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Homotopy classes that are trivial mod $\mathcal{F}$

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Abstract If $\mathcal{F}$ is a collection of topological spaces, then a homotopy class α in $[X, Y]$ is called $\mathcal{F}$-trivial if

$$\alpha_* = 0 : [A, X] \longrightarrow [A, Y]$$

for all $A \in \mathcal{F}$. In this paper we study the collection $Z_{\mathcal{F}}(X, Y)$ of all $\mathcal{F}$-trivial homotopy classes in $[X, Y]$ when $\mathcal{F} = \mathcal{S}$, the collection of spheres, $\mathcal{F} = \mathcal{M}$, the collection of Moore spaces, and $F = \Sigma$, the collection of suspensions. Clearly

$$Z_{\Sigma}(X, Y) \subseteq Z_{\mathcal{M}}(X, Y) \subseteq Z_{\mathcal{S}}(X, Y),$$

and we find examples of *finite complexes* X and Y for which these inclusions are strict. We are also interested in $Z_{\mathcal{F}}(X) = Z_{\mathcal{F}}(X, X)$, which under composition has the structure of a semigroup with zero. We show that if X is a finite dimensional complex and $\mathcal{F} = \mathcal{S}$, $\mathcal{M}$ or Σ, then the semigroup $Z_{\mathcal{F}}(X)$ is nilpotent. More precisely, the nilpotency of $Z_{\mathcal{F}}(X)$ is bounded above by the $\mathcal{F}$-killing length of X, a new numerical invariant which equals the number of steps it takes to make X contractible by successively attaching cones on wedges of spaces in $\mathcal{F}$, and this in turn is bounded above by the $\mathcal{F}$-cone length of X. We then calculate or estimate the nilpotency of $Z_{\mathcal{F}}(X)$ when $\mathcal{F} = \mathcal{S}$, $\mathcal{M}$ or Σ for the following classes of spaces: (1) projective spaces (2) certain Lie groups such as $SU(n)$ and $Sp(n)$. The paper concludes with several open problems.

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1 Introduction

A map $f : X \longrightarrow Y$ is said to be detected by a collection $\mathcal{F}$ of topological spaces if there is a space $A \in \mathcal{F}$ such that the induced map $f_* : [A, X] \longrightarrow [A, Y]$ of homotopy sets is nontrivial. It is a standard technique in homotopy theory to use certain simple collections $\mathcal{F}$ to detect essential homotopy classes. In studying the entire homotopy set $[X,Y]$ using this approach, one is led naturally to consider the set of homotopy classes which are *not* detected by $\mathcal{F}$, called the $\mathcal{F}$-trivial homotopy classes. For example, if $\mathcal{S}$ is the collection of spheres, then $f : X \longrightarrow Y$ is detected by $\mathcal{S}$ precisely when some induced homomorphism of homotopy groups $\pi_k(f) : \pi_k(X) \longrightarrow \pi_k(Y)$ is nonzero. The $\mathcal{S}$-trivial homotopy classes are those that induce zero on all homotopy groups. It is also important to determine induced maps on homotopy sets. For this, one needs to understand composition of $\mathcal{F}$-trivial homotopy classes. With this in mind, we study two basic questions in this paper for a fixed collection $\mathcal{F}$: (1) What is the set of all $\mathcal{F}$-trivial homotopy classes in $[X,Y]$? and (2) In the special case $X = Y$, how do $\mathcal{F}$-trivial homotopy classes behave under composition? We are particularly interested in the collections $\mathcal{S}$ of spheres, $\mathcal{M}$ of Moore spaces and Σ of suspensions.

Some of these ideas have appeared earlier. The paper [2] considers the special case $\mathcal{F} = \mathcal{S}$. Furthermore, Christensen has studied similar questions in the stable category [5].

We next briefly summarize the contents of this paper. We write $Z_{\mathcal{F}}(X,Y)$ for the $\mathcal{F}$-trivial homotopy classes in $[X,Y]$ and set $Z_{\mathcal{F}}(X) = Z_{\mathcal{F}}(X,X)$. After some generalities on $Z_{\mathcal{F}}(X,Y)$, we observe in Section 2 that $Z_{\mathcal{F}}(X)$ is a semigroup under composition. Its nilpotency, denoted $t_{\mathcal{F}}(X)$, is a new numerical invariant of homotopy type. For the collection of suspensions, we prove that $t_\Sigma(X) \leq \lceil \log_2(\mathrm{cat}(X)) \rceil$. In Section 3 we relate $t_{\mathcal{F}}(X)$ to other numerical invariants for arbitrary collections $\mathcal{F}$. The $\mathcal{F}$-killing length of X, denoted $\mathrm{kl}_{\mathcal{F}}(X)$ (resp., the $\mathcal{F}$-cone length of X, denoted $\mathrm{cl}_{\mathcal{F}}(X)$), is the least number of steps needed to go from X to a contractible space (resp., from a contractible space to X) by successively attaching cones on wedges of spaces in $\mathcal{F}$. We prove that $t_{\mathcal{F}}(X) \leq \mathrm{kl}_{\mathcal{F}}(X)$, and, if $\mathcal{F}$ is closed under suspension, that $\mathrm{kl}_{\mathcal{F}}(X) \leq \mathrm{cl}_{\mathcal{F}}(X)$. We also show that $\mathrm{kl}_{\mathcal{F}}(X)$ behaves subadditively with respect to cofibrations. It is clear that for any X and Y, $Z_\Sigma(X,Y) \subseteq Z_{\mathcal{M}}(X,Y) \subseteq Z_{\mathcal{S}}(X,Y)$, and we ask in Section 4 if these containments can be strict. It is easy to find infinite complexes with strict containment. However, in Section 4 we solve the more difficult problem of finding finite complexes with this property. From this, we deduce that containment can be strict for finite complexes when $X = Y$. The next two sections are devoted to determining $Z_{\mathcal{F}}(X)$ and $t_{\mathcal{F}}(X)$ for certain classes of spaces. In Section 5 we calculate $Z_{\mathcal{F}}(X)$ and $t_{\mathcal{F}}(X)$ for $\mathcal{F} = \mathcal{S}, \mathcal{M}$ and Σ when X is any real or complex projective space, or is the quaternionic projective space $\mathbf{HP}^n$ with $n \leq 4$. In Section 6 we consider $t_\Sigma(Y)$ for certain Lie groups Y. We show that $2 \leq t_\Sigma(Y)$ when $Y = SU(n)$ or $Sp(n)$ by proving that the groups $[Y,Y]$ are not abelian. In addition, we compute $t_\Sigma(SO(n))$ for $n = 3$ and 4. The paper concludes in Section 7 with a list of open problems.

For the remainder of this section, we give our notation and terminology. All topological spaces are based and connected, and have the based homotopy type of CW complexes. All maps and homotopies preserve base points. We do not usually distinguish notationally between a map and its homotopy class. We let $*$ denote the base point of a space or a space consisting of a single point. In addition to standard notation, we use $\equiv$ for same homotopy type, $0 \in [X, Y]$ for the constant homotopy class and $\mathrm{id} \in [X, X]$ for the identity homotopy class.

For an abelian group G and an integer $n \geq 2$, we let $M(G, n)$ denote the Moore space of type (G, n), that is, the space with a single non-vanishing reduced homology group G in dimension n. If G is finitely generated, we also define $M(G, 1)$ as a wedge of circles S^1 and spaces obtained by attaching a 2-cell to S^1 by a map of degree m. The n^{th} homotopy group of X with coefficients in G is $\pi_n(X; G) = [M(G, n), X]$. A map $f : X \longrightarrow Y$ induces a homomorphism $\pi_n(f; G) : \pi_n(X; G) \longrightarrow \pi_n(Y; G)$, and $\pi_*(f; G)$ denotes the set of all such homomorphisms. If $G = \mathbf{Z}$, we write $\pi_n(X)$ and $\pi_n(f)$ for the n^{th} homotopy group and induced map, respectively.

We use unreduced Lusternik-Schnirelmann category of a space X, denoted $\mathrm{cat}(X)$. Thus $\mathrm{cat}(X) \leq 2$ if and only if X is a co-H-space. By an H-space, we mean a space with a homotopy-associative multiplication and homotopy inverse, i.e., a group-like space.

For a positive integer n, the cyclic group of order n is denoted $\mathbf{Z}/n$. If X is a space or an abelian group, we use the notation $X_{(p)}$ for the localization of X at the prime p [13]. We let $\lambda : X \longrightarrow X_{(p)}$ denote the natural map from X to its localization.

A semigroup is a set S with an associative binary operation, denoted by juxtaposition. We call S a pointed semigroup if there is an element $0 \in S$ such that $x0 = 0x = 0$ for each $x \in S$. A pointed semigroup is nilpotent if there is an integer n such that the product $x_1 \cdots x_n$ is 0 whenever $x_1, \ldots, x_n \in S$. The least such integer n is the nilpotency of S. If S is not nilpotent, then we say its nilpotency is ∞. Finally, if x is a real number, then $\lceil x \rceil$ denotes the least integer $n \geq x$.

2 $\mathcal{F}$-trivial homotopy classes

Let $\mathcal{F}$ be any collection of spaces.

Definition 2.1 A homotopy class $f : X \longrightarrow Y$ is $\mathcal{F}$*-trivial* if the induced map $f_* : [A, X] \longrightarrow [A, Y]$ is trivial for each $A \in \mathcal{F}$. We denote by $Z_{\mathcal{F}}(X, Y)$ the subset of $[X, Y]$ consisting of all $\mathcal{F}$-trivial homotopy classes. We denote $Z_{\mathcal{F}}(X, X)$ by $Z_{\mathcal{F}}(X)$.

We study $Z_{\mathcal{F}}(X, Y)$ and $Z_{\mathcal{F}}(X)$ for certain collections $\mathcal{F}$. The following are some interesting examples.

Examples

(a) $\mathcal{S} = \{S^n \mid n \geq 1\}$, the collection of spheres. In this case $f \in Z_{\mathcal{S}}(X, Y)$ if and only if $\pi_*(f) = 0$.

(b) $\mathcal{M} = \{M(\mathbf{Z}/m, n) \mid m \geq 0, n \geq 1\}$, the collection of Moore spaces $M(\mathbf{Z}/m, n)$. Here $f \in Z_{\mathcal{M}}(X, Y)$ if and only if $\pi_*(f; G) = 0$ for any finitely generated abelian group G.

(c) $\Sigma = \{\Sigma A\}$, the collection of all suspensions. In this case $f \in Z_{\Sigma}(X, Y)$ if and only if $f_* = 0 : [\Sigma A, X] \longrightarrow [\Sigma A, Y]$ for every space A.

(d) $\mathcal{P}$ is the collection of all finite dimensional complexes. Then $f \in Z_{\mathcal{P}}(X, Y)$ if and only if $f : X \longrightarrow Y$ is a phantom map [19].

In this paper our main interest is in the collections $\mathcal{S}$, $\mathcal{M}$ and Σ. In Section 7 we will mention a few other collections. However, we begin with some simple, general facts about arbitrary collections.

Lemma 2.2

(a) *If* $\mathcal{F} \subseteq \mathcal{F}'$, *then* $Z_{\mathcal{F}'}(X, Y) \subseteq Z_{\mathcal{F}}(X, Y)$ *for any* X *and* Y.

(b) *If* $X_\alpha \in \mathcal{F}$ *for each* α *in some index set, then* $Z_{\mathcal{F}}(\bigvee X_\alpha, Y) = 0$ *for every* Y.

(c) *For any* X, $Z_{\mathcal{F}}(X)$ *is a pointed semigroup under the binary operation of composition of homotopy classes, and with zero the constant homotopy class.*

Any map $f : X \longrightarrow Y$ gives rise to functions $\widetilde{f} : X \times Y \longrightarrow X \times Y$ and $\overline{f} : X \vee Y \longrightarrow X \vee Y$ defined by the diagrams

$$\begin{tikzcd} X \times Y \arrow[r, "\widetilde{f}"] \arrow[d, "p_1"'] & X \times Y \\ X \arrow[r, "f"] & Y \arrow[u, "i_2"'] \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} X \vee Y \arrow[r, "\overline{f}"] \arrow[d, "q_1"'] & X \vee Y \\ X \arrow[r, "f"] & Y. \arrow[u, "j_2"'] \end{tikzcd}$$

Clearly, if $f \in Z_{\mathcal{F}}(X, Y)$, then $\widetilde{f} \in Z_{\mathcal{F}}(X \times Y)$ and $\overline{f} \in Z_{\mathcal{F}}(X \vee Y)$.

The following lemma, whose proof is obvious, will be used frequently.

Lemma 2.3 *The functions*

$$\theta : Z_{\mathcal{F}}(X,Y) \longrightarrow Z_{\mathcal{F}}(X \times Y) \quad \text{and} \quad \phi : Z_{\mathcal{F}}(X,Y) \longrightarrow Z_{\mathcal{F}}(X \vee Y),$$

defined by $\theta(f) = \widetilde{f}$ and $\phi(f) = \overline{f}$, are injective. Thus, $Z_{\mathcal{F}}(X,Y) \neq 0$ implies that $Z_{\mathcal{F}}(X \times Y) \neq 0$ and $Z_{\mathcal{F}}(X \vee Y) \neq 0$

We conclude this section with some basic results about $Z_\Sigma(X,Y)$ and $Z_\Sigma(X)$. Recall that a map $f : X \longrightarrow Y$ has *essential category weight* at least n, written $E(f) \geq n$, if for every space A with $\mathrm{cat}(A) \leq n$, we have $f_* = 0 : [A,X] \longrightarrow [A,Y]$ [30, 25].

Lemma 2.4 *For any two spaces X and Y,*

$$\begin{aligned} Z_\Sigma(X,Y) &= \{f \mid f \in [X,Y],\, E(f) \geq 2\} \\ &= \{f \mid f \in [X,Y],\, \Omega f = 0\}. \end{aligned}$$

Proof Let $f \in Z_\Sigma(X,Y)$. If $\mathrm{cat}(A) \leq 2$ then the canonical map ν: $\Sigma\Omega A \longrightarrow A$ has a section s. Thus the diagram

$$\begin{tikzcd} {[A,X]} \arrow[r, "f_*"] \arrow[d, "\nu^*"'] & {[A,Y]} \arrow[d, "\nu^*"] \\ {[\Sigma\Omega A, X]} \arrow[r, "f_{**}"] & {[\Sigma\Omega A, Y]} \end{tikzcd}$$

commutes, and so $f_* = s^* f_{**} \nu^* = 0$. Since the reverse implication is trivial, this establishes the first equality.

Now assume that $f_* = 0 : [\Sigma B, X] \longrightarrow [\Sigma B, Y]$ for every space B. Taking $B = \Omega X$, we find that $f \circ \nu = 0 : \Sigma\Omega X \longrightarrow Y$. Since this map is adjoint to Ωf, we conclude that $\Omega f = 0$. Conversely, if $\Omega f = 0$, then $\Omega f_* = 0 : [B, \Omega X] \longrightarrow [B, \Omega Y]$, which means that $f_* = 0 : [\Sigma B, X] \longrightarrow [\Sigma B, Y]$. This completes the proof. □

Remarks

(a) Since $\mathrm{cat}(A) \leq 2$ if and only if A is a co-H-space, a map $f : X \longrightarrow Y$ has $E(f) \geq 2$ if and only if $f_* = 0 : [A,X] \longrightarrow [A,Y]$ for every co-H-space A.

(b) By Lemma 2.4, we can regard the set $Z_\Sigma(X,Y)$ as the kernel of the looping function $\Omega : [X,Y] \longrightarrow [\Omega X, \Omega Y]$. We see from (a) that $\ker \Omega = 0$ if X is a co-H-space. The function Ω has been extensively studied in special cases, e.g., when Y is an Eilenberg-MacLane space, then Ω is just the cohomology suspension [32, Chap. VII].

Proposition 2.5 *Let X be a space of finite category, and let $n \geq \log_2(\mathrm{cat}(X))$. If $f_1, \ldots, f_n \in Z_\Sigma(X)$, then $f_1 \circ \cdots \circ f_n = 0$. Thus the nilpotency of the semigroup $Z_\Sigma(X)$ is at most $\lceil \log_2(\mathrm{cat}(X)) \rceil$, the least integer greater than or equal to $\log_2(\mathrm{cat}(X))$.*

Proof Since $f_i \in Z_\Sigma(X)$, Lemma 2.4 shows that $E(f_i) \geq 2$. By the product formula for essential category weight [30, Thm. 9], $E(f_1 \circ \cdots \circ f_n) \geq E(f_1) \cdots E(f_n) \geq 2^n \geq \mathrm{cat}(X)$. From the definition of essential category weight, $f_1 \circ \cdots \circ f_n = 0$. □

Remark We shall see later that the semigroup $Z_{\mathcal{S}}(X)$ is nilpotent if X is a finite dimensional complex. It follows that this is true for $Z_{\mathcal{M}}(X)$ and $Z_\Sigma(X)$ (Remark (b) following Theorem 3.3).

Definition 2.6 For any collection $\mathcal{F}$ of spaces and any space X, we define $t_{\mathcal{F}}(X)$, the *nilpotency of X mod $\mathcal{F}$* as follows: If X is contractible, set $t_{\mathcal{F}}(X) = 0$; Otherwise, $t_{\mathcal{F}}(X)$ is the nilpotency of the semigroup $Z_{\mathcal{F}}(X)$.

Thus $t_{\mathcal{F}}(X) = 1$ if and only if X is not contractible and $Z_{\mathcal{F}}(X) = 0$.

The set $Z_{\mathcal{S}}(X)$ and the integer $t_{\mathcal{S}}(X)$ were considered in [2], where they were written $Z_\infty(X)$ and $t_\infty(X)$. Since $\mathcal{S} \subseteq \mathcal{M} \subseteq \Sigma$, we have

$$0 \leq t_\Sigma(X) \leq t_{\mathcal{M}}(X) \leq t_{\mathcal{S}}(X) \leq \infty$$

for any space X.

Since $\mathrm{cat}(\Sigma A_1 \times \cdots \times \Sigma A_r) \leq r+1$ [16, Prop. 2.3], we have the following result.

Corollary 2.7 *For any r spaces $A_1, \ldots, A_r$,*

$$t_\Sigma(\Sigma A_1 \times \cdots \times \Sigma A_r) \leq \lceil \log_2(r+1) \rceil.$$

This paper is devoted to a study of the sets $Z_{\mathcal{F}}(X,Y)$, with emphasis on the nilpotency of spaces mod $\mathcal{F}$ for $\mathcal{F} = \mathcal{S}, \mathcal{M}$ and Σ.

3 $\mathcal{F}$-killing length and $\mathcal{F}$-cone length

Proposition 2.5 shows that $\lceil \log_2(\mathrm{cat}(X)) \rceil$ is an upper bound for $t_\Sigma(X)$. In this section, we obtain upper bounds on $t_{\mathcal{F}}(X)$ for arbitrary collections $\mathcal{F}$. We begin with the main definitions of this section.

Definition 3.1 Let $\mathcal{F}$ be a collection of spaces and X a space. Suppose there is a sequence of cofibrations

$$L_i \longrightarrow X_i \longrightarrow X_{i+1}$$

for $0 \leq i < m$ such that each L_i is a wedge of spaces which belong to $\mathcal{F}$. If $X_0 \equiv X$ and $X_m \equiv *$, then this is called an *$\mathcal{F}$-killing length decomposition* of X with length m. If $X_0 \equiv *$ and $X_m \equiv X$, then this is an *$\mathcal{F}$-cone length decomposition* with length m. Define the *$\mathcal{F}$-killing length* and the *$\mathcal{F}$-cone length* of X, denoted by $\mathrm{kl}_{\mathcal{F}}(X)$ and $\mathrm{cl}_{\mathcal{F}}(X)$, respectively, as follows. If $X \equiv *$, then $\mathrm{kl}_{\mathcal{F}}(X) = 0$; otherwise, $\mathrm{kl}_{\mathcal{F}}(X)$ is the smallest integer m such that there exists an $\mathcal{F}$-killing length decomposition of X with length m. The $\mathcal{F}$-cone length of X is defined analogously.

The main result of this section is that $\mathrm{kl}_{\mathcal{F}}(X)$ is an upper bound for $t_{\mathcal{F}}(X)$. We need a lemma.

Lemma 3.2 *If $X \xrightarrow{f} Y \xrightarrow{g} Z$ is a sequence of spaces and maps, then there is a cofiber sequence of mapping cones $C_f \longrightarrow C_{gf} \longrightarrow C_g$, where the maps are induced by f and g.*

The proof is elementary, and hence omitted.

Theorem 3.3 *If $\mathcal{F}$ is any collection of spaces and X is any space, then*

$$t_{\mathcal{F}}(X) \leq \mathrm{kl}_{\mathcal{F}}(X).$$

If $\mathcal{F}$ is closed under suspensions, then $\mathrm{kl}_{\mathcal{F}}(X) \leq \mathrm{cl}_{\mathcal{F}}(X)$.

Proof Assume that $\mathrm{kl}_{\mathcal{F}}(X) = m > 0$ with $\mathcal{F}$-killing length decomposition

$$L_i \xrightarrow{f_i} X_i \xrightarrow{p_i} X_{i+1}$$

for $0 \leq i < m$. Let $g_0, \ldots, g_{m-1} \in Z_{\mathcal{F}}(X)$ and consider the following diagram, with dashed arrows to be inductively defined below:

$$\begin{tikzcd}
L_0 \arrow[r, "f_0"] & X_0 \equiv X \arrow[r, "g_0"] \arrow[d, "p_0"'] & X \arrow[r, "g_1"] & X \arrow[r, "g_2"] & \cdots \arrow[r, "g_{m-2}"] & X \arrow[r, "g_{m-1}"] & X \\
L_1 \arrow[r, "f_1"] & X_1 \arrow[d, "p_1"'] \arrow[ur, dashed, "g_0'"'] & & & & & \\
L_2 \arrow[r, "f_2"] & X_2 \arrow[d, "p_2"'] \arrow[uurr, dashed, "g_1'"'] & & & & & \\
 & \vdots \arrow[d, "p_{m-2}"'] & & & & & \\
L_{m-1} \arrow[r, "f_{m-1}"] & X_{m-1} \arrow[d, "p_{m-1}"'] \arrow[uuuurrrr, dashed, "g_{m-2}'"'] & & & & & \\
 & X_m. \arrow[uuuuurrrrr, dashed, "g_{m-1}'"'] & & & & &
\end{tikzcd}$$

Since L_0 is a wedge of spaces in $\mathcal{F}$ and $g_0 \in Z_{\mathcal{F}}(X)$, we have $g_0 \circ f_0 = 0$ by Lemma 2.2(b). Thus there is a map $g_0' : X_1 \longrightarrow X$ extending g_0. The same argument inductively defines g_i' for each i, and shows $g_{m-1} \cdots g_1 g_0 = g_{m-1}' \circ (p_{m-1} \cdots p_1 p_0)$. Now $g_{m-1} \cdots g_1 g_0 = 0$ since $X_m \equiv *$. This proves the first assertion.

Next we let $m = \mathrm{cl}_{\mathcal{F}}(X)$, and show that $\mathrm{kl}_{\mathcal{F}}(X) \le m$. Let

$$L_i \xrightarrow{f_i} X_i \xrightarrow{p_i} X_{i+1}$$

for $0 \le i < m$ be an $\mathcal{F}$-cone length decomposition of X. Set

$$h_i = (p_{m-1} p_{m-2} \cdots p_{i+1}) \circ p_i : X_i \longrightarrow X_m \equiv X$$

for $i < m$ and $h_m = \mathrm{id}$. Since $h_i = h_{i+1} \circ p_i$, Lemma 3.2 yields cofiber sequences

$$C_{p_i} \longrightarrow C_{h_i} \longrightarrow C_{h_{i+1}},$$

for $0 \le i < m$. This is a killing length decomposition of X. To see this, observe that $C_{p_i} \equiv \Sigma L_i$, which is a wedge of spaces in $\mathcal{F}$ because $\mathcal{F}$ is closed under suspension. Furthermore, $h_0 : X_0 \equiv * \longrightarrow X$, so $C_{h_0} \equiv X_m \equiv X$. Finally, $C_{h_m} \equiv 0$ because $h_m = \mathrm{id} : X \longrightarrow X$. $\square$

Remarks

(a) The notion of cone length has been extensively studied. The version in Definition 3.1 is similar to the one given by Cornea in [7] (see (c) below).

It is precisely the same as the definition of $\mathcal{F}$-Cat given by Sheerer and Tanré [26]. The $\mathcal{F}$-cone length $\mathrm{cl}_{\mathcal{F}}(X)$ can be regarded as the minimum number of steps needed to build the space X up from a contractible space by attaching cones on wedges of spaces in $\mathcal{F}$. The notion of $\mathcal{F}$-killing length is new and also appears in [2] for the case $\mathcal{F} = \mathcal{S}$. It can be regarded as the minimum number of steps needed to destroy X (i.e. go from X to a contractible space) by attaching cones on wedges of spaces in $\mathcal{F}$. We note that Theorem 3.3 appears in [2, Thm. 3.4] for the case $\mathcal{F} = \mathcal{S}$. For the collection $\mathcal{S}$, it was shown in [2, Ex. 6.8] that the inequalities in Theorem 3.3 can be strict.

(b) A space need not have a finite $\mathcal{F}$-killing length or $\mathcal{F}$-cone length decomposition. For example, $\mathrm{kl}_{\Sigma}(\mathbf{CP}^{\infty}) = \infty$ because all 2^n-fold cup products vanish in a space X with $\mathrm{kl}_{\Sigma}(X) \leq n$. However, if X is a finite dimensional complex, then the process of attaching i-cells to the $(i-1)$-skeleton provides X with a $\mathcal{S}$-cone length decomposition. Thus in this case, $\mathrm{kl}_{\mathcal{S}}(X) \leq \mathrm{cl}_{\mathcal{S}}(X) \leq \dim(X)$. Since $\mathcal{S} \subseteq \mathcal{M} \subseteq \Sigma$, it follows that $\mathrm{kl}_{\Sigma}(X) \leq \mathrm{kl}_{\mathcal{M}}(X) \leq \mathrm{kl}_{\mathcal{S}}(X)$ and $\mathrm{cl}_{\Sigma}(X) \leq \mathrm{cl}_{\mathcal{M}}(X) \leq \mathrm{cl}_{\mathcal{S}}(X)$, and so $\dim(X)$ is an upper bound for all of these integers. If X is a 1-connected finite dimensional complex, then a better upper bound for $\mathrm{cl}_{\mathcal{M}}(X)$ is the number of nontrivial positive-dimensional integral homology groups of X. This can be seen by taking a homology decomposition of X [12, Chap. 8].

(c) It follows from work of Cornea [7] that the cone length of a space X, denoted $\mathrm{cl}(X)$, can be defined exactly like the Σ-cone length $\mathrm{cl}_{\Sigma}(X)$ above, except that one does not require $L_0 \in \Sigma$. It follows immediately that $\mathrm{cl}(X) \leq \mathrm{cl}_{\Sigma}(X)$.

(d) The inequality $\mathrm{kl}_{\mathcal{F}}(X) \leq \mathrm{cl}_{\mathcal{F}}(X)$ also follows from work of Sheerer and Tanré since the function $\mathrm{kl}_{\mathcal{F}}$ satisfies the axioms for $\mathcal{F}$-Cat [26, Thm. 2].

We conclude this section by giving a few properties of killing length.

Theorem 3.4 *If $\mathcal{F}$ is any collection of spaces and $X \xrightarrow{j} Y \xrightarrow{q} Z$ is a cofiber sequence, then*

$$\mathrm{kl}_{\mathcal{F}}(Y) \leq \mathrm{kl}_{\mathcal{F}}(X) + \mathrm{kl}_{\mathcal{F}}(Z).$$

Proof Write $\mathrm{kl}_{\mathcal{F}}(X) = m$ and $\mathrm{kl}_{\mathcal{F}}(Z) = n$. Let

$$L_i \xrightarrow{f_i} X_i \longrightarrow X_{i+1}$$

for $0 \leq i < m$ be a $\mathcal{F}$-killing length decomposition of X. Set $g_0 = j : X_0 \equiv X \longrightarrow Y$ and define Y_1 by the cofibration $L_0 \xrightarrow{g_0 f_0} Y \longrightarrow Y_1$. By Lemma 3.2, there is an auxilliary cofibration

$$\begin{tikzcd} C_{f_0} \arrow[r] \arrow[d, equal] & C_{g_0 f_0} \arrow[r] \arrow[d, equal] & C_{g_0} \arrow[d, equal] \\ X_1 \arrow[r, "g_1"] & Y_1 \arrow[r] & Z \end{tikzcd}$$

which defines g_1. We proceed by induction: given $g_i : X_i \longrightarrow Y_i$, let Y_{i+1} be the cofiber of the map $g_i f_i : L_0 \longrightarrow Y_i$ and use Lemma 3.2 to construct an auxilliary cofibration

$$\begin{tikzcd} C_{f_i} \arrow[r] \arrow[d, equal] & C_{g_i f_i} \arrow[r] \arrow[d, equal] & C_{g_i} \arrow[d, equal] \\ X_{i+1} \arrow[r, "g_{i+1}"] & Y_{i+1} \arrow[r] & Z \end{tikzcd}$$

which defines g_{i+1}. This defines cofiber sequences of the form $L_j \to Y_j \to Y_{j+1}$ with $0 \leq j < m$. Since $X_m \equiv *$, the $(m+1)^{st}$ cofiber sequence, $X_m \to Y_m \to Z$, shows that $Y_m \equiv Z$. Now adjoin the n cofiber sequences of a minimal $\mathcal{F}$-killing length decomposition of Z to the first m cofiber sequences to obtain an $\mathcal{F}$-killing length decomposition for Y with length $n + m$. $\square$

Finally, we obtain an upper bound for $\mathrm{kl}_\Sigma(X)$ and hence an upper bound for $t_\Sigma(X)$. This provides a useful complement to Proposition 2.5 when $\mathrm{cat}(X)$ is not known.

Proposition 3.5 *Let X be an N-dimensional complex which is $(n-1)$-connected for some $n \geq 1$. Then*

$$\mathrm{kl}_\Sigma(X) \leq \left\lceil \log_2 \left(\frac{N+1}{n} \right) \right\rceil .$$

Proof We argue by induction on $\left\lceil \log_2 \left(\frac{N+1}{n} \right) \right\rceil$. If $\left\lceil \log_2 \left(\frac{N+1}{n} \right) \right\rceil = 1$, then $N \leq 2n-1$. It is well known that this implies that X is a suspension, which means that $\mathrm{kl}_\Sigma(X) = 1$. Now suppose $\left\lceil \log_2 \left(\frac{N+1}{n} \right) \right\rceil = r$ and the result is known for all smaller values. Let X^k denote the k-skeleton of X, and consider the cofiber sequence

$$X^{2n-1} \longrightarrow X \longrightarrow X/X^{2n-1}.$$

By Theorem 3.4, $\mathrm{kl}_\Sigma(X) \leq \mathrm{kl}_\Sigma(X^{2n-1}) + \mathrm{kl}_\Sigma(X/X^{2n-1})$. The inductive hypothesis applies to X^{2n-1} and to X/X^{2n-1}, so $\mathrm{kl}_\Sigma(X) \leq 1 + (r-1) = r$. $\square$

4 Distinguishing $Z_{\mathcal{F}}$ for different $\mathcal{F}$

We have a chain of pointed sets

$$Z_\Sigma(X,Y) \subseteq Z_{\mathcal{M}}(X,Y) \subseteq Z_{\mathcal{S}}(X,Y).$$

Simple examples show that each of these containments can be strict. There are nontrivial phantom maps $\Sigma\mathbf{CP}^\infty \longrightarrow S^4$ [19]. These all lie in $Z_{\mathcal{M}}(\Sigma\mathbf{CP}^\infty, S^4)$ because $\mathcal{M} \subseteq \mathcal{P}$ (see Examples in Section 2), but not in $Z_\Sigma(\Sigma\mathbf{CP}^\infty, S^4)$, by Lemma 2.2(b). For the other containment, the Bockstein applied to the fundamental cohomology class of $M(\mathbf{Z}/p,n)$ [3] corresponds to a map $f : M(\mathbf{Z}/p,n) \longrightarrow K(\mathbf{Z}/p,n+1)$. If p is an odd prime, then $\pi_{n+1}(M(\mathbf{Z}/p,n)) = 0$ [3, pp. 268–69] so f is in $Z_{\mathcal{S}}(M(\mathbf{Z}/p,n), K(\mathbf{Z}/p,n+1))$. Since it is essential, f cannot lie in $Z_{\mathcal{M}}(M(\mathbf{Z}/p,n), K(\mathbf{Z}/p,n+1))$.

In these examples either the domain or the target is an infinite CW complex. Thus they leave open the possibility that if X and Y are finite complexes, all of the pointed sets above are the same. We will give examples which show that, even for finite complexes, these inclusions can be strict. These examples are more difficult to find and verify. They are inspired by an example (due to Fred Cohen) from [10].

Recall that if p is an odd prime, then $S^{2n+1}_{(p)}$ is an H-space [1]. Moreover, if f is in the abelian group $[\Sigma^2 X, S^{2n+1}]$ then the order of $\lambda \circ f \in [\Sigma^2 X, S^{2n+1}_{(p)}]$ is either infinite or a power of p.

Lemma 4.1 *Let X be a finite complex and let $h : X \longrightarrow S^{2n+1}$ be a map such that for some odd prime p, $\lambda \circ \Sigma^2 h$ is nonzero and has finite order divisible p. Then there is an $s > 0$ such that the composite*

$$X \xrightarrow{h} S^{2n+1} \xrightarrow{i} M(\mathbf{Z}/p^s, 2n+1),$$

where i is the inclusion, is essential.

Proof Consider the diagram

$$\begin{tikzcd}
 & S^{2n+1} \arrow[r, "\lambda"] \arrow[d, "p^s"] & S^{2n+1}_{(p)} \arrow[d, "p^s"] \\
X \arrow[r, "h"] \arrow[dr, "i\circ h"'] & S^{2n+1} \arrow[r, "\lambda"] \arrow[d, "i"] & S^{2n+1}_{(p)} \arrow[d, "j"] \\
 & M(\mathbf{Z}/p^s, 2n+1) \arrow[r, "="] & M(\mathbf{Z}/p^s, 2n+1)
\end{tikzcd}$$

in which the vertical sequences are cofibrations and p^s denotes the map with degree p^s. If $i \circ h = 0$, then $j \circ \lambda \circ h = 0$. It can be shown that $\Sigma(\lambda \circ h)$ lifts through the map $p^s : S^{2n+2}_{(p)} \longrightarrow S^{2n+2}_{(p)}$. Suspending once more, we obtain a lift given by the dashed line in the diagram

$$\begin{tikzcd} & & S^{2n+3}_{(p)} \arrow[d, "p^s"] \\ \Sigma^2 X \arrow[r, "\Sigma^2 h"'] \arrow[urr, dashed] & S^{2n+3} \arrow[r, "\lambda"'] & S^{2n+3}_{(p)}. \end{tikzcd}$$

Since X is a finite complex, the torsion in $[\Sigma^2 X, S^{2n+3}_{(p)}]$ is p-torsion and has an exponent e. Since $S^{2n+3}_{(p)}$ is an H-space, the map p^s induces multiplication by p^s on $[\Sigma^2 X, S^{2n+3}_{(p)}]$. If $s \geq e$, then the image of $p^s : [\Sigma^2 X, S^{2n+3}_{(p)}] \longrightarrow [\Sigma^2 X, S^{2n+3}_{(p)}]$ cannot contain any nontrivial torsion. But $\lambda \circ \Sigma^2 h$ is nonzero and has finite order. Therefore the lift cannot exist, and so $i \circ h \neq 0$. □

Theorem 4.2 *Let X be a finite complex, let p be an odd prime and let $g : X \longrightarrow S^{2n+1}$ be an essential map.*

(a) *Assume that $\pi_{2n+1}(X)$ is a finite group, and that $\lambda \circ \Sigma^2 g$ is nonzero with finite order divisible by p^{n+1}. Then there is an $s > 0$ such that the composite*

$$\begin{tikzcd} X \arrow[r, "g"'] \arrow[rrr, bend left, "l"] & S^{2n+1} \arrow[r, "p^n"'] & S^{2n+1} \arrow[r, "i"'] & M(\mathbf{Z}/p^s, 2n+1) \end{tikzcd}$$

is essential and $\pi_*(l) = 0$.

(b) *Assume that $\pi_k(X) = 0$ for $k = 2n$ and $2n+1$, and that $\lambda \circ \Sigma^2 g$ is nonzero with finite order divisible by p^{2n+1}. Then there is an $s > 0$ such that the composite*

$$\begin{tikzcd} X \arrow[r, "g"'] \arrow[rrr, bend left, "f"] & S^{2n+1} \arrow[r, "p^{2n}"'] & S^{2n+1} \arrow[r, "i"'] & M(\mathbf{Z}/p^s, 2n+1) \end{tikzcd}$$

is essential, and $\pi_*(f; G) = 0$ for any finitely generated abelian group G.

Proof In part (a), the composition $\lambda \circ \Sigma^2(p^n \circ g)$ has finite order divisible by p. Therefore Lemma 4.1 shows that $l = i \circ p^n \circ g$ is essential if s is large enough.

Similarly, if s is large enough, the map f in part (b) is essential. From now on, we assume that s has been so chosen. We use the commutative diagram

$$\begin{tikzcd} X \arrow[r, "g"] \arrow[dr, "\lambda\circ g"'] & S^{2n+1} \arrow[r, "p^k"] \arrow[d, "\lambda"] & S^{2n+1} \arrow[r, "i"] \arrow[d, "\lambda"] & M(\mathbf{Z}/p^s, 2n+1) \arrow[d, "="] \\ & S^{2n+1}_{(p)} \arrow[r, "p^k"] & S^{2n+1}_{(p)} \arrow[r, "j"] & M(\mathbf{Z}/p^s, 2n+1). \end{tikzcd}$$

We take $k = n$ in part (a) and $k = 2n$ in part (b).

Proof of (a) Since $M(\mathbf{Z}/p^s, 2n+1)$ is p-local, there is only p-torsion to consider. By results of Cohen, Moore and Neisendorfer [6, Cor. 3.1] the p-torsion in $\pi_*(S^{2n+1}_{(p)})$ has exponent n. Since $S^{2n+1}_{(p)}$ is an H-space, $p^n : S^{2n+1}_{(p)} \longrightarrow S^{2n+1}_{(p)}$ annihilates all p-torsion in homotopy groups. Thus $\pi_*(l)$ can be nonzero only in dimension $2n+1$. But $\pi_{2n+1}(g)$ is a homomorphism from a finite group to $\mathbf{Z}$, so $\pi_*(l) = 0$.

Proof of (b) It suffices to show that $\pi_m(f; G) = 0$ for any cyclic group G; by part (a) we need only consider $G = \mathbf{Z}/p^r$. For each $r \geq 1$ and each $m \geq 0$, there is the exact coefficient sequence [12, Chap. 5]

$$0 \longrightarrow \mathrm{Ext}(\mathbf{Z}/p^r, \pi_{m+1}(Y)) \longrightarrow \pi_m(Y; \mathbf{Z}/p^r) \longrightarrow \mathrm{Hom}(\mathbf{Z}/p^r, \pi_m(Y)) \longrightarrow 0.$$

Let $Y = S^{2n+1}_{(p)}$. Since the p-torsion in $\pi_*(S^{2n+1}_{(p)})$ has exponent n [6], the exact sequence shows that the p-torsion in $\pi_m(S^{2n+1}_{(p)}; \mathbf{Z}/p^r)$ has exponent at most $2n$ if $m \neq 2n$. Thus the map $p^{2n} : S^{2n+1}_{(p)} \longrightarrow S^{2n+1}_{(p)}$ induces 0 on the m^{th} homotopy groups with coefficients in any finite abelian group if $m \neq 2n$. Taking $Y = X$ in the coefficient sequence, we have $\pi_{2n}(X; \mathbf{Z}/p^r) = 0$. Therefore $\pi_*(f; G) = 0$ for any finitely generated abelian group G. □

We apply this theorem to construct examples of finite complexes which distinguish the various $Z_{\mathcal{F}}$.

Our first example shows that $Z_{\mathcal{M}}(X)$ can be different from $Z_{\mathcal{S}}(X)$ even when X is a finite complex. Using the coefficient exact sequence for homotopy groups, we find that

$$[M(\mathbf{Z}/p^r, 2n), S^{2n+1}] = \pi_{2n}(S^{2n+1}; \mathbf{Z}/p^r) \cong \mathbf{Z}/p^r$$

for each r; this is a stable group. Therefore, if $r > n$, there are essential maps $g : M(\mathbf{Z}/p^r, 2n) \longrightarrow S^{2n+1}$ with finite order divisible by p^{n+1}. Applying part (a) of Theorem 4.2, we have the following example.

Example Let $r > n > 1$. For p an odd prime and s large enough, there are essential maps

$$l : M(\mathbf{Z}/p^r, 2n) \longrightarrow M(\mathbf{Z}/p^s, 2n+1)$$

such that $\pi_*(l) = 0$. Therefore by Lemma 2.3,

$$Z_{\mathcal{S}}(M(\mathbf{Z}/p^r, 2n) \vee M(\mathbf{Z}/p^s, 2n+1)) \neq 0$$

while, of course,

$$Z_{\mathcal{M}}(M(\mathbf{Z}/p^r, 2n) \vee M(\mathbf{Z}/p^s, 2n+1)) = 0$$

by Lemma 2.2 (b). It can be shown that any $s \geq r$ will suffice in this example.

Freyd's generating hypothesis [11] is the conjecture that no stably nontrivial map between finite complexes can induce zero on stable homotopy groups. The map l in this example is stably nontrivial, but our argument does *not* show that large suspensions of l induce zero on homotopy groups; the difficulty is that after two suspensions, l factors through $p^n : S^{2n+3}_{(p)} \longrightarrow S^{2n+3}_{(p)}$, which need not annihilate all p-torsion.

Our second example is a map $f : \Sigma^{2n-2}\left(\mathbf{C}\mathrm{P}^{p^{2n+1}}/S^2\right) \longrightarrow M(\mathbf{Z}/p^s, 2n+1)$ which we use to show that $Z_{\mathcal{M}}(X)$ can be different from $Z_{\Sigma}(X)$ when X is a finite complex. We need some preliminary results to show that Theorem 4.2 applies to this situation.

Lemma 4.3 *Let $f : \Sigma^{n+1}\mathbf{C}\mathrm{P}^m \longrightarrow S^{n+3}$. The degree of $f|_{\Sigma^{n+1}S^2}$ is divisible by $\mathrm{lcm}(1, \ldots, m)$, the least common multiple of $1, \ldots, m$.*

Proof We may assume that f is in the stable range. If $f|_{\Sigma^{n+1}S^2}$ has degree d, then

$$\Sigma^{n+1}\mathbf{C}\mathrm{P}^m \xrightarrow{f} S^{n+3} \hookrightarrow \Sigma^{n+1}\mathbf{C}\mathrm{P}^m.$$

has degree d in $H_{n+3}(\Sigma^{n+1}\mathbf{C}\mathrm{P}^m)$ and is trivial in all other dimensions. According to McGibbon [19, Thm. 3.4], d is divisible by $\mathrm{lcm}(1, \ldots, m)$. $\square$

Proposition 4.4 *The image of the n-fold suspension map*

$$\Sigma^n : [\mathbf{C}\mathrm{P}^{p^t}/S^2, S^3] \longrightarrow [\Sigma^n(\mathbf{C}\mathrm{P}^{p^t}/S^2), S^{n+3}]$$

contains elements of order p^t for every $n \geq 1$ and $t \geq 1$.

Proof Write $m = p^t$ and examine the commutative diagram

$$\begin{tikzcd}
{[\Sigma \mathbf{CP}^m, S^3]} \arrow[r] \arrow[d, "\Sigma^n"] & {[\Sigma S^2, S^3]} \arrow[r] \arrow[d, "\Sigma^n"] & {[\mathbf{CP}^m/S^2, S^3]} \arrow[d, "\Sigma^n"] \\
{[\Sigma^{n+1}\mathbf{CP}^m, S^{n+3}]} \arrow[r] \arrow[d, "\lambda_*"] & {[\Sigma^{n+1} S^2, S^{n+3}]} \arrow[r] \arrow[d, "\lambda_*"] & {[\Sigma^n(\mathbf{CP}^m/S^2), S^{n+3}]} \arrow[d, "\lambda_*"] \\
{[\Sigma^{n+1}\mathbf{CP}^m, S^{n+3}_{(p)}]} \arrow[r] & {[\Sigma^{n+1} S^2, S^{n+3}_{(p)}]} \arrow[r] & {[\Sigma^n(\mathbf{CP}^m/S^2), S^{n+3}_{(p)}].}
\end{tikzcd}$$

To show that the image of $\lambda_* \circ \Sigma^n : [\mathbf{CP}^m/S^2, S^3] \longrightarrow [\Sigma^n(\mathbf{CP}^m/S^2), S^{n+3}_{(p)}]$ contains elements of order p^t, we modify the above diagram as follows: the image and cokernel of $[\Sigma\mathbf{CP}^m, S^3] \longrightarrow [\Sigma S^2, S^3] \cong \mathbf{Z}$ are $k\mathbf{Z}$ and $\mathbf{Z}/k$, respectively, for some integer k; similarly for $[\Sigma^{n+1}\mathbf{CP}^m, S^{n+3}] \longrightarrow [\Sigma^{n+1}S^2, S^{n+3}]$ and $[\Sigma\mathbf{CP}^m, S^3_{(p)}] \longrightarrow [\Sigma S^2, S^3_{(p)}]$. Thus we have a commutative diagram with exact rows

$$\begin{tikzcd}
k\mathbf{Z} \arrow[r] \arrow[d] & \mathbf{Z} \arrow[r] \arrow[d, "\cong"] & \mathbf{Z}/k \arrow[d] \\
l\mathbf{Z} \arrow[r] \arrow[d, "\lambda"] & \mathbf{Z} \arrow[r] \arrow[d, "\lambda"] & \mathbf{Z}/l \arrow[d, "\lambda"] \\
l_p\mathbf{Z}_{(p)} \arrow[r] & \mathbf{Z}_{(p)} \arrow[r] & \mathbf{Z}/l_p
\end{tikzcd}$$

for some integers k, l and l_p, where l_p is the largest power of p which divides l. Lemma 4.3 shows that l_p is divisible by p^t. The composite $\mathbf{Z}/k \longrightarrow \mathbf{Z}/l \longrightarrow \mathbf{Z}/l_p$ is surjective, and this completes the proof. □

It follows from Proposition 4.4 that part (b) of Theorem 4.2 applies to the space $\Sigma^{2n-2}(\mathbf{CP}^{p^{2n+1}}/S^2)$ for each $n > 1$, and so we obtain our second example.

Example For each odd prime p and each $n \geq 1$, there is an $s > 0$ such that there are essential maps

$$f : \Sigma^{2n-2}\left(\mathbf{CP}^{p^{2n+1}}/S^2\right) \longrightarrow M(\mathbf{Z}/p^s, 2n+1)$$

which induce zero on homotopy groups with coefficients. Therefore,

$$Z_{\mathcal{M}}\left(\Sigma^{2n-2}\left(\mathbf{CP}^{p^{2n+1}}/S^2\right) \vee M\left(\mathbf{Z}/p^s, 2n+1\right)\right) \neq 0$$

while, of course,

$$Z_{\Sigma}\left(\Sigma^{2n-2}\left(\mathbf{CP}^{p^{2n+1}}/S^2\right) \vee M\left(\mathbf{Z}/p^s, 2n+1\right)\right) = 0.$$

The map f can be chosen to be stably nontrivial. As in the previous example, the suspensions of f might not be trivial on homotopy groups with coefficients.

Finally, let $A = \Sigma^{2n-2}(\mathbf{C}\mathrm{P}^{p^{2n+1}}/S^2)$, $B = M(\mathbf{Z}/p^r, 2n)$ and $C = M(\mathbf{Z}/p^s, 2n+1)$ for s large. Then

$$Z_\Sigma(A \vee B \vee C) < Z_{\mathcal{M}}(A \vee B \vee C) < Z_{\mathcal{S}}(A \vee B \vee C),$$

so both of these inequalities can be strict for a single finite complex.

5 Projective spaces

We show that for projective spaces $\mathbf{F}\mathrm{P}^n$ with $\mathbf{F} = \mathbf{R}, \mathbf{C}$ or $\mathbf{H}$,

$$Z_\Sigma(\mathbf{F}\mathrm{P}^n) = Z_{\mathcal{M}}(\mathbf{F}\mathrm{P}^n) = Z_{\mathcal{S}}(\mathbf{F}\mathrm{P}^n),$$

and we completely determine these sets for $\mathbf{F} = \mathbf{R}$ and $\mathbf{C}$ and all n. We also determine $t_{\mathcal{S}}(\mathbf{H}\mathrm{P}^n)$, for $n \leq 4$.

5.1 General facts

We first prove some general results that will be applied later.

Proposition 5.1 *If $\Sigma\Omega X \equiv \bigvee S^{n_\alpha}$, then $Z_{\mathcal{S}}(X,Y) = Z_{\mathcal{M}}(X,Y) = Z_\Sigma(X,Y)$ for any space Y.*

Proof Let $f \in Z_{\mathcal{S}}(X,Y)$. The map Ωf is adjoint to the composition $\Sigma\Omega X \xrightarrow{\nu} X \xrightarrow{f} Y$. Since $\Sigma\Omega X \equiv \bigvee S^{n_\alpha}$, $f \circ \nu = 0$, and so $\Omega f = 0$. Thus $f \in Z_\Sigma(X,Y)$. □

By Lemma 2.4, the condition $Z_{\mathcal{S}}(X,Y) = Z_\Sigma(X,Y)$ is equivalent to the condition that if $f : X \longrightarrow Y$ induces zero on homotopy groups, then $\Omega f = 0$.

Proposition 5.1 applies to $X = S^{n+1}$ because, by James [14], $\Sigma\Omega S^{n+1} \equiv \bigvee_{k=1}^\infty S^{nk+1}$. Of course $Z_{\mathcal{S}}(S^{n+1}, Y) = 0$. Since $\Sigma(A \times B) \equiv \Sigma A \vee \Sigma B \vee \Sigma(A \wedge B)$ for any A and B [12, 11.10], James's result allows us to apply Proposition 5.1 to any space whose loop space splits as a finite type product of spheres and loop spaces on spheres. Moreover, if X and X' both satisfy the hypotheses of Proposition 5.1, then so does $X \times X'$.

For $\mathbf{F} = \mathbf{R}, \mathbf{C}$ or $\mathbf{H}$, let $d = 1, 2$ or 4, respectively. For each $n \geq 1$ there is a homotopy equivalence $\Omega\mathbf{F}\mathrm{P}^n \equiv S^{d-1} \times \Omega S^{(n+1)d-1}$. This is a direct consequence of [8, Thm. 5.2], which applies even in the case $d = 1$.

Corollary 5.2 *For $\mathbf{F} = \mathbf{R}, \mathbf{C}$ or $\mathbf{H}$ and each $n \geq 1$, $Z_{\mathcal{S}}(\mathbf{F}\mathrm{P}^n) = Z_{\mathcal{M}}(\mathbf{F}\mathrm{P}^n) = Z_{\Sigma}(\mathbf{F}\mathrm{P}^n)$.*

Another corollary of Proposition 5.1 applies to intermediate wedges of spheres. For spaces $X_1, X_2, \ldots, X_n$, the elements $(x_1, \ldots, x_k) \in X_1 \times \cdots \times X_k$ with at least j coordinates equal to the base point form a subspace $T_j(X_1, \ldots, X_k) \subseteq X_1 \times \cdots \times X_k$. Porter has shown [24, Thm. 2] that $\Omega T_j(S^{n_1}, \ldots, S^{n_k})$ has the homotopy type of a product of loop spaces of spheres for each $0 \leq j \leq k$. Our previous discussion establishes the following.

Corollary 5.3 *For any $n_1, \ldots, n_k \geq 1$ and any $0 \leq j \leq k$,*

$$Z_{\mathcal{S}}(T_j(S^{n_1}, \ldots, S^{n_k})) = Z_{\mathcal{M}}(T_j(S^{n_1}, \ldots, S^{n_k})) = Z_{\Sigma}(T_j(S^{n_1}, \ldots, S^{n_k})).$$

Remarks

(a) Taking $j = 0$ in Corollary 5.3, we deduce from Corollary 2.7 that

$$t_{\mathcal{S}}(S^{n_1} \times \cdots \times S^{n_k}) \leq \lceil \log_2 (k+1) \rceil.$$

This reproves [2, Prop. 6.2] by a different method.

(b) It is proved in [2, Prop. 6.5] that for any positive integer n, there is a finite product of spheres X with $t_{\mathcal{S}}(X) = n$. By Corollary 5.3, the same is true for $t_{\Sigma}(X)$ and $t_{\mathcal{M}}(X)$. Thus the integers $t_{\mathcal{F}}(X)$ for $\mathcal{F} = \mathcal{S}, \mathcal{M}$ or Σ and any X take on all positive integer values.

Finally, we observe that the splitting of $\Omega\mathbf{F}\mathrm{P}^n$ gives a useful criterion for deciding when a map $f : \mathbf{F}\mathrm{P}^n \longrightarrow Y$ lies in $Z_{\mathcal{S}}(\mathbf{F}\mathrm{P}^n, Y)$.

Proposition 5.4 *Let i be the inclusion $S^d = \mathbf{F}\mathrm{P}^1 \hookrightarrow \mathbf{F}\mathrm{P}^n$, and let $p : S^{(n+1)d-1} \longrightarrow \mathbf{F}\mathrm{P}^n$ be the Hopf fiber map. Then the map*

$$(i,p) : S^d \vee S^{(n+1)d-1} \longrightarrow \mathbf{F}\mathrm{P}^n$$

induces a surjection on homotopy groups. Therefore, a map $f : \mathbf{F}\mathrm{P}^n \longrightarrow Y$ satisfies $\pi_*(f) = 0$ if and only if $f \circ i = 0$ and $f \circ p = 0$.

5.2 Complex projective spaces

Next we show that certain skeleta X of Eilenberg-MacLane spaces have the property that $Z_{\mathcal{S}}(X) = 0$. We apply this to $\mathbf{C}\mathrm{P}^n$ and $\Sigma^n\mathbf{C}\mathrm{P}^2$ for each n.

Let G be a finitely generated abelian group. Give the Eilenberg-MacLane space $K(G,n)$ with $n \geq 2$ a homology decomposition [12, Chap. 8] and denote the m^{th} section by $K(G,n)_m$. Thus $K(G,n)$ is filtered

$$* \subseteq K(G,n)_n \subseteq \cdots \subseteq K(G,n)_m \subseteq \cdots \subseteq K(G,n)$$

and there are cofiber sequences

$$M(H_{m+1}(K(G,n)),m) \longrightarrow K(G,n)_m \longrightarrow K(G,n)_{m+1}.$$

Theorem 5.5 *If the group $H_m(K(G,n))$ is torsion free and $H_{m+1}(K(G,n)) = 0$, then $Z_{\mathcal{S}}(K(G,n)_m) = 0$.*

Proof We write $X = K(G,n)_m$. Then $H_k(K(G,n), X) = 0$ for $k \leq m+1$. By Whitehead's theorem [32, Thm. 7.13], the induced map $\pi_k(X) \longrightarrow \pi_k(K(G,n))$ is an isomorphism for $k \leq m$. Since $H_m(K(G,n))$ is torsion free, X has dimension at most m, and so X has a CW decomposition

$$\bigvee S^n = X^n \subseteq X^{n+1} \subseteq \cdots \subseteq X^m \equiv X.$$

For $f \in Z_{\mathcal{S}}(X)$ we prove by induction on k that f factors through X/X_k for each $k \leq m$. The first step is trivial since $\pi_n(f) = 0$ implies $f|_{X_n} = 0$. Inductively, assume that f factors through X/X_k with $n \leq k < m$. There is a cofibration

$$\bigvee S^{k+1} \equiv X_{k+1}/X_k \longrightarrow X/X_k \longrightarrow X/X_{k+1}.$$

Since $n < k+1 \leq m$, it follows that $\pi_{k+1}(X) \cong \pi_{k+1}(K(G,n)) = 0$, so f extends to X/X_{k+1}. Taking $k = m$, we find $f = 0$. □

Remark Clearly, $\pi_k(K(G,n)_m) = 0$ for $n < k < m$. The hypotheses in Theorem 5.5 are needed to conclude further that $\pi_m(K(G,n)_m) = 0$.

As an application of Theorem 5.5, we have the following calculations.

Theorem 5.6

(a) $Z_{\mathcal{F}}(\mathbf{C}\mathrm{P}^n) = 0$ *for each* $n \geq 1$ *and each* $\mathcal{F} = \Sigma, \mathcal{M}$ *or* $\mathcal{S}$.

(b) $Z_{\mathcal{F}}(\Sigma^n \mathbf{C}\mathrm{P}^2) = 0$ *for each* $n \geq 1$ *and each* $\mathcal{F} = \Sigma, \mathcal{M}$ *or* $\mathcal{S}$.

Proof By Proposition 5.1 it suffices to consider the case $\mathcal{F} = \mathcal{S}$. Since $\mathbf{C}\mathrm{P}^\infty = K(\mathbf{Z}, 2)$ and the $\mathbf{C}\mathrm{P}^n$ are the sections of a homology decomposition of $\mathbf{C}\mathrm{P}^\infty$, part (a) follows from Theorem 5.5. Recall from [9] that for $n \geq 2$

$$H_k(K(\mathbf{Z},n)) = \begin{cases} \mathbf{Z} & \text{if } k = n \text{ or } n+2 \\ 0 & \text{if } k = n+1 \text{ or } n+3. \end{cases}$$

Since Sq^2 is nontrivial on $H^n(K(\mathbf{Z},n);\mathbf{Z}/2)$, we have $K(\mathbf{Z},n)_{n+2} \equiv \Sigma^{n-2}\mathbf{C}\mathrm{P}^2$. Thus Theorem 5.5 applies to $\Sigma^{n-2}\mathbf{C}\mathrm{P}^2$. $\square$

This theorem immediately shows that $t_{\mathcal{F}}(\mathbf{C}\mathrm{P}^n) = t_{\mathcal{F}}(\Sigma^n\mathbf{C}\mathrm{P}^2) = 1$ for $\mathcal{F} = \Sigma, \mathcal{M}$ or $\mathcal{S}$ and each $n \geq 1$.

5.3 Real projective spaces

In this subsection we completely calculate $Z_{\mathcal{S}}(\mathbf{R}\mathrm{P}^n)$. By the Hopf-Whitney theorem [32, Cor. 6.19] $[\mathbf{R}\mathrm{P}^{2n}, S^{2n}] \cong H^{2n}(\mathbf{R}\mathrm{P}^{2n}) \cong \mathbf{Z}/2$. The unique nontrivial map $q : \mathbf{R}\mathrm{P}^{2n} \longrightarrow S^{2n}$ is the quotient map obtained by factoring out $\mathbf{R}\mathrm{P}^{2n-1}$. Let f_{2n} denote the composite $\mathbf{R}\mathrm{P}^{2n} \xrightarrow{q} S^{2n} \xrightarrow{p} \mathbf{R}\mathrm{P}^{2n}$ where p is the universal covering map.

Theorem 5.7 *For $\mathcal{F} = \Sigma, \mathcal{M}$ or $\mathcal{S}$ and each $n \geq 1$,*

(a) $Z_{\mathcal{F}}(\mathbf{R}\mathrm{P}^{2n-1}) = 0$

(b) $Z_{\mathcal{F}}(\mathbf{R}\mathrm{P}^{2n}) = \{0, f_{2n}\}$.

Proof Let $f : \mathbf{R}\mathrm{P}^m \longrightarrow \mathbf{R}\mathrm{P}^m$ with $\pi_1(f) = 0$. Because $\pi_k(\mathbf{R}\mathrm{P}^m) = 0$ for $1 < k < m$, an argument similar to the proof of Theorem 5.5 shows that f must factor through $q : \mathbf{R}\mathrm{P}^m \longrightarrow S^m$. For $m > 1$, any map $S^m \longrightarrow \mathbf{R}\mathrm{P}^m$ lifts through $p : S^m \longrightarrow \mathbf{R}\mathrm{P}^m$. Thus there is a map $g : S^m \longrightarrow S^m$ of degree d which makes the following diagram commute

$$\begin{tikzcd} S^m \arrow[r, "p"] \arrow[dr, "q\circ p"'] & \mathbf{R}\mathrm{P}^m \arrow[r, "f"] \arrow[d, "q"] & \mathbf{R}\mathrm{P}^m \\ & S^m \arrow[r, "g"] & S^m. \arrow[u, "p"'] \end{tikzcd}$$

First let $m = 2n-1$. We may assume $n > 1$. The composite $q \circ p : S^{2n-1} \longrightarrow S^{2n-1}$ is known to have degree 2. Since $\pi_i(p)$ is an isomorphism for $i > 1$, $f \circ p$ represents $2d \in \mathbf{Z} \cong \pi_{2n-1}(\mathbf{R}\mathrm{P}^{2n-1})$. If $f \in Z_{\mathcal{S}}(\mathbf{R}\mathrm{P}^{2n-1})$, then d must be 0, and so $f = 0$. This proves (a).

Now take $m = 2n$. The composite $q \circ p : S^{2n} \longrightarrow S^{2n}$ is trivial because it is zero on homology. Therefore $f \circ p = 0$, and since $\pi_1(f) = 0$, Proposition 5.4 shows that $f \in Z_{\mathcal{S}}(\mathbf{R}\mathrm{P}^{2n})$. Since $\mathbf{R}\mathrm{P}^{2n}$ is connected, there is a bijection

$$p_* : [\mathbf{R}\mathrm{P}^{2n}, S^{2n}] \xrightarrow{\cong} \{f \mid f \in [\mathbf{R}\mathrm{P}^{2n}, \mathbf{R}\mathrm{P}^{2n}],\ \pi_1(f) = 0\} = Z_{\mathcal{S}}(\mathbf{R}\mathrm{P}^{2n}).$$

Since $[\mathbf{R}\mathrm{P}^{2n}, S^{2n}] = \{0, q\}$ as noted above, $Z_{\mathcal{S}}(\mathbf{R}\mathrm{P}^{2n}) = \{0, f_{2n}\}$, where $f_{2n} = p \circ q$. $\square$

Remark This argument actually shows that, if $\pi_k(Y) = 0$ for $1 < k < 2n+1$, there is a bijection between $Z_{\mathcal{S}}(\mathbf{R}\mathrm{P}^{2n+1}, Y)$ and the set of elements $\alpha \in \pi_{2n+1}(Y)$ such that $2\alpha = 0$.

Corollary 5.8 *For each* $n \geq 1$,

(a) $t_{\mathcal{F}}(\mathbf{R}\mathrm{P}^{2n-1}) = 1$ *for* $\mathcal{F} = \Sigma, \mathcal{M}$ *and* $\mathcal{S}$.

(b) $t_{\mathcal{F}}(\mathbf{R}\mathrm{P}^{2n}) = 2$ *for* $\mathcal{F} = \Sigma, \mathcal{M}$ *and* $\mathcal{S}$.

Proof It suffices to prove part (b) for $\mathcal{F} = \mathcal{S}$. Since $Z_{\mathcal{S}}(\mathbf{R}\mathrm{P}^{2n}) \neq 0$, $t_{\mathcal{S}}(\mathbf{R}\mathrm{P}^{2n}) \geq 2$. The only possibly nonzero product in this semigroup is $f_{2n} \circ f_{2n}$. But this is zero because $Z_{\mathcal{S}}(\mathbf{R}\mathrm{P}^{2n})$ is nilpotent by Theorem 3.3. □

5.4 Quaternionic projective spaces

The quaternionic projective spaces are not skeleta of Eilenberg-MacLane spaces, and it is much more difficult to compute their nilpotency.

Let $f \in [\mathbf{H}\mathrm{P}^{n+1}, \mathbf{H}\mathrm{P}^{n+1}]$, and assume that f is cellular. Then $f|_{\mathbf{H}\mathrm{P}^n} : \mathbf{H}\mathrm{P}^n \longrightarrow \mathbf{H}\mathrm{P}^n$ and the homotopy class $f|_{\mathbf{H}\mathrm{P}^n}$ is well defined.

Lemma 5.9 *If* $f \in Z_{\mathcal{S}}(\mathbf{H}\mathrm{P}^{n+1})$, *then* $f|_{\mathbf{H}\mathrm{P}^n} \in Z_{\mathcal{S}}(\mathbf{H}\mathrm{P}^n)$.

Proof Let $f \in Z_{\mathcal{S}}(\mathbf{H}\mathrm{P}^{n+1})$ and let $g = f|_{\mathbf{H}\mathrm{P}^n}$. Consider the diagram

$$\begin{tikzcd}
 & S^{4n+3} \arrow[r, dashed, "h"] \arrow[d, "p"] & S^{4n+3} \arrow[d, "p"] & \\
S^4 \arrow[r, "i"] & \mathbf{H}\mathrm{P}^n \arrow[r, "g"] \arrow[d, "j"] & \mathbf{H}\mathrm{P}^n \arrow[d, "j"] \arrow[dr, "l"] & \\
 & \mathbf{H}\mathrm{P}^{n+1} \arrow[r, "f"] \arrow[d, "q"] & \mathbf{H}\mathrm{P}^{n+1} \arrow[r, "m"] \arrow[d, "q"] & \mathbf{H}\mathrm{P}^{\infty} \\
 & S^{4n+4} \arrow[r, dashed, "\Sigma h"] & S^{4n+4}. &
\end{tikzcd}$$

where i, j, m and l are inclusions. Since $S^{4n+3} \xrightarrow{p} \mathbf{H}\mathrm{P}^n \xrightarrow{l} \mathbf{H}\mathrm{P}^{\infty}$ can be regarded as a fibration and $l \circ (g \circ p) = m \circ (f \circ (j \circ p)) = 0$, it follows that $g \circ p$ lifts to the map h. Since $f \in Z_{\mathcal{S}}(\mathbf{H}\mathrm{P}^{n+1})$, f induces zero on $H^4(\mathbf{H}\mathrm{P}^{n+1})$, and hence is zero in cohomology. Therefore Σh is zero in cohomology and hence is trivial. Thus $h = 0$, so $g \circ p = 0$. Also, $g \circ i = 0$, so $g \in Z_{\mathcal{S}}(\mathbf{H}\mathrm{P}^n)$ by Proposition 5.4. □

Next we indicate how we will apply Lemma 5.9. If $Z_{\mathcal{S}}(\mathbf{HP}^n) = 0$ and $f \in Z_{\mathcal{S}}(\mathbf{HP}^{n+1})$, then $f|_{\mathbf{HP}^n} = 0$, so f factors through $q : \mathbf{HP}^{n+1} \longrightarrow S^{4n+4}$. By Proposition 5.4, if $i : S^4 \hookrightarrow \mathbf{HP}^{n+1}$, then $\pi_{4n+4}(i)$ is surjective, so f factors as in the diagram

$$\begin{tikzcd} \mathbf{HP}^{n+1} \arrow[r, "f"] \arrow[d, "q"'] & \mathbf{HP}^{n+1} \arrow[r, "m"] & \mathbf{HP}^{\infty} \\ S^{4n+4} \arrow[r, "g"] & S^4. \arrow[u, "i"] \arrow[ur, "l"'] & \end{tikzcd}$$

By cellular approximation, f is essential if and only if $m \circ f$ is essential. The map $l \circ g : S^{4n+4} \longrightarrow \mathbf{HP}^{\infty}$ is adjoint to a map $g' : S^{4n+3} \longrightarrow S^3$, which in turn is adjoint to $l \circ \Sigma g'$. By cellular approximation again, $i \circ g = i \circ \Sigma g'$, so we may assume that g is in the image of the suspension $\Sigma : \pi_{4n+3}(S^3) \longrightarrow \pi_{4n+4}(S^4)$.

The proof of our main result about quaternionic projective spaces requires some detailed information about homotopy groups of spheres. Since we refer to Toda's book [31] for this information, we use his notation here. For example, $\eta_k : S^{k+1} \longrightarrow S^k$ and $\nu_k : S^{k+3} \longrightarrow S^k$ are suspensions of the Hopf fiber maps.

Theorem 5.10

(a) $Z_{\mathcal{F}}(\mathbf{HP}^n) = 0$ *for* $\mathcal{F} = \mathcal{S}, \mathcal{M}$ *or* Σ *and* $n = 1, 2$ *and* 3

(b) $Z_{\mathcal{F}}(\mathbf{HP}^4) \neq 0$ *for* $\mathcal{F} = \mathcal{S}, \mathcal{M}$ *or* Σ.

Proof First $\mathbf{HP}^1 = S^4$, so $Z_{\mathcal{S}}(\mathbf{HP}^1) = 0$. If $f \in Z_{\mathcal{S}}(\mathbf{HP}^2)$, then there is a commutative diagram

$$\begin{tikzcd} \mathbf{HP}^{2} \arrow[r, "f"] \arrow[d, "q"'] & \mathbf{HP}^{2} \arrow[r, "m"] & \mathbf{HP}^{\infty} \\ S^{8} \arrow[r, "g"] \arrow[d, "\nu_5"'] & S^4 \arrow[u, "i"] \arrow[ur, "l"'] & \\ S^5 \arrow[ur, "\eta_4"'] & & \end{tikzcd}$$

in which the vertical sequence is a cofibration. If $g = 0$, then $f = 0$, so we may assume that $g \neq 0$. We know that $\pi_8(\mathbf{HP}^{\infty}) \cong \pi_7(S^3) \cong \mathbf{Z}/2$, generated by $\eta_3 \circ \nu_4$ [31, p. 43–44]. Thus we can take $g = \eta_4 \circ \nu_5$. Since $\nu_5 \circ q = 0$, we conclude that $g \circ q = 0$, so $f = 0$. This shows that $Z_{\mathcal{F}}(\mathbf{HP}^2) = 0$.

The proof that $Z_{\mathcal{S}}(\mathbf{HP}^3) = 0$ is similar. Let $f \in Z_{\mathcal{S}}(\mathbf{HP}^3)$ and apply Lemma 5.9 to get a similar factorization. The resulting map $g : S^{12} \longrightarrow S^4$ is either $\Sigma\epsilon_3$

or 0 [31, Thm. 7.1]. If $g = \Sigma\epsilon_3$, then results of [15, (2.20a)] and [31, Thm. 7.4] show that $f \circ p \neq 0$. Thus $g = 0$ and so $f = 0$.

For part (b), we make use of the diagram preceding Theorem 5.10 and the fact that g can be taken to be a suspension map. If $f \in Z_{\mathcal{S}}(\mathbf{H}\mathrm{P}^4)$, then we have

$$\begin{tikzcd}
S^{19} \arrow[r, "p"] \arrow[dr] & \mathbf{H}\mathrm{P}^4 \arrow[r, "f"] \arrow[d, "q"'] & \mathbf{H}\mathrm{P}^4 \arrow[r, "m"] & \mathbf{H}\mathrm{P}^\infty \\
 & S^{16} \arrow[r, "g"] & S^4. \arrow[u, "i"'] \arrow[ur, "l"'] &
\end{tikzcd}$$

According to Toda [31], $\pi_{16}(S^4) = \Sigma(\pi_{15}(S^3)) \cong \mathbf{Z}/2 \oplus \mathbf{Z}/2$. By [15, (2.20a)], $q \circ p$ is $4\nu_{16} \in \pi_{19}(S^{16})$. Then $g \circ (4\nu_{16}) = 4g \circ \nu_{16}$ because ν_{16} is a suspension. Since $4g = 0$, any map $\mathbf{H}\mathrm{P}^4 \longrightarrow \mathbf{H}\mathrm{P}^4$ which factors through q lies in $Z_{\mathcal{S}}(\mathbf{H}\mathrm{P}^4)$. Marcum and Randall show in [18] that the map

$$((i \circ \Sigma\nu') \circ \mu_7) \circ q : \mathbf{H}\mathrm{P}^4 \longrightarrow \mathbf{H}\mathrm{P}^4$$

is essential, where $\nu' \in \pi_6(S^3)$ generates the 2-torsion and $\mu_7 \in \pi_{16}(S^7)$ generates a $\mathbf{Z}/2$ summand [31, Thm. 7.2]. Thus $Z_{\mathcal{S}}(\mathbf{H}\mathrm{P}^4) \neq 0$, and so $t_{\mathcal{S}}(\mathbf{H}\mathrm{P}^4) \geq 2$. □

As before, we obtain the nilpotency.

Corollary 5.11

(a) $t_{\mathcal{F}}(\mathbf{H}\mathrm{P}^1) = t_{\mathcal{F}}(\mathbf{H}\mathrm{P}^2) = t_{\mathcal{F}}(\mathbf{H}\mathrm{P}^3) = 1$ *for* $\mathcal{F} = \mathcal{S}, \mathcal{M}$ *or* Σ

(b) $t_{\mathcal{F}}(\mathbf{H}\mathrm{P}^4) = 2$ *for* $\mathcal{F} = \mathcal{S}, \mathcal{M}$ *or* Σ.

Proof It suffices to prove that $t_{\mathcal{S}}(\mathbf{H}\mathrm{P}^4) \leq 2$. Suppose $f, g \in Z_{\mathcal{S}}(\mathbf{H}\mathrm{P}^n)$. The proof of Theorem 5.10 shows that f factors through S^{16}. Now $g \circ f = 0$ because $g \in Z_{\mathcal{S}}(\mathbf{H}\mathrm{P}^4)$. □

6 H-spaces

In this section we study the nilpotency of H-spaces Y mod Σ. We make calculations for specific Lie groups such as $SU(n)$ and $Sp(n)$ and show that Z_Σ is nontrivial in these cases. If Y is an H-space, the Samelson product of $\alpha \in \pi_m(Y)$ and $\beta \in \pi_n(Y)$ is written $\langle \alpha, \beta \rangle \in \pi_{n+m}(Y)$ [32, Chap. X].

We first give a few general results which are needed later.

Lemma 6.1 *If Y is an H-space and $\langle \alpha, \beta \rangle \neq 0$ for some $\alpha \in \pi_m(Y)$ and $\beta \in \pi_n(Y)$, then $[S^m \times S^n, Y]$ is not abelian.*

Proof The quotient map $q : S^m \times S^n \longrightarrow S^m \wedge S^n \equiv S^{m+n}$ induces a monomorphism $q^* : [S^m \wedge S^n, Y] \longrightarrow [S^m \times S^n, Y]$ such that $q^*\langle \alpha, \beta \rangle = [\alpha p_1, \beta p_2]$, the commutator of αp_1 and αp_2. $\square$

It is well known that if an H-space Y is a finite complex, then it has the same rational homotopy type as a product of spheres $S^{2n_1-1} \times \cdots \times S^{2n_r-1}$ with $n_1 \leq \cdots \leq n_r$. If p is an odd prime such that

$$Y_{(p)} \equiv \left(S^{2n_1-1} \times \cdots \times S^{2n_r-1}\right)_{(p)} \equiv S^{2n_1-1}_{(p)} \times \cdots \times S^{2n_r-1}_{(p)},$$

then p is called a *regular prime* for Y. If Y is a simply-connected compact Lie group, then p is regular for Y if and only if $p \geq n_r$ [17, Sec. 9-2].

We need a second product decomposition for p-localized Lie groups. By [21, Sec. 2] there are fibrations $S^{2k+1} \longrightarrow B_k(p) \longrightarrow S^{2k+2p-1}$ for $k = 1, 2, \ldots$. An odd prime p is called *quasi-regular* for the H-space Y if

$$Y_{(p)} \equiv \left(\prod_i S^{2n_i-1} \times \prod_j B_{m_j}(p)\right)_{(p)}.$$

6.1 The groups $SU(n)$ and $Sp(n)$

We apply the notions of regular and quasi-regular primes to the Lie group $SU(n)$, which has the rational homotopy type of $S^3 \times S^5 \times \cdots \times S^{2n-1}$, and to the Lie group $Sp(n)$, which has the rational homotopy type of $S^3 \times S^7 \times \cdots \times S^{4n-1}$. It is well known [21, Thm. 4.2] that if p is an odd prime then

(a) p is regular for $SU(n)$ if and only if $p \geq n$; p is quasi-regular for $SU(n)$ if and only if $p > \frac{n}{2}$

(b) p is regular for $Sp(n)$ if and only if $p \geq 2n$; p is quasi-regular for $Sp(n)$ if and only if $p > n$.

It is also known [4, Thm. 1] that if $n \geq 3$ and $r+s+1 = n$, there are generators $\alpha \in \pi_{2r+1}(SU(n)) \cong \mathbf{Z}$, $\beta \in \pi_{2s+1}(SU(n)) \cong \mathbf{Z}$ and $\gamma \in \pi_{2n}(SU(n)) \cong \mathbf{Z}/n!$ such that $\langle \alpha, \beta \rangle = r!s!\gamma$. If p is an odd prime and $\alpha' \in \pi_{2r+1}(SU(n)_{(p)})$, $\beta' \in \pi_{2s+1}(SU(n)_{(p)})$ and $\gamma' \in \pi_{2n}(SU(n)_{(p)})$ are the images of α, β and γ under the localization homomorphism $\lambda_* : \pi_*(SU(n)) \longrightarrow \pi_*(SU(n)_{(p)}) \cong \pi_*(SU(n))_{(p)}$, then

$$\langle \alpha', \beta' \rangle = r!s!\gamma' \in \pi_{2n}(SU(n))_{(p)} \cong \mathbf{Z}/n! \otimes \mathbf{Z}_{(p)}.$$

Now we prove the main result of this section.

Theorem 6.2 *The groups*

(a) $[SU(n), SU(n)]$ *for* $n \geq 5$ *and*

(b) $[Sp(n), Sp(n)]$ *for* $n \geq 2$

are not abelian.

Proof Consider $SU(n)$ for $n \geq 5$ and let p be the largest prime such that $\frac{n}{2} < p < n$. If $n \geq 12$, then it follows from Bertrand's postulate [28, p. 137] that there are two primes p and q such that $\frac{n}{2} < q < p < n$. This implies that $2n+6 < 4p$. For $5 \leq n < 12$, and $n \neq 5, 7, 11$, it is easily verified that $2n+6 < 4p$.

Assume that $n \geq 5$ and that $n \neq 5, 7$ or 11. Since $p > \frac{n}{2}$, it follows that p is quasi-regular for $SU(n)$. Since $2n+6 < 4p$, the spheres $S^{2n-2p+3}$ and S^{2p-3} both appear in the resulting product decomposition. Thus we have

$$SU(n)_{(p)} \equiv \left(B_1(p) \times \cdots \times B_{n-p}(p) \times S^{2n-2p+3} \times \cdots \times S^{2p-3} \times S^{2p-1}\right)_{(p)}.$$

Assume $[SU(n), SU(n)]$ is abelian. Then $[SU(n)_{(p)}, SU(n)_{(p)}]$ is abelian, and therefore $[S^{2n-2p+3} \times S^{2p-3}, SU(n)_{(p)}]$ is abelian.

There are $\alpha' \in \pi_{2n-2p+3}(SU(n)_{(p)})$, $\beta' \in \pi_{2p-3}(SU(n)_{(p)})$ and $\gamma' \in \pi_{2n}(SU(n)_{(p)})$ so that

$$\langle \alpha', \beta' \rangle = (n-p+1)!(p-2)!\gamma'$$

in $\pi_{2n}(SU(n)_{(p)}) \cong \mathbf{Z}/n! \otimes \mathbf{Z}_{(p)} \cong \mathbf{Z}/p$. Since γ' is a generator of $\mathbf{Z}/p$, we have $\langle \alpha', \beta' \rangle \neq 0$. By Lemma 6.1, $[S^{2n-2p+3} \times S^{2p-3}, SU(n)_{(p)}]$ is not abelian, and so $[SU(n), SU(n)]$ is not abelian.

It remains to prove that $[SU(n), SU(n)]$ is not abelian for $n = 5, 7, 11$. The argument we now give applies to $SU(p)$ for any prime $p \geq 5$. Notice that p is regular for $SU(p)$, so it suffices to show that $[S^3 \times S^{2p-3}, SU(p)_{(p)}]$ is nonabelian. Since p is a regular prime for $SU(p)$, we choose generators $\alpha \in \pi_3(SU(p))$, $\beta \in \pi_{2p-3}(SU(p))$ and $\gamma \in \pi_{2p}(SU(p))$ so that

$$\langle \alpha', \beta' \rangle = (p-2)!\gamma' \neq 0 \in \mathbf{Z}/p! \otimes \mathbf{Z}_{(p)} \cong \mathbf{Z}/p.$$

Therefore $[S^3 \times S^{2p-3}, SU(p)_{(p)}]$ is nonabelian by Lemma 6.1.

The proof that $[Sp(n), Sp(n)]$ is not abelian for $n \geq 2$ is analogous: one uses Bott's result for Samelson products in $\pi_*(Sp(n))$ [4, Thm. 2] together with a quasi-regular decomposition for $Sp(n)$. We omit the details. $\square$

Corollary 6.3

(a) *For $n \geq 5$, $Z_\Sigma(SU(n)) \neq 0$, and $2 \leq t_\Sigma(SU(n)) \leq \lceil \log_2(n) \rceil$.*

(b) *For $n \geq 2$, $Z_\Sigma(Sp(n)) \neq 0$, and $2 \leq t_\Sigma(Sp(n)) \leq \lceil 2\log_2(n+1) \rceil$.*

Proof For an H-space Y, a commutator in $[X, Y]$ is an element of $Z_\Sigma(X, Y)$ [30, Thm. 7]. Thus, if $[Y, Y]$ is nonabelian, $2 \leq t_\Sigma(Y)$. The upper bound for $t_\Sigma(SU(n))$ comes from Proposition 2.5 since Singhof has shown that $\mathrm{cat}(SU(n)) = n$ [29]. The upper bound on $t_\Sigma(Sp(n))$ follows from Proposition 3.5. □

Schweitzer [27, Ex. 4.4] has shown that $\mathrm{cat}(Sp(2)) = 4$, so it follows from Proposition 2.5 that $t_\Sigma(Sp(2)) = 2$.

6.2 Some low dimensional Lie groups

Here we consider the Lie groups $SU(3)$, $SU(4)$, $SO(3)$ and $SO(4)$ and make estimates of t_Σ by either quoting known results or by ad hoc methods. We first deal with $SU(3)$ and $SU(4)$.

Proposition 6.4 *The groups $[SU(3), SU(3)]$ and $[SU(4), SU(4)]$ are not abelian.*

Proof For the group $SU(3)$ this follows from results of Ooshima [22, Thm. 1.2]. For $SU(4)$, observe that the prime 5 is regular for both $SU(4)$ and $Sp(2)$, so

$$SU(4)_{(5)} \equiv (S^3 \times S^5 \times S^7)_{(5)} \quad \text{and} \quad Sp(2)_{(5)} \equiv (S^3 \times S^7)_{(5)}.$$

If $[SU(4), SU(4)]$ is abelian, then so is $[SU(4)_{(5)}, SU(4)_{(5)}] \cong [S^3 \times S^5 \times S^7, SU(4)_{(5)}]$, and thus $[S^3 \times S^7, SU(4)_{(5)}]$ is abelian.

If $\alpha' \in \pi_3(Sp(2)_{(5)})$ and $\beta' \in \pi_7(Sp(2)_{(5)})$ are the images of generators of $\pi_3(Sp(2)) \cong \mathbf{Z}$ and $\pi_7(Sp(2)) \cong \mathbf{Z}$ then it follows from [4] that $\langle \alpha', \beta' \rangle \neq 0 \in \pi_{10}(Sp(2)_{(5)}) \cong \mathbf{Z}/5! \otimes \mathbf{Z}_{(5)} \cong \mathbf{Z}/5$.

Now we relate $SU(4)$ to $Sp(2)$ via the fibration $Sp(2) \xrightarrow{i} SU(4) \longrightarrow S^5$. The exact homotopy sequence of a fibration shows that $\pi_{10}(i)$ is an isomorphism after localizing at any odd prime. Since i is an H-map,

$$\langle i_*(\alpha'), i_*(\beta') \rangle = i_* \langle \alpha', \beta' \rangle \neq 0 \in \pi_{10}(SU(4)_{(5)}).$$

Thus $[S^3 \times S^7, SU(4)_{(5)}]$ is not abelian, so $[SU(4), SU(4)]$ cannot be abelian. □

Corollary 6.5

(a) $Z_\Sigma(SU(3)) \neq 0$, *and* $t_\Sigma(SU(3)) = 2$

(b) $Z_\Sigma(SU(4)) \neq 0$, *and* $t_\Sigma(SU(4)) = 2$.

Proof Since the groups $[SU(n), SU(n)]$ are not abelian for $n = 3$ and 4, $t_\Sigma(SU(3))$ and $t_\Sigma(SU(4))$ are at least 2. But $\operatorname{cat}(SU(n)) = n$ by [29], so the reverse inequalities follow from Proposition 2.5. □

Next we investigate the nilpotence of $SO(3)$ and $SO(4)$. This provides us with examples of non-simply-connected Lie groups.

Proposition 6.6 $Z_\Sigma(SO(3)) = 0$ *and* $Z_\Sigma(SO(4)) \neq 0$.

Proof Since $SO(3)$ is homeomorphic to $\mathbf{R}P^3$, the first assertion follows from Theorem 5.7. For the second assertion, recall that $SO(4)$ is homeomorphic to $S^3 \times SO(3)$. For notational convenience, we write $X = SO(3)$ and $Y = S^3$. We show that $Z_\Sigma(X \times Y) \neq 0$. Let $j : X \vee Y \longrightarrow X \times Y$ be the inclusion and $q : X \times Y \longrightarrow X \wedge Y$ be the quotient map. Consider

$$q^* : [X \wedge Y, X \times Y] \longrightarrow [X \times Y, X \times Y].$$

Notice that $\operatorname{Im}(q^*) \subseteq Z_\Sigma(X \times Y)$ because $q \in Z_\Sigma(X \times Y, X \wedge Y)$, so q induces a function $q^{**} : [X \wedge Y, X \times Y] \longrightarrow Z_\Sigma(X \times Y)$. Consider the exact sequence of groups

$$[\Sigma(X \times Y), X \times Y] \xrightarrow{\Sigma j^*} [\Sigma(X \vee Y), X \times Y] \longrightarrow [X \wedge Y, X \times Y] \xrightarrow{q^*} [X \times Y, X \times Y].$$

Since Σj^* has a left inverse, $\ker(q^*) = 0$. Thus q^{**} is one-one, so it suffices to show that $[X \wedge Y, X \times Y] \cong [\Sigma^3 SO(3), SO(3)] \oplus [\Sigma^3 SO(3), S^3]$ is nonzero. This follows from [33, Cor. 2.12], where it is shown that $[\Sigma^3 SO(3), S^3] \cong \mathbf{Z}/4 \oplus \mathbf{Z}/12$. □

Corollary 6.7 $t_\Sigma(SO(3)) = 1$ *and* $t_\Sigma(SO(4)) = 2$.

Proof We only have to show that $t_\Sigma(SO(4)) \leq 2$. The remark following Theorem 5.7 shows that if $f \in Z_\Sigma(X \times Y)$ then $f|_{X \vee Y} = 0$, so q^{**} is onto. Thus f factors through a sphere, so we can proceed as in the proof of Corollary 5.11. □

6.3 The group $\mathcal{E}_\Omega(Y)$

We conclude the section by relating $Z_\Sigma(Y)$ to a certain group of homotopy equivalences of Y. For any space X, let $\mathcal{E}_\Omega(X) \subseteq [X,X]$ be the group of homotopy equivalences $f: X \longrightarrow X$ such that $\Omega f = \mathrm{id}$. This group has been studied by Felix and Murillo [10] and by Pavesic [23]. We note that if Y is an H-space, then the function

$$\theta : Z_\Sigma(Y) \longrightarrow \mathcal{E}_\Omega(Y)$$

defined by $\theta(g) = \mathrm{id} + g$ is a bijection of pointed sets. In general θ does not preserve the binary operation in $Z_\Sigma(Y)$ and $\mathcal{E}_\Omega(Y)$. Thus $\mathcal{E}_\Omega(Y)$ is nontrivial whenever $Z_\Sigma(Y)$ is nontrivial.

Proposition 6.8 *The groups $\mathcal{E}_\Omega(Y)$ are nontrivial in the following cases: $Y = SU(n)$, $n \geq 3$; $Y = Sp(n)$, $n \geq 2$; and $Y = SO(4)$. The groups $\mathcal{E}_\Omega(Y)$ are trivial in the following cases: $Y = SU(2)$, $Sp(1)$, $SO(2)$ and $SO(3)$.*

7 Problems

In this brief section we list, in no particular order, a number of problems which extend the previous results or which have been suggested by this material.

1. Calculate $t_{\mathcal{F}}(X)$ for $\mathcal{F} = \mathcal{S}, \mathcal{M}$ or Σ and various spaces X. In particular, what is $t_\Sigma(\mathbf{HP}^n)$ for $n > 4$, and $t_\Sigma(Y)$ for compact Lie groups Y not considered in Section 6?
2. Find general conditions on a space X such that $Z_\Sigma(X,Y) = Z_{\mathcal{S}}(X,Y)$. One such was given in Section 5. Is $Z_\Sigma(Y) = Z_{\mathcal{S}}(Y)$ if Y is a compact simply-connected Lie group without homological torsion, such as $SU(n)$ or $Sp(n)$?
3. Find lower bounds for $t_{\mathcal{F}}(X)$ in the cases $\mathcal{F} = \mathcal{S}, \mathcal{M}$ or Σ in terms of other numerical invariants of homotopy type.
4. With $\mathcal{F} = \mathcal{S}, \mathcal{M}$ or Σ, characterize those spaces X such that $Z_{\mathcal{F}}(X) = 0$.
5. What is the relation between $\mathrm{kl}_\Sigma(X)$ and $\lceil \log_2(\mathrm{cat}(X)) \rceil$? In particular, if $\mathrm{kl}_\Sigma(X) < \infty$, does it follow that $\mathrm{cat}(X) < \infty$? Notice that both of these integers are upper bounds for $t_\Sigma(X)$.
6. Find an example of a finite H-complex Y such that $Z_\Sigma(Y) \neq Z_{\mathcal{S}}(Y)$ (see Section 4). In the notation of Section 6, this would yield a finite complex Y for which $\mathcal{E}_\Omega(Y) \neq \mathcal{E}_{\mathcal{S}}(Y)$. Such an example which is not a finite complex was given in [10].

7. Examine $t_{\mathcal{F}}(X), \mathrm{kl}_{\mathcal{F}}(X)$ and $\mathrm{cl}_{\mathcal{F}}(X)$ for various collections $\mathcal{F}$ such as the collection of p-local spheres or the collection of all cell complexes with at most two positive dimensional cells.
8. Investigate the Eckman-Hilton dual of the results of this paper. One defines a map $f : X \longrightarrow Y$ to be *$\mathcal{F}$-cotrivial* if $f^* = 0 : [Y, A] \longrightarrow [X, A]$ for all $A \in \mathcal{F}$. One could then study the set $Z^{\mathcal{F}}(X, Y)$ of all $\mathcal{F}$-cotrivial maps $X \longrightarrow Y$ and , in particular, the semigroup $Z^{\mathcal{F}}(X) = Z^{\mathcal{F}}(X, X)$.

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