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The Homflypt skein module of a connected sum of 3-manifolds

Patrick M. Gilmer
Jianyuan K. Zhong

Abstract If M is an oriented 3-manifold, let $S(M)$ denote the Homflypt skein module of M. We show that $S(M_1 \# M_2)$ is isomorphic to $S(M_1) \otimes S(M_2)$ modulo torsion. In fact, we show that $S(M_1 \# M_2)$ is isomorphic to $S(M_1) \otimes S(M_2)$ if we are working over a certain localized ring. We show the similar result holds for relative skein modules. If M contains a separating 2-sphere, we give conditions under which certain relative skein modules of M vanish over specified localized rings.

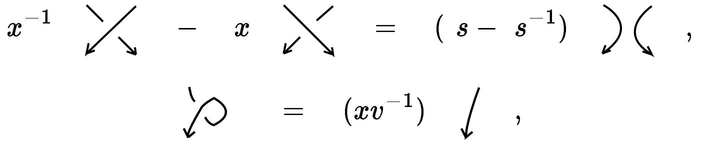
AMS Classification 57M25

Keywords Young diagrams, relative skein module, Hecke algebra

1 Introduction

We will be working with framed oriented links. By this we mean links equipped with a string orientation together with a nonzero normal vector field up to homotopy. The links described by figures in this paper will be assigned the "blackboard" framing which points to the right when travelling along an oriented strand.

Definition 1 **The Homflypt skein module** Let k be a commutative ring containing $x^{\pm 1}$, $v^{\pm 1}$, $s^{\pm 1}$, and $\frac{1}{s-s^{-1}}$. Let M be an oriented 3-manifold. The Homflypt skein module of M over k, denoted by $S_k(M)$, is the k-module freely generated by isotopy classes of framed oriented links in M including the empty link, quotiented by the Homflypt skein relations given in the following figure.



$$L \sqcup \bigcirc \quad = \quad \frac{v^{-1}-v}{s-\ s^{-1}} \quad L \quad .$$

The last relation follows from the first two in the case L is nonempty.

Remark (1) An embedding $f : M \to N$ of 3-manifolds induces a well defined homomorphism $f_* : S_k(M) \to S_k(N)$. (2) If N is obtained by adding a 3-handle to M, the embedding $i : M \to N$ induces an isomorphism $i_* : S_k(M) \to S_k(N)$. (3) If N is obtained by adding a 2-handle to M, the embedding $i : M \to N$ induces an epimorphism $i_* : S_k(M) \to S_k(N)$. (4) If $M_1 \sqcup M_2$ is the disjoint union of 3-manifolds M_1 and M_2, then $S_k(M_1 \sqcup M_2) \cong S_k(M_1) \otimes S_k(M_2)$.

Associated to a partition of n, $\lambda = (\lambda_1 \geq \dots \lambda_p \geq 1)$, $\lambda_1 + \dots + \lambda_p = n$, is associated a Young diagram of size $|\lambda| = n$, which we denote also by λ. This diagram has n cells indexed by $\{(i,j),\ 1 \leq i \leq p,\ 1 \leq j \leq \lambda_i\}$. If c is the cell of index (i,j) in a Young diagram λ, its content $cn(c)$ is defined by $cn(c) = j - i$. Define

$$c_{\lambda,\mu} = v(s^{-1} - s)\sum_{c\in\mu} s^{-2cn(c)} + v^{-1}(s - s^{-1})\sum_{c\in\lambda} s^{2cn(c)}$$

Let $\mathcal{I}$ denote the submonoid of the multiplicative monoid of $\mathbb{Z}[v,s]$ generated by v, s, $s^{2n}-1$ for all integers $n > 0$, and $c_{\lambda,\mu}$ for all pairs of Young diagrams λ, and μ, with $|\lambda| = |\mu|$, and $|\mu| \neq 0$. Let $\mathcal{R}$ be $\mathbb{Z}[v,s]$ localized at $\mathcal{I}$. [5, 7.2]

Theorem 1

$$S_{\mathcal{R}[x,x^{-1}]}(M_1 \# M_2) \cong S_{\mathcal{R}[x,x^{-1}]}(M_1) \otimes S_{\mathcal{R}[x,x^{-1}]}(M_2).$$

Remark J. Przytycki has proved the analog of this result for the Kauffman bracket skein module [9]. Our proof follows the same general outline. We thank J. Przytycki for suggesting the problem of obtaining a similar result for the Homflypt skein module.

Let $\mathcal{I}'$ denote the submonoid of the multiplicative monoid of $\mathcal{R}$ generated by $v^4 - s^{2n}$, for all n. Let $\mathcal{R}'$ be $\mathcal{R}$ localized at $\mathcal{I}'$. It follows from [4], $S_{\mathcal{R}'[x,x^{-1}]}(S^1 \times S^2)$ is the free $\mathcal{R}'[x,x^{-1}]$-module generated by the empty link.

Corollary 1 *$S_{\mathcal{R}'[x,x^{-1}]}(\#^m S^1 \times S^2)$ is a free module generated by the empty link.*

Remark This allows us to define a "Homflypt rational function" f in $\mathcal{R}'$ for oriented framed links in $\#^m S^1 \times S^2$. If L is such a link, one defines $f(L)$ by $L = f(L)\phi \in S_{\mathcal{R}'}(\#^m S^1 \times S^2)$. A specific example is given in section 5.

Let $l = \mathcal{R}$ with $x = v$, then $S_l(M)$ is a version of the Homflypt skein module for unframed links. The next two corollaries follows from the universal coefficient property for skein modules which has been described by J. Przytycki [9] for the Kauffman bracket skein module. The proof given there holds generally for essentially any skein module.

Corollary 2 $S_l(M_1 \# M_2) \cong S_l(M_1) \otimes S_l(M_2)$.

Let $l' = \mathcal{R}'$ with $x = v$.

Corollary 3 $S_{l'}(\#^m S^1 \times S^2)$ *is a free l'-module generated by the empty link.*

Definition 2 The relative Homflypt skein module Let $X = \{x_1, x_2, \cdots, x_n\}$ be a finite set of input framed points in ∂M, and let $Y = \{y_1, y_2, \cdots, y_n\}$ be a finite set of output framed points in the boundary ∂M. Define the relative skein module $S_k(M, X, Y)$ to be the k-module generated by relative framed oriented links in $(M, \partial M)$ such that $L \cap \partial M = \partial L = \{x_i, y_i\}$ with the induced framing, considered up to an ambient isotopy fixing ∂M, quotiented by the Homflypt skein relations.

Let $S(M)$ denote $S_{\mathcal{R}[x,x^{-1}]}(M,)$ and let $S(M,X,Y)$ denote $S_{\mathcal{R}[x,x^{-1}]}(M,X,Y)$. We have the following version of Theorem 1 for relative skein modules. At this point we must work over the field of fractions of $\mathbb{Z}[x,v,s]$ which we denote by $\mathcal{F}$. This is because we do not know whether the relative skein module of a handlebody is free. We conjecture that it is free. In the proof of Theorem 1, we use the absolute case first obtained by Przytycki [8]. We state Theorem 2 over $\mathcal{F}$, but conjecture it over $\mathcal{R}'[x, x^{-1}]$.

Theorem 2 *Let M_1 and M_2 be connected oriented 3-manifolds. Let $X_i = \{x_1, x_2, \cdots, x_n\}$ be a finite set of input framed points in ∂M_i, and let $Y_i = \{y_1, y_2, \cdots, y_n\}$ be a finite set of output framed points in the boundary ∂M_i. Let $X = X_1 \cup X_2$, and $Y = Y_1 \cup Y_2$, then*

$$S_{\mathcal{F}}(M_1 \# M_2, X, Y) \cong S_{\mathcal{F}}(M_1, X_1, Y_1) \otimes S_{\mathcal{F}}(M_2, X_2, Y_2).$$

We also have the following related result. Let $\mathcal{I}_r$ denote the submonoid of the multiplicative monoid of $\mathbb{Z}[x,v,s]$ generated by x, v, s, $s^{2n}-1$ for all integers

$n>0$, and $x^r-1-c_{\lambda,\mu}$ for all pairs of Young diagrams λ, and μ, such that $|\lambda|-|\mu|=r$, and $|\mu|$ and $|\lambda|$ are not both zero. Let k_r be $\mathbb{Z}[x,v,s]$ localized at $\mathcal{I}_r$. Note $k_0=\mathcal{R}[x,x^{-1}]$.

Theorem 3 *Suppose M is connected and contains a 2-sphere Σ, such that $M-\Sigma$ has two connected components. Let M' be one of these components. If $|X\cap M'|-|Y\cap M'|=r\neq 0$, then $S_{k_r}(M,X,Y)=0$.*

In section 2, we prove that there is an epimorphism from $S(\mathcal{H}_{m_1})\otimes S(\mathcal{H}_{m_2})$ to $S(\mathcal{H}_{m_1}\#\mathcal{H}_{m_2})$. Here and below, we let $\mathcal{H}_m$ denote a handlebody of genus m. In section 3, we prove Theorem 1 in the case of handlebodies. We prove Theorem 1 in the general case in section 4. Section 5 describes the class of a certain link in the $S^1\times S^2\#S^1\times S^2$. Section 6 gives a proof of a lemma needed in section 2. In section 7, we discuss the proofs of Theorems 2 and 3.

2 Epimorphism for Handlebodies

2.1 The nth Hecke algebra of Type A

We will use the related work of C. Blanchet [2], A. Aiston and H. Morton [1] on the nth Hecke algebra of Type A. This is summarized in section 3 of [4] whose conventions we follow. For the convenience of the reader, we give the basic definitions in this subsection.

Note that $s^{2n}-1$ is invertible in $\mathcal{R}$ for integers $n>0$. It follows that the quantum integers $[n]=\dfrac{s^n-s^{-n}}{s-s^{-1}}$ for $n>0$ are invertible in k. Let $[n]!=\prod_{j=1}^n[j]$, so $[n]!$ is invertible for $n>0$.

The Hecke category The k-linear Hecke category H is defined as follows. An object in this category is a disc D^2 equipped with a set of framed points. If $\alpha=(D^2,l)$ and $\beta=(D^2,l')$ are two objects, the module $Hom_H(\alpha,\beta)$ is $\mathcal{S}(D^2\times[0,1],l\times 1,l'\times 0)$. The notation $H(\alpha,\beta)$ and H_α will be used for $Hom_H(\alpha,\beta)$ and $H(\alpha,\alpha)$ respectively. The composition of morphisms are by stacking the first one on the top of the second one.

Let $\otimes$ denote the monoid structure on H given by embedding two disks D^2 side by side into one disk. For a Young diagram λ, by assigning each cell of λ a point equipped with the horizontal (to the left) framing, we obtain an object of the category H denoted by $\square_\lambda$. When λ is the Young diagram with a single row of n cells, $H_{\square_\lambda}$ will be denoted by H_n, which is the nth Hecke algebra of type A [7], [10].

For each permutation $\pi \in S_n$, a positive permutation braid, w_π, is a braid which realizes the permutation π with all crossings positive [6]. Let $\sigma_i \in H_n$, $i = 1, \dots, n-1$, be the positive permutation corresponding to the transposition $(i\ i+1)$. As in [1], define

$$f_n = \frac{1}{[n]!} s^{-\frac{n(n-1)}{2}} \sum_{\pi \in S_n} (xs^{-1})^{-l(\pi)} \omega_\pi$$

and

$$g_n = \frac{1}{[n]!} s^{\frac{n(n-1)}{2}} \sum_{\pi \in S_n} (-xs)^{-l(\pi)} \omega_\pi$$

Here $l(\pi)$ is the length of π.

Idempotents in the Hecke Algebra [1] For a Young diagram λ of size n, let F_λ be the element in $H_{\square_\lambda}$ formed with one copy of $[\lambda_i]!f_{\lambda_i}$ along the row i, for $i = 1, \dots, p$. We let $\lambda^\vee$ denote the Young diagram whose rows are the columns of λ. Let G_λ be the element in $H_{\square_\lambda}$ formed with one copy of $[\lambda_j^\vee]!g_{\lambda_j^\vee}$ along the column j, for $j = 1, \dots, q$. Let $\tilde{y}_\lambda = F_\lambda G_\lambda$, then $\tilde{y}_\lambda$ is a quasi-idempotent. Let y_λ be the normalized idempotent from $\tilde{y}_\lambda$.

A Basis for the nth Hecke Algebra H_n A standard tableau t with the shape of a Young diagram $\lambda = \lambda(t)$ is a labeling of the cells, with the integers 1 to n increasing along the rows and the columns. Let t' be the tableau obtained by deleting the cell numbered by n. Note the cell numbered by n in a standard tableau is an *extreme cell.* C. Blanchet defines $\alpha_t \in H(n, \square_\lambda)$ and $\beta_t \in H(\square_\lambda, n)$ inductively by

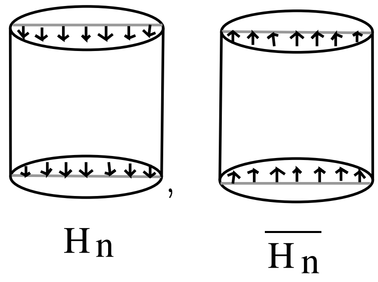
$$\begin{aligned} \alpha_1 &= \beta_1 = 1_1, \\ \alpha_t &= (\alpha_{t'} \otimes 1_1)\rho_t y_\lambda, \\ \beta_t &= y_\lambda \rho_t^{-1}(\beta_{t'} \otimes 1_1). \end{aligned}$$

Here $\rho_t \in H(\square_{\lambda(t')} \otimes 1, \square_\lambda)$ is the isomorphism given by an arc joining the added point to its place in λ in the standard way.

Note that $\beta_\tau \alpha_t = 0$ if $\tau \neq t$, and $\beta_t \alpha_t = y_{\lambda(t)}$.

Theorem 4 (Blanchet) *The family $\alpha_t \beta_\tau$ for all standard tableaux t, τ such that $\lambda(t) = \lambda(\tau)$ for all Young diagrams λ with $|\lambda| = n$ forms a basis for H_n.*

Let $\overline{H_n}$ denote H_n with the reversed string orientation.

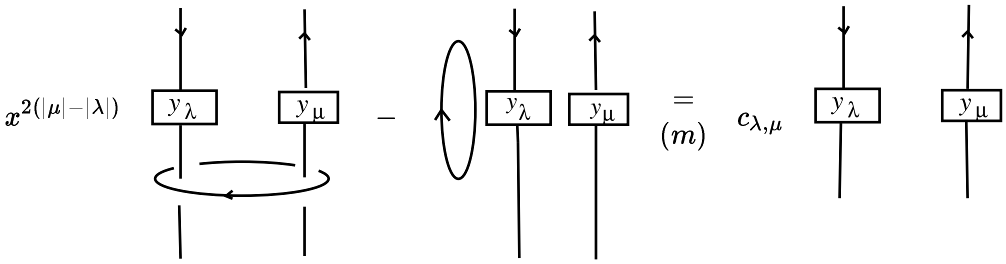


2.2 The Epimorphism on the Handlebodies

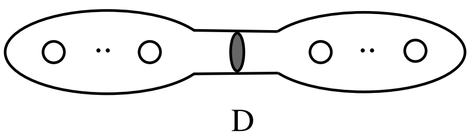
If X_m is a set of m distinguished framed points in $D^2 \times \{1\}$ and Y_m be a set of m distinguished framed points in $D^2 \times \{0\}$, Let $\underset{(m)}{=}$ denote equality in $S(D^2 \times I, X, Y)$ modulo the submodule $\mathcal{L}(m)$ generated by links which intersects $D^2 \times \{\frac{1}{2}\}$ in less than m points.

In section 6, we derive:

Lemma 2.1 *Let λ, μ be two Young diagrams, and $m = |\lambda| + |\mu|$.*



Let $\mathcal{H}_m$ be a handlebody of genus m. Let D be a separating meridian disc of $\mathcal{H}_m$, let $\gamma = \partial D$. Let $(\mathcal{H}_m)_\gamma$ be the manifold obtained by adding a 2-handle to $\mathcal{H}_m$ along γ.

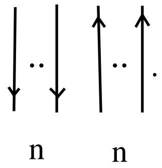


Let $V_D = [-1, 1] \times D$ be the regular neighborhood of D in $\mathcal{H}_m$, V_D can be projected into a disc $D_p = [-1, 1] \times [0, 1]$.

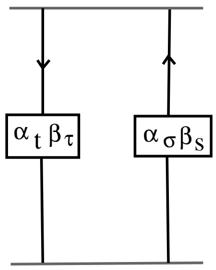
Lemma 2.2 (The Epimorphism Lemma) *The embedding $i: \mathcal{H}_m - D \to (\mathcal{H}_m)_\gamma$ induces an epimorphism:*

$$i_* : S(\mathcal{H}_m - D) \twoheadrightarrow S((\mathcal{H}_m)_\gamma).$$

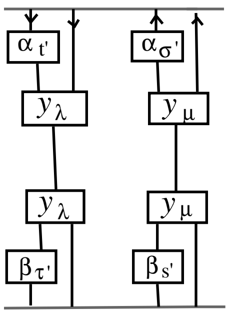
Proof Let z_n be a link in $\mathcal{H}_m$ in general position with D and cutting D $2n$ times, let $z'_n = z_n \cap V_D$, i.e.,

$z'_n =$ 

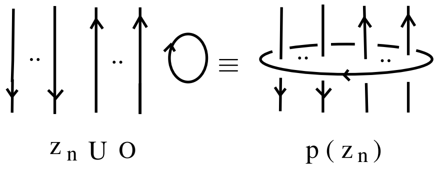
Note $z'_n \in H_n \otimes \overline{H_n}$. Using the basis elements $\alpha_t\beta_\tau$ of H_n given in the previous theorem, z'_n can be written as a linear combination of the elements $\alpha_t\beta_\tau \otimes \alpha_\sigma{\beta_s}^*$, where $\alpha_\sigma{\beta_s}^*$ is $\alpha_\sigma\beta_s$ with the reversed orientation. A diagram of $\alpha_t\beta_\tau \otimes \alpha_\sigma{\beta_s}^*$ is given by the following:

$\alpha_t\beta_\tau \otimes \alpha_\sigma{\beta_s}^* =$ 

By the inductive definition of α_t, β_τ, α_σ, β_s, an alternative diagram of $\alpha_t\beta_\tau \otimes \alpha_\sigma{\beta_s}^*$ is given by:

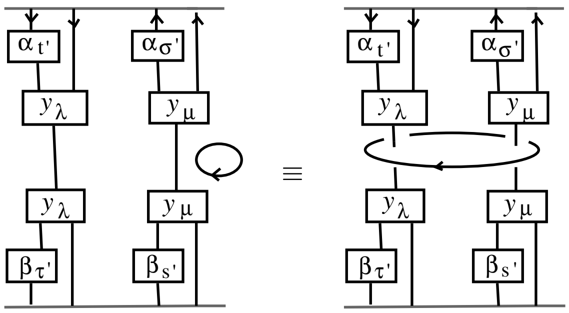
$\alpha_t\beta_\tau \otimes \alpha_\sigma{\beta_s}^* =$ 

We will consider the sliding relation given by:



(I)

From the above observation, we will be interested in the following relation:



(II)

From Relation II, and Lemma 2.1, as $|\lambda| = |\mu|$, in $S\left((D^2 \times I)_\gamma\right)$ we have

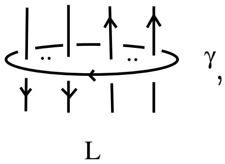
$$c_{\lambda,\mu}\left(\alpha_t\beta_\tau \otimes \alpha_\sigma{\beta_s}^*\right) \in \mathcal{L}_{|\lambda|+|\mu|}.$$

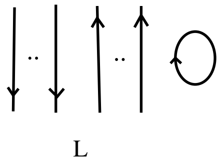
As $c_{\lambda,\mu}$ is invertible in $\mathcal{R}$, we have that $\alpha_t\beta_\tau \otimes \alpha_\sigma{\beta_s}^* \in \mathcal{L}(|\lambda| + |\mu|)$. By induction, we can eliminate all elements of $(\mathcal{H}_m)_\gamma$ which cut the 2-disk D_γ non-trivially. Thus i_* is an epimorphism. □

3 Isomorphism for handlebodies

Recall that $(\mathcal{H}_m)_\gamma$ is obtained by adding a 2-handle to $\mathcal{H}_m$ along γ. From [4] section 2, we have $S((\mathcal{H}_m)_\gamma) \cong S(\mathcal{H}_m)/R$, where R is the submodule of $S(\mathcal{H}_m)$ given by the collection $\{\Phi'(z) - \Phi''(z) \mid z \in S(\mathcal{H}_m, A, B)\}$. Here A, B are two points on γ, which decompose γ into two intervals γ' and γ'', z is any element of the relative skein module $S(\mathcal{H}_m, A, B)$ with A an input point and B an output point, and $\Phi'(z)$ and $\Phi''(z)$ are given by capping off with γ' and γ'', respectively, and pushing the resulting links back into $\mathcal{H}_m$.

Let I_0 be the submodule of $S(\mathcal{H}_m)$ given by the collection $\{p_D(L) - L \sqcup O \mid L \in S(\mathcal{H}_m)\}$, where O is the unknot. Locally, we have the following diagram description.

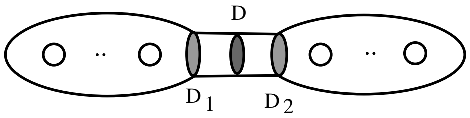
$p_D(L) =$ 

$L \sqcup O =$ 

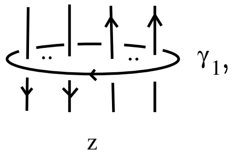
Lemma 3.1 $R = I_0$.

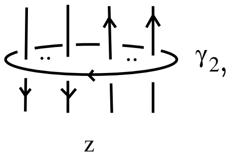
Proof First note $R \supseteq I_0$. We need only show that $R \subseteq I_0$. Let π be the projection map $\pi : S(\mathcal{H}_m) \to S(\mathcal{H}_m)/I_0$. We will show that $\pi(R) = 0$ in $S(\mathcal{H}_m)/I_0$, i.e. $R \subseteq I_0$. We show this by proving now that $\pi(\Phi'(z)) = \pi(\Phi''(z))$ for any $z \in S(\mathcal{H}_m, A, B)$.

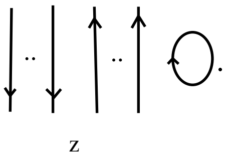
Recall that $V_D = [-1,1] \times D$ is the regular neighborhood of D in $\mathcal{H}_m$. Let $D_1 = \{-1\} \times D$ and $D_2 = \{1\} \times D$. Let $\gamma_1 = \partial D_1$ and $\gamma_2 = \partial D_1$, note γ_1 and γ_2 are parallel to γ.



Let $I_1 = \{p_{D_1}(z) - z \sqcup O \mid z \in S(\mathcal{H}_m, A, B)\}$, $I_2 = \{p_{D_2}(z) - z \sqcup O \mid z \in S(\mathcal{H}_m, A, B)\}$, where locally

$p_{D_1}(z) =$ 

$p_{D_2}(z) =$ 

$z \sqcup O =$ 

Let $\pi_{A,B}$ be the projection map $\pi_{A,B} : S(\mathcal{H}_m, A, B) \to S(\mathcal{H}_m, A, B)/(I_1 + I_2)$. Note that $\Phi'(I_i) = I_0$ and $\Phi''(I_i) = I_0$ for $i = 1, 2$.

Let $z \in S(\mathcal{H}_m - (D_1 \cup D_2), A, B)$, then $\Phi'(z) = \Phi''(z)$ in $S(\mathcal{H}_m - D_1 - D_2)$, since $V_D = [-1,1] \times D$ is a 3-disc and closing a relative link along γ by γ' and γ'' in V_D gives isotopic links. Let $\Phi : S(\mathcal{H}_m - (D_1 \cup D_2), A, B) \to S(\mathcal{H}_m - D_1 - D_2)$ denote the map which sends z to $\Phi'(z) = \Phi''(z)$.

In general, let $z \in S(\mathcal{H}_m, A, B)$. Now consider the following commutative diagram,

$$\begin{array}{ccccc}
S(\mathcal{H}_m - (D_1 \cup D_2), A, B) & \xrightarrow{j_1} & S(\mathcal{H}_m, A, B) & \xrightarrow{\pi_{A,B}} & S(\mathcal{H}_m, A, B)/(I_1 + I_2) \\
\Phi \downarrow & & \Phi' \downarrow & & \bar{\Phi}' \downarrow \\
S(\mathcal{H}_m - (D_1 \cup D_2)) & \xrightarrow{j_2} & S(\mathcal{H}_m) & \xrightarrow{\pi} & S(\mathcal{H}_m)/I_0 \\
\Phi \uparrow & & \Phi'' \uparrow & & \bar{\Phi}'' \uparrow \\
S(\mathcal{H}_m - (D_1 \cup D_2), A, B) & \xrightarrow{j_1} & S(\mathcal{H}_m, A, B) & \xrightarrow{\pi_{A,B}} & S(\mathcal{H}_m, A, B)/(I_1 + I_2)
\end{array}$$

Here j_1 and j_2 are induced by inclusion maps. Also $\bar{\Phi}'$, and $\bar{\Phi}''$ are induced by Φ', and Φ'' respectively. By an argument similar to the proof of Lemma 2.2, the composition map $\pi_{A,B} j_1 : S(\mathcal{H}_m - (D_1 \cup D_2), A, B) \to S(\mathcal{H}_m, A, B)/(I_1 + I_2)$ is an epimorphism.

Take $z \in S(\mathcal{H}_m, A, B)$, then $\pi_{A,B}(z) \in S(\mathcal{H}_m, A, B)/(I_1 + I_2)$. As $\pi_{A,B} j_1$ is an epimorphism, there exists $z' \in S(\mathcal{H}_m - (D_1 \cup D_2), A, B)$ such that $\pi_{A,B} j_1(z') = \pi_{A,B}(z)$. By the commutativity of the diagram, $\pi j_2(\Phi(z')) = \pi(\Phi'(z))$ and $\pi j_2(\Phi(z')) = \pi(\Phi''(z))$. Thus $\pi(\Phi'(z)) = \pi(\Phi''(z))$. □

Corollary 4 *The embedding $\mathcal{H}_m \to (\mathcal{H}_m)_\gamma$ induces an isomorphism*

$$S(\mathcal{H}_m)/I_0 \cong S((\mathcal{H}_m)_\gamma).$$

Now we want to show that the embedding $\mathcal{H}_m - D \to (\mathcal{H}_m)_\gamma$ induces an isomorphism

$$S(\mathcal{H}_m - D) \cong S((\mathcal{H}_m)_\gamma).$$

Lemma 3.2

$$S(\mathcal{H}_m - D) \cap I_0 = 0.$$

Proof Przytycki [8] calculated the unframed Homflypt skein module of a handlebody. It follows from this, the universal coefficient property of skein modules and an argument of Morton in [6] section (6.2) that $S(H_m)$ is free. As $S(\mathcal{H}_m - D)$ is free, the map $S(\mathcal{H}_m - D) \to S_{\mathcal{F}}(\mathcal{H}_m - D)$, induced by $\mathcal{R}[x, x^{-1}] \to \mathcal{F}$ is injective. Let $\mathcal{I}_0 = \{p_D(L) - L \sqcup O \mid L \in S_{\mathcal{F}}(\mathcal{H}_m)\}$. It is enough to show $S_{\mathcal{F}}(\mathcal{H}_m - D) \cap \mathcal{I}_0 = 0$.

Let ψ be the map from $S_{\mathcal{F}}(\mathcal{H}_m) \to S_{\mathcal{F}}(\mathcal{H}_m)$ given by $\psi(L) = p_D(L) - L \sqcup O$ for $L \in S_{\mathcal{F}}(\mathcal{H}_m)$. $Image(\psi) = \mathcal{I}_0$.

It also follows from Przytycki's basis that the map induced by inclusion $S(\mathcal{H}_m - D) \to S(H_m)$ is injective. Let $\mathcal{B}_0$ be the image of a free basis for the module $S(\mathcal{H}_m - D)$ in $S(H_m)$. $\mathcal{B}_0$ also a basis for injective image of $S_{\mathcal{F}}(\mathcal{H}_m - D)$ in $S_{\mathcal{F}}(H_m)$. Let B_n be the subspace of $S_{\mathcal{F}}(\mathcal{H}_m)$ generated by framed oriented links in $\mathcal{H}_m$ which intersect the disk $D \leq 2n$ times. Then we have a chain of vector spaces:

$$B_0 \subset B_1 \subset B_2 \subset \cdots \subset B_n \subset \cdots$$

$\mathcal{B}_0$ is a basis for B_0. The vector space B_n/B_{n-1} is generated by elements of the form $\alpha_t\beta_\tau \otimes \alpha_\sigma{\beta_s}^*$ in a neighborhood of D, where $|\lambda| = |\mu|$ is n. Let $\mathcal{B}_n$ be a basis B_n/B_{n-1}, constructed by taking a maximal linearly independent subset of the above generating set. By the proof of Lemma 2.2, each element of $\mathcal{B}_n$, where $n > 0$, is an eigenvector for ψ with nonzero eigenvalue. $\mathcal{B} = \cup_{n\geq 0}\mathcal{B}_n$ is a basis for $S_{\mathcal{F}}(\mathcal{H}_m)$. Let $\mathcal{B}' = \mathcal{B} - \mathcal{B}_0$. Note $\psi(\mathcal{B}_0) = 0$. So $\mathcal{I}_0 = Image(\psi) = \psi(\mathcal{B}')$.

It follows that ψ induces a one to one map: $B_n/B_{n-1} \to B_n/B_{n-1}$. Thus $\psi(< \mathcal{B}' >) \cap S_{\mathcal{F}}(\mathcal{H}_m - D) = 0$. The result follows. □

Theorem 5 *The embedding $\mathcal{H}_m - D \to (\mathcal{H}_m)_\gamma$ induces an isomorphism*

$$S(\mathcal{H}_m - D) \cong S((\mathcal{H}_m)_\gamma).$$

Proof From the above, we have the following commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & S(\mathcal{H}_m - D) \cap I_0 & \longrightarrow & S(\mathcal{H}_m - D) & \longrightarrow & S((\mathcal{H}_m)_\gamma) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & I_0 & \longrightarrow & S(\mathcal{H}_m) & \longrightarrow & S((\mathcal{H}_m)_\gamma) & \longrightarrow & 0 \end{array}$$

□

$\mathcal{H}_{m_1}\#\mathcal{H}_{m_2}$ is equal to $\mathcal{H}_{m_1+m_2}$ with a 2-handle added along the boundary of the meridian disc D separating $\mathcal{H}_{m_1}$ from $\mathcal{H}_{m_2}$. Let $\gamma = \partial D$. Therefore we can consider $\mathcal{H}_1\#\mathcal{H}_2 = (\mathcal{H}_m)_\gamma$. As $\mathcal{H}_{m_1+m_2} - D = \mathcal{H}_{m_1} \sqcup \mathcal{H}_{m_2}$, the above theorem says:

Corollary 5 *Let B_1 and B_2 denote the 3-balls we remove from $\mathcal{H}_{m_1}$ and $\mathcal{H}_{m_2}$ while forming $\mathcal{H}_{m_1}\#\mathcal{H}_{m_2}$. The embedding $(\mathcal{H}_{m_1} - B_1) \sqcup (\mathcal{H}_{m_2} - B_2) \to \mathcal{H}_{m_1}\#\mathcal{H}_{m_2}$ induces*

$$S(\mathcal{H}_{m_1}) \otimes S(\mathcal{H}_{m_2}) \cong S(\mathcal{H}_{m_1}\#\mathcal{H}_{m_2})$$

4 The general case for absolute skein modules

A connected oriented 3-manifold with nonempty boundary may be obtained from the handlebody $\mathcal{H}$ by adding some 2-handles. If M is closed, we will also need one 3-handle. As removing 3-balls from the interior of a 3-manifold does not change its Homflypt skein module, we may reduce Theorem 1 to the case that M_1 and M_2 are connected 3-manifolds with boundary.

In this case, each M_i is obtained from the handlebody $\mathcal{H}_{m_i}$ by adding some 2-handles. Let $m = m_1+m_2$. Let N be the manifold obtained by adding both sets of 2-handles to the boundary connected sum of $\mathcal{H}_{m_1}$ and $\mathcal{H}_{m_2}$ which we identify with $\mathcal{H}_m$. Let D be the disc in $\mathcal{H}_m$ separating $\mathcal{H}_{m_1}$ from $\mathcal{H}_{m_2}$. Let $\gamma = \partial D$, so $\mathcal{H}_{m_1} \# \mathcal{H}_{m_2} = (\mathcal{H}_m)_\gamma$. Here and below P_δ denotes the result of adding a 2-handle to a 3-manifold P along a curve δ in ∂N. We can consider $M_1 \# M_2$ as obtained from $(\mathcal{H}_m)_\gamma$ by adding those 2-handles. Thus $N - D = M_1 \sqcup M_2$, and $M_1 \# M_2 = N_\gamma$.

Theorem 6 *The embedding $N - D \to N_\gamma$ induces an isomorphism*

$$S(N - D) \cong S(N_\gamma).$$

Proof We proceed by induction on n, the number of the 2-handles to be added to $(\mathcal{H}_m)_\gamma$ to obtain N_γ. If $n = 0$, we are done by Theorem 5. If $n \geq 1$, let N' be the 3-manifold obtained from $(\mathcal{H}_m)_\gamma$ by adding $(n-1)$ of those 2-handles added to $(\mathcal{H}_m)_\gamma$. Suppose the result is true for N', i.e.

$$S(N' - D) \cong S(N'_\gamma).$$

Suppose that the nth 2-handle is added along a curve γ^* in the boundary of $(\mathcal{H}_m)$, where γ^* is disjoint from γ and the curves where the other $(n-1)$ 2-handles are attached. Let A' and B' be two points on γ^*. By the proof of the Epimorphism Lemma 2.2,

$$S(N' - D, A', B') \twoheadrightarrow S(N'_\gamma, A', B').$$

Using [4, section 2], we have the following commutative diagram with exact rows.

$$\begin{array}{ccccccc}
S(N' - D, A', B') & \longrightarrow & S(N' - D) & \longrightarrow & S((N' - D)_{\gamma^*}) & \longrightarrow & 0 \\
\text{onto}\big\downarrow & & \cong\big\downarrow & & \big\downarrow & & \\
S(N'_\gamma, A', B') & \longrightarrow & S(N'_\gamma) & \longrightarrow & S((N'_\gamma)_{\gamma^*}) & \longrightarrow & 0
\end{array}$$

The vertical map on the right is an isomorphism by the five-lemma. N is obtained from N' by adding the nth 2-handle along γ^*. Thus $(N' - D)_{\gamma^*} = N - D$ and $(N'_\gamma)_{\gamma^*} = N_\gamma$. □

Corollary 6 *Let B_1 and B_2 denote the 3-balls we remove from M_1 and M_2 while forming $M_1 \# M_2$. The embedding $(M_1 - B_1) \sqcup (M_2 - B_2) \to M_1 \# M_2$ induces an isomorphism*

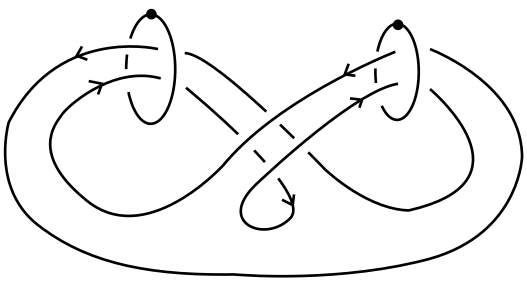
$$S(M_1) \otimes S(M_2) \cong S(M_1 \# M_2).$$

Proof Since $S(M - D) \cong S(M_1) \otimes S(M_2)$. $\square$

The above corollary holds whether or not M_1 or M_2 have boundary.

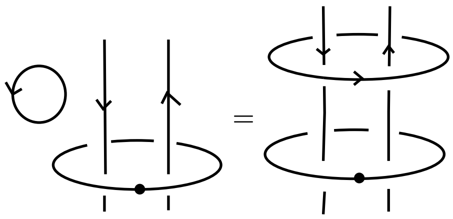
5 An example in $S^1 \times S^2 \# S^1 \times S^2$

In [4], we showed that $S(S^1 \times S^2)$ is a free $\mathcal{R}[x, x^{-1}]$-module generated by the empty link. It follows that $S(S^1 \times S^2 \# S^1 \times S^2)$ is also a free module generated by the empty link. Let K be a knot in $S^1 \times S^2 \# S^1 \times S^2$ pictured by the following diagram:

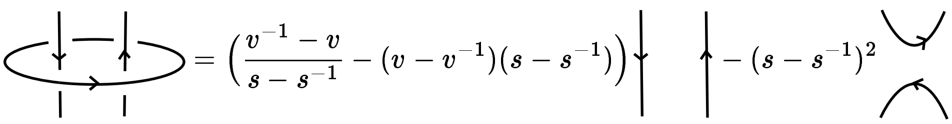


Here the two circles with a dot are a framed link description of $S^1 \times S^2 \# S^1 \times S^2$. Note this same knot was studied with respect to the Kauffman Bracket skein modules in [3].

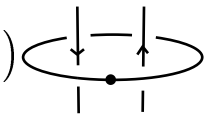
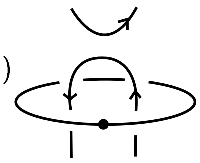
In $S(S^2 \times S^1 \# D^3, 4pts)$, isotopy yields,



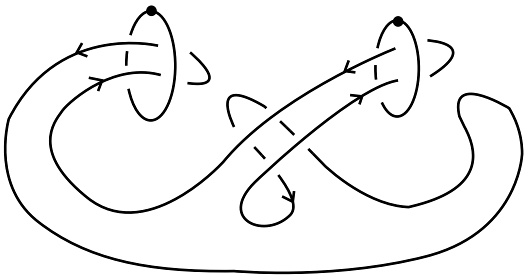
Using the Homflypt skein relations in $S(D^2 \times I, 4pts)$,



Therefore, in $S(S^2 \times S^1 \# D^3, 4pts)$, we have:

$(v - v^{-1})$  $\equiv -(s - s^{-1})$ 

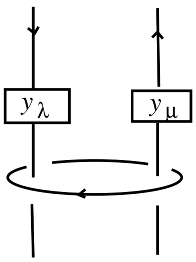
Thus

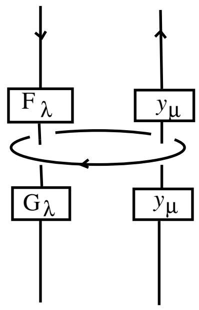
$(v - v^{-1})^2 K = (s - s^{-1})^2$ 

$= (s - s^{-1})^2 \frac{v^{-1}-v}{s-s^{-1}}\phi$. i.e. $K = \dfrac{s - s^{-1}}{v^{-1} - v}\phi$ in $S(S^1 \times S^2 \# S^1 \times S^2)$.

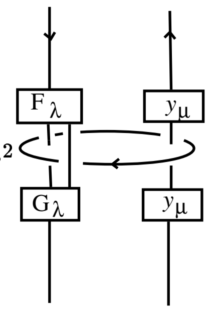
6 Proof of Lemma 2.1

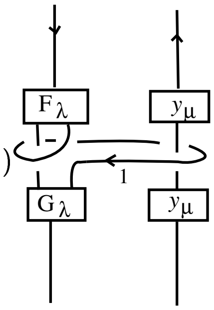
Note $y_\lambda = F_\lambda G_\lambda$. We start with the following:



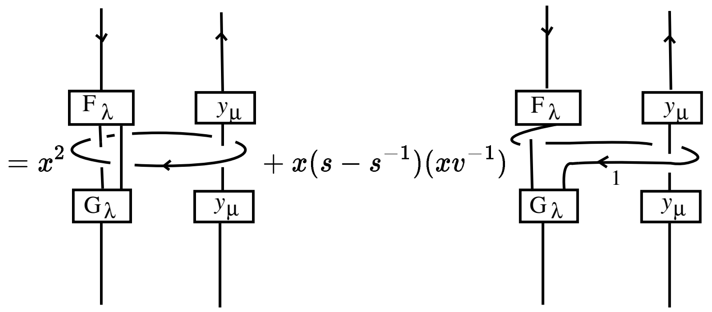
$=$


$=$

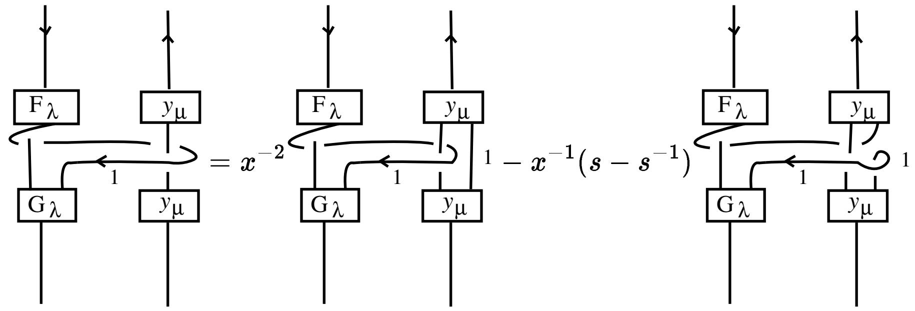
x^2 

$+ x(s - s^{-1})$ 

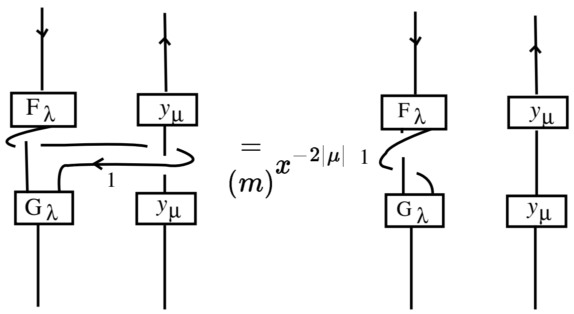
(I*)



We pulled out the string corresponding to the last cell in the last row of λ. Therefore in the above diagram, a 1 by the side of the string indicates the string related to the last cell in the last row of λ. Applying the Homflypt skein relation to the last diagram:

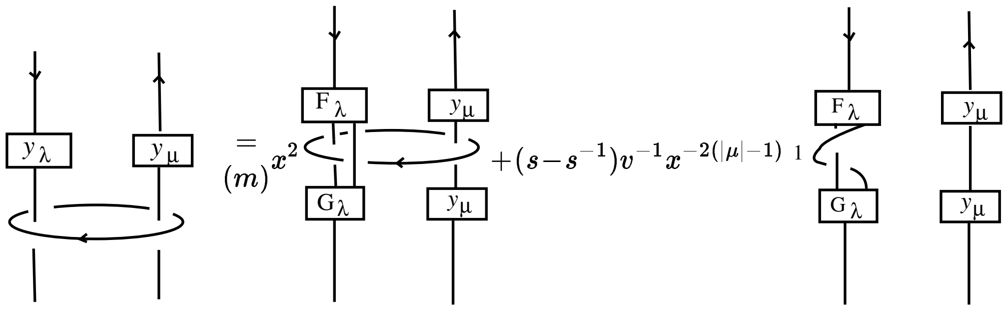


We pulled out the string corresponding to the last cell in the last row of μ. Continuing to pull out strings which correspond to cells of μ, working to the left through columns and upward through the rows of μ, we obtain:

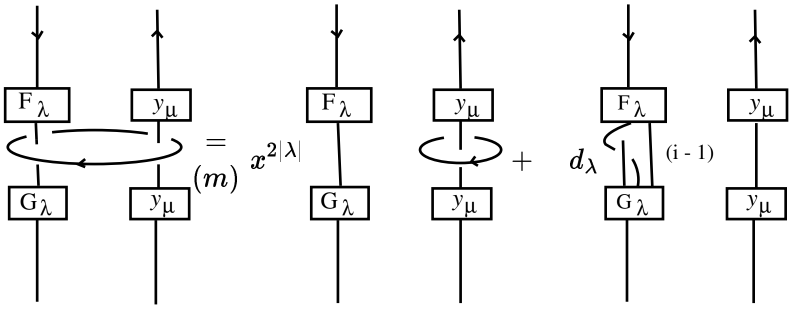


where the string corresponding to the last cell in the last row of λ encircles the remaining $|\lambda| - 1$ strings as shown.

In this way Equation (I*) becomes:

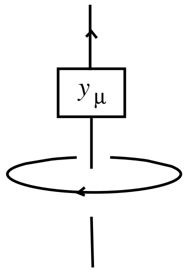


We continue in this way, pulling the encircling component successively through the vertical strings corresponding to cells of λ, working to the left through columns and upward through the rows of λ. We obtain:

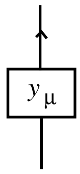


where d_λ denotes $v^{-1}(s-s^{-1})\sum_{i=1}^{|\lambda|} x^{-2(|\mu|-i)}$. In the last diagram, the $i-1$ vertical strings are related to the last $i-1$ cells of λ by the index order, and the ith string encircles the remaining $|\lambda|-i$ strings. Lemma 2.1 follows from the following lemma and Lemma 6.2 (a) below.

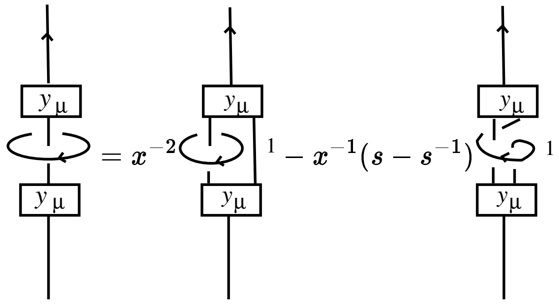
Lemma 6.1 *Let μ be a Young diagram of size n,*



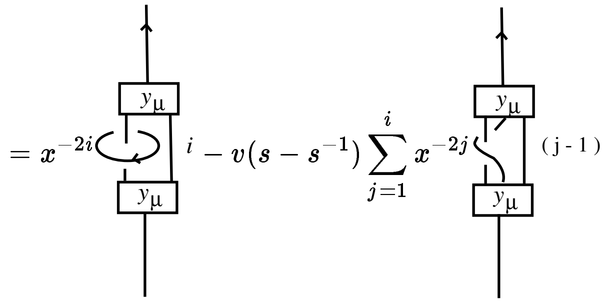
$$= x^{-2|\mu|}\left(\frac{v^{-1}-v}{s-s^{-1}} - v(s-s^{-1})\sum_{c\in\mu} s^{-2cn(c)}\right)$$

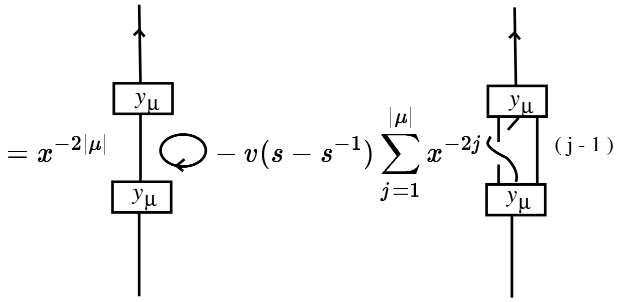


Proof We consider



Here we start with the string corresponding to the last cell in the last row of μ, we pull the encircling component successively through the vertical strings, working to the left through columns and upward through the rows. Repeating the above process, for $i \geq 2$:

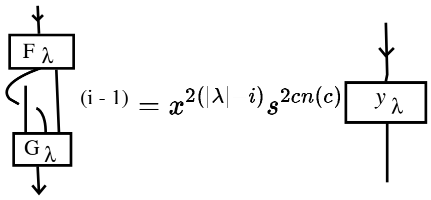




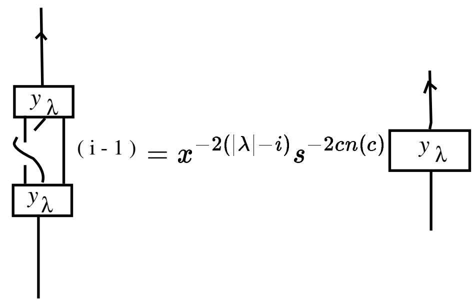
The result follows from Lemma 6.2 (b) below. □

Lemma 6.2 *Let λ be a Young diagram and (h, l) be the index of the cell after which $(i-1)$ cells of λ follow.*

(a)



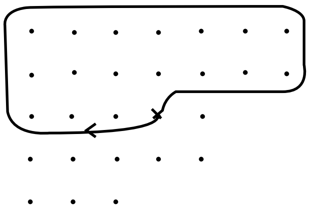
(b)



Remark The techniques used in this proof are similar to the proof of the framing factor in section 5 of [1] by H. Morton and A. Aiston.

Proof (a) We will borrow the notation of H. Morton and A. Aiston and use a schematic dot diagram to represent the element in the Hecke category $H_{\square_\lambda}$, which is between F_λ and G_λ as shown on the left-hand side.

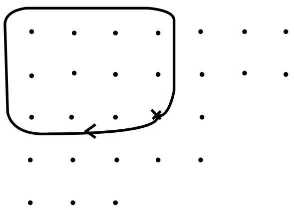
Recall that $y_\lambda = F_\lambda G_\lambda$. Now in the diagram of the left-hand side of (a), introduce a schematic picture T as follows:

$T =$ 

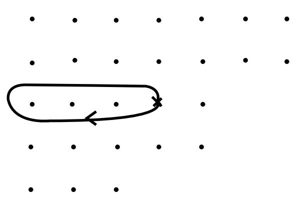
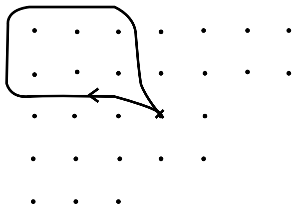
This indicates that the last $i-1$ strings were pulled out, the ith string marked by $\times$ starts and finishes at (h,l). The arrow on the ith string shows the string orientation when we look at it from above. The ith string encircles the remaining $|\lambda| - i$ strings in the clockwise direction. Here all strings shown by single dots are going vertical. The left-hand side of (a) can be expressed as $F_\lambda T G_\lambda$. We will be working on $F_\lambda T G_\lambda$. Using the Homflypt skein relations and the inseparability in Lemma 16 of [1], we have,

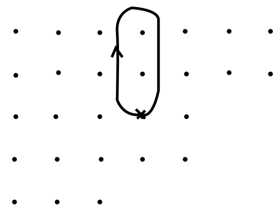
$$F_\lambda T G_\lambda = x^{2(|\lambda|-(i-1)-hl)} F_\lambda S G_\lambda$$

Where S is given by:

$S =$ 

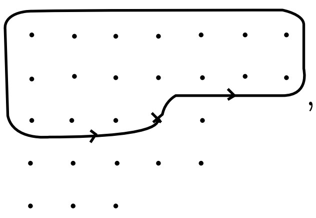
Since $S = S_1 T_1 S_2$, where:

$S_1 =$  $T_1 =$ 

$S_2 =$ 

First we have $F_\lambda S_1 = (xs)^{2(l-1)} F_\lambda$ by the property $\sigma_i f_m = xs f_m$; secondly, $S_2 G_\lambda = (-xs^{-1})^{2(h-1)} G_\lambda$ by the property $g_m \sigma_i = -xs^{-1} g_m$, [1, Lemma 8]) . It follows that $F_\lambda S G_\lambda = x^{2(h+l-2)} s^{2(l-h)} F_\lambda T_1 G_\lambda$. By a similar argument as in the proof of Theorem 17 in [1], $F_\lambda T_1 G_\lambda = x^{2(l-1)(h-1)} F_\lambda G_\lambda$. Thus $F_\lambda T G_\lambda = x^{2(|\lambda|-i)} s^{2(l-h)} F_\lambda G_\lambda = x^{2(|\lambda|-i)} s^{2cn(c)} y_\lambda$, where c is the cell indexed by (h, l).

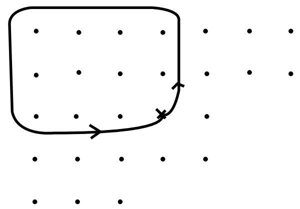
(b) We prove the result with all string orientations reversed. As string reversal defines a skein module isomorphism, this suffices. As $y_\lambda = F_\lambda G_\lambda$, we can use the following schematic picture to denote the left-hand side of (b) as $F_\lambda G_\lambda T^{-1} F_\lambda G_\lambda$, where

$T^{-1} =$  ,

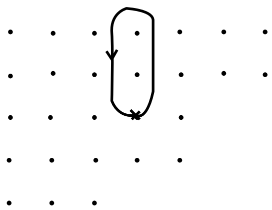
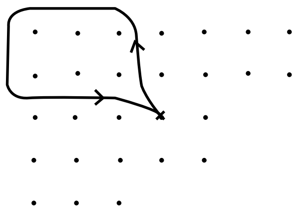
the ith string is indexed by (h, l) and circles the remaining strings in the clockwise direction. Again, we have

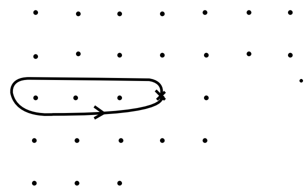
$$G_\lambda T^{-1} F_\lambda = x^{-2(|\lambda|-(i-1)-hl)} G_\lambda S^{-1} F_\lambda.$$

Where S^{-1} is given by:

$S^{-1} =$ 

Since $S^{-1} = S_2^{-1}T_1^{-1}S_1^{-1}$, where:

$S_2^{-1} =$  , $T_1^{-1} =$ 

$S_1^{-1} =$ 

We have $G_\lambda S_2^{-1} = (-x^{-1}s)^{2(h-1)}G_\lambda$ and $S_1^{-1}F_\lambda = (x^{-1}s^{-1})^{2(l-1)}F_\lambda$ by the properties $\sigma_i f_m = xsf_m$ and $g_m\sigma_i = -xs^{-1}g_m$.

We have

$$G_\lambda S^{-1}F_\lambda = x^{-2(h+l-2)}s^{2(h-l)}G_\lambda T_1^{-1}F_\lambda = x^{-2(h+l-2)}s^{2(h-l)}x^{-2(h-1)(l-1)}G_\lambda F_\lambda.$$

It follows that $G_\lambda T^{-1}F_\lambda = x^{-2(|\lambda|-i)}s^{-2cn(c)}G_\lambda F_\lambda$. By the idempotent property, $F_\lambda G_\lambda T^{-1}F_\lambda G_\lambda = x^{-2(|\lambda|-i)}s^{-2cn(c)}F_\lambda G_\lambda$. The result follows. □

7 Discussion of the proofs of Theorems 2 & 3

The proof of Theorem 2 is basically the same as the proof of Theorem 1. However as noted in the introduction we do not yet know that the relative Homflypt skein of a handlebody is free. So we must work over $\mathcal{F}$.

For the proof of Theorem 3, we note that every relative link in (M, X, Y) is isotopic to a link which intersects a tubular neighborhood of Σ with m

straight strands going in one direction and $m+r$ straight strands going the other direction. We will write such elements as linear combinations of $\alpha_t\beta_\tau \otimes \alpha_\sigma{\beta_s}^*$, and t and τ are standard tableaux of a Young diagram λ, σ and s are standard tableaux of a Young diagram μ with $|\lambda| = m$, and $|\mu| = m+r$. As $x^{2r} - 1 - c_{\lambda,\mu}$ is invertible over k_r, we have that $\alpha_t\beta_\tau \otimes \alpha_\sigma{\beta_s}^* \in \mathcal{L}(|\lambda| + |\mu|)$. We may repeat this argument until the class of our original relative link is represented by a linear combination of links each of which intersects Σ less than r times. This must be the empty linear combination.

References

[1] A. K. Aiston and H. R. Morton, *Idempotents of Hecke algebras of type A*, J. of Knot Theory and Ram. 7 No 4 (1998), 463-487.

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Department of Mathematics, Louisiana State University
Baton Rouge, LA 70803, USA
and
Program of Mathematics and Statistics
Louisiana Tech University, Ruston, LA 71272, USA

Email: `gilmer@math.lsu.edu, kzhong@coes.LaTech.edu`