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Leafwise smoothing laminations

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Abstract We show that every topological surface lamination Λ of a 3–manifold M is isotopic to one with smoothly immersed leaves. This carries out a project proposed by Gabai in [2]. Consequently any such lamination admits the structure of a *Riemann surface lamination*, and therefore useful structure theorems of Candel [1] and Ghys [3] apply.

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1 Basic notions

Definition 1.1 A *lamination* is a topological space which can be covered by open charts U_i with a local product structure $\phi_i : U_i \to \mathbb{R}^n \times X$ in such a way that the manifold–like factor is preserved in the overlaps. That is, for $U_i \cap U_j$ nonempty,

$$\phi_j \circ \phi_i^{-1} : \mathbb{R}^n \times X \to \mathbb{R}^n \times X$$

is of the form

$$\phi_j \circ \phi_i^{-1}(t, x) = (f(t, x), g(x))$$

The maximal continuations of the local manifold slices $\mathbb{R}^n \times$ point are the *leaves* of the lamination. A *surface lamination* is a lamination locally modeled on $\mathbb{R}^2 \times X$. We usually assume that X is locally compact.

Definition 1.2 A lamination is *leafwise C^n* for $n \geq 2$ if the leafwise transition functions $f(t, x)$ can be chosen in such a way that the mixed partial derivatives in t of orders less than or equal to n exist for each x, and vary continuously as functions of x.

A *leafwise C^n structure* on a lamination Λ induces on each leaf λ of Λ a C^n manifold structure, in the usual sense.

Definition 1.3 An embedding of a leafwise C^n lamination $i : \Lambda \to M$ into a manifold M is an C^n *immersion* if, for some C^n structure on M, for each leaf λ of Λ the embedding $\lambda \to M$ is C^n.

Note that if $i : \Lambda \to M$ is an embedding with the property that the image of each leaf $i(\lambda)$ is locally a C^n submanifold, and these local submanifolds vary continuously in the C^n topology, then there is a unique leafwise C^n structure on Λ for which i is a C^n immersion.

A foliation of a manifold is an example of a lamination. For a foliation to be leafwise C^n is *a priori* weaker than to ask for it to be C^n immersed.

Example 1.4 Let M be a manifold which is not stably smoothable, and N a compact smooth manifold. Then $M \times N$ has the structure of a leafwise smooth foliation (by parallel copies of N), but there is no smooth structure on $M \times N$ for which the embedding of the foliation is a smooth immersion, since there is no smooth structure on $M \times N$ at all.

Remark 1.5 For readers unfamiliar with the notion, the "tangent bundle" of a topological manifold (i.e. a regular neighborhood of the diagonal in $M \times M$) is stably (in the sense of K–theory) classified by a homotopy class of maps $f : M \to BTOP$ for a certain topological space $BTOP$. There is a fibration $p : BO \to BTOP$, and the problem of lifting f to $\widehat{f} : M \to BO$ such that $p\widehat{f} = f$ represents an obstruction to finding a smooth structure on M. For N smooth as above, the composition

$$M \to M \times \text{point} \subset M \times N \to BTOP$$

is homotopic to f, and therefore no lifting of the structure exists on $M \times N$ if none existed on M. For a reference, see [4], or the very readable [6].

With notation as above, the tangential quality of $\mathcal{F}$ is controlled by the quality of $f(\cdot, x)$ for each fixed x, for f the first component of a transition function. For sufficiently large k and $n - k$ questions of ambiently smoothing *foliated manifolds* come down to obstruction theory and classical surgery theory, as for example in [4]. But in low dimensions, the situation is more elementary and more hands–on.

2 Some 3–manifold topology

Let M be a topological 3–manifold. It is a classical theorem of Moise (see [5]) that M admits a PL or smooth structure, unique up to conjugacy.

Lemma 2.1 *Let Σ be a topological surface. Let S^1_j be a countable collection of circles, and let $\Phi : \coprod_j S^1_j \times I \to \Sigma$ be a map with the following properties:*

(1) *For each $t \in I$, $\Phi(\cdot, t) : S^1_j \to \Sigma$ is an embedding.*

(2) *For each $t \in I$ and each pair j, k the intersection*

$$\Phi(S^1_j, t) \cap \Phi(S^1_k, t)$$

is finite, and its cardinality is constant as a function of t away from finitely many values.

(3) *For every compact subset $K \subset \Sigma$ the set of j for which $\Phi(S^1_j, t) \cap K$ is nonempty for some t is finite.*

Then there is a PL (resp. smooth) structure on $\Sigma \times I$ such that the graph of each map $\Gamma_j(\Phi) : S^1_j \times I \to \Sigma \times I$ is PL (resp. smooth).

Here the *graph* $\Gamma_j(\Phi)$ *of* Φ is the function $\Gamma_j(\Phi) : S^1_j \times I \to \Sigma \times I$ defined by

$$\Gamma_j(\Phi)(\theta, t) = (\Phi(\theta), t)$$

Proof The conditions imply that the image of $\coprod_j S^1_j$ in Σ for a fixed t is topologically a locally finite graph. Such a structure in a 2 manifold is locally flat, and the combinatorics of any finite subgraph is locally constant away from isolated values of t. It is therefore straightforward to construct a PL (resp. smooth) structure on a collar neighborhood of the image of $\coprod_j S^1_j \times I$ in $\Sigma \times I$. This can be extended canonically to a PL (resp. smooth) structure on $\Sigma \times I$, by the relative version of Moise's theorem (see [5]). □

Lemma 2.2 *Let $\Phi : S^1_j \times I \to \Sigma$ satisfy the conditions of lemma 2.1. Let $\Psi_0 : S^1 \to \Sigma$ and $\Psi_1 : S^1 \to \Sigma$ be homotopic embeddings such that $\Psi_0(S^1)$ intersects finitely many circles in $\Phi(\cdot, 0)$ in finitely many points, and similarly for $\Psi_1(S^1)$. Then there is a map $\Psi : S^1 \times I \to \Sigma$ which is a homotopy between Ψ_0 and Ψ_1 so that*

$$\Phi \cup \Psi : \Big(\coprod_i S^1_i \coprod S^1\Big) \times I \to \Sigma$$

satisfies the conditions of lemma 2.1.

Proof Since the combinatorics of the image of Φ is locally finite, and since the image of Ψ is bounded, it suffices to treat the case when Φ is constant as a function of t.

Choose a PL structure on Σ for which the image of $\Phi(\cdot,0)$ and Ψ_0 are polygonal. Then produce a polygonal homotopy from Ψ_0 (with respect to this polygonal structure) to a new polygonal Ψ_0' such that $\Psi_0'(S^1)$ and $\Psi_1(S^1)$ intersect the image of $\Phi(\cdot,t)$ in a finite set of points in the same combinatorial configuration. Then Ψ_0' is isotopic to Ψ_1 rel. its intersection with the image of $\Phi(\cdot,t)$. □

3 Surface laminations of 3–manifolds

Definition 3.1 Let $\mathcal{F}$ be a codimension one foliation of a 3–manifold M. A *snake* in M is an embedding $\phi : D^2 \times I \to M$ where D^2 denotes the open unit disk, and I the open unit interval, which extends to an embedding of the closure of $D^2 \times I$, in such a way that each horizontal disk gets mapped into a leaf λ of $\mathcal{F}$. That is, $\phi : D^2 \times t \to \lambda$.

The terminology suggests that we are typically interested in snakes which are reasonably small and thin in the leafwise direction, and possibly large in the transverse direction.

A collection of snakes in a foliated manifold intersect a leaf λ of $\mathcal{F}$ in a locally finite collection of open disks. For a snake S, let $\partial_v\overline{S}$ denote the "vertical boundary" of the closed ball $\overline{S}$; this is topologically an embedded closed cylinder transverse to $\mathcal{F}$, intersecting each leaf in an inessential circle.

We say that an open cover of M by finitely many snakes S_i is *combinatorially tame* if the embeddings $\partial_v\overline{S_i} \to M$ are locally of the form described in lemma 2.1.

Note that the induced pattern on each leaf λ of $\mathcal{F}$ of the circles $\partial_v\overline{S_i} \cap \lambda$ is topologically conjugate to the transverse intersection of a locally finite collection of polygons.

Lemma 3.2 *A codimension one foliation $\mathcal{F}$ of a closed 3–manifold M admits a combinatorially tame open cover by finitely many snakes.*

Proof Since M is compact, any cover by snakes contains a finite subcover; any such cover induces a locally finite cover of each leaf. We prove the lemma by induction.

Let S_i be a collection of snakes in M which is combinatorially tame. Let $C_i = \partial_v\overline{S_i}$ be their vertical boundaries, and let S be another snake with vertical boundary C. We will show that there is a snake S' containing S such that the collection $\{S_i\} \cup \{S'\}$ is combinatorially tame.

Let λ_t for $t \in I$ parameterize the foliation of $\overline{S}$. Let $E_i(t)$ denote the pattern of circles $C_i \cap \lambda_t$ in a neighborhood of $E(t) = C \cap \lambda_t$. By hypothesis, the C_i can be thought of as polygons with respect to a PL structure on λ_t. Then $E(t)$ can be *straightened* to a polygon $E(t)'$ in general position with respect to the $E_i(t)$ in a small neighborhood, where the interior of the region in λ_t bounded by $E(t)'$ contains $E(t)$. If λ_t does not intersect the horizontal boundary of any $\overline{S_i}$, then the combinatorial pattern of intersections of the $E_i(t)$ is locally generic — i.e. the pattern might change, but it changes by the graph of a generic PL isotopy, by lemma 2.1.

It follows that we can extend the straightening of $E(t)$ to $E(t)'$ for some collar neighborhood of $t = 0$. In general, a straightening of $E(t)$ to $E(t)'$ can be extended in the positive direction until a t_0 which contains some lower horizontal boundary of an $\overline{S_i}$. The straightening can be extended past an upper horizontal boundary of an $\overline{S_i}$ without any problems, since the combinatorial pattern of intersections becomes simpler: circles disappear.

The straightening of $E(t)$ over all t can be done by *welding* straightenings centered at the finitely many values of t which contain horizontal boundary of some $\overline{S_i}$. Call these critical values t_j. So we can produce a finite collection of straightenings $E(t) \to E(t)'_j$ each valid on the open interval $t \in (t_{j-1}, t_{j+1})$. To weld these straightenings together at intermediate values s_j where $t_j < s_j < t_{j+1}$, we insert a PL isotopy from $E(s_j)'_j$ to $E(s_j)'_{j+1}$ in a little collar neighborhood of s_j, by appealing to lemma 2.2. So these welded straightenings give a straightening of $E(t)$ for all $t \in I$, and they bound a snake S' with the requisite properties.

To prove the lemma, cover M with finitely many snakes S_i, and apply the induction step to straighten S_j while fixing S_k with $k < j$. Since snakes can be straightened by an arbitrarily small (in the C^0 topology) homotopy, the union of straightened snakes can also be made to cover M, and we are done. □

Lemma 3.3 *Let M be a 3–manifold, and $\mathcal{F}$ a foliation of M by surfaces. Then $\mathcal{F}$ is isotopic to a foliation such that all leaves are PL or smoothly immersed, and the images of leaves vary locally continuously in the C^∞ topology.*

Proof If S_i is a combinatorially tame cover of $\mathcal{F}$ by snakes, the image of the union $\cup_i \partial\overline{S_i}$ can be taken to be a PL or smooth 2 complex Σ in M, whose complementary regions are polyhedral 3 manifolds. Each complementary region is foliated as a product by $\mathcal{F}$. We can straighten $\mathcal{F}$ cell–wise inductively on its intersection with the skeleta of Σ. First, we keep $\mathcal{F} \cap \Sigma^1$ constant. Then the foliation of $\mathcal{F} \cap (\Sigma^2 \backslash \Sigma^1)$ by lines can be straightened to be PL or smooth, and this straightened foliation extended in a PL or smooth manner over the product complementary regions in $M - \Sigma$. □

Theorem 3.4 *Let Λ be a surface lamination in a 3–manifold M. Then Λ is isotopic to a lamination such that all leaves are PL or smoothly immersed, and the images of leaves vary locally continuously in the C^∞ topology.*

Proof By the definition of a lamination, there is an open cover of Λ by balls B_i such that $\Lambda \cap B_i$ is a product lamination, which can be extended to a product foliation. It is straightforward to produce an open submanifold N with $\Lambda \subset N \subset M$ such that N can be foliated by a foliation $\mathcal{F}$ which contains Λ as a sublamination. Then the open manifold N can be given a PL or smooth structure in which $\mathcal{F}$, and hence Λ, is PL or smoothly immersed, by lemma 3.3. This PL or smooth structure can be extended compatibly over $M - N$ by Moise's theorem. □

Corollary 3.5 *Let Λ be a surface lamination in a 3–manifold M. Then Λ admits a leafwise PL or smooth structure.*

In particular, such a lamination admits the structure of a Riemannian surface lamination. In Gabai's problem list [2], he lists theorem 3.4 as a "project". The corollary allows us to apply the technology of complex analysis and algebraic geometry to such laminations; in particular, the following theorems of Candel and Ghys from [1] and [3] apply:

Theorem 3.6 (Candel) *Let $\mathcal{F}$ be an essential Riemann surface lamination of an atoroidal 3–manifold. Then there exists a continuously varying path metric on $\mathcal{F}$ for which the leaves of $\mathcal{F}$ are locally isometric to $\mathbb{H}^2$.*

Theorem 3.7 (Ghys) *Let $\mathcal{F}$ be a taut foliation of a 3–manifold M with Riemann surface leaves. Then there is an embedding $e : M \to \mathbb{CP}^n$ for some n which is leafwise holomorphic. That is, $e : \lambda \to \mathbb{CP}^n$ is holomorphic for each leaf λ.*

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