively, by $* : \mathcal{V} \times \mathcal{A} \longrightarrow \mathcal{A}$ along with α and λ, for which we have the adjunction (2.1) for each pair (X, A); and we now consider what more may suffice to ensure a right adjoint for $f : \mathcal{V} \longrightarrow [\mathcal{A}, \mathcal{A}]$.

Recall that $e(A) : [\mathcal{A}, \mathcal{A}] \longrightarrow \mathcal{A}$ is the functor given by evaluation at $A \in \mathcal{A}$. The composite $e(A)f : \mathcal{V} \longrightarrow \mathcal{A}$, which is $- * A$, is a left adjoint by (2.1), and hence preserves whatever colimits exist in $\mathcal{V}$; that is, a colimit cone $q_j : Xj \longrightarrow Y$ in $\mathcal{V}$ gives for each A a colimit cone $q_j * A : Xj * A \longrightarrow Y * A$ in $\mathcal{A}$. Since diagrams in $[\mathcal{A}, \mathcal{A}]$ admit colimits formed pointwise when their evaluations admit colimits in $\mathcal{A}$, such a colimit cone q_j in $\mathcal{V}$ gives a colimit cone $fq_j : fXj \longrightarrow fY$ in $[\mathcal{A}, \mathcal{A}]$; in other words, $f : \mathcal{V} \longrightarrow [\mathcal{A}, \mathcal{A}]$ preserves whatever colimits exist in $\mathcal{V}$. To say that f has a right adjoint g is to say that, for each $T \in [\mathcal{A}, \mathcal{A}]$, the functor $h = [\mathcal{A}, \mathcal{A}](f-, T) : \mathcal{V}^{\mathrm{op}} \longrightarrow \mathcal{SET}$ is representable. Note that this functor h takes its values in $\mathcal{SET}$—indeed in some legitimate full subcategory **SET** of $\mathcal{SET}$—but not in general in the category **Set** of small sets; for $[\mathcal{A}, \mathcal{A}]$ need not be locally small even when $\mathcal{A}$ is so. Observe that h preserves all limits that exist in $\mathcal{V}^{\mathrm{op}}$, since f preserves colimits. So, by the Special Adjoint Functor Theorem as given in Ch.V, Section 8 of Mac Lane's [ML], h is representable if

(i) $\mathcal{V}$ is cocomplete and locally small;

(ii) $\mathcal{V}$ admits all cointersections (even large ones, if need be) of epimorphisms;

(iii) there is a small subset of the objects of $\mathcal{V}$ that constitutes a generator;

and

(%) $h : \mathcal{V}^{\mathrm{op}} \longrightarrow \mathcal{SET}$ takes its values in the full subcategory **Set** of small sets.

Note that (ii) is implied by (i) if $\mathcal{V}$ is cowellpowered; and that one cannot omit the condition (%), even though it is often left implicit when the theorem is stated. We shall now show, however, that (%) is a consequence of the following three conditions:

(iv) $\mathcal{A}$ is locally small;

(v) there is a small subset of the objects of $\mathcal{A}$ that constitutes a generator;

and

(vi) each $fX : \mathcal{A} \longrightarrow \mathcal{A}$ is a left adjoint.

To establish (%), we are to show that each $[\mathcal{A}, \mathcal{A}](fX, T)$ is a small set, where $X \in \mathcal{V}$ and $T : \mathcal{A} \longrightarrow \mathcal{A}$. Write S for fX, which by (vi) is a left adjoint. Let the small subset $\mathcal{K}$ of the set of objects of $\mathcal{A}$ be the generator given by (v); to say that it is a generator is to say that, for each A in $\mathcal{A}$, the family $\mathcal{P}$ of all morphisms $p : K \longrightarrow A$ with codomain A and with domain $K = K(p)$ in $\mathcal{K}$ is jointly epimorphic. Because $S : \mathcal{A} \longrightarrow \mathcal{A}$ is a left adjoint, the family given by the $Sp : SK(p) \longrightarrow SA$ for p in $\mathcal{P}$ is again jointly epimorphic. For a functor $T : \mathcal{A} \longrightarrow \mathcal{A}$ and a natural transformation $\alpha : S \longrightarrow T$, therefore, it follows from the naturality square $\alpha_A.Sp = Tp.\alpha_K$ that the function $[\mathcal{A}, \mathcal{A}](S, T) \longrightarrow \Pi_{K \in \mathcal{K}} \mathcal{A}(SK, TK)$ sending α to $\{\alpha_K |\ K \in \mathcal{K}\}$ is injective; which shows $[\mathcal{A}, \mathcal{A}](S, T)$ to be a small set.

So the six conditions (i) – (vi) suffice for the existence of a right adjoint g for f. The first five of these conditions bear only on the categories $\mathcal{V}$ and $\mathcal{A}$; note that, by a separate application of the Special Adjoint Functor Theorem, (i), (ii), and (iii) imply that $\mathcal{V}$ is complete, and similarly (iv) and (v) imply that $\mathcal{A}$ is complete whenever it is cocomplete and admits all cointersections of epimorphisms. Now let us re-cast (vi), which is a condition on the action itself, expressing it in terms of the functor $* : \mathcal{V} \times \mathcal{A} \longrightarrow \mathcal{A}$. Since $fX = X * -$, condition (vi) is the requirement that, in addition to the adjunction $- * A \dashv \mathbf{A}(A, -)$ of (2.1), we also have for each $X \in \mathcal{V}$ an adjunction

$$\mathcal{A}(X * A, B) \ \cong\ \mathcal{A}(A, |X, B|); \tag{3.1}$$

then, of course, $|\ ,\ |$ is a functor $\mathcal{V}^{\mathrm{op}} \times \mathcal{A} \longrightarrow \mathcal{A}$.

When $\mathcal{A}$ is $\mathcal{V}$ itself, with $* = \otimes$, (3.1) takes the form

$$\mathcal{V}(X \otimes Y, Z) \ \cong\ \mathcal{V}(Y, [[X, Z]]); \tag{3.2}$$

a monoidal $\mathcal{V}$ with such an adjunction is of course said to be *left closed.* A monoidal $\mathcal{V}$ admitting both the adjunctions (2.2) and (3.2) was formerly said to be *biclosed*, but is now more commonly said to be simply *closed.* Recall that $\mathcal{V}^{\mathrm{rev}}$ is the monoidal category which is $\mathcal{V}$ as a category, with the same unit object I, but which has the "reverse" tensor product $\otimes_{\mathrm{rev}}$ given by

$$A \otimes_{\mathrm{rev}} B = B \otimes A; \tag{3.3}$$

of course $\mathcal{V}^{\mathrm{rev}}$ is right closed when $\mathcal{V}$ is left closed, and conversely. When the monoidal $\mathcal{V}$ is braided, we have an isomorphism $\mathcal{V}^{\mathrm{rev}} \cong \mathcal{V}$ of monoidal categories, which is involutory when $\mathcal{V}$ is actually symmetric; in these cases a right closed $\mathcal{V}$ is also left closed, and thus closed.

Recall too that each $\mathcal{V}$-category $\mathbf{A}$ gives rise to a $\mathcal{V}^{\mathrm{rev}}$-category $\mathbf{A}^*$, which has the same objects as $\mathbf{A}$ but has

$$\mathbf{A}^*(A, B) = \mathbf{A}(B, A); \tag{3.4}$$

the ordinary category $(\mathbf{A}^*)_0$ underlying $\mathbf{A}^*$ is of course $(\mathbf{A}_0)^{\mathrm{op}}$. When $\mathcal{V}$ is braided, the isomorphism $\mathcal{V}^{\mathrm{rev}} \cong \mathcal{V}$ takes us from the $\mathcal{V}^{\mathrm{rev}}$-category $\mathbf{A}^*$ to a $\mathcal{V}$-category called $\mathbf{A}^{\mathrm{op}}$, having $\mathbf{A}^{\mathrm{op}}(A, B) = \mathbf{A}(B, A)$ and $(\mathbf{A}^{\mathrm{op}})_0 = (\mathbf{A}_0)^{\mathrm{op}}$; but when $\mathcal{V}$ is only braided and not symmetric, the canonical isomorphism $\mathbf{A} \cong (\mathbf{A}^{\mathrm{op}})^{\mathrm{op}}$ ceases to be an identity.

In the presence of the adjunction (2.1), to ask for the existence of an adjunction (3.1) is equally to ask for the existence of an adjunction

$$\mathcal{A}(A, |X, B|) \;\cong\; \mathcal{V}(X, \mathbf{A}(A, B)). \tag{3.5}$$

Since we can rewrite this as

$$\mathcal{A}^{\mathrm{op}}(|X, B|, A) \;\cong\; \mathcal{V}^{\mathrm{rev}}(X, \mathbf{A}^*(B, A)), \tag{3.6}$$

it is equivalent, when $\mathcal{V}$ is left closed, to the assertion that the $\mathcal{V}^{\mathrm{rev}}$-category $\mathbf{A}^*$ is tensored. In the case of a braided closed $\mathcal{V}$, and in particular of a symmetric closed $\mathcal{V}$, this is equally to say that the $\mathcal{V}$-category $\mathbf{A}^{\mathrm{op}}$ is tensored, or equivalently that the $\mathcal{V}$-category $\mathbf{A}$ is *cotensored*, in the sense that we have for each X and B an isomorphism

$$\mathbf{A}(A, |X, B|) \;\cong\; [X, \mathbf{A}(A, B)] \tag{3.7}$$

which is $\mathcal{V}$-natural in A.

Note that the conditions (i), (ii), and (iii), requiring $\mathcal{V}$ to be cocomplete and locally small, to admit all cointersections of epimorphisms, and to have a small generating set, are satisfied by all the examples of symmetric monoidal closed categories mentioned in [KB]; so that these are not severe restrictions in practice. (Note too that, when $\mathcal{V}$ has a small *strongly generating* set, it suffices for our use of the Special Adjoint Functor Theorem that it admit all cointersections only of *strong* epimorphisms; we shall not explicitly mention in future this variant, which is less useful in practice.) Again, if $\mathbf{A}$ is a $\mathcal{V}$-category, it is automatic that $\mathbf{A}_0$ is locally small. Recall from [KB] (where only the case of a symmetric closed $\mathcal{V}$ is discussed) that a $\mathcal{V}$-category $\mathbf{A}$ is said to be *cocomplete* when $\mathbf{A}$ is tensored and the ordinary category $\mathbf{A}_0$ is cocomplete (and of course to be *complete* when $\mathbf{A}$ is cotensored and $\mathbf{A}_0$ is complete). With that said, we can sum up as follows.

3.1. Theorem. *Let the monoidal $\mathcal{V}$ be cocomplete and locally small, admit all cointersections of epimorphisms, and have a small generating set (and so be complete as well). Let the action $F = (f, \widetilde{f}, f^\circ) : \mathcal{V} \longrightarrow [\mathcal{A}, \mathcal{A}]$ of $\mathcal{V}$ on the category $\mathcal{A}$ correspond as above to the functor $* : \mathcal{V} \times \mathcal{A} \longrightarrow \mathcal{A}$. Then for f, and hence F and $\mathrm{Mon}\, F$, to admit a right adjoint, it suffices that $\mathcal{A}$ be locally small with a small generating set, and that $- * A$ and $X * -$ admit right adjoints as in (2.1) and (3.1). When the monoidal $\mathcal{V}$ is also closed, to give a category $\mathcal{A}$ and an action of $\mathcal{V}$ on $\mathcal{A}$ with the properties above is equally to give a tensored $\mathcal{V}$-category $\mathbf{A}$ for which $\mathbf{A}^*$ is tensored and for which the underlying ordinary category $\mathbf{A}_0$ has a small generating set. When the monoidal closed $\mathcal{V}$ is braided, and in particular when it is symmetric, we can replace "$\mathbf{A}^*$ is tensored" in the preceding sentence by "$\mathbf{A}$ is cotensored"; such an $\mathbf{A}$ is in fact complete and cocomplete if $\mathbf{A}_0$ is cocomplete and admits all cointersections of epimorphisms.*

In particular, we have as a special case:

3.2. Corollary. *Let the monoidal closed $\mathcal{V}$ be cocomplete and locally small, admit all cointersections of epimorphisms, and have a small generating set (and hence be complete as well); then the $f : \mathcal{V} \longrightarrow [\mathcal{V}, \mathcal{V}]$ sending X to $fX = X \otimes -$ has a right adjoint g, while the $\mathrm{Mon}\, F : \mathrm{Mon}\, \mathcal{V} \longrightarrow \mathrm{Mon}[\mathcal{V}, \mathcal{V}] = \mathrm{Mnd}\, \mathcal{V}$ sending a monoid M in $\mathcal{V}$ to the monad $M \otimes -$ on $\mathcal{V}$ has a right adjoint $\mathrm{Mon}\, G$ sending the monad (T, m, i) to a monoid whose underlying object is gT.*

3.3. Remark. There is a connexion between the example above and some old results of [KT] concerning M-algebras in various contexts, proved under mild conditions on $\mathcal{V}$—namely, right closedness and the existence of pullbacks. First, it was shown in [KT, Prop. 23.2] that, if $M \in \mathrm{Mon}\, \mathcal{V}$ is the free monoid on $X \in \mathcal{V}$, then fM (or more properly $(\mathrm{Mon}\, F)M$) is the free monad on the endfunctor $X \otimes -$; secondly, it was shown in [KT, Prop. 28.2] that $\mathrm{Mon}\, F$ preserves such colimits as exist. Of course these are trivial consequences of the existence of the right adjoints in Corollary 3.2, when $\mathcal{V}$ satisfies the stronger conditions of this corollary. It is also observed in [KT, Section 23.2] that each of the results above fails when $\mathcal{V}$ is the non-right-closed monoidal category $(\mathbf{Set}, +, 0)$.

Again, we may single out the simple case where $\mathcal{V}$ is the category $\mathbf{Set}$ of small sets:

3.4. Corollary. *Let the locally-small category $\mathcal{A}$ admit small multiples and small powers and have a small generating set; then the functor $f : \mathbf{Set} \longrightarrow [\mathcal{A}, \mathcal{A}]$ sending the set X to the endofunctor $X \cdot -$ (where $X \cdot A$ denotes the X-fold multiple of A) has a right adjoint g, which underlies a right adjoint to the functor $\mathbf{Mon} \longrightarrow \mathrm{Mnd}\, \mathcal{A}$ sending the monoid M to the monad $M \cdot -$ on $\mathcal{A}$; here $\mathbf{Mon} = \mathrm{Mon}\, (\mathbf{Set})$ is the category of monoids in the usual sense.*

When the monoidal $\mathcal{V}$ is closed, there is a further adjunction we can consider, alongside the adjunction $F \dashv G : [\mathcal{A}, \mathcal{A}] \longrightarrow \mathcal{V}$. For simplicity we restrict ourselves now to the most important of the closed-$\mathcal{V}$ cases, namely that where $\mathcal{V}$ is a symmetric monoidal closed category; and we suppose $\mathcal{V}$ and $\mathcal{A}$ to satisfy the conditions of Theorem 3.1, so that $\mathbf{A}$ is a tensored and cotensored $\mathcal{V}$-category. Then, since the right side of (2.5) is the value of a $\mathcal{V}$-functor $\mathbf{V}^{\mathrm{op}} \otimes \mathbf{A}^{\mathrm{op}} \otimes \mathbf{A} \longrightarrow \mathbf{V}$, so the representing object $X * A$ on the left side is the value of a $\mathcal{V}$-functor $\mathrm{Ten} : \mathbf{V} \otimes \mathbf{A} \longrightarrow \mathbf{A}$, whose underlying ordinary functor is of course the $* : \mathcal{V} \times \mathcal{A} \longrightarrow \mathcal{A}$ corresponding to $f : \mathcal{V} \longrightarrow [\mathcal{A}, \mathcal{A}]$. In particular, each $fX = X * -$ in $[\mathcal{A}, \mathcal{A}]$ underlies a $\mathcal{V}$-functor $\mathrm{Ten}(X, -) : \mathbf{A} \longrightarrow \mathbf{A}$, and each $fx = x * - : (X * -) \longrightarrow (X' * -)$ induced by $x : X \longrightarrow X'$ in $\mathcal{V}$ underlies a $\mathcal{V}$-natural transformation $\mathrm{Ten}(x, -) : \mathrm{Ten}(X, -) \longrightarrow \mathrm{Ten}(X', -)$. That is to say, $f : \mathcal{V} \longrightarrow [\mathcal{A}, \mathcal{A}]$ factorizes as a composite

$$\mathcal{V} \xrightarrow[q]{\qquad} (\mathbf{A}, \mathbf{A}) \xrightarrow[u]{\qquad} [\mathcal{A}, \mathcal{A}], \tag{3.8}$$

where $(\mathbf{A}, \mathbf{A})$ is the (ordinary) category of $\mathcal{V}$-functors $\mathbf{A} \longrightarrow \mathbf{A}$ and $\mathcal{V}$-natural transformations between these (as distinct from the $\mathcal{V}$-category $[\mathbf{A}, \mathbf{A}]$ of [KB, Section 2.2], which rarely exists when $\mathbf{A}$ is large), and u is the functor sending the $\mathcal{V}$-functor $H : \mathbf{A} \longrightarrow \mathbf{A}$ to its underlying ordinary functor $uH = H_0 : \mathcal{A} \longrightarrow \mathcal{A}$ and sending the $\mathcal{V}$-natural $\alpha : H \longrightarrow K$ to the underlying natural transformation $\alpha_0 : H_0 \longrightarrow K_0$; here, of course, we have $qX = \mathrm{Ten}(X, -)$ and $qx = \mathrm{Ten}(x, -)$. Since α_0 has the same components $(\alpha_0)_A = \alpha_A$ as α, the functor u is faithful; recall from [KB, Section 1.3] that u is fully faithful when I is a generator of $\mathcal{V}$. Indeed still more is true : (3.8) underlies a factorization

$$\mathcal{V} \xrightarrow[Q=(q,\widetilde{q},q^\circ)]{} (\mathbf{A}, \mathbf{A}) \xrightarrow[U=(u,\widetilde{u},u^\circ)]{} [\mathcal{A}, \mathcal{A}] \tag{3.9}$$

of the strong monoidal functor F into strong monoidal functors Q and U : here $\widetilde{q}_{xy}$ is the isomorphism $\mathrm{Ten}(X, \mathrm{Ten}(Y, -)) \cong \mathrm{Ten}(X \otimes Y, -)$ and q° is the isomorphism $1 \cong \mathrm{Ten}(I, -)$, while $\widetilde{u}_{HK}$ is the isomorphism $K_0 H_0 \cong (KH)_0$ and u° is the isomorphism $1_{\mathcal{A}} = (1_{\mathbf{A}})_0$.

3.5. Theorem. *Let the symmetric monoidal closed $\mathcal{V}$ and its action $F : \mathcal{V} \longrightarrow [\mathcal{A}, \mathcal{A}]$ on $\mathcal{A}$ satisfy the hypotheses of Theorem 3.1, and let $\mathbf{A}$ be the corresponding $\mathcal{V}$-category as in that theorem. Then q has a right adjoint r, so that Q has in $\mathcal{MONCAT}$ a right adjoint R, and* $\mathrm{Mon}\, Q : \mathrm{Mon}\, \mathcal{V} \longrightarrow \mathrm{Mon}(\mathbf{A}, \mathbf{A})$ *has a right adjoint* $\mathrm{Mon}\, R$.

Proof. As in the proof of Theorem 3.1, we use the Special Adjoint Functor Theorem. We still have on $\mathcal{V}$ the hypothesis (i)—(iii) above, and q like f preserves small colimits : for one easily sees that such colimits in $(\mathbf{A}, \mathbf{A})$, like those in $[\mathcal{A}, \mathcal{A}]$, are formed pointwise from those in $\mathcal{A}$. Moreover each $(\mathbf{A}, \mathbf{A})(qX, H)$ is a small set, being in effect a subset of $[\mathcal{A}, \mathcal{A}](uqX, uH) = [\mathcal{A}, \mathcal{A}](fX, uH) \cong \mathcal{V}(X, guH)$. ■

3.6. Remark. In the situation of Theorem 3.5, as a mate of the equality $f = uq$ we have the canonical comparison $\zeta : r \longrightarrow gu$, as well as such a comparison $\xi : R \longrightarrow GU$ and the resulting $\mathrm{Mon}\, \xi : \mathrm{Mon}\, R \longrightarrow (\mathrm{Mon}\, G)(\mathrm{Mon}\, U)$. For a $\mathcal{V}$-functor $H : \mathbf{A} \longrightarrow \mathbf{A}$, the comparison morphism

$$\zeta_H : rH \longrightarrow guH \tag{3.10}$$

is a monomorphism in $\mathcal{V}$, since $\mathcal{V}(X, rH)$ is the set of $\mathcal{V}$-natural transformations from $\mathrm{Ten}(X, A)$ to HA, which is a subset of $\mathcal{V}(X, guH)$, this being the set of (merely) natural transformations $X * A \longrightarrow H_0 A$. Since $\mathcal{V}$-naturality reduces to mere naturality when I is a generator of $\mathcal{V}$, we have $rH = guH$ in this case, for a $\mathcal{V}$-functor $H : \mathbf{A} \longrightarrow \mathbf{A}$. (Of course this does not reduce g to r : for gT is defined even for a mere functor $T : \mathcal{A} \longrightarrow \mathcal{A}$.) Note that we can make similar comments about ξ_H, when now H is a $\mathcal{V}$-monad on $\mathbf{A}$; for then $(\xi_H)_A = (\zeta_H)_A$ for each $A \in \mathbf{A}$. In particular we have such results in the special case where $\mathcal{A}$ is $\mathcal{V}$ with the canonical action, so that $\mathbf{A}$ is $\mathbf{V}$. As for the other special case given by $\mathcal{V} = \mathbf{Set}$, there is here no difference between $\mathbf{A}$ and $\mathcal{A}$, or between r and g.

4. An explicit formula for the right adjoints g and r.

We continue to suppose that $F = (f, \widetilde{f}, f^\circ) : \mathcal{V} \longrightarrow [\mathcal{A}, \mathcal{A}]$, corresponding to $* : \mathcal{V} \times \mathcal{A} \longrightarrow \mathcal{A}$ along with α and λ, is an action of $\mathcal{V}$ on $\mathcal{A}$ for which we have the adjunction (2.1). For $X \in \mathcal{V}$ and $T : \mathcal{A} \longrightarrow \mathcal{A}$, the set $[\mathcal{A}, \mathcal{A}](fX, T) = [\mathcal{A}, \mathcal{A}](X * -, T)$ is the set of natural transformations $(\beta_A : X * A \longrightarrow TA)$, which is isomorphic to the set of natural transformations $(\gamma_A : X \longrightarrow \mathbf{A}(A, TA))$—now in the generalized sense of [EC]; and to say that this set admits a representation of the form $\mathcal{V}(X, Y)$ is, by the definition of an end, to say that the end $\int_{A \in \mathcal{A}} \mathbf{A}(A, TA)$ exists. In other words:

4.1. Proposition. *For an action $F = (f, \widetilde{f}, f^\circ)$ of $\mathcal{V}$ on $\mathcal{A}$, the functor $f : \mathcal{V} \longrightarrow [\mathcal{A}, \mathcal{A}]$ has a right adjoint g if and only if, for each endofunctor T of $\mathcal{A}$, the end $\int_{A \in \mathcal{A}} \mathbf{A}(A, TA)$ exists; whereupon we have*

$$gT = \int_{A \in \mathcal{A}} \mathbf{A}(A, TA). \tag{4.1}$$

4.2. Remark. This is a *large* end, in so far as the category $\mathcal{A}$ is seldom small in the examples of interest : so that the existence of this end does not follow from completeness of $\mathcal{V}$. The formula (4.1) does nothing to establish the existence of the right adjoint g, but merely gives an explicit formula for it when its existence is otherwise assured, as for instance by the hypotheses of Theorem 3.1.

4.3. Remark. We observed in the Introduction that, when f has a right adjoint g, then also F has a right adjoint G in $\mathcal{MONCAT}$, so that $\mathrm{Mon}\, F : \mathrm{Mon}\, \mathcal{V} \longrightarrow \mathrm{Mnd}\, \mathcal{A}$ has a right adjoint $\mathrm{Mon}\, G$. This last, as we said, agrees with g on objects : one says loosely that a monoidal functor, such as G, "takes monoids to monoids". It is easy to see how this goes in the present case, using the formula (4.1). Let $T = (T, m, i)$ be a monad on $\mathcal{A}$, and write

$$\tau_A : gT \longrightarrow \mathbf{A}(A, TA) \tag{4.2}$$

for the counit of the end (4.1); that is to say, $\tau = (\tau_A)$ is universal among natural transformations into $\mathbf{A}(A, TA)$. Now the natural transformation given by the composite

$$I * A \xrightarrow[\lambda_A]{} A \xrightarrow[i_A]{} TA \tag{4.3}$$

gives by the adjunction a natural transformation

$$j_A : I \longrightarrow \mathbf{A}(A, TA), \tag{4.4}$$

which factorizes as $\tau_A j$ for a unique morphism

$$j : I \longrightarrow gT. \tag{4.5}$$

Next, the composite

$$gT \otimes gT \xrightarrow[\tau_{TA} \otimes \tau_A]{} \mathbf{A}(TA, TTA) \otimes \mathbf{A}(A, TA) \xrightarrow[M]{} \mathbf{A}(A, TTA) \xrightarrow[\mathbf{A}(A, m_A)]{} \mathbf{A}(A, TA) \tag{4.6}$$

is natural in A by the Eilenberg-Kelly calculus of [EC], so that it factorizes as $\tau_A n$ for a unique morphism

$$n : gT \otimes gT \longrightarrow gT; \tag{4.7}$$

and it is (gT, n, j) that is the monoid $(\mathrm{Mon}\, G)T$ of $\mathrm{Mon}\, \mathcal{V}$.

There is a similar formula for the right adjoint r of $q : \mathcal{V} \longrightarrow (\mathbf{A}, \mathbf{A})$, where now $\mathcal{V}$ is symmetric monoidal closed; once again, it involves a large end, and thus is of value only when r is independenly known to exist, as under the hypotheses of Theorem 3.5. Here $(\mathbf{A}, \mathbf{A})(qX, H) = (\mathbf{A}, \mathbf{A})(\mathrm{Ten}(X, -), H)$ is the set of $\mathcal{V}$-natural transformations $(\rho_A : \mathrm{Ten}(X, A) \longrightarrow HA)$, which is isomorphic to the set of $\mathcal{V}$-natural transformations $(\sigma_A : X \longrightarrow \mathbf{A}(A, HA))$. Accordingly—see Section 2.2 of [KB]—we have:

4.4. Proposition. *In the situation of Theorem 3.5, for a $\mathcal{V}$-functor $H : \mathbf{A} \longrightarrow \mathbf{A}$ we have*

$$rH = \int_{A \in \mathbf{A}} \mathbf{A}(A, HA)\ ; \tag{4.8}$$

moreover, when $H = (H, m, i)$ is a $\mathcal{V}$-monad on $\mathbf{A}$, we find $(\mathrm{Mon}\, R)H$ *as the object (4.8) of $\mathcal{V}$ with a monoid-structure formed as in Remark 4.3.*

4.5. Remark. As in Section 2.2 of [KB], we may write (4.8) in the form

$$rH = [\mathbf{A}, \mathbf{A}](1, H)\ , \tag{4.9}$$

even though, for large $\mathbf{A}$, the functor category $[\mathbf{A}, \mathbf{A}]$ may not exist as a $\mathcal{V}$-category; the point is that the *particular* hom-object (4.9) does exist in $\mathcal{V}$ under the hypotheses of Theorem 3.5. Note the special case where H is the identity functor $1 = 1_{\mathbf{A}} : \mathbf{A} \longrightarrow \mathbf{A}$; here (4.9) becomes

$$r1_{\mathbf{A}} = [\mathbf{A}, \mathbf{A}](1, 1) = \int_{A \in \mathbf{A}} \mathbf{A}(A, A)\ , \tag{4.10}$$

which is the monoid in $\mathcal{V}$ usually called the *centre* of the $\mathcal{V}$-category $\mathbf{A}$. In the case where $\mathcal{A}$ is $\mathcal{V}$ itself with the canonical action, so that $\mathbf{A} = \mathbf{V}$, the $\mathcal{V}$-functor $1_{\mathbf{V}} : \mathbf{V} \longrightarrow \mathbf{V}$ is represented by the object I of $\mathbf{V}$, so that the enriched Yoneda lemma allows us to rewrite (4.9) as

$$rH = HI \ \text{ for } \ H : \mathbf{V} \longrightarrow \mathbf{V} \ ; \tag{4.11}$$

in particular we have

$$r1_{\mathbf{V}} \ = \ I \ . \tag{4.12}$$

The canonical comparison $\zeta : r \longrightarrow gu$ of Remark 3.6 has for ζ_H the monomorphism

$$[\mathbf{A}, \mathbf{A}](1, H) = \int_{A \in \mathbf{A}} \mathbf{A}(A, HA) \longrightarrow \int_{A \in \mathbf{A}_0 = \mathcal{A}} \mathbf{A}(A, HA) \ ; \tag{4.13}$$

we observed that this is invertible when I is a generator of $\mathcal{V}$, so that $\mathcal{V}$-naturality reduces to mere naturality. To see that it is not invertible in general, even when $\mathcal{A} = \mathcal{V}$ for a well-behaved $\mathcal{V}$, take $\mathcal{V}$ to be the symmetric monoidal closed category of graded abelian groups, and take H to be $1_{\mathbf{V}}$. Then the domain of (4.13) is I by (4.12), so that (4.13) takes the form

$$\zeta : I \longrightarrow \int_{A \in \mathcal{V}} [A, A] \ . \tag{4.14}$$

To see that (4.14) is not invertible in $\mathcal{V}$, consider its image under the functor $\mathcal{V}(I, -) : \mathcal{V} \longrightarrow \mathbf{Set}$. Of course we have $I_0 = \mathbf{Z}$ and $I_n = 0$ for $n \neq 0$; so a morphism $f : I \longrightarrow I$ in $\mathcal{V}$ has $f_n = 0$ for $n \neq 0$ while $f_0 : \mathbf{Z} \longrightarrow \mathbf{Z}$ is multiplication by n for some $n \in \mathbf{Z}$; in this way, we have $\mathcal{V}(I, I) \cong \mathbf{Z}$. On the other hand, to give an element of $\mathcal{V}(I, \int_{A \in \mathcal{V}} [A, A])$ is to give a family $(\alpha_A : I \longrightarrow [A, A])_{A \in \mathcal{V}}$ which is natural in A; equivalently, it is to give a family $(\beta_A : A \cong I \otimes A \longrightarrow A)$ that is natural in A, or an element β of the centre of the ordinary category $\mathcal{V}$. For any sequence $(k_n)_{n \in \mathbf{Z}}$ of integers, we obtain such a family by setting $(\beta_A)_n(a) = k_n a$ for $a \in A_n$; and consideration of those A having for some m the components $A_n = 0$ for $n \neq m$ and $A_m = \mathbf{Z}$ shows that every natural family (β_A) is of this kind. Thus the monoid $\mathcal{V}(I, \int_{A \in \mathcal{V}} [A, A])$ is the power $\mathbf{Z}^{\mathbf{Z}}$ of $\mathbf{Z}$, and (4.14) is not invertible.

4.6. Remark. When $\mathcal{V} = \mathbf{Set}$, there is, as we said, no difference between $\mathbf{A}$ and $\mathcal{A}$ or between r and g, and for any $T : \mathcal{A} \longrightarrow \mathcal{A}$ we have the formula

$$gT = \int_{A \in \mathcal{A}} \mathcal{A}(A, TA) = [\mathcal{A}, \mathcal{A}](1, T) \ ; \tag{4.15}$$

in particular $g1_{\mathcal{A}}$ is the *centre* $[\mathcal{A}, \mathcal{A}](1, 1)$ of the category $\mathcal{A}$. In the still more special case where $\mathcal{A}$ too is $\mathbf{Set}$, so that f is the functor $\mathbf{Set} \longrightarrow [\mathbf{Set}, \mathbf{Set}]$ sending X to $X \times -$, the formula (4.15) for the right adjoint $g : [\mathbf{Set}, \mathbf{Set}] \longrightarrow \mathbf{Set}$ reduces, since $1 : \mathbf{Set} \longrightarrow \mathbf{Set}$ is represented by 1, to

$$gT \; = \; T1\,. \tag{4.16}$$

Note what this becomes when we pass to the monoids and consider the functor $\mathrm{Mon}\,F : \mathbf{Mon} \longrightarrow \mathrm{Mnd}(\mathbf{Set})$ sending the monoid M to the monad $M\times -$ on $\mathbf{Set}$; the right adjoint $\mathrm{Mon}\,G : \mathrm{Mnd}(\mathbf{Set}) \longrightarrow \mathbf{Mon}$ sends the monad $T = (T, m, i)$ on $\mathbf{Set}$, which we may think of as a (perhaps infinitary) Lawvere theory, to the monoid $T1$ of its unary operations.

4.7. Remark. Consider again, in the case $\mathcal{V} = \mathbf{Set}$, the functor $\mathrm{Mon}\,F : \mathbf{Mon} \longrightarrow \mathrm{Mnd}\,\mathcal{A}$; we can compose it with the inclusion functor $m : \mathbf{Grp} \longrightarrow \mathbf{Mon}$, which considers each group as a monoid, to obtain a functor $h = (\mathrm{Mon}\,F)m : \mathbf{Grp} \longrightarrow \mathrm{Mnd}\,\mathcal{A}$ whose domain is the category $\mathbf{Grp}$ of groups. Since $\mathrm{Mon}\,F$ has a right adjoint $\mathrm{Mon}\,G$ when $\mathcal{A}$ satisfies the conditions of Corollary 3.4, and since m has a right adjoint n sending a monoid to the group of its invertible elements, so h too has a right adjoint $k = n(\mathrm{Mon}\,G)$. By (4.15), its explicit value at a monad T on $\mathcal{A}$ is the group of invertible elements in the monoid $[\mathcal{A}, \mathcal{A}](1, T)$; in the case $\mathcal{A} = \mathbf{Set}$, this is the group of invertible elements of $T1$. For a group X, observe that hX is the monad $X \cdot -$ on $\mathcal{A}$. The case $\mathcal{A} = \mathbf{Grp}$ is a familiar one : an $(X \cdot -)$-algebra is a group A along with a group homomorphism $X \longrightarrow \mathrm{Aut}\,A$, or equally an action $a : X \times A \longrightarrow A$ of X on the group A. As is well known, the category of such algebras is isomorphic to that of pairs $(u : B \longrightarrow X,\; v : X \longrightarrow B)$ of group-homomorphisms with $uv = 1$; here B is the semi-direct product of X with (A, a), while conversely A is the kernel of u with the appropriate action.

4.8. Remark. Consider again the case $\mathcal{A} = \mathcal{V}$, and suppose again for simplicity that $\mathcal{V}$ is symmetric monoidal closed; so that $f : \mathcal{V} \longrightarrow [\mathcal{V}, \mathcal{V}]$ factorizes as

$$\mathcal{V} \xrightarrow[q]{} (\mathbf{V}, \mathbf{V}) \xrightarrow[u]{} [\mathcal{V}, \mathcal{V}], \tag{4.17}$$

and $\mathrm{Mon}\,F$ factorizes as

$$\mathrm{Mon}\,\mathcal{V} \xrightarrow[\mathrm{Mon}\,Q]{} (\mathcal{V}\text{-Mnd})\mathbf{V} \xrightarrow[\mathrm{Mon}\,U]{} \mathrm{Mnd}\,\mathcal{V}, \tag{4.18}$$

where $(\mathcal{V}\text{-Mnd})\mathbf{V}$ is the category of $\mathcal{V}$-monads on $\mathbf{V}$ and $\mathcal{V}$-natural monad maps. The functor q is fully faithful; for a $\mathcal{V}$-natural family $(\beta_Z : X \otimes Z \longrightarrow Y \otimes Z)$ corresponds by the adjunction (2.2) to a $\mathcal{V}$-natural family $(\gamma_Z : Z \longrightarrow [X, Y \otimes Z])$, which in turn (since we have the representation $Z \cong [I, Z]$ of the $\mathcal{V}$-functor $1_{\mathbf{V}} : \mathbf{V} \longrightarrow \mathbf{V}$) corresponds by the enriched Yoneda lemma to a morphism $b : X \longrightarrow Y$ of $\mathcal{V}$, so that $\beta_Z = b \otimes Z$ for this unique b. In fact $\mathrm{Mon}\,Q$ too is fully faithful; for, when X and Y are monoids, b is a monoid map when the $\beta_Z = b \otimes Z$ constitute a monad map, as we see on taking $Z = I$. We observed in Section 3 that the functor u is always faithful, and is moreover fully faithful whenever I is a generator of $\mathcal{V}$. It follows that $\mathrm{Mon}\,U$ too is always faithful; and indeed it too is fully faithful when I is a generator of $\mathcal{V}$—for a $\mathcal{V}$-natural $\beta : H \longrightarrow K$ between $\mathcal{V}$-monads satisfies the conditions to be a monad map when the underlying $\beta_0 : H_0 \longrightarrow K_0$ does so. Thus in this case $\mathcal{A} = \mathcal{V}$ the functor $f : \mathcal{V} \longrightarrow [\mathcal{V}, \mathcal{V}]$ is always faithful, and is fully faithful when I is a generator of $\mathcal{V}$; and the same is true of $\mathrm{Mon}\,F : \mathrm{Mon}\,\mathcal{V} \longrightarrow \mathrm{Mnd}\,\mathcal{V}$.

4.9. Example. Even for a symmetric monoidal closed category $\mathcal{V}$ that is locally finitely presentable (both as a category, and as a closed category in the sense of [KA]), neither $f : \mathcal{V} \longrightarrow [\mathcal{V}, \mathcal{V}]$ nor $\mathrm{Mon}\, F : \mathrm{Mon}\, \mathcal{V} \longrightarrow \mathrm{Mnd}\, \mathcal{V}$ need be full. It suffices to prove the second of these : for $\mathrm{Mon}\, F$ is full when f is so, since if X and Y are monoids in $\mathcal{V}$ and $z : X \longrightarrow Y$ in $\mathcal{V}$ is such that $z \otimes - : (X \otimes -) \longrightarrow (Y \otimes -)$ is a monad map, taking the I-component $z \otimes I : X \otimes I \longrightarrow Y \otimes I$ shows z itself to be a monoid map. For our counter-example we take $\mathcal{V}$ to be the symmetric monoidal closed category of differential graded abelian groups—that is, of chain complexes of abelian groups—and chain maps. Here I is the chain complex given by $I_0 = \mathbf{Z}$, $I_n = 0$ for $n \neq 0$; and $fI = I \otimes -$ is (to within isomorphism) the identity endofunctor $1_{\mathcal{V}}$ of $\mathcal{V}$. In fact I is trivially a *monoid* in $\mathcal{V}$, and $(\mathrm{Mon}\, F)I \cong 1_{\mathcal{V}}$ is the identity *monad* on $\mathcal{V}$. Now write J for the chain complex having $J_{-1} = \mathbf{Z}$ and $J_n = 0$ for $n \neq -1$, with of course the zero boundary map; and observe that $DA = J \otimes A$ is the *desuspension* of A, given by $(DA)_n = A_{n+1}$, the boundary map $(DA)_n \longrightarrow (DA)_{n-1}$ being the negative of the boundary map $A_{n+1} \longrightarrow A_n$. An algebra for the endofunctor D of $\mathcal{V}$ is a pair (A, a) where $A \in \mathcal{V}$ and $a : DA \longrightarrow A$; these form a category D-Alg with a forgetful functor $U_D : D\text{-Alg} \longrightarrow \mathcal{V}$. Write $\Phi : \mathcal{V} \longrightarrow D\text{-Alg}$ for the functor sending $A \in \mathcal{V}$ to the D-algebra (A, a), where $a : DA \longrightarrow A$ has the components $a_n : (DA)_n \longrightarrow A_n$ given by the boundary map $A_{n+1} \longrightarrow A_n$ of A; clearly a is indeed a chain map, and Φ is a functor with $U_D \Phi = 1 : \mathcal{V} \longrightarrow \mathcal{V}$. Let $z : J \longrightarrow M$ exhibit M as the free monoid in $\mathcal{V}$ on the object J of $\mathcal{V}$; we don't really need the explicit value of M, but of course $M = \sum_{n \geq 0} J^{\otimes n}$ has $M_n = \mathbf{Z}$ for $n \leq 0$ and $M_n = 0$ for $n > 0$, with zero as the boundary map of M. Because $\mathcal{V}$ is right closed, it follows from Proposition 23.2 of [KT] that z induces an isomorphism, commuting with the forgetful functors,

$$(z \otimes -)^* : (M \otimes -)\text{-Alg} \longrightarrow (J \otimes -)\text{-Alg} = D\text{-Alg}, \tag{4.19}$$

where $(M \otimes -)$-Alg is the category of algebras for the monad $M \otimes -$, while $(J \otimes -)$-Alg is the category of algebras for the mere endofunctor $J \otimes - = D$. So there is a functor $\Psi : \mathcal{V} \longrightarrow (M \otimes -)\text{-Alg}$, commuting with the forgetful functors, for which $(z \otimes -)^* \Psi = \Phi$. Now $\mathcal{V}$ is the category $1_{\mathcal{V}}$-Alg of algebras for the identity monad $1_{\mathcal{V}} \cong I \otimes -$ on $\mathcal{V}$, so that Ψ is a functor $(I \otimes -)\text{-Alg} \longrightarrow (M \otimes -)\text{-Alg}$ commuting with the forgetful functors; therefore, by a classical result (see for example Proposition 3.4 of [KS]), $\Psi = \theta^*$ for a unique monad map $\theta : (M \otimes -) \longrightarrow (I \otimes -)$. If $\mathrm{Mon}\, F$ were full, θ would be $w \otimes -$ for some monad map $w : M \longrightarrow I$; which, since M is the free monoid on J, corresponds to some mere morphism $t = wz : J \longrightarrow I$. Now we have

$$\Phi = (z \otimes -)^* \Psi = (z \otimes -)^* \theta^* = (z \otimes -)^* (w \otimes -)^* = (t \otimes 1)^*. \tag{4.20}$$

But this cannot be so : the only morphism $t : J \longrightarrow I$ in $\mathcal{V}$ is the zero morphism, so that $(t \otimes -)^* A$ for $A \in \mathcal{V}$ is $(A,\, 0 : DA \longrightarrow A)$, which is not equal to ΦA in general. So $\mathrm{Mon}\, F$ in this example is not full.

5. Alternative expressions for g and $\mathrm{Mon}\, G$ as large limits.

It turns out that, for an action $F : \mathcal{V} \longrightarrow [\mathcal{A}, \mathcal{A}]$ admitting the adjunction (2.1), one can easily describe gT for certain special endofunctors T of $\mathcal{A}$; we now show that each endofunctor T is canonically a limit of these special ones, so that gT, when it exists, may be expressed as a limit of the gP for the special P. Once again, since the limit involved is a large one, this expression for gT does nothing to establish its existence, for which we must refer to such results as Theorem 3.1. Note that, although $(\mathrm{Mon}\, G)T$ for a monad T is just gT with the appropriate monoid-structure, we elect also to give below a second limit-formula for $(\mathrm{Mon}\, G)T$, which makes sense only when T is a monad, but then gives a tidier result, expressing $(\mathrm{Mon}\, G)T$ as a "smaller" large limit. So each of our considerations in this section will have two forms : one for endofunctors and one for monads.

First, recall again the classical result used in the last section, asserting that, for monads T and S on a category $\mathcal{A}$, every functor T-Alg $\longrightarrow$ S-Alg commuting with the forgetful functors to $\mathcal{A}$ is of the form θ^* for a unique monad map $\theta : S \longrightarrow T$. We now give an analogue of this in which endofunctors of $\mathcal{A}$ replace monads.

If T is a mere endofunctor of $\mathcal{A}$ it is usual to employ the name T-*algebra*, as we did in Example 4.9 above, for a pair (A, a) where $A \in \mathcal{A}$ and $a : TA \longrightarrow A$ is any morphism in $\mathcal{A}$. Such T-algebras do not serve our purpose here : rather we define a T-*quasi-algebra* to be a triple (A, a, C) where $A, C \in \mathcal{A}$ and $a : TA \longrightarrow C$ is a morphism in $\mathcal{A}$ (called the *quasi-action*). Then a map $(A, a, C) \longrightarrow (B, b, D)$ of T-quasi-algebras is a pair $(u : A \longrightarrow B,\ v : C \longrightarrow D)$ making commutative the diagram

$$\begin{tikzcd} TA \arrow[r, "a"] \arrow[d, "Tu"'] & C \arrow[d, "v"] \\ TB \arrow[r, "b"'] & D \end{tikzcd} ;$$

so that T-quasi-algebras and their maps form a category T-QAlg, with a forgetful functor to $\mathcal{A}^2 = \mathcal{A} \times \mathcal{A}$. Of course, T-QAlg is just the comma category $T/1$; but our present notation is chosen to emphasize the analogy between it and T-Alg. For any morphism $\theta : S \longrightarrow T$ of endofunctors, we obtain a functor $\theta^\dagger : T\text{-QAlg} \longrightarrow S\text{-QAlg}$, commuting with the forgetful functors to $\mathcal{A}^2$: it sends the T-quasi-algebra $(A, a : TA \longrightarrow C, C)$ to the S-quasi-algebra $(A, a.\theta_A : SA \longrightarrow C, C)$ whose quasi-action is the composite of $\theta_A : SA \longrightarrow TA$ and $a : TA \longrightarrow C$.

5.1. Lemma. *Every functor $\Phi : T$-QAlg $\longrightarrow$ S-QAlg commuting with the forgetful functors is of the form $\theta^\dagger$ for a unique morphism $\theta : S \longrightarrow T$ in $[\mathcal{A}, \mathcal{A}]$.*

Proof. Given Φ, we define θ_A by setting

$$(A, \theta_A : SA \longrightarrow TA, TA) = \Phi(A, 1_{TA} : TA \longrightarrow TA, TA);$$

then θ is a natural transformation because Φ sends the morphism

$$(u, Tu) : (A, 1, TA) \longrightarrow (B, 1, TB) \quad \text{to} \quad (u, Tu) : (A, \theta_A, TA) \longrightarrow (B, \theta_B, TB),$$

giving $Tu.\theta_A = \theta_B.Su$. If now $(A, a : TA \longrightarrow C, C)$ is a general T-quasi-algebra, sent by Φ to $(A, \overline{a} : SA \longrightarrow C, C)$, then Φ sends the morphism

$$(1_A, a) : (A, 1, TA) \longrightarrow (A, a, C) \quad \text{to} \quad (1_A, a) : (A, \theta_A, TA) \longrightarrow (A, \overline{a}, C),$$

giving $\overline{a} = a.\theta_A$, as desired. ∎

To introduce the "special" endofunctors and "special" monads we have referred to, we begin by recalling some classical actions. First, for each category $\mathcal{A}$, we have the action of the monoidal $[\mathcal{A}, \mathcal{A}]$ on $\mathcal{A}$ corresponding to the identity functor $[\mathcal{A}, \mathcal{A}] \longrightarrow [\mathcal{A}, \mathcal{A}]$; in its "functor of two variables" form it is the evaluation functor $E : [\mathcal{A}, \mathcal{A}] \times \mathcal{A} \longrightarrow \mathcal{A}$ sending (T, A) to TA; the partial functor $E(-, A)$ is what we earlier called $e(A) : [\mathcal{A}, \mathcal{A}] \longrightarrow \mathcal{A}$. Provided that, as we henceforth suppose, $\mathcal{A}$ is complete and locally small, this action admits an adjunction of the form (2.1), namely

$$\mathcal{A}(TA, B) \;\cong\; [\mathcal{A}, \mathcal{A}](T, \langle A, B\rangle) \,, \tag{5.1}$$

where $\langle A, B\rangle C$ is defined as the power

$$\langle A, B\rangle C = B^{\mathcal{A}(C,A)} \,;$$

which is equivalently to define $\langle A, B\rangle : \mathcal{A} \longrightarrow \mathcal{A}$ as the right Kan extension of $B : 1 \longrightarrow \mathcal{A}$ along $A : 1 \longrightarrow \mathcal{A}$. The counit and the unit of the adjunction (2.1) take, in the present case (5.1), the forms $\epsilon : \langle A, B\rangle A \longrightarrow B$ and $\delta : T \longrightarrow \langle A, TA\rangle$; and the $M : \mathbf{A}(B, C) \otimes \mathbf{A}(A, B) \longrightarrow \mathbf{A}(A, C)$ of Section 2 here becomes a multiplication $\mu : \langle B, C\rangle\langle A, B\rangle \longrightarrow \langle A, C\rangle$, while the $j : I \longrightarrow \mathbf{A}(A, A)$ becomes $\iota : 1 \longrightarrow \langle A, A\rangle$. In particular, we have for each $A \in \mathcal{A}$ a monad $(\langle A, A\rangle, \mu, \iota)$: moreover, if T is any monad on $\mathcal{A}$, and if the morphism $a : TA \longrightarrow A$ corresponds under the adjunction (5.1) to the natural transformation $\alpha : T \longrightarrow \langle A, A\rangle$, then one easily verifies that a is an *action* of T on A precisely when α is a *map of monads.*

Besides the above, we also need a "morphism" level : we consider the evident action $[\mathcal{A}, \mathcal{A}] \longrightarrow [\mathcal{A}^{\mathbf{2}}, \mathcal{A}^{\mathbf{2}}]$ of the monoidal $[\mathcal{A}, \mathcal{A}]$ on the arrow category $\mathcal{A}^{\mathbf{2}}$, where $\mathbf{2}$ here denotes the category $0 \longrightarrow 1$; as a functor $[\mathcal{A}, \mathcal{A}] \times \mathcal{A}^{\mathbf{2}} \longrightarrow \mathcal{A}^{\mathbf{2}}$, this action sends $(T, u : A \longrightarrow B)$ to $Tu : TA \longrightarrow TB$. Here too the action admits an adjunction of the form (2.1), namely

$$\mathcal{A}^{\mathbf{2}}(Tu, v) \;\cong\; [\mathcal{A}, \mathcal{A}](T, \{u, v\}) \,, \tag{5.2}$$

where $\{u, v\}$ is given by the pullback

$$\begin{tikzcd} & \langle A, C\rangle \arrow[dr, "\langle A,v\rangle"] & \\ \{u, v\} \arrow[ur, "\sigma_{u,v}"] \arrow[dr, "\tau_{u,v}"'] & & \langle A, D\rangle \,; \\ & \langle B, D\rangle \arrow[ur, "\langle u,D\rangle"'] & \end{tikzcd} \tag{5.3}$$

for to give a morphism $\gamma : T \longrightarrow \{u, v\}$ is to give morphisms $\alpha : T \longrightarrow \langle A, C\rangle$ and $\beta : T \longrightarrow \langle B, D\rangle$ having $\langle A, v\rangle\alpha = \langle u, D\rangle\beta$, and hence to give morphisms $a : TA \longrightarrow C$ and $b : TB \longrightarrow D$ making commutative

$$\begin{tikzcd} TA \arrow[r, "a"] \arrow[d, "Tu"'] & C \arrow[d, "v"] \\ TB \arrow[r, "b"'] & D \end{tikzcd} .$$

We can express the above in terms of T-quasi-algebras : to give a morphism $\alpha : T \longrightarrow \langle A, C\rangle$ is just to give a quasi-action $a : TA \longrightarrow C$, while to give a morphism $\gamma : T \longrightarrow \{u, v\}$ is just to give a map $(u, v) : (A, a, C) \longrightarrow (B, b, D)$ of T-quasi-algebras.

Moreover, just as each $\mathbf{A}(A, A)$ in the context of the adjunction (2.1) is a monoid in $\mathcal{V}$, so here each $\{u, u\}$ is a monad on $\mathcal{A}$, and for a general monad T on $\mathcal{A}$, a morphism $\gamma : T \longrightarrow \{u, u\}$ is a monad map precisely when

$$\begin{tikzcd} TA \arrow[r, "a"] \arrow[d, "Tu"'] & A \arrow[d, "u"] \\ TB \arrow[r, "b"'] & B \end{tikzcd}$$

is an action of the monad T on u—which is equally to say that a and b are actions of the monad T on A and on B for which the square above commutes (that is, for which u is a map of T-algebras). To give such a monad map $\gamma : T \longrightarrow \{u, u\}$ is, of course, equally to give monad maps $\alpha : T \longrightarrow \langle A, A\rangle$ and $\beta : T \longrightarrow \langle B, B\rangle$ for which $\langle A, u\rangle\alpha = \langle u, B\rangle\beta$; whereupon $\alpha = \sigma_{u,u}\gamma$ and $\beta = \tau_{u,u}\gamma$. If here we take in particular T to be $\{u, u\}$ and γ the identity, we see that $\sigma_{u,u}$ and $\tau_{u,u}$ are monad maps in the special case of (5.3) given by

$$\begin{tikzcd} & \langle A, A\rangle \arrow[dr, "\langle A,u\rangle"] & \\ \{u, u\} \arrow[ur, "\sigma_{u,u}"] \arrow[dr, "\tau_{u,u}"'] & & \langle A, B\rangle \\ & \langle B, B\rangle \arrow[ur, "\langle u,B\rangle"'] & \end{tikzcd} . \tag{5.4}$$

We shall now show that an arbitrary endofunctor T of $\mathcal{A}$ is canonically a limit in $[\mathcal{A}, \mathcal{A}]$ of endofunctors of the special forms $\langle A, C\rangle$ and $\{u, v\}$; and that an arbitrary monad T on $\mathcal{A}$ is canonically a limit in $\mathrm{Mnd}\,\mathcal{A}$ of monads of the special forms $\langle A, A\rangle$ and $\{u, u\}$. (The referee has kindly pointed out that, since $\{1_A, 1_C\} = \langle A, C\rangle$, every endofunctor (or monad) is in fact a limit of those of the form $\{u, v\}$ (or $\{u, u\}$).) To describe the domain of these limit-diagrams, we need to recall the notion of a *derived category* (in the sense of Kan).

To each category $\mathcal{K}$ we associate a *derived category* $\mathcal{K}'$; namely the free category on the following graph (which we may also, loosely, call $\mathcal{K}'$). For each object K of $\mathcal{K}$, there is an object K' of $\mathcal{K}'$, and for each morphism $u : K \longrightarrow L$ of $\mathcal{K}$ there is an object $(K, u, L)'$ of $\mathcal{K}'$, which might be written just as u' if it is agreed that a morphism determines its domain and codomain; there are no other objects of $\mathcal{K}'$. For each morphism $u : K \longrightarrow L$ of $\mathcal{K}$, there are edges $s_u : (K, u, L)' \longrightarrow K'$ and $t_u : (K, u, L)' \longrightarrow L'$ of the graph $\mathcal{K}'$; and there are no other edges of this graph. The free category $\mathcal{K}'$ generated by this graph has of course as morphisms, besides these edges, in addition the identity morphisms $1_{K'}$ and $1_{u'}$—but that is all. (Note that an identity morphism $1_K : K \longrightarrow K$ in $\mathcal{K}$ just gives rise, like any other morphism, to an object $(K, 1_K, K)'$ of $\mathcal{K}'$ and the associated edges $s_{1_K}, t_{1_K} : (K, 1_K, K)' \longrightarrow K'$ of $\mathcal{K}'$.) Of course $\mathcal{K}'$ is large whenever $\mathcal{K}$ is so.

For each endofunctor T of $\mathcal{A}$ (still supposed to be complete and locally small) we define as follows a functor

$$\widehat{T} : (T\text{-QAlg}\,)' \longrightarrow [\mathcal{A}, \mathcal{A}]\,. \tag{5.5}$$

For a T-quasi-algebra (A, a, C) we take $\widehat{T}(A, a, C)'$ to be the endofunctor $\langle A, C \rangle$ of $\mathcal{A}$; for a map $(u, v) : (A, a, C) \longrightarrow (B, b, D)$ of T-quasi-algebras we take

$$\widehat{T}((A, a, C), (u, v), (B, b, D))'$$

to be the endofunctor $\{u, v\}$ of $\mathcal{A}$, and we ask $\widehat{T}$ to send

$$s_{(u,v)} : ((A, a, C), (u, v), (B, b, D))' \longrightarrow (A, a, C)'$$

and

$$t_{(u,v)} : ((A, a, C), (u, v), (B, b, D))' \longrightarrow (B, b, D)'$$

to the natural transformations $\sigma_{u,v}$ and $\tau_{u,v}$ of (5.3). Next, we describe a cone ϕ^T in $[\mathcal{A}, \mathcal{A}]$, with vertex T, over the functor $\widehat{T}$ of (5.5). For its component $\phi^T_{(A,a,C)'} : T \longrightarrow \langle A, C \rangle$ we take the natural transformation α corresponding to the quasi-action $a : TA \longrightarrow C$; and for its component $\phi^T_{((A,a,C),(u,v),(B,b,D))'} : T \longrightarrow \{u, v\}$ we take the natural transformation γ corresponding as above to the morphism $(u, v) : (A, a, C) \longrightarrow (B, b, D)$ of T-quasi-algebras. Clearly ϕ^T is a cone, since $\sigma_{u,v}\phi^T_{(u,v)'} = \phi^T_{(A,c,C)'}$ and $\tau_{u,v}\phi^T_{(u,v)'} = \phi^T_{(B,b,D)'}$.

Alongside the functor $\widehat{T}$ of (5.5) and the cone ϕ^T of vertex T over $\widehat{T}$, we have the following analogue adapted to monads : for each monad T on $\mathcal{A}$ we define as follows a functor

$$\widetilde{T} : (T\text{-Alg})' \longrightarrow \quad \mathrm{Mnd}\,\mathcal{A}\,. \tag{5.6}$$

For a T-algebra $(A, a) = (A, a : TA \longrightarrow A)$ we take $\widetilde{T}(A, a)'$ to be the monad $\langle A, A \rangle$ on $\mathcal{A}$; for a map $u : (A, a) \longrightarrow (B, b)$ of T-algebras we take $\widetilde{T}((A, a), u, (b, B))'$ to be the monad $\{u, u\}$ on $\mathcal{A}$; and we ask $\widetilde{T}$ to send $s_u : ((A, a), u, (B, b))' \longrightarrow (A, a)'$ and $t_u : ((A, a), u, (B, b))' \longrightarrow (B, b)'$ to the monad maps $\sigma_{u,u}$ and $\tau_{u,u}$ of (5.4). Then we describe a cone ψ^T in $\mathrm{Mnd}\,\mathcal{A}$, with vertex T, over the functor $\widetilde{T}$ of (5.6) : its component $\psi^T_{(A,a)'} : T \longrightarrow \langle A, A\rangle$ is the monad map α corresponding to the action $a : TA \longrightarrow A$, and its component $\psi_{((A,a),u,(B,b))'} : T \longrightarrow \{u, u\}$ is the monad map γ corresponding as above to the map $u : (A, a) \longrightarrow (B, b)$ of T-algebras; once again, ψ^T is clearly a cone, having $\sigma_{u,u}\psi^T_{u'} = \psi^T_{(A,a)'}$, and $\tau_{u,u}\psi^T_{u'} = \psi^T_{(B,b)'}$.

The following observations seem to be new:

5.2. Proposition. *For a complete and locally-small category $\mathcal{A}$, the cone ϕ^T is a limit cone in $[\mathcal{A}, A]$ for each endofunctor T of $\mathcal{A}$, and the cone ψ^T is a limit cone in* $\mathrm{Mnd}\,\mathcal{A}$ *for each monad T on $\mathcal{A}$. Thus each endofunctor of $\mathcal{A}$ is canonically a limit of those of the special forms $\langle A, C\rangle$ and $\{u, v\}$, while each monad on $\mathcal{A}$ is canonically a limit of those of the special forms $\langle A, A\rangle$ and $\{u, u\}$.*

Proof. We prove only the monad version : the proof for the endofunctor version follows exactly the same pattern, using Lemma 5.1 in place of the classical result about monad maps. So let T be a monad on $\mathcal{A}$, and consider a cone ρ in $\mathrm{Mnd}\,\mathcal{A}$ over the functor $\widetilde{T} : (T\text{-Alg})' \longrightarrow \mathrm{Mnd}\,\mathcal{A}$, the vertex of this cone being the monad S. For each T-algebra (A, a), the component $\rho_{(A,a)'} : S \longrightarrow \langle A, A\rangle$ of ρ corresponds to an action $\overline{a} : SA \longrightarrow A$ of the monad S on A, and hence to an S-algebra $(A, \overline{a})$. For each map $u : (A, a) \longrightarrow (B, b)$ of T-algebras, the component $\rho_{u'} : S \longrightarrow \{u, u\}$ of ρ corresponds to an action

$$\begin{tikzcd} SA \arrow[r, "\widetilde{a}"] \arrow[d, "Su"'] & A \arrow[d, "u"] \\ SB \arrow[r, "\widetilde{b}"'] & B \end{tikzcd}$$

of the monad S on u. Since, however, the components of ρ satisfy $\sigma_{u,u}\rho_{u'} = \rho_{(A,a)'}$ and $\tau_{u,u}\rho_{u'} = \rho_{(B,b)'}$, the $\widetilde{a}$ and the $\widetilde{b}$ above are in fact the $\overline{a}$ and the $\overline{b}$ of the S-algebras $(A, \overline{a})$ and $(B, \overline{b})$. So $(A, a) \longmapsto (A, \overline{a})$ gives a functor T-Alg $\longrightarrow S$-Alg commuting with the forgetful functors to $\mathcal{A}$, which therefore has the form θ^* for a unique monad map $\theta : S \longrightarrow T$. Equivalently, the cone ρ has the form $\psi^T\theta$ for a unique $\theta : S \longrightarrow T$, showing ψ^T to be a limit cone. ∎

Still with $\mathcal{A}$ complete and locally small, let us once again consider an action $F : \mathcal{V} \longrightarrow [\mathcal{A}, \mathcal{A}]$ of a monoidal $\mathcal{V}$ on $\mathcal{A}$, which we further suppose to admit an adjunction (2.1). Composing F with the action $[\mathcal{A}, \mathcal{A}] \longrightarrow [\mathcal{A}^\mathbf{2}, \mathcal{A}^\mathbf{2}]$ above gives an action $\mathcal{V} \longrightarrow [\mathcal{A}^\mathbf{2}, \mathcal{A}^\mathbf{2}]$, which in its form $\mathcal{V} \times \mathcal{A}^\mathbf{2} \longrightarrow \mathcal{A}^\mathbf{2}$ sends $(X, u : A \longrightarrow B)$ to $(X * u : X * A \longrightarrow X * B)$. Provided that $\mathcal{V}$ has pullbacks, this too admits an adjunction of the form (2.1), namely

$$\mathcal{A}^\mathbf{2}(X * u, v) \;\cong\; \mathcal{V}(X, \overline{\mathbf{A}}(u, v)) , \tag{5.7}$$

where $\overline{\mathbf{A}}(u,v)$ is given by the pullback

$$\begin{tikzcd}[column sep=small] & \mathbf{A}(A,C) \arrow[dr, "\mathbf{A}(A,v)"] & \\ \overline{\mathbf{A}}(u,v) \arrow[ur, "\overline{\sigma}_{u,v}"] \arrow[dr, "\overline{\tau}_{u,v}"'] & & \mathbf{A}(A,D) \\ & \mathbf{A}(B,D) \arrow[ur, "\mathbf{A}(u,D)"'] & \end{tikzcd} \tag{5.8}$$

in $\mathcal{V}$; note that (5.2) and (5.3) are just the special cases of (5.7) and (5.8) obtained by taking $\mathcal{V}$ to be $[\mathcal{A},\mathcal{A}]$ and $F : \mathcal{V} \longrightarrow [\mathcal{A},\mathcal{A}]$ to be the identity.

A right adjoint g to $f : \mathcal{V} \longrightarrow [\mathcal{A},\mathcal{A}]$ certainly exists *locally* at the endofunctor $\langle A, C\rangle$; for (5.1) and (2.1) give us, naturally in X, the composite isomorphism

$$[\mathcal{A},\mathcal{A}](fX, \langle A,C\rangle) \cong \mathcal{A}((fX)A, C) = \mathcal{A}(X * A, C) \cong \mathcal{V}(X, \mathbf{A}(A,C)), \tag{5.9}$$

showing that we can take

$$g\langle A,C\rangle = \mathbf{A}(A,C) \tag{5.10}$$

as the local value of the right adjoint; the counit $fg\langle A,C\rangle \longrightarrow \langle A,C\rangle$ of this representation (5.7) is of course the morphism $f\mathbf{A}(A,C) \longrightarrow \langle A,C\rangle$ corresponding to $\epsilon : (f\mathbf{A}(A,C))A = \mathbf{A}(A,C) * A \longrightarrow C$.

The right adjoint g also exists locally at the endomorphism $\{u,v\}$; for (5.2) and (5.7) give us, naturally in X, the composite isomorphism

$$[\mathcal{A},\mathcal{A}](fX, \{u,v\}) \cong \mathcal{A}^{\mathbf{2}}((fX)u, v)) = \mathcal{A}^{\mathbf{2}}(X * u, v) \cong \mathcal{V}(X, \overline{\mathbf{A}}(u,v)), \tag{5.11}$$

showing that we can take

$$g\{u,v\} = \overline{\mathbf{A}}(u,v) \tag{5.12}$$

as the local value of the right adjoint, with the evident value of the counit $fg\{u,v\} \longrightarrow \{u,v\}$ which we leave the reader to make explicit. On those endofunctors T for which gT exists, g is of course functorial—it is a “partial functor”. In particular, one easily sees that the partial functor g carries the diagram (5.3) into the diagram (5.8).

Consider now the functor

$$\widehat{T}^* : (T\text{-QAlg})' \longrightarrow \mathcal{V}$$

which sends $(A,a,C)'$ to $\mathbf{A}(A,C)$, sends $(u,v)'$ to $\overline{\mathbf{A}}(u,v)$, sends $s_(u,v)$ to $\overline{\sigma}_{u,v}$, and sends $t_{(u,v)}$ to $\overline{\tau}_{u,v}$; it follows from the above, since g is to be a right adjoint, that :

5.3. THEOREM. *Let the action $F : \mathcal{V} \longrightarrow [\mathcal{A}, \mathcal{A}]$ of the monoidal $\mathcal{V}$, which we suppose to admit pullbacks, on the complete and locally small $\mathcal{A}$ admit an adjunction (2.1); then a right adjoint g of f is given locally at an endofunctor T of $\mathcal{A}$ by*

$$gT = \lim(\widehat{T}^* : (T\text{-}QAlg)' \longrightarrow \mathcal{V}) ,$$

either side existing if the other does.

Although gT has of course a monoid structure when T is a monad, there is a more direct alternative formula in the monad case, the domain of the limit now being the "simpler" (although still large) category $(T\text{-Alg})'$. For when $C = A$ in (5.9) and X is a monoid, an element α of $[\mathcal{A}, \mathcal{A}](fX, \langle A, A\rangle)$ is a monad map precisely when the corresponding element a of $\mathcal{A}((fX)A, A) = \mathcal{A}(X * A, A)$ is an action, and thus precisely when the corresponding element $\kappa(a)$ of $\mathcal{V}(X, \mathbf{A}(A, A))$ is a monoid map; so that (5.10) becomes an equality

$$(\mathrm{Mon}\, G)\langle A, A\rangle \;=\; \mathbf{A}(A, A)$$

of monoids, and similarly (5.12) becomes for $v = u$ an equality of monoids

$$(\mathrm{Mon}\, G)\{u, u\} = \overline{\mathbf{A}}(u, u) .$$

In fact the left half of (5.8) becomes a diagram

$$\begin{tikzcd}[column sep=small] & \mathbf{A}(A, A) \\ \overline{\mathbf{A}}(u, u) \arrow[ur, "\overline{\sigma}_{u,u}"] \arrow[dr, "\overline{\tau}_{u,u}"'] & \\ & \mathbf{A}(B, B) \end{tikzcd} \tag{5.13}$$

of monoids, describing a functor

$$\widetilde{T}^* : (T\text{-Alg})' \longrightarrow \ \mathrm{Mon}\, \mathcal{V} .$$

Now the corresponding argument gives :

5.4. THEOREM. *With the same hypothesis as in Theorem 5.3, a right adjoint* $\mathrm{Mon}\, G$ *to* $\mathrm{Mon}\, F$ *is given locally at a monad T on $\mathcal{A}$ by*

$$(\mathrm{Mon}\, G)T = \lim(\widetilde{T}^* : (T\text{-Alg})' \longrightarrow \ \mathrm{Mon}\, \mathcal{V}) ,$$

either side existing if the other does.

5.5. Remark. We saw in Example 4.9 that, even for a locally-finitely-presentable symmetric monoidal closed $\mathcal{V}$, when we consider the canonical action $F : \mathcal{V} \longrightarrow [\mathcal{V}, \mathcal{V}]$ of $\mathcal{V}$ on itself, neither f nor $\mathrm{Mon}\, F$ need be fully faithful. Equivalently, the unit $X \longrightarrow g(X \otimes -)$ of the adjunction $f \dashv g$ is not in general invertible, even when X is a monoid in $\mathcal{V}$. It follows that there is no analogue of Proposition 5.2 in which $[\mathcal{A}, \mathcal{A}]$ is replaced by a general monoidal category $\mathcal{V}$, even a well-behaved one. That is to say, a monoid X in $\mathcal{V}$ is *not* the $(X\text{-Alg})'$–indexed limit of the special monoids of the forms $\mathbf{V}(A, A)(= [A, A])$ and $\overline{\mathbf{V}}(u, u)$; for, by Theorem 5.4, this limit is precisely $g(X \otimes -)$.

5.6. Remark. We have not found limit-formulae for rH and $(\mathrm{Mon}\, R)H$ analogous to Theorem 5.3 and 5.4. Certainly the action $u : (\mathbf{A}, \mathbf{A}) \longrightarrow [\mathcal{A}, \mathcal{A}]$, which as a functor $(\mathbf{A}, \mathbf{A}) \times \mathcal{A} \longrightarrow \mathcal{A}$ sends (H, A) to HA, admits an adjunction of the form (2.1), namely

$$\mathcal{A}(HA, B) \cong (\mathbf{A}, \mathbf{A})(H, \langle\langle A, B\rangle\rangle) , \tag{5.14}$$

where the $\mathbf{V}$-functor $\langle\langle A, B\rangle\rangle : \mathbf{A} \longrightarrow \mathbf{A}$ is the right Kan extension of the $\mathcal{V}$-functor $B : \mathbf{I} \longrightarrow \mathbf{A}$ along the $\mathcal{V}$-functor $A : \mathbf{I} \longrightarrow \mathbf{A}$, with $\mathbf{I}$ being the unit $\mathcal{V}$-category having one object $*$ and having $\mathbf{I}(*, *) = I$; this right Kan extension can be described in terms of the cotensor product by $\langle\langle A, B\rangle\rangle C = |\mathbf{A}(C, A), B|$, and there is a canonical comparison $\langle\langle A, B\rangle\rangle_0 \longrightarrow \langle A, B\rangle$. Moreover, for the composite action

$$(\mathbf{A}, \mathbf{A}) \longrightarrow [\mathcal{A}, \mathcal{A}] \longrightarrow [\mathcal{A}^{\mathbf{2}}, \mathcal{A}^{\mathbf{2}}]$$

we again have an adjunction of the form (2.1), namely

$$\mathcal{A}^{\mathbf{2}}(Hu, v) \cong (\mathbf{A}, \mathbf{A})(H, \{\{u, v\}\}) , \tag{5.15}$$

where $\{\{u, v\}\}$ is defined by a pullback like (5.3) but with $\langle\langle A, C\rangle\rangle$ replacing $\langle A, C\rangle$ and so on; of course there is induced a canonical comparison $\{\{u, v\}\}_0 \longrightarrow \{u, v\}$. Putting $H = qX$ in (5.14) we see, using the analogue of (5.9), that

$$r\langle\langle A, C\rangle\rangle = \mathbf{A}(A, C) ; \tag{5.16}$$

and similarly (5.15) gives

$$r\{\{u, v\}\} = \overline{\mathbf{A}}(u, v) . \tag{5.17}$$

There is an evident functor (to take the monad-adapted case)

$$\widetilde{H} : (H\text{-Alg}_0)' \longrightarrow (\mathbf{A}, \mathbf{A}),$$

sending $(A, a)'$ to $\langle\langle A, A\rangle\rangle$ and $((A, a), u, (B, b))'$ to $\{\{u, u\}\}$, and a cone over this in $\mathrm{Mon}(\mathbf{A}, \mathbf{A}) = (\mathcal{V}\text{-Mnd})\mathbf{A}$ with vertex H. However the argument of Proposition 5.2 no longer applies to prove this a limit-cone : for we only get mere naturality, where $\mathcal{V}$-naturality is needed. And in fact it is *not* a limit cone; for if it were, $(\mathrm{Mon}\, R)H$ would be, by (5.16) and (5.17), the limit in $\mathrm{Mon}\, \mathcal{V}$ of the diagram (5.13); but the latter limit is $(\mathrm{Mon}\, G)H_0$ by Theorem 5.4, and we have seen in Remark 4.5 that the canonical comparison $(\mathrm{Mon}\, R)H \longrightarrow (\mathrm{Mon}\, G)H_0$ is not invertible in general. So if each $\mathcal{V}$-monad H is to be a limit of special monads, it must be the limit of some richer diagram making better use of the $\mathcal{V}$-structure.

6. Appendix on tensored $\mathcal{V}$-categories

We justify here the claim made in Section 2 that, for a right-closed monoidal $\mathcal{V}$, to give a category $\mathcal{A}$ and an action of $\mathcal{V}$ on $\mathcal{A}$ admitting the adjunction (2.1) is essentially the same thing as to give a tensored $\mathcal{V}$-category $\mathbf{A}$. In fact we showed in Section 2 exactly how to produce, from the action of $\mathcal{V}$ on $\mathcal{A}$ and the adjunction (2.1), the $\mathcal{V}$-category $\mathbf{A}$ and the $\mathcal{V}$-natural isomorphism $k : \mathbf{A}(X * A, -) \longrightarrow [X, \mathbf{A}(A, -)]$ of (2.5) which constitutes the tensoring of $\mathbf{A}$; as our next step, we establish some further properties of the isomorphism $k = k_{XAB} : \mathbf{A}(X * A, B) \longrightarrow [X, \mathbf{A}(A, B)]$ so produced, which we saw to be natural in each variable, as well as being $\mathcal{V}$-natural in B. First consider the following two diagrams, which have the same top edge and the same bottom edge, and thus can be thought of as being pasted together along these to form a circular cylinder : we omit, to save space, the names of the interior objects, replacing each by the symbol @—the reader will have no trouble reconstructing them, guided by the names of the morphisms and our remarks below. (The arguments which follow were first used in [KC].) The diagrams are:

$$\begin{tikzcd}
\mathcal{A}(Z*(Y*(X*A)),B) \arrow[rrr, "\kappa"] \arrow[d, "\mathcal{A}(1*\alpha,B)"'] & & & \mathcal{V}(Z,\mathbf{A}(Y*(X*A),B)) \arrow[d, "\mathcal{V}(Z,\mathbf{A}(\alpha,B))"] \\
\mathcal{A}(Z*((Y\otimes X)*A),B) \arrow[rrr, "\kappa"] \arrow[d, "\mathcal{A}(\alpha,B)"'] & & & \mathcal{V}(Z,\mathbf{A}((Y\otimes X)*A,B)) \arrow[d, "\mathcal{V}(Z,k)"] \\
\mathcal{A}((Z\otimes(Y\otimes X))*A,B) \arrow[r, "\kappa"] \arrow[d, "\mathcal{A}(a*1,B)"'] & @ \arrow[rr, "\pi"] \arrow[d, "\mathcal{V}(a,\mathbf{A}(A,B))"] & & \mathcal{V}(Z,[Y\otimes X,\mathbf{A}(A,B)]) \arrow[d, "\mathcal{V}(Z,p)"] \\
\mathcal{A}(((Z\otimes Y)\otimes X)*A,B) \arrow[r, "\kappa"'] & @ \arrow[r, "\pi"'] & @ \arrow[r, "\pi"'] & \mathcal{V}(Z,[Y,[X,\mathbf{A}(A,B)]])
\end{tikzcd} \quad , \tag{6.1}$$

$$\begin{tikzcd}
\mathcal{A}(Z*(Y*(X*A)),B) \arrow[rrr, "\kappa"] \arrow[d, "\mathcal{A}(\alpha,B)"'] & & & \mathcal{V}(Z,\mathbf{A}(Y*(X*A),B)) \arrow[d, "\mathcal{V}(Z,k)"] \\
\mathcal{A}((Z\otimes Y)*(X*A),B) \arrow[rr, "\kappa"] \arrow[d, "\mathcal{A}(\alpha,B)"'] & & @ \arrow[r, "\pi"] \arrow[d, "\mathcal{V}(Z\otimes Y,k)"'] & \mathcal{V}(Z,[Y,\mathbf{A}(X*A,B)]) \arrow[d, "\mathcal{V}(Z,[Y,k])"] \\
\mathcal{A}(((Z\otimes Y)\otimes X)*A,B) \arrow[r, "\kappa"'] & @ \arrow[r, "\pi"'] & @ \arrow[r, "\pi"'] & \mathcal{V}(Z,[Y,[X,\mathbf{A}(A,B)]])
\end{tikzcd} \quad . \tag{6.2}$$

In these diagrams, each of the rectangular regions commutes by naturality, and each of the four pentagonal regions is an instance of (2.6)—except that in one of these we are in the situation $\mathcal{A} = \mathcal{V}$, so that the k in question becomes the p of (2.9), while κ becomes π

and α becomes a. However the left edges of (6.1) and (6.2) coincide by (1.1), so that the right edges coincide too; whence, by the Yoneda lemma, we have commutativity in the diagram

$$\begin{tikzcd}
\mathbf{A}(Y * (X * A), B) \arrow[rr, "k"] \arrow[d, "\mathbf{A}(\alpha,B)"'] & & {[Y, \mathbf{A}(X * A, B)]} \arrow[d, "{[Y,k]}"] \\
\mathbf{A}((Y \otimes X) * A, B) \arrow[r, "k"'] & {[Y \otimes X, \mathbf{A}(A,B)]} \arrow[r, "p"'] & {[Y, [X, \mathbf{A}(A,B)]]}\ .
\end{tikzcd} \tag{6.3}$$

Now consider similarly the following two commutative diagrams:

$$\begin{tikzcd}
\mathcal{A}(Y * A, B) \arrow[rr, "\kappa"] \arrow[d, "\mathcal{A}(Y*\lambda,B)"'] & & \mathcal{V}(Y, \mathbf{A}(A,B)) \arrow[d, "{\mathcal{V}(Y,\mathbf{A}(\lambda,B))}"] \\
\mathcal{A}(Y * (I * A), B) \arrow[rr, "\kappa"] \arrow[d, "\mathcal{A}(\alpha,B)"'] & & \mathcal{V}(Y, \mathbf{A}(I * A, B)) \arrow[d, "{\mathcal{V}(Y,k)}"] \\
\mathcal{A}((Y \otimes I) * A, B) \arrow[r, "\kappa"'] & \mathcal{V}(Y \otimes I, \mathbf{A}(A,B)) \arrow[r, "\pi"'] & \mathcal{V}(Y, [I, \mathbf{A}(A,B)])\ ,
\end{tikzcd} \tag{6.4}$$

$$\begin{tikzcd}
\mathcal{A}(Y * A, B) \arrow[rr, "\kappa"] \arrow[d, "\mathcal{A}(r*A,B)"'] & & \mathcal{V}(Y, \mathbf{A}(A,B)) \arrow[d, "{\mathcal{V}(Y,i)}"] \arrow[dl, "{\mathcal{V}(r,1)}"] \\
\mathcal{A}((Y \otimes I) * A, B) \arrow[r, "\kappa"'] & \mathcal{V}(Y \otimes I, \mathbf{A}(A,B)) \arrow[r, "\pi"'] & \mathcal{V}(Y, [I, \mathbf{A}(A,B)])\ ,
\end{tikzcd} \tag{6.5}$$

where again the two quadrangles commute by naturality, and the pentagon is an instance of (2.6); it is classical that the triangle commutes, i here being an instance of the canonical isomorphism $i : X \longrightarrow [I, X]$. Now the commutativity of (1.3) gives that of the diagram

$$\begin{tikzcd}
\mathbf{A}(I * A, B) \arrow[rr, "k"] & & {[I, \mathbf{A}(A,B)]} \\
& \mathbf{A}(A,B)\ . \arrow[ul, "{\mathbf{A}(\lambda,1)}"] \arrow[ur, "i"'] &
\end{tikzcd} \tag{6.6}$$

Finally, consider the two diagrams

$$\begin{tikzcd}
\mathcal{A}(X*A,B) \arrow[d, "\mathcal{A}(\lambda,B)"'] \arrow[drrrr, "\theta"] & & & & \\
\mathcal{A}(I*(X*A),B) \arrow[rr, "\kappa"] \arrow[d, "\mathcal{A}(\alpha,B)"'] & & \mathcal{V}(I,\mathbf{A}(X*A,B)) \arrow[d, "\mathcal{V}(I,k)"'] & = & \mathbf{A}_0(X*A,B) \arrow[d, "\mathcal{V}(I,k)"'] \\
\mathcal{A}((I\otimes X)*A,B) \arrow[r, "\kappa"'] & \mathcal{V}(I\otimes X,\mathbf{A}(A,B)) \arrow[r, "\pi"'] & \mathcal{V}(I,[X,\mathbf{A}(A,B)]) & = & \mathbf{V}_0(X,\mathbf{A}(A,B))
\end{tikzcd} \quad , \tag{6.7}$$

$$\begin{tikzcd}
\mathcal{A}(X*A,B) \arrow[r, "\kappa"] \arrow[d, "\mathcal{A}(\ell*1,B)"'] & \mathcal{V}(X,\mathbf{A}(A,B)) \arrow[d, "\mathcal{V}(\ell,\mathbf{A}(A,B))"] \arrow[drrr, "\phi"] & & & \\
\mathcal{A}((I\otimes X)*A,B) \arrow[r, "\kappa"'] & \mathcal{V}(I\otimes X,\mathbf{A}(A,B)) \arrow[r, "\pi"'] & \mathcal{V}(I,[X,\mathbf{A}(A,B)]) & = & \mathbf{V}_0(X,\mathbf{A}(A,B))
\end{tikzcd} \quad , \tag{6.8}$$

where $\theta : \mathcal{A} \cong \mathbf{A}_0$ is the isomorphism of (2.4), where $\phi : \mathcal{V} \cong \mathbf{V}_0$ is the special case of θ got by taking $\mathcal{A}$ to be $\mathcal{V}$, and where the pentagonal region in (6.7) is an instance of (2.6) while the rectangle in (6.8) commutes by naturality. Here the commutativity of (1.2) gives

$$\mathcal{V}(I,k)\theta = \phi\kappa \;; \tag{6.9}$$

in other words, if we use θ and ϕ to identify $\mathcal{A}$ with $\mathbf{A}_0$ and $\mathcal{V}$ with $\mathbf{V}_0$, then $\kappa : \mathcal{A}(X*A,B) \longrightarrow \mathcal{V}(X,\mathbf{A}(A,B))$ is the bijection $\mathcal{V}(I,k)$ underlying the isomorphism $k : \mathbf{A}(X*A,B) \longrightarrow [X,\mathbf{A}(A,B)]$ in $\mathcal{V}$—which we may also express by saying that k is a *lifting of the bijection* κ to an isomorphism in $\mathcal{V}$. In the light of this, we may describe (6.3) as a lifting from $\mathcal{SET}$ to $\mathcal{V}$ of (2.6), and similarly describe (6.6) as a lifting of (2.4). Note in particular the special case of (6.9) obtained when $\mathcal{A} = \mathcal{V}$, namely

$$\mathcal{V}(I,p)\phi = \phi\kappa \,. \tag{6.10}$$

Now we may describe the association, for a right-closed monoidal $\mathcal{V}$, between actions of $\mathcal{V}$ admitting an adjunction (2.1) and tensored $\mathcal{V}$-categories. For the first of these, the data consist of the category $\mathcal{A}$, the functor $*$, the natural isomorphisms α and λ satisfying (1.1) and (1.3), the functor $\mathbf{A} : \mathcal{A}^{\mathrm{op}} \times \mathcal{A} \longrightarrow \mathcal{V}$ (which we may write as $\mathbf{A}(-,-)$, to distinguish it from the $\mathcal{V}$-category $\mathbf{A}$), and the natural isomorphism κ of (2.1). The data for the second consist of a $\mathcal{V}$-category $\mathbf{A}$ and, for each pair X, A, an object $X * A$ of $\mathbf{A}$ together with a $\mathcal{V}$-natural isomorphism

$$k_{XA} : \mathbf{A}(X*A,-) \longrightarrow [X,\mathbf{A}(A,-)]$$

as in (2.8). The reader will recall from Section 2 the process—let us call it ξ—leading from the first to the second : we construct the $\mathcal{V}$-category $\mathbf{A}$ with $\mathrm{ob}\,\mathbf{A} = \mathrm{ob}\,\mathcal{A}$ by giving it the hom-objects $\mathbf{A}(A,B)$ from the adjunction κ of (2.1), with the composition operation M given as the mate under κ of the composite $\epsilon_{BC}(1 * \epsilon_{AB})\alpha$ of (2.3), and with the unit operation $j : I \longrightarrow \mathbf{A}(A,A)$ which is the image under κ of λ; then we use (2.6) to define k_{XAB}, whose $\mathcal{V}$-naturality in B follows from Lemma 2.1, after our checking that the functor $\mathbf{A} : \mathcal{A}^{\mathrm{op}} \times \mathcal{A} \longrightarrow \mathcal{V}$ does coincide, to within the isomorphisms (2.4), with the functor $\hom_{\mathbf{A}}$.

We now describe a process—let us call it η—going in the other direction. We take for the category $\mathcal{A}$ the underlying category $\mathbf{A}_0$ of the given $\mathcal{V}$-category $\mathbf{A}$, so that $\mathcal{A}(A,B) = \mathcal{V}(I, \mathbf{A}(A,B))$, and we use $\mathbf{A} : \mathcal{A}^{\mathrm{op}} \times \mathcal{A} \longrightarrow \mathcal{V}$ as another name for the functor $\hom_{\mathbf{A}} : \mathbf{A}_0^{\mathrm{op}} \times \mathbf{A}_0 \longrightarrow \mathcal{V}$ arising from the $\mathcal{V}$-category $\mathbf{A}$. We take for κ the composite bijection

$$\mathcal{A}(X * A, B) = \mathcal{V}(I, \mathbf{A}(X * A, B)) \xrightarrow[\mathcal{V}(I,k)]{} \mathcal{V}(I, [X, \mathbf{A}(A,B)]) \xrightarrow[\phi^{-1}]{} \mathcal{V}(X, \mathbf{A}(A,B)), \quad (6.11)$$

where ϕ (as in (6.8)) denotes the isomorphism $\pi\mathcal{V}(\ell, 1)$, which is the case $\mathcal{A} = \mathcal{V}$ of (2.4); thus, modulo the isomorphism ϕ, the bijection κ is that underlying the isomorphism k of $\mathcal{V}$. Since k is $\mathcal{V}$-natural in the variable B, it is certainly natural in B, so that the same is true of κ; accordingly $X * A$, so far defined only on objects, extends to a functor $* : \mathcal{V} \times \mathcal{A} \longrightarrow \mathcal{A}$ in a unique way that makes $\kappa = \phi^{-1}\mathcal{V}(I,k)$ natural in X and A as well as in B. For α we take the unique morphism—clearly invertible—that makes (2.6) commute for each B; and for λ we take the unique morphism—again clearly invertible—that makes (2.4) the identity, so that $\kappa\mathcal{A}(\lambda, 1) = 1$. It remains to see that α and λ satisfy the coherence conditions (1.1) and (1.3). Consider first the diagram (6.3) : because each arrow therein is $\mathcal{V}$-natural in B, each leg has the form of a $\mathcal{V}$-natural transformation $\rho : \mathbf{A}(K, -) \longrightarrow T$ where $K = Y * (X * A)$ and $T = [Y, [X, \mathbf{A}(A, -)]]$. By the enriched Yoneda lemma, two such $\mathcal{V}$-natural transformation ρ and ρ' coincide if and only if

$$I \xrightarrow[j_K]{} \mathbf{A}(K,K) \xrightarrow[\rho_K]{} TK \qquad = \qquad I \xrightarrow[j_K]{} \mathbf{A}(K,K) \xrightarrow[\rho'_K]{} TK. \quad (6.12)$$

On the other hand, we have the ordinary natural transformations

$$\mathcal{V}(I,\rho), \mathcal{V}(I,\rho') : \mathcal{A}(K,-) = \mathcal{V}(I, \mathbf{A}(K,-)) \longrightarrow \mathcal{V}(I, TK);$$

and by the ordinary Yoneda Lemma, these coincide if and only if $\mathcal{V}(I,\rho_K)1_K = \mathcal{V}(I,\rho'_K)1_K$; which is the same criterion as (6.12). That is to say, $\rho = \rho'$ if and only if $\mathcal{V}(I,\rho) = \mathcal{V}(I,\rho')$. We conclude that (6.3) commutes if its image under $\mathcal{V}(I,-)$ does so; but by (6.10) and (6.11), this image is essentially the commutative diagram (2.6). So (6.3) does commute; and now we can read (6.1) and (6.2) in the reverse direction, to conclude that (1.1) commutes. Similarly, each morphism in (6.6) is $\mathcal{V}$-natural in B, so that (6.6) commutes if its image under $\mathcal{V}(I,-)$ does so; but since $\mathcal{A}(A,B) = \mathcal{V}(I, \mathbf{A}(A,B))$, the commutativity of this image reduces, using (6.11), to the trivially-verified fact that $\mathcal{V}(I,i) : \mathcal{V}(I,X) \longrightarrow \mathcal{V}(I,[I,X])$ is just an instance of $\phi = \pi\mathcal{V}(\ell,1)$. Now reading in the reverse order the diagrams (6.7) and (6.8), in which θ is here an identity, gives the desired commutativity of (1.3).

Suppose now that, starting with the data $(\mathcal{A}, *, \alpha, \lambda, \mathbf{A}(-,-), \kappa)$, we first apply the process ξ to arrive at the data $(\mathbf{A}, k)$, and then the process η to arrive at the data $(\mathcal{A}', *', \alpha', \lambda', \mathbf{A}'(-,-), \kappa')$. We observed already in Section 2 that we have an isomorphism $\theta = \kappa\mathcal{A}(\lambda, 1) : \mathcal{A} \longrightarrow \mathbf{A}_0 = \mathcal{A}'$ given by (2.4), and that the functor $\mathbf{A}'(-,-) = \hom_{\mathbf{A}}$ agrees with $\mathbf{A}(-,-)$ modulo θ. Since $\kappa' : \mathcal{A}'(X * A, B) \cong \mathcal{V}(X, \mathbf{A}(A,B))$ is $\phi^{-1}\mathcal{V}(I,k)$ by (6.11), it follows from (6.9) that $\kappa = \kappa'\theta$. Accordingly $*'$ agrees with $*$ modulo θ. Since $\theta = \kappa\mathcal{A}(\lambda, 1)$ while λ' is defined by $\kappa'\mathcal{A}'(\lambda', 1) = 1$, we see that λ' agrees with λ modulo θ. Finally, since k is defined in terms of α by (2.6), while α' is defined in terms of k by the primed version of (2.6), the observation above that $\kappa = \kappa'\theta$ implies that α' agrees with α modulo θ. Thus θ constitutes, in an evident sense, an isomorphism between the data $(\mathcal{A}, *, \alpha, \lambda, \mathbf{A}(-,-), \kappa)$ and the image of this under the process $\eta\xi$.

Suppose on the other hand that, starting with the data $(\mathbf{A}, k)$, we first apply the process η to arrive at the data $(\mathcal{A}, *, \alpha, \lambda, \mathbf{A}(-,-), \kappa)$, and then the process ξ to arrive at the data $(\mathbf{A}'', k'')$. Here $\mathrm{ob}\,\mathbf{A}'' = \mathrm{ob}\,\mathcal{A} = \mathrm{ob}\,\mathbf{A}$, and $\mathcal{A}''(A,B) = \mathbf{A}(-,-)(A,B) = \hom_{\mathbf{A}}(A,B) = \mathbf{A}(A,B)$. With κ defined as $\phi^{-1}\mathcal{V}(I,k)$ as in (6.11), recall that α is defined in terms of k by (2.6), while k'' is then defined in terms of α by (2.6); it follows that $k'' = k$. Since λ is so defined that the $\kappa\mathcal{A}(\lambda, 1)$ of (2.4) is the identity, the image of λ under $\kappa : \mathcal{A}(I * A, A) \longrightarrow \mathcal{V}(I, \mathbf{A}(A,A))$ is $j : I \longrightarrow \mathbf{A}(A,A)$; but j'' is this image, so that $j'' = j$. We shall therefore have $(\mathbf{A}'', k'') = (\mathbf{A}, k)$ if we can show that $M'' = M$. Given the definition of M'' using the process ξ, we are to show, for any tensored $\mathcal{V}$-category $\mathbf{A}$, the commutativity of the diagram

$$\begin{tikzcd}
(\mathbf{A}(B,C) \otimes \mathbf{A}(A,B)) * A \arrow[r, "M*1"] \arrow[d, "\alpha"] & \mathbf{A}(A,C) * A \arrow[dd, "\epsilon_{AC}"] \\
\mathbf{A}(B,C) * (\mathbf{A}(A,B)) * A) \arrow[d, "1*\epsilon_{AB}"] & \\
\mathbf{A}(B,C) * B \arrow[r, "\epsilon_{BC}"'] & C
\end{tikzcd} \tag{6.13}$$

wherein the ϵ_{AB} and so on are the counits of the adjunction k, or equally of the adjunction κ of (6.11). A simple way of establishing this commutativity is the following. When we see $X * A$ as the value of the $\mathcal{V}$-functor $\mathrm{Ten} : \mathbf{V} \otimes \mathbf{A} \longrightarrow \mathbf{A}$, the isomorphism k_{XAB} : $\mathbf{A}(X * A, B) \longrightarrow [X, \mathbf{A}(A,B)]$ is $\mathcal{V}$-natural in each of the variables X, A, B : see Sections 1.10 and 1.11 of [KB]. By these same sections, the counit $\epsilon_{AB} : \mathbf{A}(A,B) * A \longrightarrow B$ of the adjunction k is $\mathcal{V}$-natural in each variable (now in the generalized sense of [EC]). Moreover we have seen that (6.3) commutes; since k and p are $\mathcal{V}$-natural in each variable, so too is α. Finally, M is $\mathcal{V}$-natural in each variable by Section 1.8(g) of [KB]. All that we in fact use of the two legs of (6.13) is their $\mathcal{V}$-naturality in the variable C. Using the $\mathcal{V}$-adjunctions k and p, we can equally see these legs as two families of morphisms $\mathbf{A}(B,C) \longrightarrow [\mathbf{A}(A,B), \mathbf{A}(A,C)]$, each $\mathcal{V}$-natural in C. By the enriched Yoneda lemma, the two legs coincide if they do so when we put $C = B$ and compose with $j : I \longrightarrow \mathbf{A}(B,B)$; but if we do this in (6.13) and use (1.2), each leg reduces to $\epsilon_{AB} : \mathbf{A}(A,B) * A \longrightarrow B$. We conclude that the process $\xi\eta$ is the identity : and this ends our appendix by providing the precise sense in which to give, for a right-closed monoidal $\mathcal{V}$, a category $\mathcal{A}$ and an action of $\mathcal{V}$ on $\mathcal{A}$ admitting an adjunction (2.1) is essentially the same thing as to give a tensored $\mathcal{V}$-category $\mathbf{A}$.

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