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On the intersection forms of spin 4–manifolds bounded by spherical 3–manifolds

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Abstract We determine the contributions of isolated singularities of spin V 4–manifolds to the index of the Dirac operator over them. From these data we derive certain constraints on the intersection forms of spin 4–manifolds bounded by spherical 3–manifolds, and also on the embeddings of the real projective planes into 4–manifolds.

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Keywords Spin 4–manifold, spherical 3–manifold, Dirac operator

The 10/8–theorem [Fu] and its V manifold version [FF] have provided several results about the intersection forms of spin 4–manifolds. For example, these theorems were used to show the homology cobordism invariance of the Neumann-Siebenmann invariant for certain Seifert homology 3–spheres in [FFU], and for all Seifert homology 3–spheres by Saveliev [Sa]. For this purpose in [FFU] we studied the index of the Dirac operator over spin V 4–manifolds, in particular those with only isolated singular points whose neighborhoods are cones over lens spaces. The spin V manifolds considered in [Sa] are also of the same type, although they are different from those considered in [FFU]. For a closed spin V 4–manifold X, the index of the Dirac operator over X is represented as

$$\operatorname{ind} D(X) = -(\operatorname{sign} X + \delta(X))/8,$$

where $\operatorname{sign} X$ is the signature of X and $\delta(X)$ is the contribution of the singular points to the index of the Dirac operator, which is determined only by the data on the neighborhoods of the singular points according to the V-index theorem [K2]. In particular if all the singular points are isolated, $\delta(X)$ is the sum of the contributions $\delta(x)$ of the singular points x. In [FFU] we showed that $\delta(x)$ for the case when the neighborhood of x is a cone over a lens space is determined by simple recursive formulae. In this paper we determine the value $\delta(x)$ for every isolated singularity x, and combining such data with the 10/8 theorem, we derive certain information on the intersection form of a spin 4–manifold bounded by a spherical 3–manifold equipped with a spin structure. We also apply this to the embeddings of the real projective plane into 4–manifolds.

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1 The V manifold version of the 10/8 theorem

First let us recall the theorem in [FF], which coincides with the 10/8 theorem for the case of non-singular spin 4–manifolds.

Theorem 1 [FF] *Let Z be a closed spin V 4–manifold with $b_1(Z)=0$. Then either $\operatorname{ind} D(Z)=0$ or*

$$1-b^-(Z)\le \operatorname{ind} D(Z)\le b^+(Z)-1.$$

Since $\operatorname{ind} D(Z)$ is even, we have $\operatorname{ind} D(Z)=0$ if $b^\pm(Z)\le 2$.

A direct application of this theorem leads us to the following result.

Proposition 1 *Suppose that a 3–manifold M with a spin structure c bounds a spin V manifold X with isolated singularities with $b_1(X)=0$.*

(1) *If M also bounds a spin 4–manifold Y, then either $\operatorname{sign}(X)+\delta(X)=\operatorname{sign}(Y)$, or both of the following inequalities hold.*

$$\begin{aligned} b^+(Y)-9b^-(Y)&\le \operatorname{sign}(X)+\delta(X)+8b^+(X)-8\\ 9b^+(Y)-b^-(Y)&\ge \operatorname{sign}(X)+\delta(X)-8b^-(X)+8. \end{aligned}$$

In either case, $\operatorname{sign}(X)+\delta(X)$ must be equal to $\operatorname{sign}(Y) \pmod{16}$, which is the Rochlin invariant $R(M,c) \pmod{16}$ of (M,c)

(2) *If both $b^+(X)\le 2$ and $b^-(X)\le 2$ and M bounds a $\mathbf{Q}$ acyclic spin 4–manifold, then $\operatorname{sign}(X)+\delta(X)=0$.*

Proof We can assume that $b_1(Y)=0$, for otherwise we can perform a spin surgery to get a new spin 4–manifold Y' with $b_1(Y')=0$, $b^\pm(Y)=b^\pm(Y')$, and $\operatorname{sign} Y=\operatorname{sign} Y'$. The first claim comes from the application of Theorem 1 to the index of the Dirac operator on $X\cup(-Y)$, given by

$$\operatorname{ind} D(X\cup(-Y))=-(\operatorname{sign} X-\operatorname{sign} Y+\delta(X))/8.$$

Note that the value in the parentheses on the right hand side must be divisible by 16 since the index (over $\mathbf{C}$) of the Dirac operator associated with the spin structure is even. To prove the second claim, suppose that M bounds a spin $\mathbf{Q}$ acyclic 4–manifold Y. Then Theorem 1 shows that either $\operatorname{sign}(X)+\delta(X) = \operatorname{sign}(Y) = 0$ or

$$-1 \le 1 - b^-(X) \le \operatorname{ind} D(X \cup (-Y)) = -(\operatorname{sign} X + \delta(X))/8 \le b^+(X) - 1 \le 1.$$

Since $\operatorname{ind} D(X \cup (-Y))$ is even, we obtain the desired result. □

In case of spherical 3–manifolds we obtain the following stronger result.

Proposition 2 *Let S be a spherical 3–manifold equipped with the spin structure c, and $\delta(S,c)$ be the contribution of the cone cS over S to the index of the Dirac operator (we will show in the next section that such a contribution is determined only by (S,c)).*

(1) *If S bounds a spin 4–manifold Y, then either $\operatorname{sign}(Y) = \delta(S,c)$ or*

$$b^+(Y) - 9b^-(Y) \le \delta(S,c) - 8 \text{ and } 9b^+(Y) - b^-(Y) \ge \delta(S,c) + 8.$$

(2) *Suppose that for some k the connected sum kS of k copies of S (equipped with the spin structure induced by c) bounds a $\mathbf{Q}$ acyclic spin 4–manifold (whose spin structure is an extension of the given one on kS), then $\delta(S,c) = 0$ (and hence $R(S,c) \equiv 0 \pmod{16}$). In particular any $\mathbf{Z}_2$ homology 3–sphere S with $\delta(S) \neq 0$ (or $R(S) \not\equiv 0 \pmod{16}$) has infinite order in the homology cobordism group $\Theta^3_{\mathbf{Z}_2}$ of $\mathbf{Z}_2$ homology 3–spheres.*

Proof Again it suffices to prove the claim for the case when $b_1(Y) = 0$. We can apply Proposition 1 by putting $X = cS$ to prove the first claim. (We will prove in the next section that c extends uniquely to the spin structure on cS.) In this case $\operatorname{sign} X = b^{\pm}(X) = 0$ and $\delta(X) = \delta(S,c)$. To prove the second claim suppose that kS bounds a spin $\mathbf{Q}$ acyclic 4–manifold Y. Then applying Theorem 1 to the closed spin V manifold Z obtained by gluing the boundary connected sum of k copies of cS and $-Y$, we have $\operatorname{ind} D(Z) = 0$ since $b^{\pm}(Z) = 0$. Since

$$\operatorname{ind} D(Z) = -(k \operatorname{sign}(cS) - \operatorname{sign} Y + k\delta(cS))/8$$

and $\operatorname{sign}(Y) = 0$, we obtain the desired result. □

2 Contributions from the cones over the spherical 3–manifolds to the index of the Dirac operator

Let Z be a closed spin V 4–manifold with a spin structure c whose singularties consist of isolated points $\{x_1,\dots,x_k\}$. Then the V index theorem [K2] shows that the V index over $\mathbf{C}$ of the Dirac operator over Z is described as

$$\operatorname{ind} D(Z)=\int_Z(-p_1(Z)/24)+\sum_{i=1}^k\delta_D(x_i),$$

where $\delta_D(x_i)$ is a contribution from the singular point x_i, which is described as follows. We omit the subscript i for simplicity. Suppose that the neighborhood $N(x)$ of x is represented as D^4/G (which is the cone over the spherical 3–manifold $S=S^3/G$). Here G is a finite subgroup of $SO(4)$ that acts freely on S^3. The restriction of D to $N(x)$ is covered by a G invariant Dirac operator $\widetilde{D}$ over D^4 and the normal bundle over x in Z is covered by a normal bundle N over 0 in D^4, which is identified with $\mathbf{C}^2$. Then we have

$$\delta_D(x)=\sum_{(g)\subset(G),g\neq1}\frac{1}{m_g}\cdot\frac{ch_g j^*\sigma(\widetilde{D})}{ch_g\lambda_{-1}(N\otimes\mathbf{C})}$$

where $j\colon\{0\}\subset D^4$ is the inclusion, m_g denotes the order of the centralizer of g in G and the sum on the right hand side ranges over all the conjugacy classes of G other than the identity. On the other hand the signature of Z (which is the index of the signature operator D_{sign} over Z) is given by

$$\operatorname{sign}(Z)=\int_Z p_1(Z)/3+\sum_{i=1}^k\delta_{D_{\text{sign}}}(x_i),$$

where the local contribution $\delta_{D_{\text{sign}}}(x)$ from x to $\operatorname{sign}(Z)$ is described as

$$\delta_{D_{\text{sign}}}(x)=\sum_{(g)\subset(G),g\neq1}\frac{1}{m_g}\cdot\frac{ch_g j^*\sigma(\widetilde{D}_{\text{sign}})}{ch_g\lambda_{-1}(N\otimes\mathbf{C})}.$$

Here D_{sign} over $N(x)$ is covered by a G invariant signature operator $\widetilde{D}_{\text{sign}}$ as before [K1]. Hence we have

$$\operatorname{ind} D(Z)=-\frac{1}{8}(\operatorname{sign} Z+\sum_{i=1}^k\delta(x_i)),$$

where

$$\delta(x)=-(\delta_{D_{\text{sign}}}(x)+8\delta_D(x))$$

and we put $\delta(Z) = \sum_{i=1}^{k} \delta(x_i)$. If $N(x)$ is the cone over $S = S^3/G$ we write $\delta(x) = \delta(S,c)$, where c denotes the spin structure on S induced from that on Z, since we will see later that $\delta(x)$ is determined completely by (S,c). In [FFU] $\delta(S,c)$ in the case when $S = L(p,q)$ is given explicitly as follows. The spin structure c on the cone $cL(p,q)$ over $L(p,q)$ is determined by the choice of the complex line bundle $\widetilde{K}$ over D^4 that is a double covering of the canoncial bundle K over $cL(p,q)$. Here $\widetilde{K}$ is the quotient space of $D^4 \times \mathbf{C}$ by the cyclic group $\mathbf{Z}_p$ of order p so that the action of the generator g of $\mathbf{Z}_p$ is given by

$$g(z_1, z_2, w) = (\zeta z_1, \zeta^q z_2, \epsilon\zeta^{-(q+1)/2} w),$$

where $\zeta = \exp(2\pi i/p)$ and $\epsilon = \pm 1$. There is a one-to-one correspondence between the choice of the spin structure on $L(p,q)$ and that of ϵ. We note that every spin structure on $L(p,q)$ extends uniquely to that on $cL(p,q)$ and we must have $\epsilon = (-1)^{q-1}$ if p is odd (see [F], [FFU]).

Definition 1 [FFU] For $L(p,q)$ with a spin structure c, which corresponds to the sign ϵ as above, $\delta(L(p,q),c)$ equals $\sigma(q,p,\epsilon)$, which is defined by

$$\sigma(q,p,\epsilon) = \frac{1}{p}\sum_{k=1}^{|p|-1}\left(\cot(\frac{\pi k}{p})\cot(\frac{\pi kq}{p}) + 2\epsilon^k \csc(\frac{\pi k}{p})\csc(\frac{\pi kq}{p})\right). \qquad (1)$$

Here p or q may be negative under the convention $L(p,q) = L(|p|, (\operatorname{sgn} p)q)$.

In [FFU] we give the following characterization of $\sigma(q,p,\epsilon)$.

Proposition 3 [FFU] *$\sigma(q,p,\epsilon)$ is an integer characterized uniquely by the following properties.*

(1) $\sigma(q+cp,p,\epsilon) = \sigma(q,p,(-1)^c\epsilon)$.

(2) $\sigma(-q,p,\epsilon) = \sigma(q,-p,\epsilon) = -\sigma(q,p,\epsilon)$.

(3) $\sigma(q,1,\epsilon) = 0$.

(4) $\sigma(p,q,-1) + \sigma(q,p,-1) = -\operatorname{sgn}(pq)$ *if* $p+q \equiv 1 \pmod 2$.

Proposition 4 [FFU] *If $p+q \equiv 1 \pmod 2$ and $|p| > |q|$ then for a unique continued fraction expansion of the form*

$$p/q = [[\alpha_1, \alpha_2 \ldots, \alpha_n]] = \alpha_1 - \cfrac{1}{\alpha_2 - \cfrac{1}{\ddots - \cfrac{1}{\alpha_n}}}$$

with α_i even and $|\alpha_i| \geq 2$, we have

$$\sigma(q,p,-1) = -\sum_{i=1}^{n} \operatorname{sgn} \alpha_i.$$

Corollary 1 *For any coprime integers p, q with p odd and q even, we have $\sigma(p,q,\pm 1) \equiv 1 \pmod 2$ and $\sigma(q,p,-1) \equiv 0 \pmod 2$.*

Proof If $|p| > |q|$ and p and q have opposite parity, then in the continued fraction expansion of p/q in Proposition 4 we can see inductively that $q \equiv n \pmod 2$, and hence $\sigma(q,p,-1) \equiv n \equiv q \pmod 2$ by Proposition 4. It follows from Proposition 3 that $\sigma(p,q,-1) \equiv p \pmod 2$. If p is odd and q is even then $\sigma(p,q,1) = \sigma(p+q,q,-1) \equiv p+q \equiv p \pmod 2$ also by Proposition 3. This proves the claim. □

Next we consider $\delta(S,c)$ for a spherical 3–manifold $S = S^3/G$ with nonabelian fundamental group G with spin structure c. Such a manifold S is a Seifert manifold over a spherical 2–orbifold $S^2(a_1,a_2,a_3)$ represented by the Seifert invariants of the form

$$S = \{(a_1,b_1),(a_2,b_2),(a_3,b_3)\}$$

with $a_i \geq 2$, $\gcd(a_i,b_i) = 1$ for $i = 1,2,3$, $\sum_{i=1}^{3} 1/a_i > 1$, $e = -\sum_{i=1}^{3} b_i/a_i \neq 0$. Here we adopt the convention in [NR] so that S is represented by a framed link L as in Figure 1.

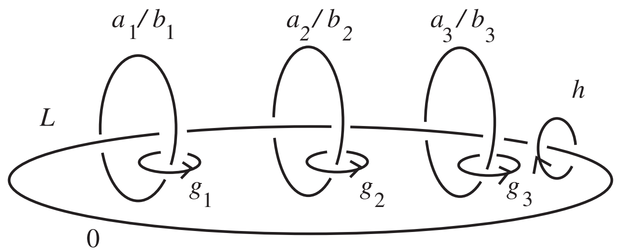

Figure 1

The meridians g_i and h in Figure 1 generate G with relations:

$$g_1^{a_1}h^{b_1} = g_2^{a_2}h^{b_2} = g_3^{a_3}h^{b_3} = g_1g_2g_3 = 1, \quad [g_i,h] = 1 \quad (i = 1,2,3).$$

The representation above is unnormalized. We can choose the other curves g_i' homologous to $g_i + c_ih$ with $\sum_{i=1}^{3} c_i = 0$, which give an alternative representation of S of the form

$$\{(a_1,b_1-a_1c_1),(a_2,b_2-a_2c_2),(a_3,b_3-a_3c_3)\}.$$

Furthermore $-S$ is represented by $\{(a_1,-b_1),(a_2,-b_2),(a_3,-b_3)\}$. Thus the class of the spherical 3–manifolds with non-abelian fundamental group up to orientation is given by the following list.

(1) $\{(2,1),(2,1),(n,b)\}$ with $n\geq 2$, $\gcd(n,b)=1$,

(2) $\{(2,1),(3,1),(3,b)\}$ with $\gcd(3,b)=1$,

(3) $\{(2,1),(3,1),(4,b)\}$ with $\gcd(4,b)=1$,

(4) $\{(2,1),(3,1),(5,b)\}$ with $\gcd(5,b)=1$.

We also note that the above class together with the lens spaces coincides with the class of the links of the quotient singularities. The orientation of S induced naturally by the complex orientation is given by choosing the signs of the Seifert invariants so that the rational Euler class e is negative.

Definition 2 [FFU] Let M be a 3–manifold represented by a framed link L, and let m_i and ℓ_i be the meridian and the preferred longitude of the component L_i of L with framing p_i/q_i. Denote by M_i the meridian of the newly attached solid torus along L_i (homologous to $p_im_i+q_i\ell_i$ in $S^3\setminus L_i$). Then according to [FFU] we describe a spin structure c on M by a homomorphism $w\in\mathrm{Hom}(H_1(S^3\setminus L,\mathbf{Z}),\mathbf{Z}_2)$ so that

$$w(M_i):=p_iw(m_i)+q_iw(\ell_i)+p_iq_i \pmod 2$$

is zero for every component L_i. Note that $w(m_i)=0$ if and only if c extends to the spin structure on the meridian disk in S^3.

Hereafter the above homomorphism w is denoted by the same symbol c as the spin structure on S if there is no danger of confusion. Thus the spin structures on $S=\{(a_1,b_1),(a_2,b_2),(a_3,b_3)\}$ correspond to the elements $c\in\mathrm{Hom}(H_1(S^3\setminus L,\mathbf{Z}),\mathbf{Z}_2)$ satisfying

$$a_ic(g_i)+b_ic(h)\equiv a_ib_i \quad (i=1,2,3),\quad \sum_{i=1}^{3}c(g_i)\equiv 0 \pmod 2 \qquad (2)$$

Proposition 5 *Every spin structure on the spherical 3–manifold S extends uniquely to that on the cone $cS=D^4/G$ over S.*

Proof The claim for a lens space was proved in [F]. We can assume that up to conjugacy G is contained in $U(2)=S^3\times S^1/\mathbf{Z}_2$ ([S]). Since the tangent frame bundle of S is trivial, the associated stable $SO(4)$ bundle is reduced to the $U(2)$ bundle, which is represented as $S^3\times U(2)/G$. A spin structure on S corresponds to the double covering $S^3 \times (S^3 \times S^1)/G \to S^3 \times U(2)/G$ for some representation $G \to S^3 \times S^1$ that covers the original representation of G to $U(2)$. Using this representation we have a double covering $D^4 \times (S^3 \times S^1)/G \to D^4 \times U(2)/G$, which gives a spin structure on the V frame bundle over $cS = D^4/G$. Passing to the determinant bundle (which is the dual of the canonical bundle of cS), we have a double covering of the representation $G \to S^1$ defined by the determinant of the element of G. Such coverings are classified by $H^1(G, \mathbf{Z}_2) = H^1(S, \mathbf{Z}_2)$. It follows that there is a one-to-one correspondence between the set of spin structures on S and that for cS. This proves the claim. □

Thus for a spin structure c on S, we also denote its unique extension to cS by c and the contribution of cS to the index of the Dirac operator by $\delta(S,c)$. To compute $\delta(S,c)$, we appeal to the vanishing theorem of the index of the Dirac operator on a certain V manifold as in [FFU]. (There is an alternative method of computing $\delta(S,c)$ by using plumbing constructions. See §3.) For this purpose we consider the V 4–manifold X with S^1 action and with $\partial X = S$, which is constructed as follows. We denote by $\pi\colon X \to X^*$ the projection to the orbit space $X^* = X/S^1$. Suppose that $S = \{(a_1,b_1),(a_2,b_2),(a_3,b_3)\}$ with the spin structure c. Then X^* has the following properties (see Figure 2).

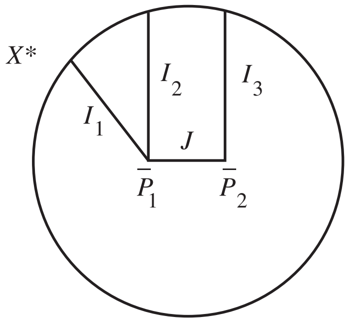

Figure 2

(1) The underlying space of X^* is the 3–ball.

(2) The image of the fixed points consists of two interior points $\overline{P}_i$ $(i = 1,2)$, and the image of the exceptional orbits in X^* consists of three segments I_j $(j = 1,2,3)$ such that I_j connects some point on ∂X^* and $\overline{P}_1$ (for $j = 1,2$) or $\overline{P}_2$ (for $j = 3$).

(3) The Seifert invariant of the orbit over any point on I_j except for $\overline{P}_i$'s is (a_j, b_j).

(4) The orbit over any point outside the union of I_j's has a trivial stabilizer.

Let D_i be the small 4-ball neighborhood of $\overline{P}_i$ for $i=1,2$. Then $\pi^{-1}(D_i)$ is the cone over $L_i=\pi^{-1}(\partial D_i)$, where L_i is represented by the framed links in Figure 3. Here L_2 is the lens space $L(b_3,a_3)$ represented by a $-b_3/a_3$ surgery along the trivial knot with meridian corresponding to h, while L_1 is the lens space $L(Q,P)$ such that

$$Q=a_1b_2+a_2b_1,\ P=a_2v_1+b_2u_1 \text{ for } u_1,\ v_1\in\mathbf{Z} \text{ with } a_1v_1-b_1u_1=1. \quad (3)$$

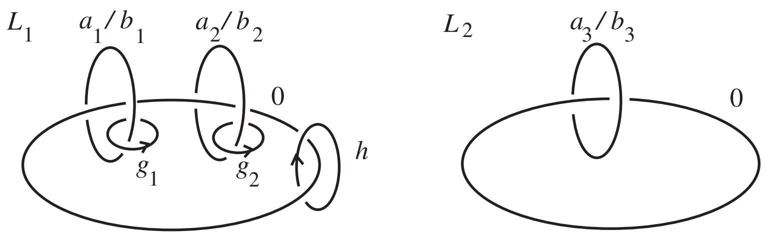

Figure 3

Note that L_1 is represented by the $-Q/P$ surgery along the trivial knot whose meridian corresponds to

$$m=u_1g_1+v_1h. \quad (4)$$

It follows that X is a V manifold with $\partial X=S$, and with two singular points $P_i=\pi^{-1}(\overline{P}_i)$ whose neighborhoods are the cones over L_i. Now according to the argument in [FFU] we can check the properties of X. (In [FFU] such a construction was considered when ∂X is a $\mathbf{Z}$ homology 3–sphere. But the argument there is valid when ∂X is a $\mathbf{Q}$ homology 3–sphere without any essential change.) Let J be the segment that connects $\overline{P}_1$ and $\overline{P}_2$ in the interior of X^* and disjoint from the interior of I_j (Figure 2). Then $S_0:=\pi^{-1}J$ is a 2–sphere and X is homotopy equivalent to S_0. Furthermore the rational self intersection number of S_0 is given by

$$S_0\cdot S_0=a_1a_2/Q+a_3/b_3, \quad (5)$$

which is nonzero if ∂X is a $\mathbf{Q}$ homology 3–sphere. It follows that $b_1(X)=0$, $b_2(X)=1$, and $\operatorname{sign} X=\operatorname{sgn} S_0\cdot S_0$. Furthermore X admits a spin structure extending c on $S=\partial X$ if and only if $c\in \operatorname{Hom}(H_1(S^1\setminus L,\mathbf{Z}),\mathbf{Z}_2)$ satisfies the following conditions.

$$a_ic(g_i)+b_ic(h)\equiv a_ib_i \quad (i=1,2,3), \quad \sum_{i=1}^{2}c(g_i)\equiv c(g_3)\equiv 0. \quad (\mathrm{mod}\ 2) \quad (6)$$

In our case $c(h)$ must be 0 since $(a_i, b_i) = (2, 1)$ for some i. We will see later that we can arrange the Seifert invariants for any given (S, c) so that they satisfy these conditions. If we put

$$\widehat{X} = X \cup (-cS),$$

then by Proposition 5 $\widehat{X}$ is a closed spin V 4–manifold with $b_1(\widehat{X}) = 0$, $b_2(\widehat{X}) = 1$ and $\operatorname{sign} \widehat{X} = \operatorname{sgn} S_0 \cdot S_0$, such that $\widehat{X}$ has (at most) three singular points whose neighborhoods are the cones over L_1, L_2, and $-S$. Here the Seifert invariants for $-S$ are given by $\{(a_1, -b_1), (b_2, -b_2), (a_3, -b_3)\}$ with respect to the curves g_i and $-h$, and we can consider the spin structure on $-S$ induced from c, which is given by the same homomorphism in $\mathrm{Hom}(H_1(S^3 \setminus L, \mathbf{Z}), \mathbf{Z}_2)$ as c and is denoted by $-c$. Then the spin structure on $-S$ induced from $\widehat{X}$ is $-c$. Moreover the spin structures on L_1 and L_2 induced from that on $\widehat{X}$ correspond to $c(h)$ and $c(m)$ respectively, where

$$c(m) \equiv u_1 c(g_1) + v_1 c(h) + u_1 v_1 \pmod 2 \tag{7}$$

(see [FFU]). Then the argument in [FFU], Proposition 3 shows that

$$\delta(L_1, c) = \sigma(P, Q, (-1)^{(c(m)-1)}), \quad \delta(L_2, c) = \sigma(a_3, b_3, (-1)^{(c(h)-1)}). \tag{8}$$

Thus from Theorem 1 [FF] we deduce

$$\begin{aligned} 0 &= \operatorname{ind} D(\widehat{X}) \\ &= -(\operatorname{sign} \widehat{X} + \sigma(P, Q, (-1)^{(c(m)-1)}) + \sigma(a_3, b_3, (-1)^{(c(h)-1)}) + \delta(-S, -c))/8, \end{aligned}$$

Thus we can see that

$$\delta(-S, -c) = -\delta(S, c) \tag{9}$$

and hence (using the fact that $c(h) = 0$),

$$\delta(S, c) = \operatorname{sgn} S_0 \cdot S_0 + \sigma(P, Q, (-1)^{(c(m)-1)}) + \sigma(a_3, b_3, -1). \tag{10}$$

Now we apply this result to compute $\delta(S, c)$ by constructing the above X associated with the Seifert invariants of S, which is rearranged if necessary. We denote by $\{(a_1, b_1), (a_2, b_2), (a_3, b_3)\}$ the rearraged Seifert invariants and the corresponding meridian curves in Figure 1 by g_i' (h remains unchanged). Hereafter we write the data of the required X by giving the rearranged Seifert invariants, the values of Q, P, m, and $S_0 \cdot S_0$.

2.1 Case 1 $S = \{(2,1),(2,1),(n,b)\}$

2.1.1 n is odd and b is even

In this case the spin structure c on S satisfies

$$c(h) \equiv c(g_3) \equiv 0, \quad c(g_1) \equiv c(g_2) = \epsilon \pmod 2 \tag{11}$$

where ϵ is arbitrary. Then X associated with $(a_1,b_1) = (a_2,b_2) = (2,1)$, $(a_3,b_3) = (n,b)$, $Q = 4$, $P = 3$, $m = g_1' + h$, $S_0 \cdot S_0 = (n+b)/b$ shows that

$$\delta(S,c) = \operatorname{sgn}(n+b)b + \sigma(3,4,(-1)^\epsilon) + \sigma(n,b,-1).$$

Since $\sigma(3,4,1) = \sigma(7,4,-1) = -\sigma(4,7,-1) - 1$ and $4/3 = [[2,2,2]]$, $7/4 = [[2,4]]$, we deduce from Propositions 3 and 4 that

$$\delta(S,c) = \begin{cases} \sigma(n,b,-1) & (\epsilon = 0, \quad -n < b < 0) \\ \sigma(n,b,-1)+2 & (\epsilon = 0, \quad b > 0 \text{ or } b < -n) \\ \sigma(n,b,-1)-4 & (\epsilon = 1, \quad -n < b < 0) \\ \sigma(n,b,-1)-2 & (\epsilon = 1, \quad b > 0 \text{ or } b < -n). \end{cases} \tag{12}$$

In either case $\delta(S,c)$ is odd by Corollary 1.

2.1.2 n and b are odd

In this case c is given by

$$c(h) = 0, \quad c(g_3) \equiv c(g_1) + c(g_2) \equiv 1 \pmod 2. \tag{13}$$

It suffices to consider the case when $c(g_1) \equiv 1$ and $c(g_2) \equiv 0 \pmod 2$, since we have a self-diffeomorphism of S mapping (g_1,g_2,g_3,h) to $(-g_2,-g_1,-g_3,-h)$. Thus X associated with $(a_1,b_1) = (a_3,b_3) = (2,1)$ and $(a_2,b_2) = (n,b)$, $Q = n+2b$, $P = n+b$, $m = g_1' + h$ and $S_0 \cdot S_0 = 4(n+b)/(n+2b)$ shows that (X has only one singular point since $L(1,2)$ is the 3–sphere)

$$\begin{aligned} \delta(S,c) &= \operatorname{sgn}(n+b)(n+2b) + \sigma(n+b,n+2b,-1) \\ &= -\sigma(n+2b,n+b,-1) = -\sigma(-n,n+b,-1) = \sigma(n,n+b,-1) \end{aligned} \tag{14}$$

Again $\delta(S,c)$ is odd in this case by Corollary 1.

2.1.3 n is even

In this case c satisfies

$$c(h) \equiv 0, \quad c(g_1) + c(g_2) + c(g_3) \equiv 0 \pmod 2. \tag{15}$$

Since at least one of $c(g_i)$ is zero and there is a self-diffeomorphism of S exchanging g_1 and g_2 up to orientation as before, it suffices to consider the following subcases.

(*i*) $c(h) \equiv c(g_3) \equiv 0, \quad c(g_1) = c(g_2) = \epsilon \pmod 2$.

Then X associated with $(a_1, b_1) = (a_2, b_2) = (2, 1)$ and $(a_3, b_3) = (n, b)$, $Q = 4$, $P = 3$, $m = g_1' + h$, and $S_0 \cdot S_0 = (n+b)/b$ shows that

$$\delta(S, c) = \operatorname{sgn}(n+b)b + \sigma(3, 4, (-1)^\epsilon) + \sigma(n, b, -1)$$

Hence $\delta(S, c)$ is represented by the same equation as in Case (2.1.1) 12.

(*ii*) $c(h) \equiv c(g_2) \equiv 0, \quad c(g_1) \equiv c(g_3) \equiv 1 \pmod 2$.

If $Q = n + 2b \neq 0$ (i.e., if $(n, b) \neq (2, -1)$) then X associated with $(a_1, b_1) = (a_3, b_3) = (2, 1)$ and $(a_2, b_2) = (n, b)$, $Q = n + 2b$, $P = n + b$, $m = g_1' + h$, $S_0 \cdot S_0 = 4(n+b)/(n+2b)$ shows that

$$\delta(S, c) = \operatorname{sgn}(n+b)(n+2b) + \sigma(n+b, n+2b, -1).$$

By Proposition 3 the right hand side equals

$$-\sigma(n+2b, n+b, -1) = -\sigma(-n, n+b, -1) = \sigma(n, n+b, -1).$$

For the case when $(n, b) = (2, -1)$, we consider another representation of S of the form $\{(2,1), (2,-3), (2,3)\}$ with respect to the curves $g_1' = g_1$, $g_2' = g_2 + 2h$, and $g_3' = g_3 - 2h$. Since $c(g_1') \equiv c(g_1) \pmod 2$ and

$$c(g_2') \equiv c(g_2) + 2c(h) + 2 \equiv c(g_2), \quad c(g_3') \equiv c(g_3) - 2c(h) - 2 \equiv c(g_3),$$

considering X with $(a_1, b_1) = (2, 1)$, $(a_2, b_2) = (2, 3)$, $(a_3, b_3) = (2, -3)$, $Q = 8$, $P = 5$, $m = g_1' + h$, and $S_0 \cdot S_0 = -1/6$, we have

$$\delta(S, c) = -1 + \sigma(5, 8, -1) + \sigma(2, -3, -1) = 0.$$

It follows that in either case

$$\delta(S, c) = \sigma(n, n+b, -1). \tag{16}$$

We also note that there are some overlaps in the above list if we also consider $(-S, -c)$. In fact we have

$$\{(2,-1), (2,-1), (n,-b)\} = \{(2,1), (2,1), (n, -2n-b)\}. \tag{17}$$

2.2 Case 2 $S = \{(2,1),(3,1),(3,b)\}$

In this case S is a $\mathbf{Z}_2$ homology 3–sphere and c is uniquely determined.

2.2.1 b is even

In this case c satisfies $c(h) \equiv c(g_3) \equiv 0$, $c(g_1) = c(g_2) \equiv 1 \pmod 2$. Thus X associated with $(a_1,b_1) = (2,1)$, $(a_2,b_2) = (3,1)$, $(a_3,b_3) = (3,b)$, $Q = 5$, $P = 4$, $m = g_1' + h$, $S_0 \cdot S_0 = (6b+15)/5b$ shows that

$$\delta(S,c) = \operatorname{sgn}(2b+5)b + \sigma(4,5,-1) + \sigma(3,b,-1).$$

Here we must have $b = 6k \pm 2$ for some k, and if $k \neq 0$,

$$(6k+2)/3 = [[2k,-2,-2]], \quad (6k-2)/3 = [[2k,2,2]]$$

and hence

$$\delta(S,c) = \begin{cases} -\operatorname{sgn} b - 1 & (b = 6k+2 \text{ for some } k), \\ -\operatorname{sgn} b - 5 & (b = 6k-2 \text{ for some } k \neq 0), \\ -6 & (b = -2). \end{cases} \tag{18}$$

2.2.2 b is odd

Consider a representation of S of the form $\{(2,-1),(3,1),(3,b+3)\}$. Then we have $c(h) \equiv c(g_3') \equiv 0$, $c(g_1') \equiv c(g_2') \equiv 1 \pmod 2$. Thus X associated with $(a_1,b_1) = (2,-1)$, $(a_2,b_2) = (3,1)$, $(a_3,b_3) = (3,b+3)$, $Q = -1$, $P = 2$, $m = -g_1' + h$, $S_0 \cdot S_0 = (3 - 6(b+3))/(b+3)$ shows that

$$\delta(S,c) = -\operatorname{sgn}(2b+5)(b+3) + \sigma(3,b+3,-1).$$

Here we must have $b = 6k \pm 1$ for some k. Since

$$(6k+4)/3 = [[2(k+1),2,2]] \ \ (k \neq -1), \quad (6k+2)/3 = [[2k,-2,-2]] \ \ (k \neq 0),$$

we can see that

$$\delta(S,c) = \begin{cases} -\operatorname{sgn} b - 3 & (b = 6k+1 \text{ for some } k), \\ -\operatorname{sgn} b + 1 & (b = 6k-1 \text{ for some } k \neq 0), \\ 0 & (b = -1). \end{cases} \tag{19}$$

We also note that

$$\{(2,-1),(3,-1),(3,-6k-2)\} = \{(2,1),(3,1),(3,-6(k+1)-1)\}. \tag{20}$$

2.3 $S=\{(2,1),(3,1),(4,b)\}$

In this case c satisfies

$$c(h)\equiv 0,\quad c(g_2)\equiv 1,\quad c(g_1)+c(g_3)\equiv 1 \pmod 2. \tag{21}$$

2.3.1 $c(g_1)\equiv c(g_2)\equiv 1$, $c(g_3)\equiv c(h)\equiv 0 \pmod 2$

Considering X with $(a_1,b_1)=(2,1)$, $(a_2,b_2)=(3,1)$, $(a_3,b_3)=(4,b)$, $Q=5$, $P=4$, $m=g_1'+h$, $S_0\cdot S_0=(6b+20)/(5b)$, we have

$$\delta(S,c)=\operatorname{sgn}(3b+10)b+\sigma(4,5,-1)+\sigma(4,b,-1).$$

Here we must have $b=8k\pm 1$ or $b=8k\pm 3$ for some k. Since

$$(8k+1)/4=[[2k,-4]],\quad (8k-1)/4=[[2k,4]],$$
$$(8k+3)/4=[[2k,-2,-2,-2]],\quad (8k-3)/4=[[2k,2,2,2]]$$

for $k\neq 0$, we can see that

$$\delta(S,c)=\begin{cases}-\operatorname{sgn}b-2 & (b=8k+1 \text{ for some } k),\\ -\operatorname{sgn}b-4 & (b=8k-1 \text{ for some } k\neq 0),\\ -\operatorname{sgn}b & (b=8k+3 \text{ for some } k),\\ -\operatorname{sgn}b-6 & (b=8k-3 \text{ for some } k\neq 0),\\ -5 & (b=-1),\\ -7 & (b=-3).\end{cases} \tag{22}$$

2.3.2 $c(g_1)\equiv c(h)\equiv 0$, $c(g_2)\equiv c(g_3)\equiv 1 \pmod 2$

In this case X associated with $(a_1,b_1)=(3,1)$, $(a_2,b_2)=(4,b)$, $(a_3,b_3)=(2,1)$, $Q=3b+4$, $P=2b+4$, $m=2g_1'+h$, $S_0\cdot S_0=(6b+20)/(3b+4)$ shows that

$$\delta(S,c)=\operatorname{sgn}(3b+4)(3b+10)+\sigma(2b+4,3b+4,-1).$$

Here $(3b+4)/(2b+4)$ equals

$$\begin{array}{ll} [[2,2,2(k+1),2,2,2]] & (\text{if } b=8k+1 \text{ for } k\neq -1), \\ [[2,2,2k,-2,-2,-2]] & (\text{if } b=8k-1 \text{ for some } k\neq 0), \\ [[2,2,2(k+1),4]] & (\text{if } b=8k+3 \text{ for } k\neq -1), \\ [[2,2,2k,-4]] & (\text{if } b=8k-3 \text{ for some } k\neq 0), \\ [[2,4,2,2]] & (\text{if } b=-7), \\ [[2,6]] & (\text{if } b=-5), \\ [[2,-2]] & (\text{if } b=-3). \end{array}$$

Hence we have

$$\delta(S,c)=\begin{cases} -\operatorname{sgn} b-4 & (b=8k+1), \\ -\operatorname{sgn} b+2 & (b=8k-1 \text{ for } k\neq 0), \\ -\operatorname{sgn} b-2 & (b=8k+3), \\ -\operatorname{sgn} b & (b=8k-3 \text{ for } k\neq 0\ ), \\ -1 & (b=-3), \\ 1 & (b=-1). \end{cases} \tag{23}$$

2.4 $S=\{(2,1),(3,1),(5,b)\}$

In this case the spin structure on S is unique.

2.4.1 b is even

In this case c satisfies $c(h)\equiv c(g_3)\equiv 0$, $c(g_1)\equiv c(g_2)\equiv 1 \pmod 2$. Then X associated with $(a_1,b_1)=(2,1)$, $(a_2,b_2)=(3,1)$, $(a_3,b_3)=(5,b)$, $Q=5$, $P=4$, $m=g_1'+h$, $S_0\cdot S_0=(6b+25)/(5b)$ shows that

$$\delta(S,c)=\operatorname{sgn}(6b+25)b+\sigma(4,5,-1)+\sigma(5,b,-1).$$

Here we must have $b=10k\pm 2$, or $10k\pm 4$ for some k. Since for $k\neq 0$

$$(10k+2)/5=[[2k,-2,2]],\quad (10k-2)/5=[[2k,2,-2]],$$
$$(10k+4)/5=[[2k,-2,-2,-2,-2]],\quad (10k-4)/5=[[2k,2,2,2,2]],$$

we have

$$\delta(S,c)=\begin{cases}-\operatorname{sgn} b-3 & (b=10k\pm 2 \text{ other than } -2),\\ -\operatorname{sgn} b+1 & (b=10k+4),\\ -\operatorname{sgn} b-7 & (b=10k-4 \text{ other than } -4),\\ -4 & (b=-2),\\ -8 & (b=-4).\end{cases} \tag{24}$$

2.4.2 b is odd

Consider the Seifert invariants of S of the form $\{(2,-1),(3,1),(5,b+5)\}$. Then $c(h)\equiv c(g_3')\equiv 0$, $c(g_1')\equiv c(g_2')\equiv 1 \pmod 2$. Hence X associated with $(a_1,b_1)=(2,-1)$, $(a_2,b_2)=(3,1)$, $(a_3,b_3)=(5,b+5)$, $Q=-1$, $P=2$, $m=-g_1'+h$, $S_0\cdot S_0=(5-6(b+5))/(b+5)$ shows that

$$\delta(S,c)=-\operatorname{sgn}(b+5)(6b+25)+\sigma(5,b+5,-1).$$

Here we must have $b=10k\pm 1$ or $10k\pm 3$ for some k. Since

$$\begin{aligned}(10k+6)/5&=[[2(k+1),2,2,2,2]] \quad (k\neq -1),\\ (10k+4)/5&=[[2k,-2,-2,-2,-2]] \quad (k\neq 0),\\ (10k+8)/5&=[[2(k+1),2,-2]] \quad (k\neq -1),\\ (10k+2)/5&=[[2k,-2,2]] \quad (k\neq 0),\end{aligned}$$

we have

$$\delta(S,c)=\begin{cases}-\operatorname{sgn} b-5 & (b=10k+1 \text{ for some } k)\\ -\operatorname{sgn} b+3 & (b=10k-1 \text{ for some } k\neq 0)\\ -\operatorname{sgn} b-1 & (b=10k\pm 3 \text{ for some } k \text{ and } b\neq -3),\\ 2 & (b=-1),\\ -2 & (b=-3).\end{cases} \tag{25}$$

Now we remove the overlaps (17, 20) from the above results by giving the data only for the spherical 3–manifolds with negative rational Euler class.

Proposition 6 *The value $\delta=\delta(S,c)$ for a spherical 3–manifold S with negative rational Euler class and its spin structure c is given by the following list. Note that $\delta(-S,-c)=-\delta(S,c)$. Except for the lens spaces, we give the list of the Seifert invariants for S, the set of the values $(c(g_1),c(g_2),c(g_3))$ of c, and δ. Here g_i and h are the meridians of the framed link associated with the*

Seifert invariants as in Figure 1. We omit $c(h)$ since it is always zero. In the list below, the data of c is omitted when S is a $\mathbf{Z}_2$ homology sphere (cases (3) and (5)), and ϵ is ± 1.

(1) $S = L(p,q)$ *with* $p > q > 0$.

In this case $\delta(L(p,q),c) = \sigma(q,p,\epsilon)$ where the relation between c and ϵ is explained in the paragraph before Definition 1. We also note that if $L(p,q)$ is represented by the $-p/q$–surgery along the trivial knot O, then the spin structure given by $c \in \mathrm{Hom}(H_1(S^3 \setminus O, \mathbf{Z}), \mathbf{Z}_2)$ explained as in Definition 2 satisfies $\epsilon \equiv c(\mu) - 1 \pmod 2$ with respect to the above correspondence, where μ is the meridian of O (see [FFU]).

(2) $S = \{(2,1),(2,1),(n,b)\}$.

	S	c	δ
$(2-1)$	n odd, b even, $-n < b < 0$	$(0,0,0)$	$\sigma(n,b,-1)$
$(2-2)$	n odd, b even, $-n < b < 0$	$(1,1,0)$	$\sigma(n,b,-1) - 4$
$(2-3)$	n odd, b even, $b > 0$	$(0,0,0)$	$\sigma(n,b,-1) + 2$
$(2-4)$	n odd, b even, $b > 0$	$(1,1,0)$	$\sigma(n,b,-1) - 2$
$(2-5)$	n, b odd, $n + b > 0$	$(\epsilon, 1-\epsilon, 1)$	$\sigma(n,n+b,-1)$
$(2-6)$	n even, $-n < b < 0$	$(0,0,0)$	$\sigma(n,b,-1)$
$(2-7)$	n even, $-n < b < 0$	$(1,1,0)$	$\sigma(n,b,-1) - 4$
$(2-8)$	n even, $b > 0$	$(0,0,0)$	$\sigma(n,b,-1) + 2$
$(2-9)$	n even, $b > 0$	$(1,1,0)$	$\sigma(n,b,-1) - 2$
$(2-10)$	n even, $n + b > 0$	$(\epsilon, 1-\epsilon, 1)$	$\sigma(n,n+b,-1)$

(3) S *is a Seifert fibration over* $S^2(2,3,3)$.

	S	δ
$(3-1)$	$\{(2,1),(3,1),(3,6k+2)\}$, $k \geq 0$	-2
$(3-2)$	$\{(2,-1),(3,-1),(3,-6k-2)\}$, $k \leq -1$	0
$(3-3)$	$\{(2,1),(3,1),(3,6k-2)\}$, $k \geq 0$	-6
$(3-4)$	$\{(2,-1),(3,-1),(3,-6k+2)\}$, $k < 0$	4
$(3-5)$	$\{(2,1),(3,1),(3,6k+1)\}$, $k \geq 0$	-4
$(3-6)$	$\{(2,-1),(3,-1),(3,-6k-1)\}$, $k < 0$	2

(4) S *is a Seifert fibration over* $S^2(2,3,4)$.

	S	c	δ
$(4-1)$	$\{(2,1),(3,1),(4,8k+1)\}$, $k\geq 0$	$(1,1,0)$	-3
$(4-2)$	$\{(2,1),(3,1),(4,8k+1)\}$, $k\geq 0$	$(0,1,1)$	-5
$(4-3)$	$\{(2,-1),(3,-1),(4,-8k-1)\}$, $k<0$	$(1,1,0)$	1
$(4-4)$	$\{(2,-1),(3,-1),(4,-8k-1)\}$, $k<0$	$(0,1,1)$	3
$(4-5)$	$\{(2,1),(3,1),(4,8k-1)\}$, $k\geq 0$	$(1,1,0)$	-5
$(4-6)$	$\{(2,1),(3,1),(4,8k-1)\}$, $k\geq 0$	$(0,1,1)$	1
$(4-7)$	$\{(2,-1),(3,-1),(4,-8k+1)\}$, $k<0$	$(1,1,0)$	3
$(4-8)$	$\{(2,-1),(3,-1),(4,-8k+1)\}$, $k<0$	$(0,1,1)$	-3
$(4-9)$	$\{(2,1),(3,1),(4,8k+3)\}$, $k\geq 0$	$(1,1,0)$	-1
$(4-10)$	$\{(2,1),(3,1),(4,8k+3)\}$, $k\geq 0$	$(0,1,1)$	-3
$(4-11)$	$\{(2,-1),(3,-1),(4,-8k-3)\}$, $k<0$	$(1,1,0)$	-1
$(4-12)$	$\{(2,-1),(3,-1),(4,-8k-3)\}$, $k<0$	$(0,1,1)$	1
$(4-13)$	$\{(2,1),(3,1),(4,8k-3)\}$, $k\geq 0$	$(1,1,0)$	-7
$(4-14)$	$\{(2,1),(3,1),(4,8k-3)\}$, $k\geq 0$	$(0,1,1)$	-1
$(4-15)$	$\{(2,-1),(3,-1),(4,-8k+3)\}$, $k<0$	$(1,1,0)$	5
$(4-16)$	$\{(2,-1),(3,-1),(4,-8k+3)\}$, $k<0$	$(0,1,1)$	-1

(5) *S is a Seifert fibration over $S^2(2,3,5)$.*

	S	δ
$(5-1-\epsilon)$	$\{(2,1),(3,1),(5,10k+2\epsilon)\}$, $k\geq 0$	-4
$(5-2-\epsilon)$	$\{(2,-1),(3,-1),(5,-10k-2\epsilon)\}$, $k<0$	2
$(5-3)$	$\{(2,1),(3,1),(5,10k+4)\}$, $k\geq 0$	0
$(5-4)$	$\{(2,-1),(3,-1),(5,-10k-4)\}$, $k<0$	-2
$(5-5)$	$\{(2,1),(3,1),(5,10k-4)\}$, $k\geq 0$	-8
$(5-6)$	$\{(2,-1),(3,-1),(5,-10k+4)\}$, $k<0$	6
$(5-7)$	$\{(2,1),(3,1),(5,10k+1)\}$, $k\geq 0$	-6
$(5-8)$	$\{(2,-1),(3,-1),(5,-10k-1)\}$, $k<0$	4
$(5-9)$	$\{(2,1),(3,1),(5,10k-1)\}$, $k\geq 0$	2
$(5-10)$	$\{(2,-1),(3,-1),(5,-10k+1)\}$, $k<0$	-4
$(5-11-\epsilon)$	$\{(2,1),(3,1),(5,10k+3\epsilon)\}$, $k\geq 0$	-2
$(5-12-\epsilon)$	$\{(2,-1),(3,-1),(5,-10k-3\epsilon)\}$, $k<0$	0

3 Some applications

Let us start with some (well-known) results for later use.

Proposition 7 (1) *Suppose that a spin 4–manifold Y is represented by a framed link L with even framings. Then the spin structure on ∂Y*

is induced from that on Y if and only if it is represented by the zero homomorphism of $\mathrm{Hom}(H_1(S^3 \setminus L, \mathbf{Z}), \mathbf{Z}_2)$.

(2) *Let M be a 3–manifold represented by a framed link L in Figure 4, whose framing for the component K is given by p/q for coprime p, q with opposite parity. Suppose that a spin structure c on M is represented by $c \in \mathrm{Hom}(H_1(S^3 \setminus L, \mathbf{Z}), \mathbf{Z}_2)$ with $c(\mu) = c(\mu') = 0$ for meridians μ of K and μ' of K'. Then the 3–manifold M' represented by a framed link L' in Figure 4, where $p/q = [[a_1, \ldots, a_k]]$ for even a_i, $a_i \neq 0$ is diffeomorphic to M, so that c corresponds to $c' \in \mathrm{Hom}(H_1(S^3 \setminus L', \mathbf{Z}), \mathbf{Z}_2)$ with $c'(\mu_i) = 0$ for any meridian μ_i of the new components of framing a_i, and $c'(\mu'') = c(\mu'')$ for a meridian μ'' of any common component of L and L'.*

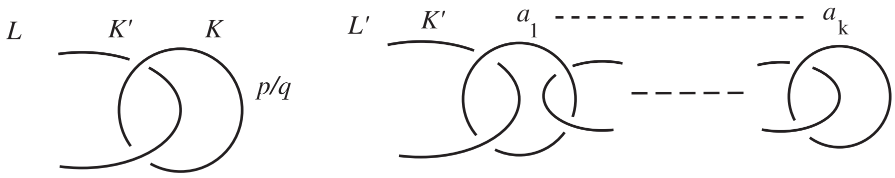

Figure 4

Proof If the spin structure c on ∂Y is induced from that on Y, then the associated element of $\mathrm{Hom}(H_1(S^3 \setminus L, \mathbf{Z}), \mathbf{Z}_2)$ is zero since c extends to that on S^3. Conversely if c is zero, c extends to the spin structure on S^3, and hence on the 4–ball, while there is no obstruction to extending c to that on the 2–handles attached to the 4-ball since all the framings are even. This proves the first claim. To see the second claim note that there is a diffeomorphism between M and M' such that μ and μ' correspond to the meridians μ_i by the following relations.

$$\begin{pmatrix} -a_i & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \mu_i \\ \mu_{i-1} \end{pmatrix} = \begin{pmatrix} \mu_{i+1} \\ \mu_i \end{pmatrix} \ (i \leq k-1), \quad \begin{pmatrix} -a_k & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \mu_k \\ \mu_{k-1} \end{pmatrix} = \begin{pmatrix} \widetilde{\mu} \\ \widetilde{\lambda} \end{pmatrix}$$

where $(\mu_1, \mu_0) = (\mu, \mu')$ and $(\widetilde{\mu}, \widetilde{\lambda})$ is a pair of a meridian and a longitude for a newly attached solid torus along K. Since all a_i are even, we have $c'(\mu_i) = 0$ for every i. □

Next we consider plumbed 4–manifolds bounded by spherical 3–manifolds. Let $P(\Gamma)$ be a plumbed 4–manifold associated with a weighted tree graph Γ. Let

x_i be the generators of $H_2(P(\Gamma), \mathbf{Z})$ corresponding to the vertices v_i $(i \in I)$ of Γ, and μ_i be the meridian of the component associated with v_i of a framed link L_Γ naturally corresponding to $P(\Gamma)$. For every spin structure c on $\partial P(\Gamma)$, there exists a Wu class w of $P(\Gamma)$ associated with c of the form $w = \sum_{i \in I} \epsilon_i x_i$ with $\epsilon_i = 0$ or 1 such that

$$w \cdot x_i \equiv x_i \cdot x_i \pmod 2 \quad (i \in I)$$

where c corresponds to an element $w \in \mathrm{Hom}(H_1(S^3 \setminus L_\Gamma, \mathbf{Z}), \mathbf{Z}_2)$ (we use the same symbol w since there is no danger of confusion) satisfying $w(\mu_i) = \epsilon_i$. The set of v_i with $\epsilon_i = 1$ in the above representation of w is called the Wu set ([Sa]). It is well known that no adjacent vertices in Γ both belong to the same Wu set. Moreover the spin structure c extends to that on the complement in $P(\Gamma)$ of the union of $P(v_i)$ for v_i in the Wu set. The following proposition is a generalization of the result for lens spaces in [Sa].

Proposition 8 *Suppose that a spherical 3–manifold S bounds a plumbed 4–manifold $P(\Gamma)$. For any spin structure c on S, we have $\delta(S, c) = \operatorname{sign} P(\Gamma) - w \cdot w$ for the associated Wu class $w \in H_2(P(\Gamma), \mathbf{Z})$. In particular if $P(\Gamma)$ is spin and c is the spin structure inherited from that on $P(\Gamma)$, we have $\delta(S, c) = \operatorname{sign} P(\Gamma)$.*

Proof It suffices to consider the case when Γ is reduced, for otherwise by blowing down processes we obtain a reduced graph Γ' such that $S = \partial P(\Gamma) = \partial P(\Gamma')$ and the Wu class w' of $P(\Gamma')$ associated with c satisfies $\operatorname{sign} P(\Gamma) - w \cdot w = \operatorname{sign} P(\Gamma') - w' \cdot w'$. In the case of lens spaces, this claim follows from the result in [Sa] under the correspondence of $\sigma(q, p, \pm 1)$ and $\delta(L(p, q), c)$. If S is not a lens space, Γ is star-shaped with just three branches. As in [Sa], we can take a disjoint union of subtrees Γ_0 containing the Wu set associated with c, such that the complement of Γ_0 in Γ is a single vertex v_0. Then $\partial P(\Gamma_0)$ is a union of the lens spaces L_i and $P(\Gamma_0)$ can be embedded into the interior of $P(\Gamma)$ so that c extends to the spin structure on the complement $X_0 = P(\Gamma) \setminus P(\Gamma_0)$ and on L_i (we denote them by the same symbol c). Next we consider the closed V manifold $\widehat{X}$ obtained from X_0 by attaching the cones cL_i over L_i and the cone cS over S (with orientation reversed). Then c on X_0 extends naturally to the spin structure on $\widehat{X}$ by Proposition 5. Since $b_1(\widehat{X}) = 0$ and $b_2(\widehat{X}) = 1$, Theorem 1 shows that

$$0 = \operatorname{ind} D(\widehat{X}) = -(\operatorname{sign} \widehat{X} + \sum \delta(L_i, c) - \delta(S, c))/8.$$

Since $\sum \delta(L_i, c) = \operatorname{sign} P(\Gamma_0) - w \cdot w$ by [Sa] and $\operatorname{sign} \widehat{X} + \operatorname{sign} P(\Gamma_0) = \operatorname{sign} P(\Gamma)$ by the additivity of the signature, we obtain the desired result. Since $w = 0$ if $P(\Gamma)$ is spin, the last claim follows. □

For any given spherical manifold S with a spin structure c, we can construct a plumbed 4–manifold bounded by (S,c) from the data of the Seifert invariants of S and obtain the Wu set explicitly. For example, from the Seifert invariants $\{(a_1,b_1),(a_2,b_2),(a_3,b_3)\}$ of S and the data $c(g_i)$, $c(h)$ given in the list in Proposition 6, we can obtain anther representation of S of the form

$$\{(1,a),(a_1,b_1'),(a_2,b_2'),(a_3,b_3')\}$$

such that a is even, a_i and b_i' have opposite parity, and c satisfies $c(g_i)=c(h)=0$ as the element of $\mathrm{Hom}(H_1S^3\setminus L,\mathbf{Z}),\mathbf{Z}_2)$, where L is a framed link in Figure 1 (obtained by replacing the framings a_i/b_i and 0 by a_i/b_i' and $-a$ respectively). Then by using the continued fraction expansions of a_i/b_i' by nonzero even numbers and by Proposition 7, we obtain a spin plumbed 4–manifold bounded by (S,c). This provides us an alternative method of computing $\delta(S,c)$. The details are omitted.

Combining the list in Proposition 6 with the 10/8 theorem we can derive certain information on the intersection form of a spin 4–manifold bounded by a spherical 3–manifold.

Theorem 2 *Let (S,c) be a spherical 3–manifold with a spin structure c.*

(1) *If $\delta(S,c)\neq 0$, then a connected sum of any copies of (S,c) does not bound a $\mathbf{Q}$ acyclic spin 4–manifold. In particular, any $\mathbf{Z}_2$ homology 3–sphere S with $\delta(S,c)\neq 0$ for a unique c has infinite order in $\Theta^3_{\mathbf{Z}_2}$.*

(2) *If $|\delta(S,c)|\leq 18$ and (S,c) bounds a spin definite 4–manifold Y, then we must have* $\mathrm{sign}(Y)=\delta(S,c)$.

Proof The claim (1) is deduced from Proposition 2. To prove (2), we note that if $|\delta|<10$ then the region of $(b^-(Y),b^+(Y))$ given by the two inequalities in Proposition 2 does not contain the part with $b^+(Y)=0$ nor $b^-(Y)=0$. If $10\leq|\delta|\leq 18$, then the intersection of the region defined by the same inequalities and the line $b^+(Y)=0$ or $b^-(Y)=0$ does not contain the point satisfying $b^+(Y)-b^-(Y)\equiv\delta \pmod{16}$, which violates the condition $\mathrm{sign}\,Y\equiv\delta(S,c) \pmod{16}$. Hence we have $\mathrm{sign}(Y)=\delta(S,c)$. □

We do not know whether a given (S,c) bounds a definite spin 4–manifold in general, but in certain cases we can give such examples explicitly (see the Addendum below). To describe them we need some notation and results.

Notation We denote the plumbed 4–manifold associated with the star-shaped diagram with three branches such that the weight of the central vertex is a and the weights of the vertices of the ith branch are given by $(a^i_1, \ldots, a^i_{k_i})$ as in Figure 5 by

$$(a; a^1_1, \ldots, a^1_{k_1}; a^2_1, \ldots, a^2_{k_2}; a^3_1, \ldots, a^3_{k_3}).$$

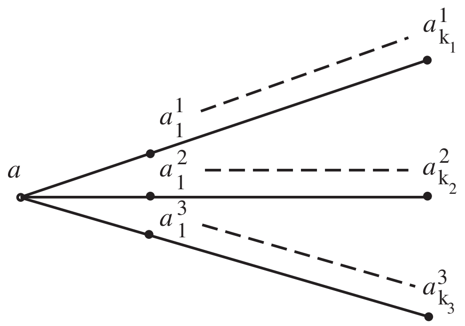

Figure 5

Proposition 9 [FS] *Consider a 3–manifold M represented by s/t surgery along a knot K in a framed link L in Figure 6. Here p, q, a, b are intergers satisfying $pa + qb = 1$, and s and t are coprime integers with opposite parity. Suppose also that M has a spin structure represented by $c \in \mathrm{Hom}(H_1(S^3 \setminus L, \mathbf{Z}), \mathbf{Z}_2)$ with $c(g_i) = w(h) = 0$. Then for a continued fraction expansion $-t/s = [[a_1, \ldots, a_k]]$ with a_i nonzero and even, (M, s) bounds a spin 4–manifold represented by a framed link L' in Figure 6. Here the component of L' on the left hand side is a (p, q) torus knot $C(p, q)$. We denote L' by $C(p, q)(pq; a_1, \ldots, a_k)$.*

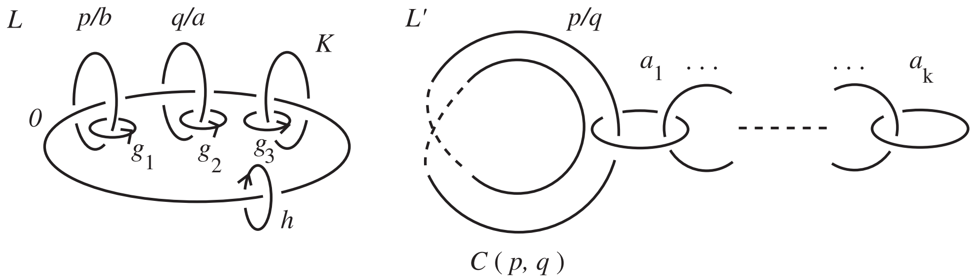

Figure 6

Proof The knot K in Figure 6 represents $C(p, q)$ in S^3, and the meridian and the preferred longitude of K is given by g_3 and $h + pqg_3$ respectively. Thus M

is realized as a $pq + s/t$ surgery along K, and hence by framed link calculus we see that M is also represented by L', where the spin structure c corresponds to the zero element in $\mathrm{Hom}(H_1(S^3 \setminus L', \mathbf{Z}), \mathbf{Z}_2)$ by Proposition 7. This proves the claim. □

Proposition 10 [FS] *Consider a knot K in Figure 7, where p_1, q_1, p_2, s_2, s_1, t_1, e, b_j are integers such that $p_1t_1 + q_1s_1 = 1$, $[[b_1, \ldots, b_s, 0, p_2/s_2]] = 1/e$, and that if we put $q_2/t_2 = [[b_s, \ldots, b_1]]$ for q_2, t_2 coprime, we have $p_2t_2 + q_2s_2 = 1$. Then K represents a knot $C(q_2 + p_2p_1q_1, p_2, C(p_1, q_1))$ in S^3. Here we denote by $C(q, p, K)$ the cable of the knot K with linking number q and winding number p. Moreover $1/u$ surgery along K in Figure 7 yields a $p_2(q_2 + p_2p_1q_1) + 1/u$ surgery along this cable knot in S^3, and the resulting manifold is a Seifert manifold of the form $\{(1, -e), (p_1, s_1), (q_1, t_1), (r_1, u_1)\}$, where*

$$r_1/u_1 = [[b_1, \ldots, b_s, -u, p_2/q_2]].$$

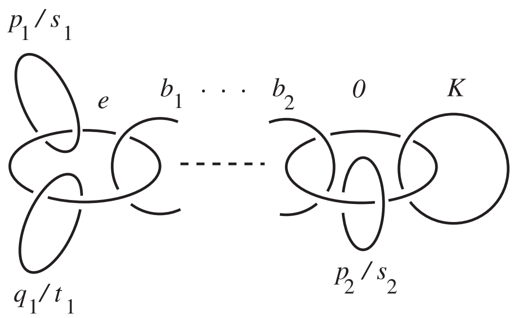

Figure 7

See [FS] for the proof of Proposition 10. We denote by $C(u, v, C(p, q))(a; b)$ the 4–manifold represented by a framed link with two components, which consists of $C(u, v, C(p, q))$ with framing a and its meridian with framing b.

Addendum For a spherical 3–manifold S with a spin structure c, there exists a definite spin 4–manifold Y with $\operatorname{sign} Y = \delta(S, c)$ bounded by (S, c) if at least (S, c) satisfies one of the following conditions.

(1) If the negative definite 4–manifold Y obtained by a minimal resolution of cS is spin and the induced spin struction on S is c, then Y satisfies the above condition. For the cases 3–5 in Proposition 6, the list of such (S, c) and Y is given by Table 1.

Table 1

(S,c)	Y	δ
(3-3)	$(-2k-2;-2;-2,-2;-2,-2)$	-6
(4-5)	$(-2k-2;-2;-4;-2,-2)$	-5
(4-13)	$(-2k-2;-2;-2,-2;-2,-2,-2)$	-7
(5-5)	$(-2k-2;-2,-2,-2;-2,-2,-2,-2)$	-8

Table 2

(S,c)	Y	δ
(3-5)	$C(2,-3)(-6;-2k-2,-2,-2)$	-4
(4-2)	$C(2,-3)(-6;-2k-2,-2,-2,-2)$	-5
(4-8)	$C(3,-4)(-12;2k,-2)$	-3
(4-10)	$C(2,-3)(-6;-2k-2,-4)$	-3
(5-1-1)	$C(2,-5)(-10;-2k-2,-2,-2)$	-4
(5-7)	$C(2,-3)(-6;-2k-2,-2,-2,-2,-2)$	-6
(3-6) with $k=-1$	$C(2,3)(8;2)$	2
(3-6) with $k=-2$	$C(13,2,C(2,3))(26;2)$	2
(4-4) with $k=-1$	$C(2,3)(8;2,2)$	3
(4-11) with $k=-1$	$C(3,-4)(-10)$	-1
(4-12) with $k=-1$	$C(2,3)(10)$	1
(4-14) with $k=0$	$C(2,-3)(-2)$	-1
(5-2-1) with $k=-1$	$C(2,5)(12;2)$	2
(5-2-1) with $k=-2$	$C(21,2,C(2,5))(42,2)$	2
(5-8) with $k=-1$	$C(2,3)(8;2,2,2)$	4
(5-11-(-1)) with $k=0$	$C(2,-3)(-4;-2)$	-2

(2) For the cases in 3–5 in Proposition 6 other than Table 1, the minimal resolution is non-spin, but (S,c) in Table 2 bounds another definite spin 4–manifold Y with $\operatorname{sign} Y = \delta(S,c)$.

Proof The first claim follows from Proposition 8. The construction of Y in Table 1 is given according to the procedure explained in the paragraph after Proposition 8. The construction of Y for the case when the associated framed link contains a torus knot component is given by the procedure in Proposition 9. To construct Y for the case (3-6) with $k=-2$, consider the knot K in Figure 8. Then the $-1/2$ surgery along K gives the Seifert manifold of type $\{(1,3),(2,1),(3,-1),(3,-1)\}$, which is S in case (3-6) with $k=-2$. We also note that K in Figure 8 gives the knot $C(1+2\cdot 6, 2, C(2,3))$ and $-1/2$ surgery on K yields the $2(1+2\cdot 6)-1/2$ surgery along the cable knot. Thus according to Proposition 10 the resulting 3–manifold bounds a 4–manifold of type $C(13, 2, C(2, 3))(42; 2)$, which is spin with signature 2. It follows that this 4–manifold induces the (unique) spin structure on S described in (3-6). The other cases are proved similarly and we omit the details. □

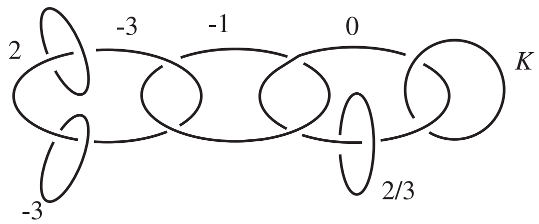

Figure 8

Remark 1 (1) If S is a Poincare homology sphere (which is the case $(5-5)$ with $k = 0$), the above result together with the classification of the unimodular forms implies that the intersection form of a spin negative definite 4–manifold bounded by S must be E_8, which is a part of Frøyshov theorem [Fr] (which was also observed by Furuta).

(2) The non-unimodular definite quadratic forms over $\mathbf{Z}$ with given rank and determinant are far from unique even if the ranks are small, in contrast to the unimodular ones. Y. Yamada [Y] pointed out that the Seifert manifold $\{(2, -1), (3, -1), (5, 8)\}$ ((5-2-1) with $k = -1$ in Table 2) bounds three 1-connected positive definite spin 4–manifolds with intersection matrices $\begin{pmatrix} 2 & 1 \\ 1 & 12 \end{pmatrix}, \begin{pmatrix} 4 & 3 \\ 3 & 8 \end{pmatrix}, \begin{pmatrix} 4 & 5 \\ 5 & 12 \end{pmatrix}$ respectively, which are not congruent.

(3) (S, c) with $\delta(S, c) > 0$ in Table 2 bounds a negative definite (non-spin) 4–manifold Z (coming from the minimal resolution) and a positive definite spin 4–manifold Y, both of which are 1-connected. Then $Z \cup (-Y)$ is a closed 1-connected negative definite 4–manifold, which is homeomorphic to a connected sum of $\overline{\mathbf{CP}}^2$'s by Donaldson and Freedman's theorem. Y. Yamada [Y] observed that such a manifold appeared in Table 2 other than the cases (3-6) and (5-2-1) (with $k = -2$) is diffeomorphic to $\sharp\overline{\mathbf{CP}}^2$.

Remark 2 In [BL] Bohr and Lee considered two invariants $m(\Sigma)$ and $\overline{m}(\Sigma)$ for a $\mathbf{Z}_2$ homology 3–sphere Σ. Here $m(\Sigma)$ is defined as the maximum of $5 \operatorname{sign} X/4 - b_2(X)$, while $\overline{m}(\Sigma)$ is defined as the minimum of $5 \operatorname{sign} X/4 + b_2(X)$, where X ranges over all spin 4–manifolds with $\partial X = \Sigma$. They proved that if

$m(\Sigma) > 0$ or $\overline{m}(\Sigma) < 0$ (or $m(\Sigma) = 0$ and $R(\Sigma) \neq 0$) then Σ has infinite order in $\Theta^3_{\mathbf{Z}_2}$ by using the 10/8 theorem for spin 4–manifolds. For example, consider $\Sigma = \{(2,-1),(3,-1),(3,-6k+2)\}$ with $k < 0$ ((3-4) in Proposition 6). Then as is remarked after Proposition 8, we can see that Σ bounds a spin plumbed 4–manifold $P(\Gamma)$ with $b^+(P(\Gamma)) = 5$ and $b^-(P(\Gamma)) = 1$. It follows that $-1 \leq m(\Sigma) \leq \overline{m}(\Sigma) \leq 11$. If Σ would bound a spin definite 4–manifold Y (we do not know whether this is the case), then we could see that $m(\Sigma) > 0$ and recover the claim (1) in Theorem 2 for this case. M. Furuta [Fu2] pointed out that if we extend the above definitions of $m(\Sigma)$ and $\overline{m}(\Sigma)$ by replacing $5\,\mathrm{sign}\,X/4 \pm b_2(X)$ by $5\,\mathrm{sign}\,X/4 + \delta(X)/4 \pm b_2(X)$, where X ranges over all spin V 4–manifolds with $\partial X = \Sigma$, then the same conclusion in [BL] is obtained by virtue of the V version of the 10/8 theorem.

Finally we give an application of Theorem 2 to embeddings of $\mathbf{RP}^2$ into 4–manifolds. The following theorem generalizes the result in [L] in the case when the embedded $\mathbf{RP}^2$ is a characteristic surface.

Theorem 3 *Let X be a closed smooth 4–manifold X with $H_1(X,\mathbf{Z}) = 0$. Suppose that there exists a smoothly embedded real projective plane F in X with $PD[F]$ (mod 2) $= w_2(X)$. Denote by $e(\nu)$ the normal Euler number of the normal bundle of the embedding $F \subset X$. Then $\mathrm{sign}\,X - e(\nu) \equiv \pm 2$ (mod 16). Furthermore*

(1) *If $\mathrm{sign}\,X - e(\nu) \equiv 2\epsilon$ (mod 16) with $\epsilon = \pm 1$, then either $e(\nu) + 2\epsilon = \mathrm{sign}\,X$ or*

$$8(1 - b^-(X)) + \mathrm{sign}\,X \leq e(\nu) + 2\epsilon \leq 8(b^+(X) - 1) + \mathrm{sign}\,X.$$

(2) *If both $b^-(X) < 3$ and $b^+(X) < 3$, then $e(\nu) = \mathrm{sign}\,X - 2\epsilon$.*

Proof Let $[F]$ be the element of $H_2(X,\mathbf{Z}_2)$ represented by F. First suppose that there exists an element $y \in H_2(X,\mathbf{Z}_2)$ with $[F]\cdot y \equiv 1$ (mod 2), and $e(\nu) \neq 0$. Put $n = e(\nu)$. Let X_0 be the complement of the tubular neighborhood $N(F)$ of F in X. Then $\partial N(F) = -\partial X_0$ is the twisted S^1 bundle over $\mathbf{RP}^2$ with normal Euler number n, which is diffeomorphic to a Seifert manifold over $S^2(2,2,|n|)$ (which we denote by S) with Seifert invariants $\{(2,1),(2,-1),(|n|,\mathrm{sgn}\,n)\}$. We fix the correspondence between $\partial N(F)$ and N as follows. Denote by g_i, h be the curves in the framed link picture L of S associated with the above Seifert invariants as in Figure 1. Also denote by Q, H be (one of) the cross section of the curve generating $H_1(\mathbf{RP}^2,\mathbf{Z}_2)$ and the fiber of the S^1 bundle $\partial N(F)$. Then we have a diffeomorphism between S and $\partial N(F)$ so that

$$Q = g_2, \quad H = g_1 + g_2 = -g_3, \quad 2Q = h$$

in the first homology group. Considering the exact sequence of the homology groups for the pair (X, X_0), we see that $H_2(X_0, \mathbf{Z}_2)$ is the set of $x \in H_2(X, \mathbf{Z}_2)$ with $x \cdot [F] \equiv \langle w_2(X), x \rangle \equiv 0 \pmod 2$. Hence X_0 admits a spin structure c (which is unique since $H_1(X_0, \mathbf{Z}_2) = 0$ by the above assumption). Since the spin structure induced on $S = -\partial X_0$ from c extends uniquely to that on the cone cS over S (Proposition 5), c extends uniquely to the spin structure on $\widehat{X} = cS \cup X_0$ (which we also denote by c). Since $H_3(X, X_0, \mathbf{Z}) = H^1(F, \mathbf{Z}) = 0$ and $H_2(X, X_0, \mathbf{Z}) = H^2(F, \mathbf{Z}) = \mathbf{Z}_2$, we have $H_2(X_0, \mathbf{Q}) = H_2(X, \mathbf{Q})$ and hence $b^{\pm}(\widehat{X}) = b^{\pm}(X)$ and $\mathrm{sign}(\widehat{X}) = \mathrm{sign}(X)$. Note that since $\langle w_2(X), y \rangle \equiv [F] \cdot y \equiv 1 \pmod 2$, the spin structure restricted on H does not extend to that on the disk fiber of $N(F)$. Under the above correspondence, this implies that if c restricted on S is represented by a homomorphism from $H_1(S^3 \setminus L, \mathbf{Z})$ to $\mathbf{Z}_2$ for a framed link L as in Definition 2, then $c(H) \equiv c(q_3) \equiv c(q_1) + c(q_2) \equiv 1 \pmod 2$.

The case when $n > 1$

Consider the representation of S by $\{(2,1), (2,1), (n, 1-n)\}$. If we denote the curves associated with the corresponding framed link picture by g_i' and h', then the correspondence between them and the original curves is given by

$$g_1' = g_1, \quad g_2' = g_2 - h, \quad g_3' = g_3 + h, \quad h' = h.$$

Now we check $\delta(S, c)$ according to the list in Proposition 6 (note that we have always $c(h) = c(h') = 0$).

The case when $(c(g_1'), c(g_2'), c(g_3')) = (0, 0, 0)$

Under the above correspondence we have $(c(g_1), c(g_2), c(g_3)) = (0, 1, 1)$ and hence $c(Q) = c(H) = 1$. Then since $n/(n-1) = \overbrace{[[2, \ldots, 2]]}^{n-1}$, (2-1) (if n is odd) or (2-6) (if n is even) in Proposition 6 shows that

$$\delta(S, c) = \sigma(n, 1-n, -1) = -\sigma(n, n-1, -1) = \sigma(n-1, n, -1) + 1 = -n + 2.$$

It follows that $\operatorname{ind} D(\widehat{X}) = -(\operatorname{sign}(\widehat{X}) + \delta(S,c))/8 = -(\operatorname{sign} X - n + 2)/8$. Thus Proposition 2 shows that $n \equiv \operatorname{sign} X + 2 \pmod{16}$, and either $n = \operatorname{sign} X + 2$ or

$$8(1 - b^-(X)) + \operatorname{sign} X \leq n - 2 \leq 8(b^+(X) - 1) + \operatorname{sign} X.$$

The case when $(c(g_1'), c(g_2'), c(g_3')) = (1,1,0)$

In this case $(c(g_1), c(g_2), c(g_3)) = (1,0,1)$ and hence $c(Q) = 0$, $c(H) = 1$. Then (2-2) or (2-7) in Proposition 6 implies that

$$\delta(S,c) = \sigma(n, 1-n, -1) - 4 = -n - 2.$$

Thus $n \equiv \operatorname{sign} X - 2 \pmod{16}$, and either $n = \operatorname{sign} X - 2$ or

$$8(1 - b^-(X)) + \operatorname{sign} X \leq n + 2 \leq 8(b^+(X) - 1) + \operatorname{sign} X.$$

The case when $(c(g_1'), c(g_2'), c(g_3')) = (\epsilon, 1-\epsilon, 1)$ **and** n **is even**

In this case we have $(c(g_1), c(g_2), c(g_3)) = (\epsilon, \epsilon, 0)$ and hence $c(H) = 0$. But this violates the above condition, and hence this case cannot occur.

The case when $n < -1$

Reversing the orientation of X we have an embedding $F \subset -X$ whose normal Euler number is $-n$. If we consider $(-\widehat{X}, -c)$ in place of $(\widehat{X}, c)$, we have $b^+(-\widehat{X}) = b^-(\widehat{X}) = b^-(X)$, $b^-(-\widehat{X}) = b^+(\widehat{X}) = b^+(X)$, $\operatorname{sign}(-\widehat{X}) = -\operatorname{sign}\widehat{X} = -\operatorname{sign} X$, and $\delta(-S,-c) = -\delta(S,c)$. Hence we derive the same result from the case when $n > 1$ by applying Proposition 2 to $-\widehat{X}$.

Next we consider the general case. Consider the internal connected sum of $F \subset X$ and k copies of the standard embedding $\mathbf{CP}^1 \subset \mathbf{CP}^2$ for some k to obtain another embedding $\widetilde{F} \subset \widetilde{X}$ of $\mathbf{RP}^2$, where $\widetilde{F} = F \sharp k\mathbf{CP}^1$ and $\widetilde{X} = X \sharp k\mathbf{CP}^2$. Then $PD[\widetilde{F}] \pmod 2 = w_2(\widetilde{X})$ and there exists an element $y \in H_2(\widetilde{X}, \mathbf{Z}_2)$ with $y \cdot \widetilde{F} \equiv 1 \pmod 2$ (for example, choose a copy of $\mathbf{CP}^1$ in one $\mathbf{CP}^2$ summand as y). Moreover the normal Euler number $e(\widetilde{\nu})$ of the embedding $\widetilde{F} \subset \widetilde{X}$ is $e(\nu) + k$ (which is greater than one for some k), $\operatorname{sign}(\widetilde{X}) = \operatorname{sign} X + k$, and $b^-(\widetilde{X}) = b^-(X)$. Thus applying the above result to $\widetilde{F} \subset \widetilde{X}$, we have $\operatorname{sign} X - e(\nu) = \operatorname{sign}(\widetilde{X}) - e(\widetilde{\nu}) \equiv \pm 2 \pmod{16}$, and obtain the inequality on the left hand side in (1). If we consider the embedding $\widetilde{F} \subset \widetilde{X}$ obtained by the internal connected sum of $F \subset X$ and k copies of the standard embedding

$\overline{\mathbf{CP}}^1 \subset \overline{\mathbf{CP}}^2$, we have $\widetilde{F} = F\sharp k\overline{\mathbf{CP}}^1$, $\widetilde{X} = X\sharp k\overline{\mathbf{CP}}^2$, the normal Euler number of $\widetilde{F} \subset \widetilde{X}$ is $e(\nu) - k$ (which is less than -1 for some k), $\operatorname{sign}\widetilde{X} = \operatorname{sign} X - k$, and $b^+(\widetilde{X}) = b^+(X)$. We also have $y \in H_2(\widetilde{X}, \mathbf{Z}_2)$ with $y \cdot \widetilde{F} \equiv 1 \pmod 2$. Thus applying the above result to this embedding we obtain the inequality on the right hand side in (1). To see (2) suppose that $b^+(X) < 3$ and $b^-(X) < 3$. Then $\operatorname{sign} X - 16 < 8(1 - b^-(X)) + \operatorname{sign} X$ and $8(b^+(X) - 1) + \operatorname{sign} X < \operatorname{sign} X + 16$. Since $e(\nu) + 2\epsilon \equiv \operatorname{sign} X \pmod{16}$, the above inequalities do not hold unless $e(\nu) + 2\epsilon = \operatorname{sign} X$. This proves (2). □

Remark 3 The claim $e(\nu) - \operatorname{sign} X \equiv \pm 2 \pmod{16}$ is also deduced from Guillou and Marin's theorem [GM], [M]. We note that for some X both of the cases when $\epsilon = \pm 1$ in (1) occur. For example, when $X = k\mathbf{CP}^2$, consider the connected sum of $\mathbf{RP}^2 \subset S^4$ with normal Euler class ± 2 and the k copies of $\mathbf{CP}^1 \subset \mathbf{CP}^2$ ([L]).

Remark 4 Acosta [A] obtained the estimate on the self-intersection number of a characteristic element x of X that is realized by a smoothly embedded 2–sphere by considering the 10/8 theorem for V 4–manifolds with $cL(p, \pm 1)$ type singularities. In our terminology, the result is derived as follows. Suppose that X is non-spin and x is realized by an embedded 2–sphere F with $x \cdot x = n > 0$. Then the complement X_0 of the tubular neighborhood $N(F)$ in X has a spin structure c, which does not extend to that on the disk fiber of $N(F)$. Since $N(F)$ is represented by the n surgery along the trivial knot O in S^3, this implies that the induced spin structure on $\partial N(F) = L(n, -1)$ corresponds to the homomorphism from $H_1(S^3 \setminus O, \mathbf{Z})$ (generated by the meridian μ of O) to $\mathbf{Z}_2$ with $c(\mu) = 1$. Consider the V manifold $\widehat{X} = cL(n, -1) \cup X_0$ with spin structure c, which is an extension of the original one. Then according to [FFU] Proposition 3, the contribution $\delta(L(n, -1), c)$ to $\operatorname{ind} D(\widehat{X})$ equals $\sigma(-1, n, (-1)^{c(\mu)-1}) = \sigma(-1, n, 1) = \sigma(n-1, n, -1) = -(n-1)$. Thus applying the 10/8 theorem to $\widehat{X}$ we obtain the inequality in [A]. The general case is proved by a similar argument as in the proof of Proposition 3 ([A]).

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