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Commensurability of graph products

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Abstract We define graph products of families of pairs of groups and study the question when two such graph products are commensurable. As an application we prove linearity of certain graph products.

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Graph products are useful and pretty generalizations of both products and free products, intimately linked with right-angled buildings. Part of their appeal is their generality: they can be studied in any category with products and direct limits.

The question that motivated the present paper was "when are the graph products of two families of groups commensurable". The inspiration came from a special case considered in [5] and from a conversation with Marc Bourdon on linearity of certain lattices in automorphism groups of right-angled buildings.

Here is an answer to the simplest version of this question. Recall first that two groups G, G^* are *commensurable* if there is a group H isomorphic to a subgroup of finite index in both G and G^*; they are *strongly commensurable* if H has the same index in both G and G^*.

Theorem 1 *Let Γ be a finite graph, $(G_v)_{v\in V}$, $(G_v^*)_{v\in V}$ be two families of groups indexed by the vertex set of Γ. Suppose that for every $v \in V$, G_v and G_v^* are strongly commensurable with the common subgroup H_v. Then the graph products $\mathbf{G} = \Pi_\Gamma(G_v)_{v\in V}$, and $\mathbf{G}^* = \Pi_\Gamma(G_v^*)_{v\in V}$ are strongly commensurable: they share a subgroup of index $\Pi_v[G_v : H_v]$.*

We will prove a slightly more general result on graph products of pairs of groups. The proof uses two complementary descriptions of right-angled building on which a graph product acts. One of them allows an easy identification of the group acting as the graph product, the other allows to compare subgroups.

Theorem 1 and its stronger version formulated in Section 4 (Corollary 4.2) have several interesting special cases discussed in Section 5.

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1 Graph products of pairs

Graphs A *graph* Γ on the vertex set $V = V(\Gamma)$ is an antireflexive symmetric relation on V. Thus our graphs have no loops and there is at most one undirected edge between two vertices. Graphs considered in this paper are always finite.

A *full subgraph* $\Gamma^+ < \Gamma$ on vertices $W \subset V$ is the restriction of the relation to W.

A graph is *complete* if there is an edge between any two vertices.

A *map of graphs* $f : \Gamma \to \Gamma^*$ is an injection of sets of vertices with the property that if there is an edge between v, w then there is an edge between $f(v), f(w)$. Thus our maps of graphs are inclusions.

Graph products Let Γ be a finite graph, with vertex set V. Suppose for each $v \in V$ one is given a pair of groups $A_v < G_v$. For S, a complete subgraph of Γ, define $G_S = \Pi_{v \in S} G_v \times \Pi_{v \in V \setminus S} A_v$. The family of groups G_S together with obvious inclusions on factors of products gives a direct system of groups directed by the poset $\mathcal{P}$ of complete subgraphs in Γ, empty set and singletons included ($G_\emptyset = \Pi_v A_v; G_{\{v\}} = G_v \times \Pi_{w \in V \setminus \{v\}} A_w$).

The direct limit of this system

$$\mathbf{G} = \lim\, (G_S)_{S \in \mathcal{P}} = \Pi_\Gamma (G_v, A_v)$$

is called *the graph product along* Γ *of the family of pairs* (G_v, A_v). To keep notation simple we will denote it for most of the time by $\mathbf{G}$. Note that for $A_v = \{e\}$ we obtain ordinary graph products.

Graph products are functorial If $g : \Gamma \to \Gamma^*$ is a map of graphs, and if there is a family of group homomorphisms $\omega_v : G_v \to G^*_{g(v)}$, such that $\omega_v(A_v) < A^*_{g(v)}$ then we have induced maps $\omega_S : G_S \to G^*_{g(S)}$ which clearly commute with the maps of direct systems and consequently induce a homomorphism

$$\omega : \mathbf{G} \to \mathbf{G}^*.$$

If g is a surjection on the vertices and ω_v are all surjections, so is the induced homomorphism ω. If g is an embedding onto a full subgraph and ω_v are injections, so is the induced homomorphism.

Remark 1.1 It follows from functoriality above that if Γ is a full subgraph of Γ^* then graph product of any family of pairs along Γ^* contains as a subgroup the graph product of that family of pairs restricted to Γ. In particular, groups G_S inject into $\mathbf{G}$. Thus we can (and will) consider G_S as subgroups of $\mathbf{G}$.

Presentations Graph products can be given in terms of generators and relations. Suppose that each group G_v is given by presentation $\langle S_v | R_v \rangle$ and that Σ_v is a set of generators for the subgroup A_v expressed in terms of generators in S_v. Then the graph product $\mathbf{G} = \Pi_\Gamma(G_v, A_v)$ is given by the presentation $\langle \cup_{v\in V} S_v | \cup_{v\in V} R_v \cup C \rangle$, where C consists of commutators $\{sts^{-1}t^{-1}\}$ whenever $s \in S_v$, $t \in S_w$ and there is an edge between v and w in Γ, or whenever $s \in S_v$, $t \in \Sigma_w$ for some $v \neq w$.

Examples

(1) Graph product of pairs $\Pi_\Gamma(G_v, A_v)$ along a complete graph Γ is the (direct) product $\Pi_{v\in V} G_v$.

(2) If Γ is an empty graph (i.e. an empty relation on the vertex set V) then the graph product $\Pi_\Gamma(G_v, A_v)$ is the free product of groups $G_{\{v\}} = G_v \times \Pi_{w\in V\setminus\{v\}} A_w$ amalgamated along their common subgroup $G_\emptyset = \Pi_{v\in V} A_v$.

(3) Graph products (with trivial subgroups A_v) of infinite cyclic groups are called right-angled Artin groups.

(4) Graph products of cyclic groups of order 2 are called right-angled Coxeter groups (i.e. Coxeter groups with exponents 2 or ∞ only).

2 The complex $D_{\mathbf{G}}$

Description of $D_{\mathbf{G}}$ Let P be the realization of the poset $\mathcal{P}$ of complete subgraphs in Γ i.e. the simplicial complex with the vertex set $\mathcal{P}$ and with simplices corresponding to flags (i.e. linearly ordered subsets) of $\mathcal{P}$. For each $S \in \mathcal{P}$ let P_S be the subcomplex of P spanned by those vertices $S' \in \mathcal{P}$ which contain S. Note that the poset of subcomplexes P_S with the reverse inclusion is isomorphic to the poset $\mathcal{P}$. Define a simplicial complex $D_{\mathbf{G}} = \mathbf{G} \times P/ \sim$ where the equivalence relation is given by $(g_1, x_1) \sim (g_2, x_2)$ iff for some $S \in \mathcal{P}$ we have $x_1 = x_2 \in P_S$ and $g_1^{-1} g_2 \in G_S \subset \mathbf{G}$. We denote the point in $D_{\mathbf{G}}$ corresponding to a pair $(g, x) \in \mathbf{G} \times P$ by $[g, x]$. Group $\mathbf{G}$ acts on the complex $D_{\mathbf{G}}$ on the left by $g \cdot [g', x] = [gg', x]$.

One should keep in mind that the complex $D_{\mathbf{G}}$ depends on the description of the group as a graph product, rather than on the group only.

Remark The $\mathbf{G}$ action on $D_{\mathbf{G}}$ need not be effective. Its kernel is the product $\Pi N_v < \Pi A_v$, where N_v is the intersection of all G_v conjugates of A_v. Dividing by the kernel of the action is geometrically sound and gives the *reduced graph product of pairs*. For example if all A_v are normal the reduced graph product is just the graph product of quotients.

Complex of groups $\mathbf{G}(\mathcal{P})$ Denote by $\mathbf{G}(\mathcal{P})$ the simple complex of groups (in the sense of [1], Chapter II.12) over the poset $\mathcal{P}$ defined by the directed system $(G_S)_{S \in \mathcal{P}}$ of groups. In view of the injectivity discussed in Remark 1.1, Theorems 12.18, 12.20 and Corollary 12.21 of [1] imply:

Proposition 2.1 *The simplicial complex $D_{\mathbf{G}}$ is isomorphic to the development of the complex of groups $\mathbf{G}(\mathcal{P})$ corresponding to the family $(i_S)_{S \in \mathcal{P}}$ of canonical inclusions $i_S : G_S \to \mathbf{G}$ into the direct limit. In particular $D_{\mathbf{G}}$ is connected and simply connected.*

Moreover the complex of groups associated to the action of $\mathbf{G}$ on $D_{\mathbf{G}}$ coincides with $\mathbf{G}(\mathcal{P})$.

$D_{\mathbf{G}}$ is a building The complex $D_{\mathbf{G}}$ is well known and is sometimes called the right-angled building associated to a graph product $\mathbf{G}$, see [4, Section 5] and [1] (section 12.30 (2)). It is indeed a Tits building whose appartments are Davis complexes of the (right-angled) Coxeter group which is the graph product of Z_2's along Γ.

3 Another description of G and $D_{\mathbf{G}}$

Associated graph product along the complete graph Given a finite graph Γ on the vertex set V and a graph product $\mathbf{G} = \Pi_\Gamma(G_v, A_v)$, denote by $\mathbf{G}^c$ the graph product of pairs (G_v, A_v) along the complete graph Γ^c on the vertex set V. Put $\omega^c : \mathbf{G} \to \mathbf{G}^c$ to be the homomorphism given by functoriality discussed in Section 1 and note that ω^c is surjective.

Let $\mathcal{P}^c$ be the poset of complete subgraphs in Γ^c (including singletons and the empty graph) and let P^c be its realization. The inclusion $\Gamma \to \Gamma^c$ clearly induces an injective simplicial map $p^c : P \to P^c$ (where P is the realization of the corresponding poset for Γ).

Complex $\Delta_{\mathbf{G}}$ and group $\widetilde{G}$ Let $D_{\mathbf{G}^c}$ be the simplicial complex associated to the graph product $\mathbf{G}^c$ as in Section 2. Denote by $\pi^c : D_{\mathbf{G}^c} \to P^c$ the simplicial map induced by the projection $\mathbf{G}^c \times P^c \to P^c$. Put $\Delta_{\mathbf{G}} := (\pi^c)^{-1}(p^c(P))$ and note that, since the action of $\mathbf{G}^c$ on $D_{\mathbf{G}^c}$ commutes with π^c, the subcomplex $\Delta_{\mathbf{G}} \subset D_{\mathbf{G}^c}$ is invariant under this action. Thus we will speak about the (restricted) action of $\mathbf{G}^c$ on $\Delta_{\mathbf{G}}$. Consider the universal cover $\widetilde{\Delta_{\mathbf{G}}}$ of $\Delta_{\mathbf{G}}$, with the action of the group $\widetilde{G}$ which is the extension (induced by the covering $\widetilde{\Delta_{\mathbf{G}}} \to \Delta_{\mathbf{G}}$) of the group $\mathbf{G}^c$ by the fundamental group $\pi_1(\Delta_{\mathbf{G}})$.

Theorem 3.1 *Groups $\widetilde{G}$ and $\mathbf{G}$ are isomorphic, simplicial complexes $D_{\mathbf{G}}$ and $\widetilde{\Delta_{\mathbf{G}}}$ are equivariantly isomorphic and the homomorphism $\widetilde{G} \to \mathbf{G}^c$ induced by the covering $\widetilde{\Delta_{\mathbf{G}}} \to \Delta_{\mathbf{G}}$ coincides with the map $\omega^c : \mathbf{G} \to \mathbf{G}^c$.*

Proof Let $f : D_{\mathbf{G}} \to \Delta_{\mathbf{G}} \subset D_{\mathbf{G}^c}$ be defined by $f([g,x]) = [\omega^c(g), p^c(x)]$. This map is easily seen to be surjective and ω^c-equivariant. It induces then a morphism $f_* : \mathbf{G}\backslash\backslash D_{\mathbf{G}} \to \mathbf{G}^c\backslash\backslash\Delta_{\mathbf{G}}$ between the complexes of groups $\mathbf{G}\backslash\backslash D_{\mathbf{G}}$ and $\mathbf{G}^c\backslash\backslash\Delta_{\mathbf{G}}$ associated to the actions of $\mathbf{G}$ on $D_{\mathbf{G}}$ and of $\mathbf{G}^c$ on $\Delta_{\mathbf{G}}$ as in [1].

Observe that for a vertex $[g,S] \in D_{\mathbf{G}}$ the isotropy subgroup of $\mathbf{G}$ at $[g,S]$ can be described as $\mathrm{Stab}(\mathbf{G},[g,S]) = gG_Sg^{-1}$. By substituting $\mathbf{G}$ with $\mathbf{G}^c$ in this observation we see that the homomorphism $\omega^c : \mathbf{G} \to \mathbf{G}^c$ maps stabilizers in $D_{\mathbf{G}}$ isomorphically to stabilizers in $D_{\mathbf{G}^c}$ and hence also in $\Delta_{\mathbf{G}}$. The morphism f_* is then isomorphic on local groups. Since moreover the map between the underlying spaces (quotient spaces of the corresponding actions) associated to the morphism f_* is a bijection, it follows that f_* is an isomorphism of complexes of groups.

Let $u : \widetilde{\Delta_{\mathbf{G}}} \to \Delta_{\mathbf{G}}$ be the universal covering map. As before, by natural equivariance, this map induces a morphism $u_* : \widetilde{G}\backslash\backslash\widetilde{\Delta_{\mathbf{G}}} \to \mathbf{G}^c\backslash\backslash\Delta_{\mathbf{G}}$ between the complexes of groups associated to the corresponding actions. It follows then from local injectivity of u that the stabilizers of $\widetilde{G}$ in $\widetilde{\Delta_{\mathbf{G}}}$ are mapped isomorphically (by the homomorphism $\widetilde{G} \to \mathbf{G}^c$ associated to the covering) to the stabilizers of $\mathbf{G}^c$ in $\Delta_{\mathbf{G}}$, hence u_* is isomorphic on local groups. Combining this with equality of the underlying quotient complexes (which follows directly from the description of $\widetilde{G}$) we see that u_* is also an isomorphism of complexes of groups.

Now, since both complexes $D_{\mathbf{G}}$ and $\widetilde{\Delta_{\mathbf{G}}}$ are connected and simply connected, it follows that they are both equivariantly isomorphic to the universal covering of the complex of groups $\Pi_{v\in V}G_v\backslash\backslash\Delta_{\mathbf{G}}$ acted upon by the fundamental group of this complex of groups. Thus the theorem follows. □

Complex $C\mathcal{X}$ Consider the family $\mathcal{X} = (X_v)_{v\in V}$ of quotients $X_v = G_v/A_v$. Denote by $\mathcal{C}$ the poset consisting of all subsets Y in the disjoint union $\cup\mathcal{X}$ having at most one common element with each of the sets X_v. We assume that the empty set $\emptyset$ is also in $\mathcal{C}$. Put $C\mathcal{X}$ to be the realization of the poset $\mathcal{C}$ i.e. a simplicial complex with simplices corresponding to linearly ordered subsets of $\mathcal{C}$. Alternatively, $C\mathcal{X}$ is the simplicial cone over the barycentric subdivision of the join of the family $\mathcal{X}$.

The complex $C\mathcal{X}$ carries the action of the group $\Pi_{v\in V}G_v$ induced from actions of the groups G_v on the sets X_v (from the left).

Proposition 3.2 *The action of $\mathbf{G}^c$ on the associated complex $D_{\mathbf{G}^c}$ is equivariantly isomorphic to the action of $\Pi_{v\in V}G_v$ on $C\mathcal{X}$.*

Proof We will construct a simplicial isomorphism $c : D_{\mathbf{G}^c} \to C\mathcal{X}$ as required, defining it first on vertices. Let $[g, S] \in D_{\mathbf{G}^c}$ be a vertex where $g = \Pi g_v \in \Pi G_v$, $g_v \in G_v$, and $S \subset V$. Put

$$c_0([g, S]) := \{g_v A_v : v \in V \setminus S\}$$

and notice the following properties:

(1) for any vertex $[g, S]$ of $D_{\mathbf{G}^c}$ its image $c_0([g, S])$ is a well defined vertex in $C\mathcal{X}$;

(2) c_0 defines a bijection between the vertex sets of the complexes $D_{\mathbf{G}^c}$ and $C\mathcal{X}$;

(3) both c_0 and c_0^{-1} preserve the adjacency relation on the vertex sets in the corresponding complexes (where two vertices are called *adjacent* when they span a 1-simplex).

Note that, by definition, both complexes $D_{\mathbf{G}^c}$ and $C\mathcal{X}$ have the following property: each set of pairwise adjacent vertices in the complex spans a simplex of this complex (complexes satisfying this property are often called flag complexes). This property, together with properties (2) and (3) above, imply that the map c_0 induces a simplicial isomorphism $c : D_{\mathbf{G}^c} \to C\mathcal{X}$.

Now, if $g' = \Pi g'_v \in \mathbf{G}^c = \Pi G_v$, with $g'_v \in G_v$, we have

$$\begin{aligned} g' \cdot c([g,S]) &= g' \cdot \{g_v A_v : v \in V \setminus S\} = \{g'_v g_v A_v : v \in V \setminus S\} \\ &= c([g'g, S]) = c(g' \cdot [g,S]), \end{aligned}$$

and hence c is equivariant. □

Alternative description of $\Delta_{\mathbf{G}}$ Denote by Q the quotient of the action of $\Pi_{v\in V} G_v$ on $C\mathcal{X}$, and by $q : C\mathcal{X} \to Q$ the associated quotient map. Q is easily seen to be the simplicial cone over the barycentric subdivision of the simplex spanned by the indexing set V of the family $\mathcal{X}$. Observe now that the equivariant isomorphism $c : D_{\mathbf{G}^c} \to C\mathcal{X}$ of Proposition 3.2 induces an isomorphism $\varepsilon : P^c \to Q$ of the quotients, and thus we have $q \circ c = \varepsilon \circ \pi^c$. In fact ε is given on vertices by $\varepsilon(S) = V \setminus S$. Define the map $\delta : P \to Q$ by $\delta := \varepsilon \circ p^c$. Proposition 3.2 implies then the following.

Corollary 3.3 *The subcomplex $q^{-1}(\delta(P)) \subset C\mathcal{X}$ is invariant under the action of the group $\Pi_{v\in V} G_v$ and the action of this group restricted to this subcomplex is equivariantly isomorphic to the action of $\mathbf{G}^c$ on $\Delta_{\mathbf{G}}$.*

Slightly departing from the main topic of the paper, we give the following interesting consequence of Theorem 3.1.

Corollary 3.4 *A graph product (along any finite graph) of pairs (G_v, A_v) is virtually torsion free iff all G_v are virtually torsion free.*

Proof Since the groups G_v inject into the graph product $\mathbf{G} = \Pi_\Gamma(G_v, A_v)$, they are clearly virtually torsion free if their graph product is. To prove the converse, observe that by Theorem 3.1 $\mathbf{G}$ is a semidirect product of the group $\mathbf{G}^c = \Pi_{v\in V} G_v$ by the fundamental group $\pi_1(\Delta_{\mathbf{G}})$. Since the space $\Delta_{\mathbf{G}}$ is finite dimensional and aspherical (its universal cover $\widetilde{\Delta_{\mathbf{G}}}$ is isomorphic to the Davis' realization of a building, and hence contractible, see [4]), its fundamental group is torsion free and the corollary follows. □

4 Large common subgroups and the proof of Theorem 1

Subgroups Let (G_v, A_v) and (G_v^*, A_v^*) be two families of pairs of groups. Denote by $\mathbf{G}$ and $\mathbf{G}^*$ the corresponding graph products of pairs along the same graph Γ, and by $\mathbf{G}^c$ and $(\mathbf{G}^*)^c$ the corresponding graph products along the complete graph Γ^c. Let $\omega^c : \mathbf{G} \to \mathbf{G}^c$ and $(\omega^*)^c : \mathbf{G}^* \to (\mathbf{G}^*)^c$ be the homomorphisms induced by functoriality from the inclusion map $\Gamma \to \Gamma^c$.

For each $v \in V$ let $H_v < G_v$ and $H_v^* < G_v^*$ be arbitrary subgroups. Denote by $\mathbf{H}$ and $\mathbf{H}^*$ preimages of subgroups $\Pi H_v < \Pi G_v = \mathbf{G}^c$ and $\Pi H_v^* < \Pi G_v^* = (\mathbf{G}^*)^c$ under the maps ω^c and $(\omega^*)^c$ respectively.

Theorem 4.1 *If the left actions of H_v on G_v/A_v and of H_v^* on G_v^*/A_v^* are equivariantly isomorphic for all $v \in V$ then the actions of $\mathbf{H}$ on $D_{\mathbf{G}}$ and of $\mathbf{H}^*$ on $D_{\mathbf{G}^*}$ are equivariantly isomorphic. In particular the subgroups $\mathbf{H}$ and $\mathbf{H}^*$ are isomorphic.*

Proof Let $\mathcal{X}$ and $\mathcal{X}^*$ be the families of the sets of cosets for the families (G_v, A_v) and (G_v^*, A_v^*) respectively. Under assumptions of the theorem, the actions of products ΠH_v on $C\mathcal{X}$ and ΠH_v^* on $C\mathcal{X}^*$ are equivariantly isomorphic. Applying Corollary 3.3 we conclude that the actions of the groups ΠH_v and ΠH_v^* on the complexes $\Delta_{\mathbf{G}}$ and $\Delta_{\mathbf{G}^*}$ respectively are equivariantly isomorphic.

Denote by $\widetilde{H}$ and $\widetilde{H}^*$ the preimages of the products ΠH_v and ΠH_v^* by the homomorphisms $\widetilde{G} \to \Pi G_v$ and $\widetilde{G}^* \to \Pi G_v^*$ respectively. It follows that the actions of $\widetilde{H}$ on $\widetilde{\Delta}_{\mathbf{G}}$ and of $\widetilde{H}^*$ on $\widetilde{\Delta}_{\mathbf{G}^*}$ are equivariantly isomorphic. But, due to Theorem 3.1, these actions are equivariantly isomorphic to the actions of $\mathbf{H}$ on $D_{\mathbf{G}}$ and of $\mathbf{H}^*$ on $D_{\mathbf{G}^*}$ respectively, hence the theorem. □

Corollary 4.2 *Let (G_v, A_v) and (G_v^*, A_v^*) be two families of group pairs indexed by the vertex set V of a finite graph Γ. Suppose that for all $v \in V$ there exist subgroups $H_v < G_v$ and $H_v^* < G_v^*$ of finite index, such that the left actions of H_v on G_v/A_v and of H_v^* on G_v^*/A_v^* are equivariantly isomorphic. Then the graph products $\mathbf{G} = \Pi_\Gamma(G_v, A_v)$ and $\mathbf{G}^* = \Pi_\Gamma(G_v^*, A_v^*)$ are commensurable.*

Proof According to Theorem 4.1 the groups $\mathbf{G}$ and $\mathbf{G}^*$ share a subgroup $\mathbf{H} = \mathbf{H}^*$, which is of finite index in both of them. □

Proof of Theorem 1 Under assumptions of Theorem 1 the left actions of the group H_v on G_v and on G_v^* are clearly equivariantly isomorphic. Then by Corollary 4.2 the graph products $\Pi_\Gamma G_v$ and $\Pi_\Gamma G_v^*$ share a subgroup $\mathbf{H}$ which is easily seen to be of index $\Pi_{v \in V}[G_v : H_v]$ in both graph products. □

5 Applications, examples and comments

Is strong commensurability a necessary assumption in Theorem 1?

Considering free products $Z_2 * Z_2$ and $Z_3 * Z_3$ shows that one needs a hypothesis stronger than commensurability to guarantee commensurability of graph products. A more delicate example is provided by a family of graph products along the pentagon, where at each vertex we put the group Z_p. Bourdon computes in [2] an invariant (conformal dimension at infinity) of the hyperbolic groups arising in this way. His invariant shows that as p varies, the graph products are not even quasiisometric, hence noncommensurable.

A more subtle reason for noncommensurability occurs for free products of surface groups. According to Whyte [9], the groups $M_g * M_g$ and $M_h * M_h$ are quasiisometric if $g, h \geq 2$. On the other hand, we have the following well known fact.

Lemma 5.1 *Free products $M_g * M_g$ and $M_h * M_h$ of surface groups are not commensurable if $g \neq h$.*

Proof Recall that Kurosh theorem asserts that if N is a subgroup of finite index i in $L_1 * L_2$, then N is a free product

$$N_1 * N_2 * \ldots * N_k * F_l,$$

where each N_j is a subgroup of finite index in either L_1 or L_2, F_l is a free group of rank l and moreover $i = k + l - 1$. Now assume L_1, L_2 are fundamental groups of orientable aspherical manifolds of the same dimension m (e.g. surface groups). One readily sees that $k = b^m(N) = rank H^m(N, Z)$ while l is the rank of the kernel in $H^1(N, Z)$ of the cup product $H^1(N, Z) \times H^{m-1}(N, Z) \to H^m(N, Z)$ interpreted as a bilinear form. Hence if one knows N, one knows the index of N as a subgroup in $L_1 * L_2$. This implies that if the free products $L_1 * L_2$ and $L_1' * L_2'$ of two such group pairs are commensurable they are strongly commensurable.

Now, if $g \neq h$ then the groups $M_g * M_g$ and $M_h * M_h$ are not strongly commensurable, because they have different Euler characteristics. It follows that these groups are not commensurable. □

Commensurability of graph products as transformation groups

As it is shown in Section 1, to each graph product $\mathbf{G}$ of group pairs there is associated a right-angled building $D_{\mathbf{G}}$ on which $\mathbf{G}$ acts canonically by automorphisms. Such buildings corresponding to different groups $\mathbf{G}$ may sometimes be isomorphic. In particular we have:

Lemma 5.2 *Let $(G_v, A_v)_{v \in V}$ and $(G_v^*, A_v^*)_{v \in V}$ be two families of groups and subgroups, indexed by a finite set V. Suppose that for each $v \in V$ the indices (not necessarily finite) $[G_v : A_v]$ and $[G_v^* : A_v^*]$ are equal. Then for any graph Γ on the vertex set V the buildings $D_{\mathbf{G}}$ and $D_{\mathbf{G}^*}$ associated to the graph products $\mathbf{G} = \Pi_\Gamma(G_v, A_v)$ and $\mathbf{G}^* = \Pi_\Gamma(G_v^*, A_v^*)$ are isomorphic.*

Proof Observe that, under assumptions of the lemma, the complexes $D_{\mathbf{G}^c}$ and $D_{(\mathbf{G}^*)^c}$, and hence also their subcomplexes $\Delta_{\mathbf{G}}$ and $\Delta_{\mathbf{G}^*}$, are isomorphic. Since by Theorem 3.1 the buildings $D_{\mathbf{G}}$ and $D_{\mathbf{G}^*}$ are the universal covers of the complexes $\Delta_{\mathbf{G}}$ and $\Delta_{\mathbf{G}^*}$, the lemma follows. □

Call two graph products *commensurable as transformation groups* if their associated buildings are isomorphic and if they contain subgroups of finite index whose actions on the corresponding buildings are equivariantly isomorphic. The arguments we give in this paper show that the graph products satisfying our assumptions are not only commensurable but also commensurable as transformation groups (see Theorem 4.1). Closer examination of these arguments shows that the strong commensurability condition of Theorem 1 (and a more general condition of Corollary 4.2) is not only sufficient, but also necessary for two graph products of groups (of group pairs respectively) to be commensurable as transformation groups. The details of this argument are not completely immediate but we omit them.

Special cases of Theorem 1

Theorem 1 has interesting special cases resulting from various examples of strongly commensurable groups. The simplest class of examples is given by finite groups of equal order. Thus:

Corollary 5.3 *Let $(G_v)_{v \in V}$ and $(G_v^*)_{v \in V}$ be two families of finite groups indexed by the vertex set V of a finite graph Γ. Suppose that for each $v \in V$ we have $|G_v| = |G_v^*|$. Then the graph products $\Pi_\Gamma G_v$ and $\Pi_\Gamma G_v^*$ are strongly commensurable.*

The infinite cyclic group Z and the infinite dihedral group D_∞ are strongly commensurable since they both contain an infinite cyclic subgroup of index two. Thus a graph product of infinite cyclic groups (right-angled Artin group) is commensurable with the corresponding graph product of infinite dihedral groups which is a right-angled Coxeter group. Thus we reprove a result from [5]:

Corollary 5.4 *Right angled Artin groups are commensurable with right-angled Coxeter groups.*

A source of strongly commensurable groups is given by subgroups of the same finite index in some fixed group. The intersection of two such subgroups has clearly the same finite index in both of them. As an example of this kind consider a natural number $g \geq 2$ and a tessellation of the hyperbolic plane H^2 by regular $4g$-gons with all angles equal to $\pi/2g$ (so that $4g$ tiles meet at each vertex). Let T be the group of all symmetries of this tessellation and $W_g < T$ be the Coxeter group generated by reflections in sides of a fixed $4g$-gon. Consider also the fundamental group M_g of the closed surface of genus g and note that this group can be viewed as a subgroup of T. Since the groups W_g and M_g have the same fundamental domain in H^2 (equal to a single $4g$-gon) they have clearly the same index in T (equal to $8g$, the number of symmetries of a $4g$-gon) and hence are strongly commensurable. Since graph products of Coxeter groups are again Coxeter groups, Theorem 1 implies:

Corollary 5.5 *Graph products of surface groups are commensurable with Coxeter groups.*

Pairs of subgroups of the same finite index in a given group (being thus strongly commensurable) are applied also in the following.

Proposition 5.6 *Graph products of arbitrary subgroups of finite index in right-angled Coxeter groups are commensurable with right-angled Coxeter groups.*

Proof Since graph products of right-angled Coxeter groups remain in this class, it is sufficient to show that a finite index subgroup in a right-angled Coxeter group W is strongly commensurable with another right-angled Coxeter group. This is clearly true for finite groups, as they are (both groups and subgroups) isomorphic to products of the group Z_2. To prove this for an infinite group W, we will exhibit in W a family $W_n : n \in N$ of subgroups, indexed by all natural numbers, with $[W : W_n] = n$, such that each of the groups W_n is also a right-angled Coxeter group.

Note that if W is infinite, it contains two generators t and s whose product ts has infinite order in W. Let D be a fundamental domain in the Coxeter-Davis complex Σ of W. D is a subcomplex in Σ with the distinguished set of "faces", so that reflections with respect to those faces are the canonical generators of W. Since the faces of the reflections t and s are disjoint, the following complex

$$D_n := \begin{cases} D \cup tD \cup stD \cup \ldots \cup (st)^k D & \text{if } n = 2k+1 \\ D \cup tD \cup stD \cup \ldots \cup t(st)^{k-1} D & \text{if } n = 2k \end{cases}$$

is a fundamental domain of a subgroup $W_n < W$ generated by reflections with respect to "faces" of this complex. By comparing fundamental domains we have $[W : W_n] = n$, and the proposition follows.

The algebraic wording of this proof is as follows. An infinite right angled Coxeter group (W, S) contains an infinite dihedral parabolic subgroup$(V, \{s,t\})$. The map of S which is the identity on $\{s,t\}$ and sends remaining generators to 1 extends to the homomorphism $r : W \to V$.The group V contains (Coxeter) subgroups generated by $s, (st)^k s (st)^{-k}$ and $s, (st)^k t (st)^{-k}$. These have indices $2k, 2k+1$ respectively. Preimages under r of these subgroups are Coxeter subgroups of W of the same indices. □

The example discussed just before Corollary 5.5 generalizes as follows. Let $(T_v)_{v \in V}$ be a family of topological groups and let $\Lambda_v \subset T_v$ and $\Lambda_v^* \subset T_v$ be two families of lattices such that volumes of the quotients T_v/Λ_v and T_v/Λ_v^* are finite and equal for all v. Suppose also that for each $v \in V$ there is $t \in T_v$ such that the intersection $t^{-1}\Lambda_v t \cap \Lambda_v^*$ has finite index in both Λ_v^* and the conjugated lattice $t^{-1}\Lambda_v t$. Then for each v the lattices Λ_v and Λ_v^* are strongly commensurable and hence the graph products $\Pi_\Gamma \Lambda_v$ and $\Pi_\Gamma \Lambda_v^*$ are commensurable for any graph Γ with the vertex set V.

For surface groups commensurability condition is a very weak one and we have the following:

Fact 5.7 *Let M and N be two 2-dimensional orbifolds which are developable. Then their fundamental groups G_M and G_N are strongly commensurable iff the orbifold Euler characteristics of M and N are equal.*

Clearly, Fact 5.7 allows to formulate the appropriate result on commensurability of graph products of 2-orbifold groups. On the other hand, combining this fact with Theorem 1 and with the argument based on Kurosh' theorem (as in the proof of Lemma 5.1) one has:

Corollary 5.8 *Under assumptions and notation of Fact 5.7 the free products $G_M * G_M$ and $G_N * G_N$ are commensurable iff the orbifold Euler characteristics of M and N are equal.*

We now pass to applications that require the full strength of Corollary 4.2 rather than that of Theorem 1.

Orthoparabolic subgroups of Coxeter groups

Recall that parabolic subgroup of a Coxeter group W is the group generated by a subset S' of the generating set S for W. An *orthoparabolic* subgroup of a Coxeter group W is a normal subgroup $J = \ker\rho$ for a homomorphism $\rho : W \to P$ to a parabolic subgroup P such that $\rho|_P = id_P$. We say that P is the *orthogonal parabolic* of J. Note that a homomorphism ρ as above, and hence also an orthoparabolic subgroup orthogonal to P, does not always exist.

Since the left actions of a group J on itself and on the cosets W/P are equivariantly isomorphic, Theorem 4.1 implies:

Corollary 5.9 *If for each $v \in V$ group J_v is an orthoparabolic subgroup in a Coxeter group W_v, orthogonal to a parabolic subgroup P_v, then the graph product $\Pi_\Gamma J_v$ is a subgroup in the graph product $\Pi_\Gamma(W_v, P_v)$. This subgroup has finite index iff the subgroups P_v are finite for all $v \in V$.*

Applying presentations of graph products from Section 1, we see that any graph product $\Pi_\Gamma(W_v, P_v)$ of pairs of a Coxeter group and its parabolic subgroup is again a Coxeter group. Thus Corollary 5.9 implies:

Corollary 5.10 *A graph product of orthoparabolic subgroups of finite index in Coxeter groups is a finite index subgroup of a Coxeter group.*

Finite cyclic groups Z_p are orthoparabolic in the dihedral groups D_p (as well as Z in D_∞). This again allows to reprove (and extend) the result of [5] (compare 5.4 above):

Corollary 5.11 *Graph products of cyclic groups (among them right-angled Artin groups) are subgroups of finite index in Coxeter groups.*

More generally, the *even* subgroup of a Coxeter group is the kernel of the homomorphism $h : W \to Z_2$ which sends all generators of W to the generator of Z_2. For example, triangle groups $T(p, q, r)$ and other rotation groups of some euclidean or hyperbolic tessellations are the even subgroups of the Coxeter reflections groups related to these tessalations. Since these groups are clearly orthoparabolic we have:

Corollary 5.12 *Graph products of even subgroups of Coxeter groups are finite index subgroups in Coxeter groups.*

Although it is fairly hard to find orthoparabolics in general Coxeter groups, they are plentiful in right-angled groups, or more generally in groups where all entries of the Coxeter matrix are even. There, for every parabolic subgroup there exist orthogonal to it orthoparabolics (usually many different ones).

Graph products of finite group pairs

Note first that by combining Corollaries 5.11 and 5.3 we obtain:

Corollary 5.13 *Graph products $\Pi_\Gamma G_v$ of finite groups G_v are commensurable with Coxeter groups.*

Next, applying Corollary 4.2 with trivial groups H_v, we have:

Corollary 5.14 *Graph products $\Pi_\Gamma(G_v, A_v)$ and $\Pi_\Gamma(G_v^*, A_v^*)$ of finite group pairs are commensurable if $[G_v : A_v] = [G_v^* : A_v^*]$ for all $v \in V$.*

An argument referring to above corollaries and using cyclic groups of orders $[G_v : A_v]$ proves then the following.

Corollary 5.15 *Graph products of finite group pairs are commensurable with Coxeter groups.*

In the rest of this subsection we prove the following slightly stronger result, under slightly stronger hypotheses:

Proposition 5.16 *Let $(G_v, A_v)_{v \in V}$ be a family of pairs of a finite group and its subgroup. Suppose that the left action of G_v on the cosets G_v/A_v is effective for each $v \in V$. Then any graph product $\Pi_\Gamma(G_v, A_v)$ is a subgroup of finite index in a Coxeter group.*

Proof Canonical action of each of the groups G_v on the cosets G_v/A_v defines a homomorphism $i_v : G_v \to S_{G_v/A_v} = S_{|G_v/A_v|}$ to the symmetric group on the set of cosets. By the assumption of the proposition this homomorphism is injective. Consider a subgroup $\mathrm{Stab}(A_v, S_{G_v/A_v}) = S_{|G_v/A_v|-1}$ and note that $i_v(A_v) \subset \mathrm{Stab}(A_v, S_{G_v/A_v})$. It follows that there is a homomorphism $i : \Pi_\Gamma(G_v, A_v) \to \Pi_\Gamma(S_{G_v/A_v}, \mathrm{Stab}(A_v, S_{G_v/A_v})) = \Pi_\Gamma(S_{|G_v/A_v|}, S_{|G_v/A_v|-1})$ between the graph products. Now for each $v \in V$ the action of G_v on G_v/A_v is easily verified to be equivariantly isomorphic (by i_v) to the action of the image group $i_v(G_v)$ on the cosets $S_{G_v/A_v}/\mathrm{Stab}(A_v, S_{G_v/A_v})$. It follows from Theorem 4.1 that the homomorphism i is injective and it maps the graph product $\Pi_\Gamma(G_v, A_v)$ to the subgroup of finite index in the graph product $\Pi_\Gamma(S_{|G_v/A_v|}, S_{|G_v/A_v|-1})$.

Symmetric group $S_{|G_v/A_v|}$ is a Coxeter group and its subgroup $S_{|G_v/A_v|-1}$ is a parabolic subgroup. By the remark before Corollary 5.10 a graph product of symmetric group pairs is a Coxeter group, and thus the proposition follows. □

Remark Removing in Proposition 5.16 the assumption of effectiveness for the actions of G_v on G_v/A_v one can obtain a similar conclusion for the reduced graph products of pairs (G_v, A_v) as defined in Section 2.

Groups of automorphisms of locally finite buildings

It is an open question (except in dimension 1, [8]) whether any two groups of automorphisms acting properly discontinuously and cocompactly on a fixed locally finite right-angled buildings are commensurable as transformation groups. The building $D_{\mathbf{G}}$ associated to a graph product $\mathbf{G} = \Pi_\Gamma(G_v, A_v)$ is locally finite iff the indices $[G_v : A_v]$ are finite for all $v \in V$. The action of $\mathbf{G}$ on $D_{\mathbf{G}}$ is then properly discontinuous iff the groups G_v are all finite. Furthermore, since we always assume that Γ is finite, this action is automatically cocompact.

We may now ask above question in the restricted class of appropriate graph products. By using Lemma 5.2 and Corollary 5.14 we have:

Corollary 5.17 *Let $\mathbf{G} = \Pi_\Gamma(G_v, A_v)$ and $\mathbf{G}^* = \Pi_\Gamma(G_v^*, A_v^*)$ be two graph products of finite group pairs along the same graph Γ. Suppose that for each $v \in V$ we have $[G_v : A_v] = [G_v^* : A_v^*]$. Then the associated buildings $D_{\mathbf{G}}$ and $D_{\mathbf{G}^*}$ are locally finite and isomorphic, and the actions on them are properly discontinuous and cocompact. Moreover, the groups $\mathbf{G}$ and $\mathbf{G}^*$ are commensurable as transformation groups.*

Remark By looking more closely one can show that the assumptions of Corollary 5.17 are necessary for the buildings $D_{\mathbf{G}}$ and $D_{\mathbf{G}^*}$ to be locally finite and isomorphic and to carry properly discontinuous actions of $\mathbf{G}$ and $\mathbf{G}^*$. Thus the question discussed in this subsection has positive answer in the class of (associated actions of) graph products. We omit the details of the argument.

Linearity of graph products

In [5] it was pointed out that commensurability of right-angled Artin groups (i.e. graph products of infinite cyclic groups) and right-angled Coxeter groups implies linearity of the former: Coxeter groups are linear and groups commensurable with linear groups are linear by inducing representation. By the same argument graph products of groups from various other classes are linear. For example, Corollaries 5.5 and 5.15 imply the following.

Corollary 5.18 *Graph products of surface groups and graph products of pairs of finite groups are linear.*

Remark Bourdon [3] using an entirely different method constructed and studied faithful linear representations of certain graph products of cyclic groups. The target of any of his representations is the Lorenz group $SO(N,1)$ and the dimension is much smaller than of ones constructed for that group using Corollary 5.18.

Without referring to commensurability we still can conclude that graph products of any subgroups in Coxeter groups are linear. This follows from the fact that graph products of Coxeter groups are Coxeter groups. The similar fact for pairs of Coxeter groups and their parabolic subgroups implies:

Corollary 5.19 *Let (W_v, P_v) be a family of pairs where W_v are Coxeter groups and P_v are their parabolic subgroups. For each $v \in V$ let H_v be a subgroup of W_v. Then any graph product of the family of pairs $(H_v, H_v \cap P_v)$ is a linear group.*

Proof A graph product $\Pi_\Gamma(H_v, H_v \cap P_v)$ is a subgroup of $\Pi_\Gamma(W_v, P_v)$ which is a Coxeter group. □

After this paper was written we've learned from John Meier about a paper of T. Hsu and D. Wise [6]. There linearity of graph products of finite groups was established by embedding them into Coxeter groups. Linearity of right-angled Artin groups has been proved by S. P. Humphries [7].

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