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On Mazurkiewicz Sets from the Measure-Theoretical Point of View

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Mazurkiewicz subsets of the Euclidean plane $\mathbf{R}^2$ are considered from the view-point of their potential measurability with respect to the class of all nonzero σ-finite translation invariant measures on $\mathbf{R}^2$.

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This paper may be treated as a continuation of [4], where some measurability properties of well-known Mazurkiewicz type subsets of the Euclidean plane $\mathbf{R}^2$ are considered. So, we follow the notation and terminology adopted in [4].

The symbol $\mathcal{M}(\mathbf{R}^2)$ stands for the family of all those nonzero σ-finite measures on $\mathbf{R}^2$ which are translation invariant (i.e., $\mathbf{R}^2$-invariant).

A set $X \subset \mathbf{R}^2$ is called negligible with respect to $\mathcal{M}(\mathbf{R}^2)$ (briefly, $\mathbf{R}^2$-negligible) if these two conditions are satisfied for X:

(*) there exists a measure $\nu \in \mathcal{M}(\mathbf{R}^2)$ such that $X \in \mathrm{dom}(\nu)$;

(**) for any measure $\mu \in \mathcal{M}(\mathbf{R}^2)$, the relation $X \in \mathrm{dom}(\mu)$ implies the equality $\mu(X) = 0$.

A set $Y \subset \mathbf{R}^2$ is called absolutely negligible with respect to $\mathcal{M}(\mathbf{R}^2)$ (briefly, $\mathbf{R}^2$-absolutely negligible) if, for every measure $\mu \in \mathcal{M}(\mathbf{R}^2)$, there exists a measure $\mu' \in \mathcal{M}(\mathbf{R}^2)$ such that the relations

$$\mu' \text{ extends } \mu, \quad Y \in \mathrm{dom}(\mu'), \quad \mu'(Y) = 0$$

hold true.

Let us remark that any $\mathbf{R}^2$-absolutely negligible set is also $\mathbf{R}^2$-negligible, but the converse assertion fails to be valid.

In what follows, the symbol ω stands for the least infinite ordinal (cardinal) number and the symbol $\mathbf{c}$ stands for the cardinality continuum.

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A set $T \subset \mathbf{R}^2$ is called almost $\mathbf{R}^2$-invariant if $\mathrm{card}(T) = \mathbf{c}$ and

$$\mathrm{card}((h + T) \triangle T) < \mathbf{c}$$

for each vector $h \in \mathbf{R}^2$ (here $\triangle$ denotes, as usual, the operation of symmetric difference of two sets).

The symbol λ_2 will be used below for denoting the classical two-dimensional Lebesgue measure on the plane $\mathbf{R}^2$.

A set $W \subset \mathbf{R}^2$ is called λ_2-thick (or λ_2-massive) in $\mathbf{R}^2$ if $B \cap W \neq \emptyset$ for every Borel set $B \subset \mathbf{R}^2$ with $\lambda_2(B) > 0$.

Let e be an arbitrary nonzero vector in $\mathbf{R}^2$.

According to the standard terminology (see, e.g., [5]), a set $A \subset \mathbf{R}^2$ is uniform in direction e if $\mathrm{card}(l \cap A) \leq 1$ for any straight line $l \subset \mathbf{R}^2$ parallel to e.

A set $A \subset \mathbf{R}^2$ is called finite in direction e (cf. [7]) if $\mathrm{card}(l \cap A) < \omega$ for every straight line $l \subset \mathbf{R}^2$ parallel to e.

A set $A \subset \mathbf{R}^2$ is called countable in direction e if $\mathrm{card}(l \cap A) \leq \omega$ for every straight line $l \subset \mathbf{R}^2$ parallel to e.

Obviously, A is uniform (finite, countable) in direction e if and only if A is uniform (finite, countable) in direction $-e$.

Recall also that $S \subset \mathbf{R}^2$ is a Mazurkiewicz set if $\mathrm{card}(l \cap S) = 2$ for each straight line l lying in $\mathbf{R}^2$.

Such a set S was first constructed by Mazurkiewicz in his remarkable paper [6]. The above definition immediately implies that, for any nonzero vector $e \in \mathbf{R}^2$, the set S is finite in direction e.

As mentioned in [4], every Mazurkiewicz set turns out to be $\mathbf{R}^2$-negligible. Moreover, there exists a measure ν on $\mathbf{R}^2$ which extends the Lebesgue measure λ_2, is invariant under the group of all isometries of $\mathbf{R}^2$, and contains in its domain all Mazurkiewicz subsets of $\mathbf{R}^2$.

Also, there exist Mazurkiewicz sets which are $\mathbf{R}^2$-absolutely negligible. The latter fact readily follows from the statement that there is a Mazurkiewicz set Z in $\mathbf{R}^2$ which simultaneously is a Hamel basis of $\mathbf{R}^2$. The transfinite construction of Z modifies, in certain respects, the usual construction of Mazurkiewicz type sets in $\mathbf{R}^2$ and is fairly standard. However, we would like to give here a detailed proof of the existence of Z.

Lemma 1: *There exists a Mazurkiewicz set $Z \subset \mathbf{R}^2$ which is a Hamel basis of $\mathbf{R}^2$.*

Proof: For our further purposes, it is convenient to introduce the following notation:

α = the least ordinal number of cardinality $\mathbf{c}$;

$\{l_\xi : \xi < \alpha\}$ = an α-sequence consisting of all straight lines in $\mathbf{R}^2$;

$\{z_\xi : \xi < \alpha\}$ = an α-sequence consisting of all points of $\mathbf{R}^2$.

We may assume, without loss of generality, that the partial family

$$\{l_\xi : \xi < \alpha \ \& \ \xi \text{ is an odd ordinal}\}$$

contains all straight lines of $\mathbf{R}^2$, and that the partial family

$$\{z_\xi : \xi < \alpha \ \& \ \xi \text{ is an even ordinal}\}$$

contains all points of $\mathbf{R}^2$.

In addition to the said earlier, we will use the symbol $l(y,z)$ for denoting the straight line in $\mathbf{R}^2$ which passes through two distinct points $y \in \mathbf{R}^2$ and $z \in \mathbf{R}^2$.

We are going to construct by the method of transfinite recursion an increasing (by inclusion) α-sequence $\{Z_\xi : \xi < \alpha\}$ of subsets of $\mathbf{R}^2$, for which the following conditions would be satisfied:

(a) $\mathrm{card}(Z_\xi) \leq \mathrm{card}(\xi) + \omega$ for any ordinal $\xi < \alpha$;

(b) every set Z_ξ is linearly independent over the field $\mathbf{Q}$ of all rational numbers;

(c) every Z_ξ is a set of points in general position in $\mathbf{R}^2$;

(d) if an ordinal $\xi < \alpha$ is odd, then

$$\mathrm{card}(Z_\xi \cap l_\xi) = 2;$$

(e) if an ordinal $\xi < \alpha$ is even, then the point z_ξ belongs to the linear hull (over $\mathbf{Q}$) of the set Z_ξ.

Suppose that, for an ordinal $\xi < \alpha$, the partial family $\{Z_\zeta : \zeta < \xi\}$ has already been constructed fulfilling the above conditions (a), (b), (c), (d), and (e). Let us put

$$Z(\xi) = \cup\{Z_\zeta : \zeta < \xi\}.$$

Observe that the set $Z(\xi)$ is linearly independent over $\mathbf{Q}$ and, at the same time, is a set of points in general position in $\mathbf{R}^2$. Also, it is clear that

$$\mathrm{card}(Z(\xi)) \leq \mathrm{card}(\xi) + \omega.$$

Now, consider the two possible cases.

1. The ordinal ξ is odd.

In this case, we take the straight line l_ξ and claim that there exists a subset T of l_ξ such that:

(i) $Z(\xi) \cup T$ is linearly independent over $\mathbf{Q}$ and, simultaneously, is a set of points in general position in $\mathbf{R}^2$;

(ii) $\mathrm{card}((Z(\xi) \cup T) \cap l_\xi) = 2$.

Indeed, the validity of our assertion follows from the relation

$$\mathrm{card}(Z(\xi)) \leq \mathrm{card}(\xi) + \omega < \mathbf{c}$$

and from the fact that the line l_ξ contains continuum many linearly independent points over $\mathbf{Q}$.

So, we may define

$$Z_\xi = Z(\xi) \cup T.$$

Notice that, by virtue of $\mathrm{card}(T) \leq 2$, we also have

$$\mathrm{card}(Z_\xi) \leq \mathrm{card}(\xi) + \omega.$$

2. The ordinal ξ is even.

In this case, we take the point z_ξ. If z_ξ belongs to the linear hull (over $\mathbf{Q}$) of the set

$$Z(\xi) = \cup\{Z_\zeta : \zeta < \xi\},$$

then we define $Z_\xi = Z(\xi)$.

It remains to consider the situation when z_ξ is linearly independent (again, over $\mathbf{Q}$) of the set $Z(\xi)$.

In such a situation, we introduce the following notation:

$$U_\xi \ = \ \text{the vector space over } \mathbf{Q} \text{ generated by Z}(\xi).$$

Evidently, we may write

$$\mathrm{card}(U_\xi) \leq \mathrm{card}(\xi) + \omega < \mathbf{c}.$$

Further, we define the three sets:

$$K_1 = \cup\{l(z, z') : z \in Z(\xi),\ z' \in Z(\xi),\ z \neq z'\},$$
$$K_2 = z_\xi + K_1,$$
$$K_3 = \cup\{l_\xi(z) : z \in Z(\xi)\},$$

where, for each point $z \in Z(\xi)$, the symbol $l_\xi(z)$ denotes the straight line in $\mathbf{R}^2$ passing through z and parallel to the nonzero vector z_ξ.

According to the above definitions, the set

$$K = K_1 \cup K_2 \cup K_3$$

is a union of straight lines in $\mathbf{R}^2$, the total number of which is strictly less than $\mathbf{c}$. This circumstance implies at once that

$$\mathrm{card}(\mathbf{R}^2 \setminus K) = \mathbf{c}$$

and, consequently,

$$\mathrm{card}(\mathbf{R}^2 \setminus (K \cup U_\xi \cup (U_\xi + z_\xi))) = \mathbf{c}.$$

Also, it can readily be seen that if z is an arbitrary point from $\mathbf{R}^2 \setminus K$, then

$$\{z\} \cup \{z - z_\xi\} \cup Z(\xi)$$

turns out to be a set of points in general position in $\mathbf{R}^2$.

Our goal now is to choose a point $z' \in \mathbf{R}^2 \setminus K$ so that the set

$$\{z'\} \cup \{z' - z_\xi\} \cup Z(\xi)$$

would be linearly independent over $\mathbf{Q}$. For this purpose, it suffices to find a point $z' \in \mathbf{R}^2 \setminus K$ having the property that, for any two rational numbers p and r, the relation

$$pz' + r(z' - z_\xi) \in U_\xi$$

necessarily implies the equalities $p = r = 0$. To see the existence of such a z', let us take an injective family of points

$$\{z_i : i \in I\} \subset \mathbf{R}^2 \setminus (K \cup U_\xi \cup (U_\xi + z_\xi)),$$

where $\mathrm{card}(I) = \mathbf{c}$ and $z_i - z_j \notin U_\xi$ for any two distinct indices i and j from the set I.

Supposing, contrary to our assertion, the non-existence of a desired z', we get

$$p_i z_i + r_i(z_i - z_\xi) \in U_\xi \qquad (i \in I),$$

where rational numbers p_i and r_i are such that $|p_i| + |r_i| > 0$ for each index $i \in I$. Since the set $\mathbf{Q} \times \mathbf{Q}$ is countable and the set I is uncountable, there are two distinct indices $i \in I$ and $j \in I$ and two rational numbers p and r satisfying the relations

$$|p| + |r| > 0,$$

$$pz_i + r(z_i - z_\xi) \in U_\xi, \quad pz_j + r(z_j - z_\xi) \in U_\xi.$$

The last two relations lead us to

$$(p + r)(z_i - z_j) \in U_\xi,$$

and we obtain a contradiction, because $p + r \neq 0$ and U_ξ is a vector space over $\mathbf{Q}$. The obtained contradiction shows that we may put

$$Z_\xi = \{z', z' - z_\xi\} \cup Z(\xi)$$

for an appropriate point z' from the family $\{z_i : i \in I\}$. Since we trivially have

$$z_\xi = z' - (z' - z_\xi),$$

the point z_ξ belongs to the linear hull (over $\mathbf{Q}$) of the set Z_ξ.

Proceeding in this manner, we will come to the family $\{Z_\xi : \xi < \alpha\}$ of subsets of $\mathbf{R}^2$, for which all conditions (a), (b), (c), (d), and (e) hold true. Now, we define

$$Z = \cup\{Z_\xi : \xi < \alpha\}.$$

Conditions (b) and (c) give us that the set Z is linearly independent over $\mathbf{Q}$ and, simultaneously, is a set of points in general position in $\mathbf{R}^2$. By virtue of condition (d), the same Z is a Mazurkiewicz subset of $\mathbf{R}^2$. Finally, in view of condition (e), the set Z is a Hamel basis of $\mathbf{R}^2$. Lemma 1 has thus been proved. □

Further, we need the following auxiliary proposition.

Lemma 2: *Let E be an uncountable vector space over the field $\mathbf{Q}$ of all rational numbers and let H be a Hamel basis of E. Then H is an E-absolutely negligible subset of E.*

Actually, the argument presented in [3] yields also a proof of Lemma 2. Notice that a more general result can be stated. For any natural number n, denote by H_n the set of all those vectors in E whose representation via the Hamel basis H contains at most n nonzero rational coefficients. Then each set H_n $(n < \omega)$ turns out to be E-absolutely negligible in E.

Lemmas 1 and 2 imply the following statement.

Theorem 3: *There exists a Mazurkiewicz subset X of $\mathbf{R}^2$ which is absolutely negligible with respect to the class $\mathcal{M}(\mathbf{R}^2)$. Therefore, for an arbitrary measure $\mu \in \mathcal{M}(\mathbf{R}^2)$, there exists a measure $\mu' \in \mathcal{M}(\mathbf{R}^2)$ extending μ and such that $X \in \mathrm{dom}(\mu')$ and $\mu'(X) = 0$.*

It was proved in [4] that, under the Continuum Hypothesis (**CH**), there are Mazurkiewicz sets which are not $\mathbf{R}^2$-absolutely negligible. Here we wish to establish the same result without using any additional set-theoretical assumptions. The method applied below is primarily taken from the paper [2]. As demonstrated in [2], there exists a set of points in general position in $\mathbf{R}^2$ which is not $\mathbf{R}^2$-absolutely negligible. However, it should be emphasized that not every set of points in general position in $\mathbf{R}^2$ is contained in an appropriate Mazurkiewicz set. There are known rather simple examples of plane sets of points in general position, which do not admit an expansion to a Mazurkiewicz set (see, for instance, [1]). For this reason, the method of [2] needs certain modifications.

We begin with the following easy auxiliary proposition.

Lemma 4: *Let e be a nonzero vector in $\mathbf{R}^2$ and let Z be a subset of $\mathbf{R}^2$ countable in direction e. Then there exist a set $Z_0 \subset \mathbf{R}^2$ and a countable family $\{h_n : n < \omega\} \subset \mathbf{R}^2$ such that:*

(1) Z_0 is uniform in direction e;

(2) $Z \subset \cup\{h_n + Z_0 : n < \omega\}$.

Proof: We may assume, without loss of generality, that the vector e is parallel to the axis $\{0\} \times \mathbf{R}$. Since Z is countable in direction e, it suffices to show that, for any function

$$\phi : \mathbf{R} \to \mathbf{R}$$

and for any disjoint countable family $\{[a_n, b_n[\ : \ n < \omega\}$ of half-open subintervals of $\mathbf{R}$ such that

$$\sum\{b_n - a_n : n < \omega\} = +\infty,$$

there exist a family $\{h_n : n < \omega\} \subset \mathbf{R}^2$ and a partial function

$$\psi : \mathbf{R} \to \mathbf{R}$$

having the property that:

(a) $\mathrm{dom}(\psi) = \cup\{[a_n, b_n[\ :\ n < \omega\}$;

(b) $\mathrm{Gr}(\phi) \subset \cup\{\mathrm{Gr}(\psi)+h_n : n < \omega\}$, where $\mathrm{Gr}(\phi)$ and $\mathrm{Gr}(\psi)$ denote, respectively, the graph of ϕ and the graph of ψ.

Now, it is not difficult to see that the existence of the required $\{h_n : n < \omega\}$ and ψ is guaranteed by the assumption $\sum\{b_n - a_n : n < \omega\} = +\infty$. □

Remark 1: Another proof of Lemma 4 is presented in [2]. Notice, by the way, that all vectors h_n $(n < \omega)$ can be taken to be parallel to the axis $\mathbf{R} \times \{0\}$.

Lemma 5: *Let G, A, and B satisfy the following conditions:*

(1) G is a subgroup of the additive group $(\mathbf{R}^2, +)$ and $\mathrm{card}(G) < \mathbf{c}$;

(2) A is a subset of $\mathbf{R}^2$ and $\mathrm{card}(A) < \mathbf{c}$;

(3) B is a λ_2-measurable subset of $\mathbf{R}^2$ with $\lambda_2(B) > 0$.

Then there exists a point $z \in B$ such that:

(i) $(G + z) \cap A = \emptyset$;

(ii) for any two distinct points $a \in A$ and $a' \in A$, the line $l(a, a')$ does not intersect the orbit $G + z$;

(iii) for any two distinct points $x \in G + z$ and $y \in G + z$, the line $l(x, y)$ does not intersect the set A.

Proof: Let us denote

$$A_1 = \cup\{l(x, y) : x \in G + A,\ y \in G + A,\ x \neq y\},$$
$$A_2 = \cup\{l_x(y) : x \in G,\ x \neq 0,\ y \in A\},$$

where the symbol $l_x(y)$ stands for the straight line passing through a point y and parallel to a nonzero vector x. Further, consider the set

$$A_3 = (G + A) \cup (G + A_1) \cup (G + A_2).$$

Obviously, this set is contained in the union of some family of lines in $\mathbf{R}^2$ whose cardinality is strictly less than $\mathbf{c}$. Since $\lambda_2(B) > 0$, we must have $B \setminus A_3 \neq \emptyset$. Pick a point $z \in B \setminus A_3$. It is not hard to check that the relations (i), (ii), and (iii) of Lemma 5 are fulfilled for $G + z$. □

Lemma 6: *Let Γ be any countably infinite non-collinear subgroup of $\mathbf{R}^2$. There exists a Mazurkiewicz set Z such that $\Gamma + Z$ has the following property: for each countable family $\{h_m : m < \omega\} \subset \mathbf{R}^2$, the set*

$$\cap\{h_m + \Gamma + Z : m < \omega\}$$

is λ_2-thick in $\mathbf{R}^2$ and the equality

$$\mathrm{card}(\cap\{h_m + \Gamma + Z : m < \omega\}) = \mathbf{c}$$

holds true.

Proof: Denote again by α the least ordinal number of cardinality $\mathbf{c}$ and let $\{G_\xi : \xi < \alpha\}$ be an α-sequence of all those countable subgroups of $\mathbf{R}^2$ which contain Γ

and satisfy the relation

$$\mathrm{card}(G_\xi/\Gamma) = \omega.$$

For every group G from the above α-sequence, define the set

$$\Xi(G) = \{\xi < \alpha : G_\xi = G\}.$$

We may assume, without loss of generality, that

$$\mathrm{card}(\Xi(G)) = \mathbf{c}.$$

Further, take a family $\{B_\xi : \xi < \alpha\}$ of Borel subsets of $\mathbf{R}^2$ such that, for any ordinal $\xi < \alpha$, the partial family $\{B_\zeta : \zeta \in \Xi(G_\xi)\}$ consists of all Borel subsets of $\mathbf{R}^2$ having strictly positive λ_2-measure. The existence of $\{B_\xi : \xi < \alpha\}$ is evident.

Finally, let $\{l_\xi : \xi < \alpha\}$ be an α-sequence of all straight lines lying in $\mathbf{R}^2$.

By using the method of transfinite recursion, let us construct a double family

$$\{z_{k,\xi} : k < \omega,\ \xi < \alpha\}$$

of points in $\mathbf{R}^2$ and a family $\{T_\xi : \xi < \alpha\}$ of subsets of $\mathbf{R}^2$ satisfying the following conditions:

(1) $(\{z_{k,\xi} : k < \omega,\ \xi < \alpha\}) \cup (\cup\{T_\xi : \xi < \alpha\})$ is a set of points in general position in $\mathbf{R}^2$;

(2) $\mathrm{card}(T_\xi) \leq 2$ for each $\xi < \alpha$;

(3) if $\xi < \alpha$, then $z_{0,\xi} \in B_\xi$;

(4) for each $\xi < \alpha$, the set $\Gamma + \{z_{k,\xi} : k < \omega\}$ coincides with $G_\xi + z_{0,\xi}$;

(5) $\mathrm{card}\Big(l_\xi \cap \big((\cup\{T_\zeta : \zeta \leq \xi\}) \cup \{z_{k,\zeta} : k < \omega,\ \zeta \leq \xi\}\big)\Big) = 2$ for each $\xi < \alpha$.

Suppose that, for an ordinal number $\xi < \alpha$, the partial families

$$\{z_{k,\zeta} : k < \omega,\ \zeta < \xi\}, \quad \{T_\zeta : \zeta < \xi\}$$

have already been constructed with the properties corresponding to (1)-(5), and put

$$G = G_\xi, \quad B = B_\xi,$$
$$A = (\{z_{k,\zeta} : k < \omega,\ \zeta < \xi\}) \cup (\cup\{T_\zeta : \zeta < \xi\}).$$

Observe that A is a set of points in general position in $\mathbf{R}^2$. Also,

$$\mathrm{card}(A) < \mathbf{c}, \quad \mathrm{card}(G) = \omega < \mathbf{c}, \quad \lambda_2(B) > 0.$$

So, Lemma 5 is applicable to G, A, and B. Let $z \in B$ be a point as in Lemma 5 and let us consider the G-orbit $G + z$ of this point. Since we have the relations

$$G = G_\xi, \quad \mathrm{card}(G_\xi/\Gamma) = \omega,$$

the set $G + z$ is the union of some pairwise disjoint Γ-orbits $\{Z_k : k < \omega\}$. Clearly, an enumeration of these Γ-orbits can be chosen so that $z \in Z_0$. Now, we put

$$z_{0,\xi} = z$$

and define a sequence of points $\{z_{k,\xi} : k < \omega\}$ in $\mathbf{R}^2$ by ordinary induction. Assume that the finite collection of points

$$z_{0,\xi} \in Z_0, \quad z_{1,\xi} \in Z_1, \ \ldots, \quad z_{k,\xi} \in Z_k$$

has already been determined and consider the Γ-orbit Z_{k+1}. Keeping in mind the fact that Γ is not a collinear subgroup of $\mathbf{R}^2$, we deduce that the set Z_{k+1} cannot be covered by finitely many straight lines in $\mathbf{R}^2$. Therefore, there exists a point $t \in Z_{k+1}$ which does not belong to any line passing through two distinct points from the finite set $\{z_{0,\xi}, z_{1,\xi}, ..., z_{k,\xi}\}$. We then put

$$z_{k+1,\xi} = t.$$

Proceeding in this manner, we finally come to the desired sequence of points $\{z_{k,\xi} : k < \omega\}$.

According to the above construction, the set

$$A^* = (\{z_{k,\zeta} : k < \omega, \ \zeta \leq \xi\}) \cup (\cup\{T_\zeta : \zeta < \xi\})$$

is again in general position in $\mathbf{R}^2$. Consider now the straight line l_ξ and the set $A^* \cap l_\xi$. Obviously, we have the inequalities

$$\mathrm{card}(A^*) < \mathbf{c}, \quad \mathrm{card}(A^* \cap l_\xi) \leq 2.$$

It is not hard to see that there exists a set $T \subset \mathbf{R}^2$ which satisfies the following relations:

(a) $\mathrm{card}(T) \leq 2$;

(b) $T \cup A^*$ is a set of points in general position;

(c) $\mathrm{card}((T \cup A^*) \cap l_\xi) = 2$.

So, putting $T_\xi = T$, we obtain the family $\{T_\zeta : \zeta \leq \xi\}$.

The transfinite process just described and continued up to the ordinal α yields the two families

$$\{z_{k,\xi} : k < \omega, \ \xi < \alpha\}, \quad \{T_\xi : \xi < \alpha\}.$$

A straightforward verification then shows that all conditions (1)-(5) are fulfilled for these families. In particular, conditions (1) and (5) imply at once that

$$Z = \{z_{k,\xi} : k < \omega, \ \xi < \alpha\} \cup (\cup\{T_\xi : \xi < \alpha\})$$

is a Mazurkiewicz subset of $\mathbf{R}^2$. In addition to this, (3) and (4) imply that, for any countable family $\{h_m : m < \omega\} \subset \mathbf{R}^2$, the set

$$\cap\{h_m + \Gamma + Z : m < \omega\}$$

is λ_2-thick in $\mathbf{R}^2$ and satisfies the equality

$$\mathrm{card}(\cap\{h_m + \Gamma + Z : m < \omega\}) = \mathbf{c}.$$

Lemma 6 has thus been proved. □

Theorem 7: *Let Z and Γ be as in Lemma 6. There exists a measure ν on $\mathbf{R}^2$ such that:*

(1) ν is an extension of λ_2;

(2) ν is translation invariant, i.e., $\mathbf{R}^2$-invariant;

(3) $(\Gamma + Z) \in \mathrm{dom}(\nu)$ and $\nu(\mathbf{R}^2 \setminus (\Gamma + Z)) = 0$.

In particular, the Mazurkiewicz set Z is not $\mathbf{R}^2$-absolutely negligible in $\mathbf{R}^2$.

Proof: Denote by $\mathcal{I}$ the $\mathbf{R}^2$-invariant σ-ideal of subsets of $\mathbf{R}^2$ generated by the one-element family $\{\mathbf{R}^2 \setminus (\Gamma + Z)\}$. By virtue of Lemma 6, the inner λ_2-measure of every element of $\mathcal{I}$ is equal to zero. So, we may apply to $\mathcal{I}$ and λ_2 Marczewski's classical method of extending invariant measures (see [8] or [9]). This method gives us the measure ν satisfying (1), (2), and (3) of the theorem. Relation (3) trivially implies that Z is not $\mathbf{R}^2$-absolutely negligible. □

Remark 2: Let e be any nonzero vector in $\mathbf{R}^2$. It is easy to see that the set $\Gamma + Z$ is countable in direction e. By Lemma 4, there exist a set $Z_0 \subset \mathbf{R}^2$ and a countable family $\{h_n : n < \omega\} \subset \mathbf{R}^2$ such that:

(a) Z_0 is uniform in direction e;

(b) $\Gamma + Z \subset \cup\{h_n + Z_0 : n < \omega\}$.

Consequently, Z_0 is $\mathbf{R}^2$-negligible but is not $\mathbf{R}^2$-absolutely negligible.

Remark 3: Assuming the Continuum Hypothesis, it was proved in [4] that, for any countably infinite non-collinear group $\Gamma \subset \mathbf{R}^2$, there exists a Mazurkiewicz set $Y \subset \mathbf{R}^2$ such that the set $\Gamma + Y$ contains some λ_2-thick almost $\mathbf{R}^2$-invariant subset of cardinality $\mathbf{c}$. This result substantially strengthens Lemma 6, but is heavily based on **CH**. It is unknown whether the same result can be established without using additional set-theoretical assumptions.

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