ON THE DECOMPOSITION OF A RESOLVING OPERATOR OF AN EQUATION WITH VARIABLE OPERATOR

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Let us consider the Cauchy problem in the Banach space X:

$$u'(t) + A(t)u(t) = f(t),\ t \in [0, t^*],\ u(0) = \varphi. \tag{1}$$

Let the operator $A(t)$ satisfy the following conditions:

1) The domain of definition of the operator $A(t)$ does not depend on t;

2) The operator $A(t) + zI$ is inversible for any z, for which $Re(z) \geq 0$, and the following inequality takes place (in the sequel everywhere by c is denoted a constant number):

$$\|(A(t) + zI)^{-1}\| \leq \frac{c}{1 + |z|};$$

3) For any t', t'' and s from the segment $[0, t^*]$, holds the inequality as follows:

$$\|[A(t') - A(t'')]A^{-1}(s)\| \leq c|t' - t''|.$$

Then there exists operator $U(t, s)$ (the resolvent operator for the problem (1)) such, that the function

$$u(t) = U(t, 0)\varphi + \int_0^t U(t, s)f(s)ds,$$

where $f(t)$ satisfies Hölder's condition, $\varphi \in D(A)$, is the solution of the problem (1) (see [1], [2]).

Let the operator $A(t)$ be represented as the sum of linear closed operators $A_1(t), \cdots, A_m(t)$ $(m \geq 2)$,

$$A(t) = A_1(t) + \cdots + A_m(t),$$

and let the following conditions be fulfilled:

4) The operator $A_j(t) + \lambda I$, $j = 1, \cdots, m$ is inversible for any $\lambda > 0$ and

$$\|(A_j(t) + \lambda I)^{-1}\| \leq \frac{1}{\lambda};$$

5) The domain of definitions of the operators $A_j(t)$, $j = 1, \cdots, m$, does not depend on t and the following inequality holds:

$$\|A_j(t)\varphi\| \leq c(\|\varphi\| + \|A(t)\varphi\|),\ \varphi \in D(A);$$

6) The domains of definition $D(A^2)$ and $D(A_j^2)$ of the operators $A^2(t)$ and $A_j^2(t)$, $j = 1, \cdots, m$, does not depend on t and the following inequality takes place:

$$\|A_j^2(t)\varphi\| \le c(\|\varphi\| + \|A(t)\varphi\| + \|A^2(t)\varphi\|),\ \varphi \in D(A^2);$$

7) For any t', t'' and s from the segment $[0, t^*]$, the inequality follows

$$\|[A^2(t') - A^2(t'')]A^{-2}(s)\| \le c|t' - t''|.$$

The following theorem takes place.

Theorem. Let the operators $A(t)$ and $A_j(t)$, $j = 1, \cdots, m$ satisfy the conditions 1) – 7) and the numbers $0 < \eta_j < 1$, $j = 1, \cdots, m$ be such that

$$\eta_1 + \eta_2 + \cdots + \eta_m = 1.$$

Then for any natural number n the following estimate holds:

$$\|(U_k(\tau, 1) - T_k(\tau, 1))\varphi\| \le c\tau\|A^2(0)\varphi\|,\ \varphi \in D(A^2),\ k = 1, \cdots, n, \qquad (2)$$

where $\tau = t^*/n$,

$$U_k(\tau, 1) = L_k(\tau)\cdots L_1(\tau),\ L_k(\tau) = (I + \tau A(t_k))^{-1},\ t_k = k\tau,$$

$$T_k(\tau, 1) = S_k(\tau)\cdots S_1(\tau),\ S_k(\tau) = \sum_{j=1}^{m} \eta_j(I + \tau\eta_j^{-1}A_j(t_k))^{-1}.$$

Proof. There takes place the following representation:

$$(U_k(\tau, 1) - T_k(\tau, 1))\varphi = \sum_{i=1}^{k} T_k(\tau, i+1)(L_i(\tau) - S_i(\tau))U_{i-1}(\tau, 1)\varphi, \qquad (3)$$

where $T_k(\tau, i) = S_k(\tau)\cdots S_i(\tau)$, $k \ge i$, $T_k(\tau, k+1) = I$, $U_0(\tau, 1) = I$.

From the conditions 5),6) with the help of 2), we obtain:

$$\|A_j(t)A^{-1}(t)\| \le c, \quad \|A_j^2(t)A^{-2}(t)\| \le c, \qquad (4)$$

where $t \in [0, t^*]$.

According to the conditions 3) and 7) and the inequality (4), there takes place the following estimates for any s and t from the segment $[0, t^*]$:

$$\|A^q(s)A^{-q}(t)\| \le c,\ q = 1, 2. \qquad (5)$$

Evidently, from the inequality (4) we can obtain the inequalities as follows:

$$\|A_j(t)A(t)\varphi\| \le c\|A^2(t)\varphi\|, \quad \|A_j^2(t)\varphi\| \le c\|A^2(t)\|, \quad \varphi \in D(A^2), \qquad (6)$$

Remark. The operator $T_k(\tau, 1)$ appears in the parallel type composition schemes with differences (see [3]).

We can discuss the nearness of the operators $L_k(\tau)$ and $S_k(\tau)$ according to the following formula:

$$L_k(\tau) - S_k(\tau) = \tau^2 \sum_{j=1}^{m} S_{j,k}(\tau)(\eta_j^{-1} A_j^2(t_k) - A_j(t_k)A(t_k))L_k(\tau), \qquad (7)$$

where $S_{j,k}(\tau) = (I + \tau\eta_j^{-1} A_j(t_k))^{-1}$.

According to the formula (7), inequalities (6) and conditions 2) and 4), there hold the inequalities for the items, which stand on the right side of the formula (3):

$$\|T_k(\tau, i+1)(L_i(\tau) - S_i(\tau))U_{i-1}(\tau, 1)\varphi\| \leq$$

$$\leq c\tau^2 \|A^2(t_i)U_{i-1}(\tau, 1)A^{-2}(0)\| \|A^2(0)\varphi\|, \ i = 1, \cdots, k.$$

The following estimate takes place (we shall prove it later):

$$\|A^2(s)U_k(\tau, l)A^{-2}(t)\| \leq c, \qquad (8)$$

where $s, t \in [0, t^*]$, $k, l = 0, 1, \cdots, n$, $U_k(\tau, l) = L_k(\tau) \cdots L_l(\tau), \quad k \geq l \geq 0$.

So

$$\|T_k(\tau, i+1)(L_i(\tau) - S_i(\tau))U_{i-1}(\tau, 1)\varphi\| \leq c\tau^2 \|A^2(0)\varphi\|, \ i = 1, \cdots, n-1. \ (9)$$

It is obvious, that from the equation (3) taking into account the estimate (9), we get the desired inequality.

Now, let us prove the estimate (8).

There holds the formula:

$$A^2(s)U_k(\tau, l)A^{-2}(t) = A^2(s)L_k^{k-l+1}(\tau)A^{-2}(t) +$$

$$\tau\Big(A^2(s)L_k^{k-l+1}(\tau)(A(t_k) - A(t_l))A^{-2}(s)\Big(A^2(s)U_l(\tau, l)A^{-2}(t)\Big) +$$

$$+A^2(s)L_k^{k-l}(\tau)(A(t_k) - A(t_{l+1}))A^{-2}(s)\Big(A^2(s)U_{l+1}(\tau, l)A^{-2}(t)\Big) +$$

$$+ \cdots + A^2(s)L_k(\tau)(A(t_k) - A(t_{k-1}))A^{-2}(s)\Big(A^2(s)U_{k-1}(\tau, l)A^{-2}(t)\Big)\Big). \quad (10)$$

Using the Danford-Taylor integral, the following estimate can be easily proved:

$$\|A(s)(I + \tau A(s))^{-n}\| \leq \frac{c}{n\tau}, \qquad (11)$$

where $\tau > 0$, n is a natural number, $s \in [0, t^*]$.

Taking into account the conditions 3), 7), the equations (4) and (11) and the equation

$$A(t_k)(A(t_k) - A(t_l))A^{-2}(t) = (A^2(t_k) - A^2(t_l))A^{-2}(t) +$$

$$+(A(t_l) - A(t_k))A^{-1}(t_l)(A^2(t_l)A^{-2}(t),$$

we obtain:

$$\|A^2(s)L_k^{k-l}(\tau)(A(t_k) - A(t_{l+1}))A^{-2}(s)\| \leq c.$$

Obviously from (10) we get:

$$\mu_k \leq c\tau(\mu_{k-1} + \cdots + \mu_l), \;\; k = l+1, \cdots, n, \tag{12}$$

where $\mu_k = \|A^2(s)U_k(\tau, l)A^{-2}(s)\|$. As $\mu_s \leq c$, we get from (12) by the induction that

$$\mu_k \leq c(1 + c\tau)^{k-l},$$

what we wanted to prove.

From the estimate (2) taking into account, that

$$\lim_{n\to\infty} U_n(\frac{t^*}{n}, 1)\varphi = u(t^*) = U(t^*, 0)\varphi$$

we get

$$\lim_{n\to\infty} T_n(\frac{t^*}{n}, 1)\varphi = U(t^*, 0)\varphi.$$

This is the formula which is the analogue of Trotter type formula in case of variable operator.

Analogously the following formula:

$$\lim_{n\to\infty} T_n(\frac{t-s}{n}, 1)\varphi = U(t, s)\varphi, \quad t \geq s \geq 0,$$

can be obtained where the operator T_n is defined by the formula, which is obtained after changing $A(t_k)$ by $A(s + \frac{k}{n}(t-s))$.

In case of constant operator, the problem of estimation of the approximation error for the resolvent operator (semigroup) of the problem (1) with the help of an operator T_n is studied in [4].

References

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