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Immersed and virtually embedded π_1-injective surfaces in graph manifolds

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Abstract We show that many 3-manifold groups have no nonabelian surface subgroups. For example, any link of an isolated complex surface singularity has this property. In fact, we determine the exact class of closed graph-manifolds which have no immersed π_1-injective surface of negative Euler characteristic. We also determine the class of closed graph manifolds which have no finite cover containing an embedded such surface. This is a larger class. Thus, manifolds M^3 exist which have immersed π_1-injective surfaces of negative Euler characteristic, but no such surface is virtually embedded (finitely covered by an embedded surface in some finite cover of M^3).

AMS Classification 57M10; 57N10, 57R40, 57R42

Keywords π_1-injective surface, graph manifold, separable, surface subgroup

1 Introduction

It is widely expected that any closed 3-manifold M^3 with infinite fundamental group contains immersed π_1-injective surfaces. In fact, standard conjectures of Waldhausen, Thurston and others imply that some finite cover of M^3 has embedded π_1-injective surfaces. If M^3 is hyperbolic — or just simple and non-Seifert-fibered, i.e., conjecturally hyperbolic by the Geometrization Conjecture — then an immersed π_1-injective surface must have negative Euler characteristic.

We show here that many 3-manifolds have no immersed π_1-injective surfaces of negative Euler characteristic and that yet more 3-manifolds have no virtually embedded ones (an immersion of a surface S in M^3 is a *virtual embedding* if it can be lifted to an embedding of a finite cover of S in some finite cover of M^3). Minimal surface theory implies that any π_1-injective surface in an irreducible 3-manifold is homotopic to an immersed one.

Our results might suggest caution for the standard conjectures about hyperbolic manifolds. But the manifolds we study are graph-manifolds, that is, 3-manifolds with no non-Seifert-fibered pieces in their JSJ-decomposition. Thus, our results probably emphasise the very different behaviour of hyperbolic manifolds and graph manifolds rather than suggesting anything about what happens for hyperbolic M^3.

It is known that an immersed π_1-injective surface S of non-negative Euler characteristic is, up to isotopy, a collection of tori and Klein bottles immersed parallel to fibers in Seifert fibered pieces of the JSJ decomposition of M^3. It follows with little difficulty that S is virtually embedded. Thus an immersed π_1-injective surface which is not virtually embedded must have negative Euler characteristic.

The fact that graph manifolds can contain immersed π_1-injective surfaces which are not virtually separable (homotopic to virtually embedded) was first shown by H. Rubinstein and S. Wang [12], who in fact give a simple necessary and sufficient criterion for a given horizontal immersed surface to be virtually separable (the surface is *horizontal* if it is transverse to the fibers of the Seifert fibered pieces of M^3; this implies π_1-injective). They also show that, if a horizontal surface is virtually separable in M^3, then it is separable: it itself (rather than just some finite covering of it) lifts up to homotopy to an embedding in some finite cover of M^3. An embedded horizontal surface in a graph manifold M^3 is a fiber of a fibration of M^3 over S^1. A necessary and sufficient condition for virtual fibration of a graph manifold was given in [7].

Any infinite surface subgroup of $\pi_1(M^3)$ comes from a π_1-injective immersion of a surface to M^3. Moreover, if the subgroup is separable (i.e., the intersection of the finite index subgroups containing it), then the surface is separable [13]. Thus our results have purely group-theoretic formulations. In particular, we see that many infinite 3-manifold groups have the property that any surface subgroup is virtually abelian. In fact, it is easy to find examples with no Klein bottles (one must just avoid Siefert fibered pieces with non-orientable base) and thus see that many infinite 3-manifold groups have no non-abelian surface subgroups. For example, we show the fundamental group of a link of an isolated complex surface singularity always has this property.

Some of the results of Niblo and Wise [11] and [10] are also of interest in this context. For example, they show that subgroup separability fails for any graph manifold which is not a Seifert manifold or covered by a torus bundle and they show that a non-separable horizontal surface in a graph manifold can only be lifted to finitely many finite covers of M^3.

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After seeing an electronic preprint of this paper [9], Buyalo brought my attention to the interesting series of papers [1], [2], [3], [4] in which he and Kobel′skiĭ study existence of metrics of various types on graph manifolds. Of particular interest is their study of what they call "isometric geometrizations" in [2]. They cite [6] to say that the existence of such a geometrization is equivalent to the existence of a metric of non-positive sectional curvature. Their conditions are rather close to the conditions arising here and in [7], but it is not clear to the author why this is so.

2 Main results

From now on we assume M^3 is a closed connected graph-manifold, that is, a closed connected manifold obtained by pasting together compact Seifert fibered 3-manifolds along boundary components. We are interested in two properties:

(I) M^3 has an immersed π_1-injective surface of negative Euler characteristic;

(VE) M^3 has a virtually embedded π_1-injective surface of negative Euler characteristic (i.e., some finite cover of M^3 has an embedded such surface).

There is no loss of generality in assuming M^3 is irreducible, since a π_1-injective surface can be isotoped to be disjoint from any embedded S^2. The properties (I) and (VE) are preserved on replacing M^3 by a finite cover, so there is also no loss of generality in assuming M^3 is orientable. Moreover, if M^3 has a cover which is a torus bundle over S^1 then M^3 fails both properties (I) and (VE) so we may assume M^3 is not of this type. It is then easy to show (see [7] p.364) that, after passing to a double cover if necessary, we may assume that M^3 can be cut along a family of disjoint embedded π_1-injective tori into Seifert fibered pieces $M_1, \dots, M_s$ satisfying:

- Each M_i is Seifert fibered over orientable[1] base-orbifold of negative orbifold Euler characteristic;
- no M_i meets itself along one of the separating tori.

We may also assume that none of the separating tori is redundant. This means:

[1]Orientability is for convenience of proof and is not actually needed for our main Theorem 2.1.

- For each separating torus T the fibers of the Seifert pieces on each side of T have non-zero intersection number (which we denote $p(T)$) in T.

Each Seifert fibered piece M_i has a linear foliation of its boundary by the Seifert fibers of the adjacent Seifert fibered pieces. The rational Euler number e_i of the Seifert fibration of M_i with respect to these foliations is defined as $e_i = e(\hat{M}_i \to \hat{F}_i)$, where $\hat{M}_i \to \hat{F}_i$ is the closed Seifert fibration obtained by filling each boundary component of M_i by a solid torus whose meridian curves match the foliation. As in [7] we define the *decomposition matrix* for M^3 to be the symmetric matrix $A(M^3) = (A_{ij})$ with

$$\begin{aligned} A_{ii} &= e_i \\ A_{ij} &= \sum_{T \subset M_i \cap M_j} \frac{1}{|p(T)|} \quad (i \neq j), \end{aligned}$$

where the sum is over components T of $M_i \cap M_j$ and $p(T)$ is, as above, the intersection number in T of fibers from the two sides of T.

$A(M^3)$ is a symmetric rational matrix with non-negative off-diagonal entries. Moreover, the graph on s vertices, with an edge connecting vertices i and j if and only if $A_{ij} \neq 0$, is a connected graph. Given any matrix A with these properties, it is easy to realise it as $A(M^3)$ for some M^3.

By reordering indices we may put $A(M^3)$ in block form

$$\begin{pmatrix} P & Z \\ Z^t & N \end{pmatrix}$$

where P has non-negative diagonal entries and N has non-positive diagonal entries[2]. Let P_- be the result of multiplying the diagonal entries of P by -1 and put

$$A_-(M^3) := \begin{pmatrix} P_- & Z \\ Z^t & N \end{pmatrix}.$$

Theorem 2.1 *M^3 satisfies condition (I), that is, M^3 has an immersed π_1-injective surface of negative Euler characteristic, if and only if either $A_-(M^3)$ has a positive eigenvalue or it is negative and indefinite and all diagonal entries of $A(M^3)$ have the same sign (in which case $A_-(M^3)$ is negative semidefinite and M^3 even satisfies (VE)).*

M^3 satisfies condition (VE), that is, M^3 has a virtually embedded π_1-injective surface of negative Euler characteristic, if and only if one of P_- or N is not negative definite.

[2]Each zero diagonal entry can be put in either P or N. Notation here therefore differs from [7] where they were collected in their own block

(It is an elementary but not completely trivial exercise to see that the algebraic condition of this theorem for (I) follows from the algebraic condition for (VE). This also follows from the proof of the theorem.)

Example 2.1 If M^3 is the link of an isolated complex surface singularity then, as discussed in [7], $A(M^3)$ is negative definite (so $A(M^3) = A_-(M^3)$), so M^3 fails condition (I), and hence also fails (VE). Since it is known by [8] that the Seifert components of such an M^3 all have orientable base, M^3 has no immersed Klein bottles, so all infinite surface subgroups of $\pi_1(M^3)$ are abelian.

Example 2.2 If $M^3 = M_1 \cup M_2$ has just two Seifert components, so the decomposition matrix is $A(M^3) = (A_{ij})_{1\le i,j\le 2}$, put $D := (A_{11}A_{22})/A_{12}^2$. Theorem 2.1 implies:

$$\begin{aligned} M^3 \text{ satisfies (I)} \quad &\Leftrightarrow \quad -1 < D \le 1;\\ M^3 \text{ satisfies (VE)} \quad &\Leftrightarrow \quad 0 \le D \le 1. \end{aligned}$$

In [7] it was shown that this M^3 is virtually fibered over S^1 (i.e., has a finite cover that is fibered) if and only if either $0 < D \le 1$ or $A_{11} = A_{22} = 0$. Moreover, M^3 itself fibers over S^1 if and only if $D = 1$. The manifolds of this example were classified up to commensurability in [7] by two rational invariants, one of which is the above D.

One can ask also about compact graph manifolds M^3 with non-empty boundary. If we assume M^3 is orientable, irreducible and not one of the trivial cases $D^2\times S^1, T^2\times I$, or I-bundle over the Klein bottle then M^3 always has virtually embedded surfaces of negative Euler characteristic by [12]. In fact, Wang and Yu [14] show more: M^3 is virtually fibered over S^1. This can also be deduced using only matrix algebra (but a little effort) from [7], where a necessary and sufficient condition for virtual fibering of a closed graph manifold is given in terms of the decomposition matrix $A(M^3)$. This approach actually proves the stronger result (we omit details):

Theorem 2.2 *If M^3 is an oriented irreducible graph manifold with nonempty boundary then there exists a fibration $\partial M^3 \to S^1$ which extends to a virtual fibration of M^3 to S^1 (that is, each fiber of the virtual fibration is a virtually embedded surface whose boundary is parallel to the given fibration of ∂M^3).* □

3 Proofs

The necessary and sufficient condition for condition (VE) was proved in [7], so in this paper we just prove the condition for (I). We start with a discussion of Seifert fibered manifolds.

If $\pi\colon M \to F$ is a Seifert fibration, a proper immersion $f\colon S \to M$ of a surface S is *horizontal* if it is transverse to all fibers of π. Equivalently, $\pi \circ f$ is a covering map of S to the orbifold F.

Suppose $\pi\colon M \to F$ is a Seifert fibration with F connected and orientable and $\chi^{\text{orb}}(F) < 0$ (orbifold Euler characteristic). Assume M has non-empty boundary. On each torus $T \subset \partial M$ let a section m_T to the Seifert fibration be given. Then $e(M \to F)$ is defined with respect to these sections. We orient each m_T consistently with ∂F. Let $f\colon S \to M$ be a horizontal immersion of a surface S. Orient S so $\pi \circ f$ preserves orientation. Denote the boundary components of S that lie in T by $c_{T1}, \dots, c_{Tk_T}$. Using f to denote a generic fiber of π we have integers $a_{T\beta}, b_{T\beta}$, $\beta = 1, \dots, k_T$, so that the following homology relations hold:

$$[c_{T\beta}] = a_{T\beta}[m_T] + b_{T\beta}[f] \in H_1(T), \quad a_{T\beta} > 0. \tag{1}$$

Lemma 3.1 *If a is the degree of $\pi \circ f\colon S \to F$ then*

$$\sum_{\beta=1}^{k_T} a_{T\beta} = a \quad \text{for each } T \subset \partial M. \tag{2}$$

Moreover,

$$\sum_{T\subset\partial M} \sum_{\beta=1}^{k_T} b_{T\beta} = ae. \tag{3}$$

Conversely, suppose that for each boundary component T_β there is given a family $c_{T1}, \dots, c_{Tk_T}$ of immersed curves transverse to the fibers of π satisfying homology relations (1), and that equations (2) and (3) are satisfied. Then there exist integers $d_0 > 0, n_0 > 0$ so that for any positive integer multiple d of d_0 and n of n_0 the family of curves $c^n_{T\beta\gamma}$, $T \subset \partial M$, $\beta = 1, \dots, k_T$, $\gamma = 1, \dots, d$, obtained as follows, bounds an immersed horizontal surface. For $\gamma = 1, \dots, d$ we take $c^n_{T\beta\gamma}$ as a copy of the immersed curve obtained by going n times around the curve $c_{T\beta}$.

Proof The left side of equation (2) is $\deg(\pi\circ f|(S\cap T)\colon S\cap T\to\partial_T F)$, which is the degree of $\pi\circ f$, proving (2). Now the sum over all T and β of the curves $c_{T\beta}$ is null-homologous in M and using equation (2) this says $\sum_T a[m_T]+\big(\sum_{T,\beta}b_{T\beta}\big)[f]=0$. Equation (3) now follows from the fact that e can be defined by the equation $\sum_{T=1}^{l}[m_T]=-e[f]$ in $H_1(M;\mathbb{Q})$.

For the converse we first apply Lemma 5.1 of [7] which implies (taking d_0e to be integral in that lemma):

There are positive integers d_0 and n_0 such that $d_0e\in\mathbb{Z}$ and such that for any multiples d of d_0 and n of n_0 there is a covering $p\colon M'\to M$ satisfying

- The lifted Seifert fibration of M' has no singular fibers (so $M'\cong F'\times S^1$, where F' is the base surface for the fibration of M'),
- p has degree dn^2,
- each boundary torus T of M is covered by d boundary tori T_γ, $\gamma=1,\dots,d$ of M', each of which is a copy of the unique connected $(\mathbb{Z}/n\times\mathbb{Z}/n)$-cover of T.

Now each curve $c_{T\beta}$ lifts to n curves in T_γ, each still of slope $a_{T\beta}/b_{T\beta}$. Pick one of these and call it $c_{T\beta\gamma}$. If we can find a horizontal surface S' in M' spanning the family of curves $\{c_{T\beta\gamma} : T\subset\partial M,\beta=1,\dots,k_T,\gamma=1,\dots,d\}$, then its image in M is the desired surface.

The identification of M' with $F'\times S^1$ gives meridian curves $m'_{T\gamma}\in T_\gamma$ and with respect to these the Euler number of $M'\to F'$ is 0. Thus the curves $c_{T\beta\gamma}$ satisfy homology relations $c_{T\beta\gamma}=a_{T\beta}[m'_{T\gamma}]+b'_{T\beta}[f']$ for some $b'_{T\beta}$ with $\sum_{T,\beta}b'_{T\beta}=0$. We are thus looking for a connected surface S' mapping to $M'\times S^1$ by a map $(g,h)\colon S'\to F'\times S^1$ such that:

- the map g is a covering of degree da and the boundary component corresponding to T_γ of F' is covered by exactly k_T boundary components $\partial_{T\beta\gamma}S',\beta=1,\dots,k_T$ of S', with degrees $a_{T1},\dots,a_{Tk_T}$;
- the map h has degree $b'_{T\beta}$ on $\partial_{T\beta\gamma}S'$.

If S' is connected then the fact that $[S',S^1]=H^1(S';\mathbb{Z})$ and the exact cohomology sequence

$$H^1(S';\mathbb{Z})\to H^1(\partial S';\mathbb{Z})\to H^2(S',\partial S';\mathbb{Z})=\mathbb{Z}.$$

shows that $h\colon S'\to S^1$ exists with degree $b'_{T\beta}$ on each $\partial_{T\beta\gamma}S'$ if and only if $\sum b'_{T\beta}=0$. Thus the only issue is finding a connected cover S' of F' with $g\colon S'\to F'$ as above.

Since F' is a dn-fold cover of the orbifold F and F' has ld boundary components, where l is the number of boundary components of M, we have $2 - 2\,\mathrm{genus}(F') = dn\chi^{\mathrm{orb}}(F) + dl$, so $\mathrm{genus}(F') > 0$ as soon as n is chosen large enough. We therefore assume $\mathrm{genus}(F') > 0$. We also choose d_0 even. The existence of a connected cover with prescribed degree and boundary behaviour then follows from the following lemma, since the parity condition of the lemma is $da\chi(F') \equiv d\sum_T k_T \pmod 2$. □

Lemma 3.2 *If F' is an orientable surface of positive genus and a degree $\alpha \geq 1$ is specified and for each boundary component a collection of degrees summing to α is also specified, then a connected α-fold covering S' of F' exists with prescribed degrees on the boundary components over each boundary component of F' if and only if the prescribed number of boundary components of the cover has the same parity as $\alpha\chi(F')$.*

Proof This lemma appears to be well known, although weaker results have appeared several times in the literature. It is assumed implicitly in the proof of Lemma 2.2 of [12] (which has a minor error, since the parity condition is overlooked). The parity condition arises because the Euler characteristic of a compact orientable surface has the same parity as the number of its boundary components. Alternatively, if one represents the cover by a homomorphism of $\pi_1(F')$ to the symmetric group Sym_α of a fiber, the parity condition arises because the product of the permutations represented by boundary components is a product of commutators and is hence an even permutation. The existence of $S' \to F'$ with the given constraints can be seen by constructing a homomorphism of $\pi_1(F')$ to Sym_α with transitive image which maps the boundary curves to permutations with the desired cycle structure. Such a homomorphism exists by the result of Jacques et al. [5] that any even permutation on n symbols is a commutator of an n-cycle and an involution. □

We now return to the graph manifold M^3 of Section 2 which is glued together from Seifert fibered manifolds $M_1, \dots M_s$. For each M_i we choose an orientation of the base surface of the Seifert fibration. We can assume we have done this so that for each separating torus T the intersection number $p(T)$ of the Seifert fibers from the two sides of T is positive. Indeed, if this is not possible, then, as pointed out in [7], we can replace M^3 by a commensurable graph manifold M' with the same decomposition matrix for which it is possible (in fact M^3 and M' have a common 2-fold cover). From now on we will therefore assume all $p(T)$ are positive.

We first prove that $A_-(M^3)$ not being negative definite is necessary for having an immersed π_1-injective surface S in M^3 of negative Euler characteristic. As is proven in [12] (Lemma 3.3), such a surface is homotopic to an immersed surface whose intersection with each M_i consists of a union of horizontal surfaces and possibly also some π_1-injective vertical annuli. We will therefore assume that our surface has already been put in this position. For the moment we assume also, for simplicity, that S is horizontal in M^3, that is, vertical annuli do not occur.

Fix an index i and consider the intersection of our immersed surface S with M_i. We orient this immersed surface in M_i so that it maps orientation preservingly to the base surface of M_i. We also choose meridian curves in the boundary tori of M_i and thus obtain a collection of integer pairs $(a_{T\beta}, b_{T\beta})$ as in Lemma 3.1 satisfying the relations of that lemma. Note that the e in that lemma is not e_i, since it is Euler number with respect to the chosen meridians rather than with respect to the Seifert fibers of neighbouring Seifert fibered pieces to M_i. We denote it therefore e_i'. We denote the degree a appearing in the lemma by a_i.

Our orientation of $S\cap M_i$ induces an orientation on each boundary curve of this surface. Each such curve also inherits an orientation from the piece of surface it bounds in a neighbouring Seifert fibered piece. Call a curve *consistent* if these two orientations agree. For fixed T denote by a_T^+ the sum of the $a_{T\beta}$'s corresponding to consistent curves and a_T^- the sum of the remaining $a_{T\beta}$'s. Define b_T^+ and b_T^- similarly. Thus equations (2) and (3) become

$$a_T^+ + a_T^- = a_i \quad \text{for each } T \subset \partial M_i, \tag{4}$$

$$\sum_{T\subset \partial M_i} (b_T^+ + b_T^-) = a_i e_i'. \tag{5}$$

For given $T \subset \partial M_i$ we denote by T' the same torus considered as a boundary component of the Seifert piece M_j adjacent to M_i across T. The pair (a_T^+, b_T^+) gives coordinates of the homology class represented by the sum of the consistent curves in T with respect to the basis of $H_1(T)$ coming from meridian and fiber in M_i. The same homology class will be given by a pair $(a_{T'}^+, b_{T'}^+)$ with respect to meridian and fiber in M_j with

$$\begin{pmatrix} a_{T'}^+ \\ b_{T'}^+ \end{pmatrix} = \begin{pmatrix} q(T) & p(T) \\ -p'(T) & -q'(T) \end{pmatrix} \begin{pmatrix} a_T^+ \\ b_T^+ \end{pmatrix}, \tag{6}$$

where the square matrix is the appropriate change-of-basis matrix. Our notation for this matrix agrees with page 366 of [7]; in particular, $p(T)$ has its meaning of intersection number of fibers of M_i and M_j in T. The matrix has determinant -1, since T has opposite orientations viewed from M_i and M_j. We also have:

$$\begin{pmatrix} a^-_{T'} \\ b^-_{T'} \end{pmatrix} = -\begin{pmatrix} q(T) & p(T) \\ -p'(T) & -q'(T) \end{pmatrix} \begin{pmatrix} a^-_T \\ b^-_T \end{pmatrix}. \tag{7}$$

The first entries of matrix equations (6) and (7) are the equations

$$a^\pm_{T'} = \pm\big(q(T)a^\pm_T + p(T)b^\pm_T\big) \tag{8}$$

that we can solve for $b^\pm_T$ in terms of $a^\pm_T$ and $a^\pm_{T'}$ to give:

$$b^\pm_T = (\pm a^\pm_{T'} - q(T)a^\pm_T)/p(T). \tag{9}$$

Equation (5) thus becomes:

$$\sum_{T\subset\partial M_i}\left(\frac{a^+_{T'} - q(T)a^+_T}{p(T)} + \frac{-a^-_{T'} - q(T)a^-_T}{p(T)}\right) = a_i e'_i. \tag{10}$$

Using equation (4) this becomes

$$\sum_{T\subset\partial M_i}\frac{a^+_{T'} - a^-_{T'}}{p(T)} = a_i\left(e'_i + \sum_{T\subset\partial M_i}\frac{q(T)}{p(T)}\right). \tag{11}$$

As discussed on page 366 of [7], $q(T)/p(T)$ is the change of Euler number $e(M_i \to F_i)$ on replacing the meridian at T by the fiber of M_j. Thus the right side of (11) is a_ie_i, so equation (11) says

$$\sum_{T\subset\partial M_i}\frac{a^+_{T'} - a^-_{T'}}{p(T)} = a_i e_i. \tag{12}$$

Consider the summands on the left with $T\subset \partial M_i \cap \partial M_j$. Since $a^+_{T'} + a^-_{T'} = a_j$ and $a^+_{T'}$ and $a^-_{T'}$ are both non-negative, each summand is no larger in magnitude than $a_j/p(T)$. Their sum is therefore no larger in magnitude than

$$a_j\left(\sum_{T\subset\partial M_i\cap\partial M_j}\frac{1}{|p(T)|}\right) = a_j A_{ij}.$$

We write their sum therefore as $-a_jA'_{ij}$ with $|A'_{ij}| \le A_{ij}$, so (12) becomes

$$-\sum_{j\ne i} A'_{ij}a_j = a_ie_i. \tag{13}$$

Recalling that $e_i = A_{ii}$ and putting $A'_{ii} = A_{ii}$ we can rewrite this as

$$\sum_{j=1}^{s} A'_{ij}a_j = 0. \tag{14}$$

We have thus shown that the decomposition matrix $A(M^3)$ has the property that it can be made to have non-trivial kernel by replacing each off-diagonal entry by some rational number of no larger magnitude. The fact that $A_-(M^3)$ is not negative definite thus follows from the following lemma.

If A is a matrix with non-negative off-diagonal entries then we will use the term *reduction of* A for a matrix A' with $|A'_{ij}| \leq A_{ij}$ for all $i \neq j$ and $A'_{ii} = A_{ii}$ for all i.

Lemma 3.3 *Let $A = (A_{ij})$ be a square symmetric matrix over $\mathbb{Q}$ with $A_{ij} \geq 0$ for $i \neq j$. Then there exists a (not necessarily symmetric) singular rational reduction $A' = (A'_{ij})$ of A if and only if the matrix A_- (obtained by replacing each positive diagonal entry of A by its negative) is not negative definite. Moreover such an A' can then be found which annihilates a non-zero vector with non-negative entries.*

We postpone the proof of this Lemma and first return to the proof of Theorem 2.1. The fact that $A_-(M^3)$ is not negative definite is not quite proved, since we assumed vertical annuli do not exist in our π_1-injective surface. If we do have vertical annuli we choose orientations on them. Then we can characterise their boundary components as consistent or non-consistent as before. Equations (4) and (5) then still hold, so the above proof goes through unchanged.

For the converse, suppose that the decomposition matrix $A(M^3) = (A_{ij})$ is not negative. We shall show that this actually implies the existence of a horizontal surface (i.e., with no vertical annuli). Our condition on $A_-(M^3)$ is that it has a positive eigenvalue, which is an open condition, so we can reduce each non-zero off-diagonal entry slightly without changing it. By the above lemma we can thus assume there exists a rational matrix (A'_{ij}) with $|A'_{ij}| < A_{ij}$ for each $i \neq j$ with $A_{ij} \neq 0$ and with $A'_{ii} = A_{ii}$ for each i such that equation (14) (or the equivalent equation (13)) holds for some non-zero vector $(a_1, \ldots, a_s)$ with non-negative rational entries. For each $i \neq j$ we then define $a^+_{T'}$ and $a^-_{T'}$, for each boundary torus T' of M_j that lies in $M_i \cap M_j$, by the equations

$$a^+_{T'} = \frac{A_{ij} - A'_{ij}}{2A_{ij}}\, a_j$$

$$a^-_{T'} = \frac{A_{ij} + A'_{ij}}{2A_{ij}}\, a_j.$$

Note that these imply that $a^\pm_{T'} > 0$ whenever $a_j \neq 0$ and

$$a^+_{T'} + a^-_{T'} = a_j$$

$$a^+_{T'} - a^-_{T'} = -(A'_{ij}/A_{ij})a_j.$$

Thus, equation (4) holds, and, working backwards via equations (12), (11) and (10) we see that (5) holds if we define $b_T^{\pm}$ by equation (9). Moreover, by multiplying our original vector (a_j) by a suitable positive integer we may assume that the $a_T^{\pm}$ and $b_T^{\pm}$ are all integral.

Now, (9) is equivalent to (8) which can also be written

$$a_T^{\pm} = \pm\bigl(q(T')a_{T'}^{\pm} + p(T')b_{T'}^{\pm}\bigr), \tag{15}$$

by exchanging the roles of T and T'. But $q(T') = q'(T)$ and $p(T') = p(T)$. In fact

$$\begin{pmatrix} q(T') & p(T') \\ -p'(T') & -q'(T') \end{pmatrix} = \begin{pmatrix} q'(T) & p(T) \\ -p'(T) & -q(T) \end{pmatrix}, \tag{16}$$

since the coordinate change matrix for T' is the inverse of the one for T. Thus (15) implies

$$a_T^{\pm} = \pm\bigl(q'(T)a_{T'}^{\pm} + p(T)b_{T'}^{\pm}\bigr). \tag{17}$$

Inserting (8) in (17) and simplifying, using the fact that $1 - q'(T)q(T) = -p'(T)p(T)$, gives $p(T)b_{T'}^{\pm} = \pm\bigl(p'(T)p(T)a_T^{\pm} + q'(T)p(T)b_T^{\pm}\bigr)$, whence

$$b_{T'}^{\pm} = \pm\bigl(p'(T)a_T^{\pm} + q'(T)b_T^{\pm}\bigr). \tag{18}$$

With equation (8) this gives the matrix equations (6) and (7) which imply that the curve $c_T^{\pm}$ in T defined by coordinates $(a_T^{\pm}, b_T^{\pm})$ with respect to meridian and fiber in M_i is the same as the curve in T' defined by $(a_{T'}^{\pm}, b_{T'}^{\pm})$ with respect to meridian and fiber in M_j. We thus have a pair of curves in each separating torus so that the curves in the boundary tori of each Seifert piece M_i satisfy the numerical conditions of Lemma 3.1. By that Lemma, we can find d and n so that if we replace each of the curves c in question by d copies of the curve c^n, then the curves span a horizontal surface in each M_i. These surfaces fit together to give the desired surface in M^3.

It remains to discuss the case that $A_-(M^3)$ is negative but not definite. We postpone this until after the proof of the lemma.

Proof of Lemma 3.3 We first note that if A has a singular reduction then it has a reduction that annihilates a vector with non-negative entries. Indeed, if we have a reduction A' that annihilates the non-trivial vector (x_i), then for each i with $x_i < 0$ we multiply the i-th row and column of A' by -1. The result is a reduction A'' which annihilates $(|x_i|)$. We next note that the property of A having a singular reduction is unchanged if we change the sign of any diagonal entry of A, since if A' is a singular reduction for A then multiplying the corresponding row of A' by -1 gives a singular reduction of the modified matrix. Thus, we may assume without loss of generality that our initial matrix A has non-positive diagonal entries.

Suppose A is symmetric with non-negative off-diagonal entries and non-positive diagonal entries and suppose A has a singular reduction A', say $A'x = 0$ with x a non-zero vector. Then $x^t(A' + (A')^t)x = 0$, so $\frac{1}{2}(A' + (A')^t)$ is an indefinite symmetric reduction of A. In [7] it is shown that a symmetric reduction of a negative definite matrix with non-negative off-diagonal entries is again negative definite. Thus A is not negative definite.

Conversely, suppose A is a rational symmetric matrix with non-negative off-diagonal entries and non-positive diagonal entries and suppose A is not negative definite. We want to show the existence of a singular rational reduction of A. If A is itself singular we are done, so we assume A is non-singular. Assume first that only one eigenvalue of A is positive. Consider a piecewise linear path in the space of reductions of A that starts with A and reduces each off-diagonal entry to zero, one after another. This path ends with the diagonal matrix obtained by making all off-diagonal entries zero, which has only negative eigenvalues, so the determinant of A changes sign along this path. It is thus zero at some point of the path. Since determinant is a linear function of each entry of the matrix, the first point where determinant is zero is at a matrix with rational entries. We have thus found a rational singular reduction of A. If A has more than one positive eigenvalue, consider the smallest principal minor of A with just one non-negative eigenvalue. First reduce all off-diagonal entries that are not in this minor to zero and then apply the above argument just to this minor. $\square$

To complete the proof of Theorem 2.1 we must discuss the case that $A_-(M^3)$ is negative indefinite. We need some algebraic preparation.

Let A be a symmetric $s \times s$ matrix. The s-vertex graph with an edge connecting vertices i and j if and only if $A_{ij} \neq 0$ will be called the *graph of* A. The submatrices of A corresponding to components of this graph will be called the *components* of A. By reordering rows and columns, A can be put in block diagonal form with its components as the diagonal blocks. If A has just one component we call A *connected.*

Proposition 3.4 *Let A be a symmetric $s \times s$ matrix with non-negative off-diagonal entries such that A is connected. Then A is negative if and only if there exists a vector $\mathbf{a} = (a_j)$ with positive entries such that $A\mathbf{a}$ has non-positive entries. Moreover, in this case A is negative definite unless $A\mathbf{a} = 0$, in which case $\mathbf{a}$ generates the kernel of A.*

Proof Suppose $\mathbf{a}$ has positive entries. For any vector $\mathbf{x} = (x_j)$ we can write

$$\mathbf{x}^t A\mathbf{x} = \sum_i a_i \left(\sum_j A_{ij}a_j\right)\left(\frac{x_i}{a_i}\right)^2 + \sum_{i<j}(-A_{ij}a_ia_j)\left(\frac{x_i}{a_i} - \frac{x_j}{a_j}\right)^2. \qquad (19)$$

If $A\mathbf{a}$ has non-positive entries then both terms on the right are clearly non-positive, proving that A is negative. Moreover, since the s-vertex graph determined by nonvanishing of A_{ij} is connected, the second term on the right vanishes if and only if $x_i/a_i = x_j/a_j$ for all i, j, that is, $\mathbf{x}$ is a multiple of $\mathbf{a}$. In this case the first term vanishes if and only if $\mathbf{x} = 0$ or $A\mathbf{a} = 0$.

Conversely, suppose A is a symmetric negative matrix with non-negative off-diagonal entries. Then its diagonal entries are non-positive, and if any diagonal entry is zero then all other entries in the corresponding row and column must be zero. If a diagonal entry is non-zero, then, since it is negative, we can add positive multiples of the row and column containing it to other rows and columns, to make zero all off-diagonal entries in its row and column. This preserves the properties of A of being a symmetric negative matrix with non-negative off-diagonal entries. It thus follows that we can reduce A to a diagonal matrix using only "positive" simultaneous row and column operations, so we have $P^tAP = D$ where P is invertible with only non-negative entries and D is diagonal. If A is non-singular then $A^{-1}=PD^{-1}P^t$ and this is a matrix with non-positive entries. Thus the negative sum of the columns of A^{-1} is a vector $\mathbf{a}$ with positive entries and $A\mathbf{a} = (-1, \dots, -1)^t$, so $\mathbf{a}$ is as required. If A is singular, then D has a zero entry, and the corresponding column of P is a non-trivial vector $\mathbf{a}$ with non-negative entries such that $A\mathbf{a} = 0$. Thus, in this case $\mathbf{a}$ is as required if we show that it has no zero entries. Suppose $\mathbf{a}$ did have zero entries. By permuting rows and columns of A we can assume they are the last few entries of $\mathbf{a}$, so $\mathbf{a} = \begin{pmatrix}\mathbf{a}_0\\ \mathbf{0}\end{pmatrix}$ with no zero entries in $\mathbf{a}_0$, and A has block form $\begin{pmatrix}A_0 & B^t\\ B & A_1\end{pmatrix}$ with $\begin{pmatrix}A_0\\ B\end{pmatrix}\mathbf{a}_0 = 0$. Since B has non-negative entries and $\mathbf{a}_0$ has only positive entries, this implies $B = 0$. This contradicts the connectedness of A. □

Corollary 3.5 *Suppose A is a symmetric connected negative matrix with non-negative off-diagonal entries. If A^s is a symmetric reduction of A such that some off-diagonal entry has been reduced in absolute value then A^s is negative definite.*

Proof Let $\mathbf{a}$ be a vector with positive entries such that $A\mathbf{a}$ has non-positive entries.

Suppose first that A^s has non-negative off-diagonal entries. Then $A^s\mathbf{a}$ has non-positive entries and is non-zero, so A^s is negative definite by the preceding proposition. In general, let A' be the reduction of A with $A'_{ij} = |A^s_{ij}|$ for $i \neq j$. Then A' is negative definite by what has just been said, and A^s is a reduction of A', so A^s is negative definite by [7]. □

Corollary 3.6 *Suppose A is a symmetric connected matrix with non-negative off-diagonal entries for which A_- (the result of multiplying positive diagonal entries of A by -1) is negative indefinite. Let A' be a singular reduction of A and A'' the result of multiplying each row of A' with positive diagonal entry by -1. Then A'' is symmetric and it satisfies $|A''_{ij}| = A_{ij}$ for all $i \neq j$.*

Proof Since A'' is singular, its symmetrization $A^s = \frac{1}{2}(A'' + (A'')^t)$ is not negative definite (since $A''\mathbf{x} = 0$ implies $\mathbf{x}^t A^s \mathbf{x} = 0$). By Corollary 3.5 the entries of A^s are therefore the same in absolute value as the entries of A. This implies that A'' was already symmetric and its entries are the same in absolute value as those of A. □

Suppose now that $A_- = A_-(M^3)$ is negative indefinite and M^3 satisfies condition (I). We want to show that the block decomposition

$$A = \begin{pmatrix} P & Z \\ Z^t & N \end{pmatrix}$$

of A is trivial. Suppose we have a reduction A' of $A = A(M^3)$ is realised by a π_1-injective surface as in the proof of the necessary condition of the main theorem. Let (a_j) be as in that proof, so it is a non-trivial vector with non-negative integer entries which A' annihilates. Let A_{ij} be a non-zero entry of the block Z. Then Corollary 3.6 implies that $A'_{ij} = -A'_{ji} = \pm A_{ij}$. If $a_j = 0$ we could replace A'_{ij} by zero, which is impossible by Corollary 3.6, so we may assume $a_j > 0$. The condition $|A'_{ij}| = A_{ij}$ implies that either all the a^+_T with $T \subset M_i \cap M_j$ are zero (if $A'_{ij} = A_{ij}$) or all the a^-_T with $T \subset M_i \cap M_j$ are zero (if $A'_{ij} = -A_{ij}$). The fact that $A'_{ji} = -A'_{ij}$ implies that the corresponding $b^\pm_T$'s are not zero. Such $(a^\pm_T, b^\pm_T) = (0, b^\pm_T)$ must come from vertical annuli. The fiber coordinates of boundary components of vertical annuli sum to zero. Using equation (9) this says

$$\sum A'_{ij} a_j = 0, \quad \text{sum over } j \text{ with } A_{ij} \text{ in } Z.$$

Subtracting this equation from equation (14) we see that the reduction of A obtained by replacing the A'_{ij} corresponding to entries of Z by zero also annihilates the vector (a_j). This contradicts Corollary 3.6, so the block decomposition of A was trivial.

Conversely, if the above block decomposition of A is trivial, that is, either $A = N$ or $A = P$, then M^3 satisfies (VE), so it certainly satisfies (I). $\square$

References

[1] Buyalo, S.V.; Kobel′skiĭ, V.L., Geometrization of graphmanifolds I. conformal geometrization, St. Petersburg Math. J. **7** (1996), 185–216.

[2] Buyalo, S.V.; Kobel′skiĭ, V.L., Geometrization of graphmanifolds. II isometric geometrization, St. Petersburg Math. J. **7** (1996), 387–404.

[3] Buyalo, S.V.; Kobel′skiĭ, V.L., Geometrization of infinite graph-manifolds, St. Petersburg Math. J. **8** (1997), 413–427.

[4] Buyalo, S.V.; Kobel′skiĭ, V.L., Generalized graphmanifolds of nonpositive curvature, St. Petersburg Math. J. **11** (2000), 251–268.

[5] Jacques, Alain; Lenormand, Claude; Lentin, André; Perrot, Jean-François, Un résultat extrémal en théorie des permutations. C. R. Acad. Sci. Paris Sér. A-B **266** (1968), A446–A448.

[6] Leeb, B., 3-manifolds with(out) metrics of nonpositive curvature, Invent. Math. **122** (1995), 277–289.

[7] Neumann, W.D., Commensurability and virtual fibration for graph manifolds. Topology **39** (1996), 355–378.

[8] Neumann, W.D., A calculus for plumbing applied to the topology of complex surface singularities and degenerating complex curves. Trans. Amer. Math. Soc. **268** (1981), 299–343.

[9] Neumann, W.D., Immersed and virtually embedded π_1-injective surfaces in graph manifolds, e-print `arxiv:math.GT/9901085:v1`, 21 Jan. 1999.

[10] Niblo, Graham A.; Wise, Daniel T., The engulfing property for 3-manifolds. Geometry and Topology Monographs **1** The Epstein Birthday Schrift (1998), 413–418.

[11] Niblo, Graham A.; Wise, Daniel T., Subgroup separability, knot groups and graph manifolds. Proc. Amer. Math. Soc. **129** (2001), no. 3, 685–693

[12] Rubinstein, J. Hyam; Wang, Shicheng, π_1-injective surfaces in graph manifolds. Comment. Math. Helv. **73** (1998), 499–515.

[13] Scott, G.P., Subgroups of surfaced groups are almost geometric. J. London Math. Soc. **17** (1978), 555–565.

[14] Wang, Shicheng; Yu, Fengchun, Graph manifolds with non-empty boundary are covered by surface bundles. Math. Proc. Cambridge Philos. Soc. **122** (1997), 447–455.

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