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A Theorem of Sanderson on Link Bordisms in Dimension 4

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Abstract

The groups of link bordism can be identified with homotopy groups via the Pontryagin–Thom construction. B.J. Sanderson computed the bordism group of 3 component surface–links using the Hilton–Milnor Theorem, and later gave a geometric interpretation of the groups in terms of intersections of Seifert hypersurfaces and their framings. In this paper, we geometrically represent every element of the bordism group uniquely by a certain standard form of a surface–link, a generalization of a Hopf link. The standard forms give rise to an inverse of Sanderson's geometrically defined invariant.

AMS Classification 57Q45

Keywords Surface Links, Link Bordism Groups, Triple Linking, Hopf 2-Links

1 Introduction

By an *n-component surface–link* we mean a disjoint union $F = K_1 \cup \ldots \cup K_n$ of closed oriented surfaces $K_1, \ldots, K_n$ embedded in $\mathbf{R}^4$ locally flatly. Each K_i is called a *component*, which may or may not be connected, and possibly empty. Two n-component surface–links $F = K_1 \cup \ldots \cup K_n$ and $F' = K'_1 \cup \ldots \cup K'_n$ are *bordant* if there is a compact oriented 3-manifold W properly embedded in $\mathbf{R}^4 \times [0,1]$ such that W has n components $W_1, \ldots, W_n$ with $\partial W_i = K_i \times \{0\} \cup (-K'_i) \times \{1\}$. Let $L_{4,n}$ be the abelian group of link bordism classes of n-component surface–links. The sum $[F] + [F']$ of $[F]$ and $[F']$ is defined by the class of the split union of F and F'. The identity is represented by a trivial n-component 2-link. The inverse of $[F]$ is represented by the mirror image of F with the opposite orientation.

Brian Sanderson [17] identified the bordism group $L_{4,n}$ of (oriented) n-component surface–links with the homotopy group $\pi_4(\vee_{i=1}^{n-1} S_i^2)$ using a generalized Pontryagin–Thom construction, and computed the group via the Hilton–Milnor Theorem. Using the transversality theory developed in [2], Sanderson [18] also gave an explicit interpretation of the invariants using Seifert hypersurfaces.

Specifically, let $F = K_1 \cup \cdots K_n$ be an n-component surface–link. It is known (cf. [19]) that each K_i bounds a Seifert hypersurface M_i, i.e., an oriented compact 3-manifold with $\partial M_i = K_i$. The *double linking number*, denoted by $\mathrm{Dlk}(K_i, K_j)$, between two components K_i and K_j is the framed intersection $M_i \cdot K_j \in \pi_4(S^3) = \mathbf{Z}_2$. The *triple linking number*, denoted by $\mathrm{Tlk}(K_i, K_j, K_k)$, among three components K_i, K_j, and K_k, is the framed intersection $M_i \cdot K_j \cdot M_k \in \pi_4(S^4) = \mathbf{Z}$. Then Sanderson's geometrically defined invariant

$$H\colon L_{4,n} \to A = (\underbrace{\mathbf{Z} \oplus \ldots \oplus \mathbf{Z}}_{\frac{n(n-1)(n-2)}{3}}) \oplus (\underbrace{\mathbf{Z}_2 \oplus \ldots \oplus \mathbf{Z}_2}_{\frac{n(n-1)}{2}})$$

is given by

$$H([F]) = ((\mathrm{Tlk}(K_i, K_j, K_k), \mathrm{Tlk}(K_j, K_k, K_i))_{1 \le i < j < k \le n}, (\mathrm{Dlk}(K_i, K_j))_{1 \le i < j \le n}),$$

and it was shown [18] that this gives an isomorphism.

The purpose of this paper is to give an inverse map

$$G\colon A = (\underbrace{\mathbf{Z} \oplus \ldots \oplus \mathbf{Z}}_{\frac{n(n-1)(n-2)}{3}}) \oplus (\underbrace{\mathbf{Z}_2 \oplus \ldots \oplus \mathbf{Z}_2}_{\frac{n(n-1)}{2}}) \to L_{4,n}$$

of H, by giving an explicit set of geometric representatives for a given value of Sanderson's invariant. The representatives are generalized Hopf links, called *Hopf 2-links* (without or with *beads*), which are defined in Sections 2 and 3.

More specifically, we identify $(\mathbf{Z} \oplus \ldots \oplus \mathbf{Z}) \oplus (\mathbf{Z}_2 \oplus \ldots \oplus \mathbf{Z}_2)$ with the abelian group which is abstractly generated by a certain family $\mathcal{F}$ of Hopf 2-links without or with beads and the homomorphism G maps each generator to its bordism class. We prove that this homomorphism G is surjective. (It is clear that $H \circ G = \pm 1$; just use the obvious Hopf solid link and/or normal 3-balls as Seifert hypersurfaces when you evaluate H.) Then G is an isomorphism which is an inverse of Sanderson's homomorphism H (up to sign). The surjectivity is a consequence of the following theorem.

Theorem 1.1 *Any n-component surface–link F is bordant to a disjoint union of Hopf 2-links without or with beads. More precisely, $[F] \equiv 0$ modulo $\langle \mathcal{F} \rangle$, where $\langle \mathcal{F} \rangle$ is the subgroup of $L_{4,n}$ generated by the classes of elements of $\mathcal{F}$.*

Our proof constructs bordisms to unions of Hopf 2-links geometrically, and thus gives a self-contained geometric proof of the Sanderson's classification theorem, without using the Hilton–Milnor theorem.

The paper is organized as follows. Section 2 contains the definition of a Hopf 2-link and its characterization in the bordism group. Hopf 2-links with beads and their roles in the bordism group are given in Section 3, which also contains the definition of the family $\mathcal{F}$ and a proof of the Theorem 1.1. We describe alternate definitions of Dlk and Tlk in Section 4.

2 Hopf 2-Links

A *Hopf disk pair* is a pair of disks D_1 and D_2 in a 3-ball B^3 such that there is a homeomorphism from B^3 to a 3-ball

$$\{(x, y, z) \mid x^2 + y^2 + z^2 \leq 9\}$$

which maps D_1 to

$$\{(x, y, z) \mid x^2 + y^2 \leq 1, z = 0\}$$

and D_2 to

$$\{(x, y, z) \mid (y-1)^2 + z^2 \leq 1, x = 0\}$$

homeomorphically. The boundary of such a disk pair is a Hopf link in B^3.

A pair of solid tori V_1, V_2 in $\mathbf{R}^4$ is called a *Hopf solid link* if there is an embedding $f\colon B^3 \times S^1 \to \mathbf{R}^4$ such that $f(D_i \times S^1) = V_i$ for $i = 1, 2$. The boundary of a Hopf solid link is called a *Hopf 2-link*, which is a pair of embedded tori in $\mathbf{R}^4$. A simple loop $f((\text{a point of } B^3) \times S^1)$ is called a *core loop* of the Hopf solid link and the Hopf 2-link. Any simple loop α in $\mathbf{R}^4$ is ambient isotopic to a standard circle in $\mathbf{R}^3 \subset \mathbf{R}^4$. Since there are only two equivalence classes of framings of α (or trivialization of $N(\alpha) \cong B^3 \times S^1$), any Hopf 2-link is deformed by an ambient isotopy of $\mathbf{R}^4$ so that the projection is one of the illustrations depicted in Figure 1. If it is deformed into the illustration on the left side, it is called a *standard Hopf 2-link*; and if it is deformed into the illustration on the right side, it is called a *twisted Hopf 2-link*.

We assume that a Hopf disk pair is oriented so that the boundary is a positive Hopf link. If the core loop is oriented, a Hopf solid link and a Hopf 2-link are assumed to be oriented by use of the orientation of the Hopf disk pair and the orientation of the core loop. (In this situation, we say that the Hopf 2-link is oriented coherently with respect to the orientation of the core loop.)

Remark 2.1 Let $F = T_1 \cup T_2$ be a Hopf 2-link whose core loop is α. Then F is ambient isotopic to $-F = -T_1 \cup -T_2$ in $N(\alpha)$, where $-F$ means F with the opposite orientation. Let $F' = -T_1 \cup T_2$ and $-F' = T_1 \cup -T_2$. Then F' and $-F'$ are ambient isotopic in $N(\alpha)$. When α is oriented, one of F and F' is oriented coherently with respect to α, and the other is oriented coherently with respect to $-\alpha$. Since α and $-\alpha$ are ambient isotopic in $\mathbf{R}^4$, F, $-F$, F' and $-F'$ are ambient isotopic in $\mathbf{R}^4$.

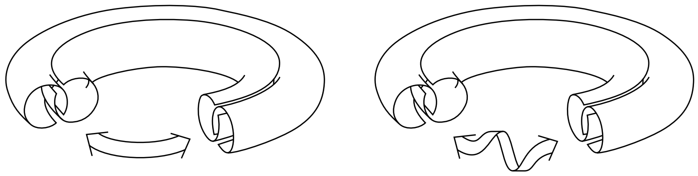

Figure 1

Lemma 2.2 *For a Hopf 2-link $F = T_1 \cup T_2$, the following conditions are mutually equivalent.*

(1) *F is standard.*

(2) $\mathrm{Dlk}(T_1, T_2) = 0$.

(3) *F is null-bordant.*

Proof Using a Hopf solid link, we see that for the left side of Figure 1, $\mathrm{Dlk}(T_1, T_2) = 0$ and for the right, $\mathrm{Dlk}(T_1, T_2) = 1$. (This is also seen by Remark 4.1.) Thus (1) and (2) are equivalent. Suppose (1). Attach 2-handles to T_1 and T_2. Then T_1 and T_2 change to 2-spheres which split by isotopy, and hence F is null-bordant. Thus (1) $\Rightarrow$ (3). It is obvious that (3) $\Rightarrow$ (2). $\square$

Let α and α' be mutually disjoint oriented simple loops in $\mathbf{R}^4$, and let $F = T_1 \cup T_2$ and $F' = T_1' \cup T_2'$ be Hopf 2-links whose core loops are α and α' such that F and F' are oriented coherently with respect to the orientations of α and α'. Let α'' be an oriented loop obtained from $\alpha \cup \alpha'$ by surgery along a band B attached to $\alpha \cup \alpha'$. Let E be a 4-manifold in $\mathbf{R}^4$ whose interior contains α, α' and B.

Lemma 2.3 *In the above situation, there is a Hopf 2-link $F'' = T_1'' \cup T_2''$ whose core loop is α'' such that F'' is bordant in E to the 2-component surface–link $(T_1 \cup T_1') \cup (T_2 \cup T_2')$.*

Proof Consider a tri-punctured sphere Σ embedded in $E \times [0,1]$ whose boundary is $\partial\Sigma = \partial_0\Sigma \cup (-\partial_1\Sigma)$ with $\partial_0\Sigma = (\alpha \cup \alpha') \times \{0\}$ and $\partial_1\Sigma = \alpha'' \times \{1\}$. There exists an identification of a regular neighborhood $N(\Sigma)$ in $E \times [0,1]$ with $B^3 \times \Sigma$ such that the Hopf 2-links F and F' in $E \times \{0\}$ correspond to $(\partial D_1 \cup \partial D_2) \times \partial_0\Sigma$, where $D_1 \cup D_2 \subset B^3$ is an oriented Hopf disk pair. The desired F'' is obtained as $(\partial D_1 \cup \partial D_2) \times \partial_1\Sigma$. □

The transformation described in Lemma 2.3 is called *fusion* between two Hopf 2-links. The inverse operation of fusion is called *fission* of a Hopf 2-link.

Lemma 2.4 *Let $F = T_1 \cup T_2$ be a twisted Hopf 2-link. The order of $[F] \in L_{4,2}$ is two.*

Proof It is a consequence of Lemmas 2.2 and 2.3. □

3 Hopf 2-Links with Beads and Proof of Theorem 1.1

Let α be a simple loop in $\mathbf{R}^4$ and let $N(\alpha) \cong B^3 \times \alpha$ be a regular neighborhood. We call a 3-disk $B^3 \times \{*\}$ $(* \in \alpha)$ a *meridian 3-disk* of α, and the boundary a *meridian 2-sphere* of α. Let $D_1 \cup D_2$ be a Hopf disk pair in a 3-disk B^3. Let $f\colon B^3 \times S^1 \to \mathbf{R}^4$ be an embedding, and let $p_1, \dots, p_m$ be points of S^1. We call the image $f(\partial D_1 \times S^1) \cup f(\partial D_2 \times S^1) \cup f(\partial B^3 \times \{p_1\}) \cup \dots \cup f(\partial B^3 \times \{p_m\})$ a *Hopf 2-link with beads*. Each meridian 2-sphere $f(\partial B^3 \times \{p_i\})$ is called a *bead*. We denote by $S_{(i,j)}$ a twisted Hopf 2-link (as an n-component surface–link) at the ith and the jth component, and by $S_{(i,j,k)}$ a standard Hopf 2-link (as an n-component surface–link) at the ith and the jth components with a bead at the kth component, respectively. Let $\mathcal{F}$ denote a family of Hopf 2-links

$$\{S_{(i,j,k)} \,|\, i < j < k\} \cup \{S_{(i,k,j)} \,|\, i < j < k\} \cup \{S_{(i,j)} \,|\, i < j\},$$

and $\langle\mathcal{F}\rangle$ the subgroup of $L_{4,n}$ generated by the classes of elements of $\mathcal{F}$.

Remark 3.1 Let $F = T_1 \cup T_2 \cup S$ be a Hopf 2-link $T_1 \cup T_2$ with a bead S whose core is α. By Remark 2.1, $-T_1 \cup T_2 \cup S$ and $T_1 \cup -T_2 \cup S$ are ambient isotopic in $N(\alpha)$. By an ambient isotopy of $\mathbf{R}^4$ carrying α to $-\alpha$, $-T_1 \cup T_2 \cup S$ is carried to $T_1 \cup T_2 \cup -S$. Therefore, any Hopf 2-link with a bead obtained from F by changing orientations of some components is ambient isotopic to $F = T_1 \cup T_2 \cup S$ or $T_1 \cup T_2 \cup -S$.

We denote by $S^-_{(i,j,k)}$ the Hopf 2-link $S_{(i,j,k)}$ such that the orientation of the bead is reversed. It is clear that $[S^-_{(i,j,k)}] = -[S_{(i,j,k)}]$ in $L_{4,n}$ (for example, use Lemmas 2.2 and 2.3). Thus, in the following proof, we do not need to take care of an orientation given to $S_{(i,j,k)}$.

Proof of Theorem 1.1 (Step 1) Let M_1 be a Seifert hypersurface for K_1 which intersects $K_2, \ldots, K_n$ transversely. For each k $(k = 2, \ldots, n)$, let A_{1k} be the intersection $M_1 \cap K_k$, which is the union of oriented simple loops in K_k (or empty). Let $N(K_k) = D^2 \times K_k$ be a regular neighborhood of K_k in $\mathbf{R}^4$. For a component α of A_{1k}, let $V_1(\alpha)$ be a solid torus $D^2 \times \alpha$ $(\subset D^2 \times K_k = N(K_k))$ in $\mathbf{R}^4$ and $T_1(\alpha)$ the boundary of $V_1(\alpha)$. They are oriented by use of the orientation of a meridian disk $D^2 \times \{*\}$ of $N(K_k)$ and the orientation of α. Then $[\alpha] \in H_1(K_k)$ corresponds to $[T_1(\alpha)] \in H_2(E(K_k))$ by the isomorphism

$$H_1(K_k) \cong H^1(K_k) \cong H^1(N(K_k)) \cong H_2(E(K_k))$$

obtained by the Poincaré and Alexander dualities, where $E(K_k)$ is the exterior of K_k. Put $V_1(A_{1k}) = \bigcup_{\alpha \in A_{1k}} V_1(\alpha)$ and $T_1(A_{1k}) = \bigcup_{\alpha \in A_{1k}} T_1(\alpha)$. The surface K_1 is bordant to $\bigcup_{k=2}^n T_1(A_{1k})$ in $\mathbf{R}^4 \backslash (\bigcup_{k=2}^n K_k)$, for they cobound a 3-manifold $\mathrm{Cl}(M_1 \backslash (\bigcup_{k=2}^n V_1(A_{1k})))$, where Cl denotes the closure. Thus, without loss of generality, we may assume that K_1 is $\bigcup_{k=2}^n T_1(A_{1k})$.

Consider a Seifert hypersurface M_k for K_k. For a component $\alpha \in A_{1k}$, let $V_2(\alpha) = N_1(\alpha; M_k)$ be a regular neighborhood of α in M_k such that the union $V_1(\alpha) \cup V_2(\alpha)$ forms a Hopf solid link in $\mathbf{R}^4$ with core α, see Figure 2 (the figure shows a section transverse to α). Let $N_2(\alpha; M_k)$ be a regular neighborhood of α in M_k with $N_1(\alpha; M_k) \subset \mathrm{int} N_2(\alpha; M_k)$ and let $C(\alpha)$ be a 3-manifold $\mathrm{Cl}(N_2(\alpha; M_k) \backslash N_1(\alpha; M_k))$. By $C(\alpha)$, the 2-component surface–link $T_1(\alpha) \cup K_k$ is bordant to $T_1(\alpha) \cup (\partial V_2(\alpha) \cup \partial(\mathrm{Cl}(M_k \backslash N_2(\alpha; M_k))))$. This 2-component surface–link is ambient isotopic to $\partial V_1(\alpha)' \cup (\partial V_2(\alpha)' \cup K_k)$, where $V_1(\alpha)' \cup V_2(\alpha)'$ is a Hopf solid link obtained from the Hopf solid link $V(\alpha) \cup V_2(\alpha)$ by pushing out along $N_1(\alpha; M_k)$ using $C(\alpha)$, see Figure 2. We denote by α' a loop obtained from α by pushing off along M_k so that α' is disjoint from M_k and it is a core of the Hopf solid link $V_1(\alpha)' \cup V_2(\alpha)'$.

Put $V_1(A_{1k})' = \bigcup_{\alpha \in A_{1k}} V_1(\alpha)'$, $V_2(A_{1k})' = \bigcup_{\alpha \in A_{1k}} V_2(\alpha)'$, and $A'_{1k} = \bigcup_{\alpha \in A_{1k}} \alpha'$. The n-component surface–link $F = K_1 \cup \ldots \cup K_n$ is bordant to $F^{(1)} = K_1^{(1)} \cup \ldots \cup K_n^{(1)}$ such that

$$\begin{cases} K_1^{(1)} = \bigcup_{k=2}^n \partial V_1(A_{1k})', \\ K_j^{(1)} = \partial V_2(A_{1j})' \cup K_j \quad \text{for } j = 2, \ldots n. \end{cases}$$

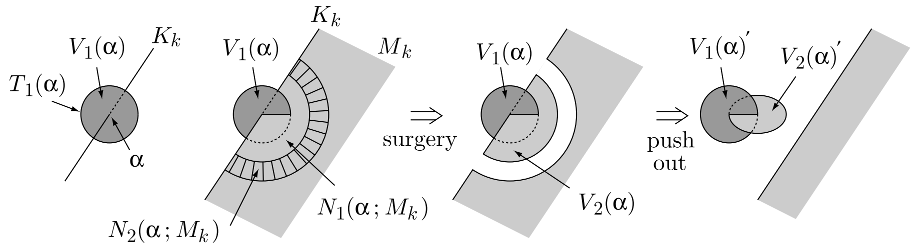

Figure 2

(Step 2) Let M_2 be the Seifert hypersurface for K_2 used in Step 1. By the construction of A'_{12}, $M_2 \cap A'_{12} = \emptyset$. For each k with $3 \leq k \leq n$, we may assume that M_2 intersects K_k and A'_{1k} transversely. Using M_2, we apply a similar argument to Step 1 to modify K_2 up to bordism. For a component α of $A_{2k} = M_2 \cap K_k$, let $V_1(\alpha)$ be a solid torus $D^2 \times \alpha$ $(\subset D^2 \times K_k = N(K_k))$ in $\mathbf{R}^4$ and $V_2(\alpha)$ a solid torus $N_1(\alpha; M_k)$. The intersection $M_2 \cap N(K_k)$ is $V_1(A_{2k}) = \bigcup_{\alpha \in A_{2k}} V_1(\alpha)$ and the intersection $M_2 \cap N(A'_{1k})$ is a union of some meridian 3-disks of A'_{1k}. Let $F^{(2)}$ be an n-component surface–link $K_1^{(2)} \cup \ldots \cup K_n^{(2)}$ such that

$$\begin{cases} K_1^{(2)} = \bigcup_{k=2}^n \partial V_1(A_{1k})', \\ K_2^{(2)} = \partial V_2(A_{12})' \cup (\bigcup_{k=3}^n \partial V_1(A_{2k})'), \\ K_j^{(2)} = \partial V_2(A_{1j})' \cup \partial V_2(A_{2j})' \cup K_j \quad \text{for } j = 3, \ldots n. \end{cases}$$

By use of a 3-manifold which is M_2 removed the above intersections, we see that $F^{(1)}$ is bordant to an n-component surface–link $F^{(2)'}$ which is the union of $F^{(2)}$ and some meridian 2-spheres of A'_{1k} with labels 2 for $k = 3, \ldots, n$. The meridian 2-spheres are the boundary of meridian 3-disks that are the intersection $M_2 \cap N(A'_{1k})$. The surface–link $F^{(2)'}$ is bordant to the union of $F^{(2)}$ and some standard Hopf 2-links $S_{(1,k,2)}$ or $S^-_{(1,k,2)}$ whose core loops are small trivial circles in $\mathbf{R}^4$. Figure 3 is a schematic picture of this process (in projection in $\mathbf{R}^3$), where an isotopic deformation and fission of a Hopf 2-link are applied. Since $[S_{(1,k,2)}]$ belongs to $\langle \mathcal{F} \rangle$, we see that $F^{(2)'}$ is bordant to $F^{(2)}$ modulo $\langle \mathcal{F} \rangle$.

(Step 3) Inductively, we see that $F^{(i-1)}$ is bordant to $F^{(i)} = K_1^{(i)} \cup \ldots \cup K_n^{(i)}$ modulo $\langle \mathcal{F} \rangle$ such that

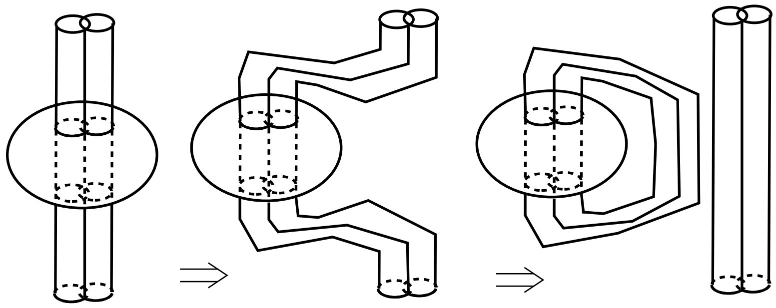

Figure 3

$$K_j^{(i)} = \begin{cases} (\bigcup_{k=1}^{j-1} \partial V_2(A_{kj})') \cup (\bigcup_{k=j+1}^{n} \partial V_1(A_{jk})') & \text{for } j \text{ with } 1 \leq j \leq i, \\ (\bigcup_{k=1}^{i} \partial V_2(A_{kj})') \cup K_j & \text{for } j \text{ with } i < j \leq n. \end{cases}$$

Specifically, we see that $F^{(i-1)}$ is bordant to an n-component surface–link which is a union of $F^{(i)}$ and some meridian 2-spheres of $A'_{k_1 k_2}$ such that $k_1 < i$ and $k_2 \neq i$. These 2-spheres are the boundary of the meridian 3-disks that are the intersection $M_i \cap N(A'_{k_1 k_2})$. Since $[S_{k_1,i,k_2}]$ belongs to $\langle \mathcal{F} \rangle$, $F^{(i-1)}$ is bordant to $F^{(i)}$ modulo $\langle \mathcal{F} \rangle$.

Thus F is bordant to $F^{(n)} = K_1^{(n)} \cup \ldots \cup K_n^{(n)}$ modulo $\langle \mathcal{F} \rangle$ such that

$$K_j^{(n)} = (\bigcup_{k=1}^{j-1} \partial V_2(A_{kj})') \cup (\bigcup_{k=j+1}^{n} \partial V_1(A_{jk})') \quad \text{for } j = 1, \ldots, n.$$

It is a union of Hopf 2-links and the link bordism class is in $\langle \mathcal{F} \rangle$. □

Recall that Sanderson's homomorphism

$$H\colon L_{4,n} \to A = (\underbrace{\mathbf{Z} \oplus \ldots \oplus \mathbf{Z}}_{\frac{n(n-1)(n-2)}{3}}) \oplus (\underbrace{\mathbf{Z}_2 \oplus \ldots \oplus \mathbf{Z}_2}_{\frac{n(n-1)}{2}})$$

is defined by

$$H([F]) = ((\mathrm{Tlk}(K_i, K_j, K_k), \mathrm{Tlk}(K_j, K_k, K_i))_{1 \leq i < j < k \leq n}, (\mathrm{Dlk}(K_i, K_j))_{1 \leq i < j \leq n})$$

for an n-component surface–link $F = K_1 \cup \ldots \cup K_n$. Let $\{ e_{ijk}, e'_{ijk} \mid i < j < k \} \cup \{ f_{ij} \mid i < j \}$ be a basis of A such that $e_{ijk} = (0, \ldots, 1, \ldots, 0)$ where 1 corresponds to $\mathrm{Tlk}(K_i, K_j, K_k)$, $e'_{ijk} = (0, \ldots, 1, \ldots, 0)$ where 1 corresponds to $\mathrm{Tlk}(K_j, K_k, K_i)$, and $f_{ij} = (0, \ldots, 1, \ldots, 0)$ where 1 corresponds to $\mathrm{Dlk}(K_i, K_j)$. Give an orientation to each $S_{(i,j,k)}$ such that

$$\begin{cases} \mathrm{Tlk}(S_i, S_j, S_k) = +1, & \mathrm{Tlk}(S_k, S_j, S_i) = -1, \\ \mathrm{Tlk}(S_j, S_k, S_i) = 0, & \mathrm{Tlk}(S_i, S_k, S_j) = 0, \\ \mathrm{Tlk}(S_k, S_i, S_j) = -1, & \mathrm{Tlk}(S_j, S_i, S_k) = +1. \end{cases}$$

We consider a homomorphism $G\colon A \to L_{4,n}$ with $G(e_{ijk}) = [S_{(i,j,k)}]$, $G(e'_{ijk}) = -[S_{(i,k,j)}]$, and $G(f_{ij}) = [S_{(i,j)}]$, which is well-defined by Lemma 2.4. Then we have $H \circ G = \mathrm{id}$. Theorem 1.1 says that G is surjective.

Remark 3.2 In general, for any n-component surface–link F, it is known (cf. [3]) that for any distinct i, j, k,

$$\mathrm{Dlk}(K_i, K_j) = \mathrm{Dlk}(K_j, K_i),$$
$$\mathrm{Tlk}(K_i, K_j, K_k) = -\mathrm{Tlk}(K_k, K_j, K_i),$$
$$\mathrm{Tlk}(K_i, K_j, K_k) + \mathrm{Tlk}(K_j, K_k, K_i) + \mathrm{Tlk}(K_k, K_i, K_j) = 0.$$

From these formulas, it follows immediately from the condition $H([F]) = H([F'])$ that $\mathrm{Dlk}(K_i, K_j) = \mathrm{Dlk}(K'_i, K'_j)$ and $\mathrm{Tlk}(K_i, K_j, K_j) = \mathrm{Tlk}(K'_i, K'_j, K'_k)$ for any distinct i, j, k.

4 Remarks on Linking Numbers

In this section we comment on different definitions and aspects of the generalized linking numbers, Dlk and Tlk. To mention an analogy to projectional definition of the classical linking number (cf. [14]), we start with a review of projections of surface–links.

Let $F = K_1 \cup \cdots \cup K_n$ be an n-component surface– and let $F^* = p(F)$ be a surface diagram of F with respect to a projection $p\colon \mathbf{R}^4 \to \mathbf{R}^3$. For details of the definition of a surface diagram, see [4, 5, 15] for example. The singularity set of F^* consists of double points and isolated branch/triple points. The singularity set is a union of immersed circles and arcs in $\mathbf{R}^3$, which is called the set of *double curves.* Two sheets intersect along a double curve, which are called *upper* and *lower* with respect to the projection direction. A double curve is *of type* (i, j) if the upper sheet comes from K_i and the lower comes from K_j. We denote by D_{ij} the union of double curves of type (i, j). For distinct i and j, D_{ij} is the union of immersed circles. (If D_{ij} contains an immersed arc, its end-points are branch points. So the upper sheet with label i and the lower sheet with label j along the arc come from the same component of

F. This contradicts $i \neq j$.) Let D_{ij}^{+} be the union of immersed circles in $\mathbf{R}^3$ obtained from D_{ij} by shifting it in a diagonal direction that is in the positive normal direction of the upper sheet and also in the positive normal direction of the lower sheet of F^* along D_{ij} so that D_{ij} and D_{ij}^{+} are disjoint. Let $\widetilde{D}_{ij}$ be a link (i.e., embedded circles) in $\mathbf{R}^3$ which is obtained from D_{ij} by a slight perturbation by a homotopy, and let $\widetilde{D}_{ij}^{+}$ be a link in $\mathbf{R}^3$ which is obtained from D_{ij}^{+} similarly. Give $\widetilde{D}_{ij}$ an orientation and $\widetilde{D}_{ij}^{+}$ the orientation which is parallel to that of $\widetilde{D}_{ij}$. The linking number between $\widetilde{D}_{ij}$ and $\widetilde{D}_{ij}^{+}$ does not depend on the perturbations and the orientation of $\widetilde{D}_{ij}$, which we call the *linking number* between D_{ij} and D_{ij}^{+}. Then we have

Remark 4.1 The double linking number $\mathrm{Dlk}(K_i, K_j)$ is equal to a value in $\mathbf{Z}_2 = \{0, 1\}$ that is the linking number between D_{ij} and D_{ij}^{+} modulo 2.

At a triple point in the projection, three sheets intersect that have distinct relative heights with respect to the projection direction, and we call them *top*, *middle*, and *bottom* sheets, accordingly. If the orientation normals to the top, middle, bottom sheets at a triple point τ matches with this order the fixed orientation of $\mathbf{R}^3$, then the sign of τ is positive and $\varepsilon(\tau) = 1$. Otherwise the sign is negative and $\varepsilon(\tau) = -1$. (See [3, 5].) A triple point is *of type* (i, j, k) if the top sheet comes from K_i, the middle comes from K_j, and the bottom comes from K_k. The following projectional interpretation of triple linking numbers was extensively used in [3] for invariants defined from quandles.

Remark 4.2 The triple linking number $\mathrm{Tlk}(K_i, K_j, K_k)$ is (up to sign) the sum of the signs of all the triple points of type (i, j, k).

Let $f\colon F_1 \cup F_2 \cup F_3 \to \mathbf{R}^4$ denote an embedding of the disjoint union of oriented surfaces F_i representing $F = K_1 \cup K_2 \cup K_3$. Define a map $L\colon F_1 \times F_2 \times F_3 \to S^3 \times S^3$ by

$$L(x_1, x_2, x_3) = \left(\frac{f(x_1) - f(x_2)}{||f(x_1) - f(x_2)||}, \frac{f(x_2) - f(x_3)}{||f(x_2) - f(x_3)||} \right)$$

for $x_1 \in F_1$, $x_2 \in F_2$ and $x_3 \in F_3$. In [12] it is observed that the degree of L is (up to sign) the triple linking number $\mathrm{Tlk}(K_1, K_2, K_3)$.

For further related topics, refer to [1, 6, 7, 8, 9, 10, 11, 13, 16].

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