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Reshetikhin–Turaev invariants of Seifert 3–manifolds and a rational surgery formula

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Abstract We calculate the RT–invariants of all oriented Seifert manifolds directly from surgery presentations. We work in the general framework of an arbitrary modular category as in [Tu], and the invariants are expressed in terms of the S– and T–matrices of the modular category. In another direction we derive a rational surgery formula, which states how the RT–invariants behave under rational surgery along framed links in arbitrary closed oriented 3–manifolds with embedded colored ribbon graphs. The surgery formula is used to give another derivation of the RT–invariants of Seifert manifolds with orientable base.

AMS Classification 57M27; 17B37, 18D10, 57M25

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1 Introduction

A major challenge in the theory of quantum invariants of links and 3–manifolds, notably the Jones polynomial of links in S^3 [Jo], is to determine relationships between these invariants and classical invariants. In 1988 Witten [Wi] gave a sort of an answer by his interpretation of the Jones polynomial (and its generalizations) in terms of quantum field theory. Witten not only gave a description of the Jones polynomial in terms of 3–dimensional topology/geometry, but he also initiated the era of quantum invariants of 3–manifolds by defining invariants $Z_k^G(M,L) \in \mathbb{C}$ of an arbitrary closed oriented 3–manifold M with an embedded colored link L by quantizing the Chern–Simons field theory associated to a simply connected compact simple Lie group G, k being an arbitrary positive integer, called the (quantum) level. The invariant $Z_k^G(M,L)$ is given by a Feynman path integral over the (infinite dimensional) space of gauge equivalence classes of connections in a G–bundle over M. This integral should be understood in a formal way since, at the moment of writing, it seems that no mathematically rigorous definition is known, cf. [JL, Sect. 20.2.A].

By using stationary phase approximation techniques together with path integral arguments Witten was able [Wi] to express the leading asymptotics of $Z_k^G(M)$ as $k \to \infty$ in terms of such topological/geometric invariants as Chern–Simons invariants, Reidemeister torsions and spectral flows, so here we see a way to extract topological information from the invariants $Z_k^G(M)$ (here $L = \emptyset$). Furthermore, a full asymptotic expansion of $Z_k^G(M)$ as $k \to \infty$ is expected on the basis of a full perturbative analysis of the Feynman path integral, see e.g. [AS1], [AS2].

Reshetikhin and Turaev [RT2] constructed invariants $\tau_r^{\mathfrak{sl}_2(\mathbb{C})}(M, L) \in \mathbb{C}$ by a mathematical approach via representations of a quantum group $U_q(\mathfrak{sl}_2(\mathbb{C}))$, $q = \exp(2\pi i/r)$, r an integer ≥ 2. Shortly afterwards, quantum invariants $\tau_r^{\mathfrak{g}}(M, L) \in \mathbb{C}$ associated to other complex simple Lie algebras $\mathfrak{g}$ were constructed using representations of the quantum groups $U_q(\mathfrak{g})$, $q = \exp(2\pi i/r)$ a 'nice' root of unity, see [TW1]. Both in Witten's approach and in the approach of Reshetikhin and Turaev the invariants are part of a so-called topological quantum field theory (TQFT). This implies that the invariants are defined for compact oriented 3–dimensional cobordisms (perhaps with some extra structure on the boundary), and satisfy certain cut-and-paste axioms, see [At], [Q], [Tu]. The TQFT of Reshetikhin and Turaev can from an algebraic point of view be given a more general formulation by using so-called modular categories [Tu]. The representation theory of $U_q(\mathfrak{g})$, $\mathfrak{g}$ an arbitrary complex simple Lie algebra, induces such a modular category if $q = \exp(2\pi i/r)$ is chosen properly, see [TW1], the appendix in [TW2], [Kir], [BK], [Sa], and [Le].

It is believed that the TQFT's of Witten and Reshetikhin–Turaev coincide. In particular it is conjectured, that Witten's leading asymptotics for $Z_k^G(M)$ should be valid for the function $r \mapsto \tau_r^{\mathfrak{g}}(M)$ in the limit $r \to \infty$ and furthermore, that this function should have a full asymptotic expansion. In this paper we initiate a verification of this conjecture for oriented Seifert manifolds by deriving formulas for the RT–invariants of these manifolds. In a subsequent paper [Ha2] we then use these formulas to calculate the large r asymptotics of the RT–invariants and thereby prove the so-called asymptotic expansion conjecture for such manifolds in the $\mathfrak{sl}_2(\mathbb{C})$–case. The precise formulation of this conjecture, which is a combination of Witten's leading asymptotics and the existence of a full asymptotic expansion of a certain type, was proposed by Andersen in [A], where he proved it for mapping tori of finite order diffeomorphisms of orientable surfaces of genus at least two using the gauge theory definition of the quantum invariants.

In the following a Seifert manifold means an oriented Seifert manifold. Calculations of quantum invariants of lens spaces and other Seifert manifolds have been done by several people [A], [G], [J], [LR], [N], [Roz], [Ta1], [Ta2], [Tu] and probably many more. The papers [G], [J], [LR], [N], [Ta1] and [Ta2] calculate and study the quantum invariants of lens spaces and other Seifert manifolds with base equal to S^2. In [Ta1], [Ta2] the so-called $P\mathfrak{sl}_n(\mathbb{C})$–invariants are calculated. The $P\mathfrak{sl}_n(\mathbb{C})$–invariant of a closed oriented 3–manifold M associated with an integer $r > n$ coprime to n is a factor of $\tau_r^{\mathfrak{sl}_n(\mathbb{C})}(M)$. Neil [N] calculates the $\mathfrak{sl}_2(\mathbb{C})$–invariants based on Lickorish skein theoretical approach [Li2]. In [LR] the $SU(2)$–invariants of certain Seifert manifolds with base S^2 are calculated and studied. The class of Seifert manifolds considered includes the Seifert manifolds which are integral homology spheres. Rozansky [Roz] derives a formula for the $SU(2)$–invariants of all Seifert manifolds with orientable base. The papers [G], [J], [LR] and [Roz] are based on Witten's approach to the invariants. Andersen [A] calculates quantum G–invariants of all mapping tori of finite order diffeomorphisms of orientable surfaces of genus at least two, where G is an arbitrary simply connected compact simple Lie group. The mapping tori of finite order diffeomorphisms of an orientable surface Σ_g of genus g are precisely the Seifert manifolds with base Σ_g and Seifert Euler number equal to zero. Turaev has calculated the RT–invariants associated to an arbitrary unimodal modular category of all graph manifolds, cf. [Tu, Sect. X.9]. These manifolds include the Seifert manifolds with orientable base (but not the ones with non-orientable base).

In this paper we extend the above results in two directions. Firstly, we calculate the RT–invariants of all Seifert manifolds. In particular we calculate the invariants of Seifert manifolds with non-orientable base. This case has to the authors knowledge not been considered before in the literature. Secondly, our calculations are done for arbitrary modular categories, cf. Theorem 4.1. We present three different calculations of the RT–invariants of Seifert manifolds with different levels of generality. In our first approach we calculate the invariants of all Seifert manifolds directly from surgery presentations only using the theory of RT–invariants of closed oriented 3–manifolds without refering to the underlying TQFT. In our second approach we calculate the RT–invariants of all Seifert manifolds with orientable base using a rational surgery formula for the RT–invariants, Theorem 5.3, derived in this paper. In these two approaches we work in the framework of an arbitrary modular category. In our third approach we use a formula for the RT–invariants of graph manifolds due to Turaev, see [Tu, Theorem X.9.3.1]. This formula is valid for all modular categories satisfying a special condition called unimodality. As mentioned above the graph manifolds include the Seifert manifolds with orientable base. We show that Turaev's formula specializes to our formula for the invariants of these Seifert manifolds.

The rational surgery formula, Theorem 5.3, states how the RT–invariants behave under rational surgeries along framed links in arbitrary closed oriented 3–manifolds with embedded colored ribbon graphs. This formula generalizes the defining formula for the RT–invariants of closed oriented 3–manifolds with embedded colored ribbon graphs (which is a surgery formula for surgeries on S^3 with embedded colored ribbon graphs along framed links). The surgery formula has the very same form as the surgery formulas presented in the Chern–Simons TQFT of Witten, see [Wi, Sect. 4], [LR], [Roz].

In the final part of the paper we analyse more carefully the $\mathfrak{sl}_2(\mathbb{C})$–case. In the general formulas for the RT–invariants of the Seifert manifolds, see Theorem 4.1, a certain factor of so-called S– and T–matrices is present. In the $\mathfrak{sl}_2(\mathbb{C})$–case the S– and T–matrices can be identified (up to normalization) with the values of a certain representation $\mathcal{R}$ of $SL(2,\mathbb{Z})$ in the standard generators of $SL(2,\mathbb{Z})$. This representation has been carefully studied by Jeffrey in [J], where an explicit formula for $\mathcal{R}(A)$ in terms of the entries of $A \in SL(2,\mathbb{Z})$ is given. We use this formula to give expressions for the RT–invariants of the Seifert manifolds in terms of the Seifert invariants, see Theorem 8.4. Theorem 8.4 generalizes results in the literature, in particular the formulas for the RT–invariants of Seifert manifolds with orientable base given in [Roz].

The paper is organized as follows. In Sect. 2 we recall the definition and classification of Seifert manifolds [Se1], [Se2]. We also present surgery presentations of the Seifert manifolds due to Montesinos [M]. In Sect. 3 we give a short introduction to the modular categories. This is a preliminary section intended to fix notation used throughout in the paper. In Sect. 4 we calculate the RT–invariants of all Seifert manifolds directly from surgery presentations. In Sect. 5 we derive the rational surgery formula for the RT–invariants of closed oriented 3–manifolds with embedded colored ribbon graphs. In Sect. 6 we calculate the RT–invariants of the Seifert manifolds with orientable base using the surgery formula. In Sect. 7 we show that Turaev's formula for the RT–invariants of graph manifolds specializes to our formula for the RT–invariants of Seifert manifolds with orientable base. In Sect. 8 we analyse the $\mathfrak{sl}_2(\mathbb{C})$–case in greater detail. Besides we have added two appendices, one comparing different normalizations of the RT–invariants used in the literature and one discussing different definitions of framed links in arbitrary closed oriented 3–manifolds.

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2 Seifert manifolds

Seifert manifolds were invented by H. Seifert in [Se1]. For an english translation, see [Se2]. We consider only oriented Seifert manifolds in this section as in the rest of the paper. These will be denoted Seifert manifolds (as in the introduction).

Oriented Seifert manifolds and their classification Let ν, μ be coprime integers with $\mu > 0$, and let $\rho\colon B^2 \to B^2$ be the rotation by the angle $2\pi(\nu/\mu)$ in the anti-clockwise direction, where $B^2 \subseteq \mathbb{C}$ is the standard oriented unit disk. The (oriented) *fibered solid torus* $T(\mu,\nu)$ is the oriented space $B^2 \times [0,1]/_R$, where R identifies $(x,1)$ with $(\rho(x),0)$, $x \in B^2$, and the orientation is given by the orientation of B^2 followed by the orientation of $[0,1]$. By this identification the lines (fibers) $\{x\} \times [0,1]$ of $B^2 \times [0,1]$, $x \in B^2 \setminus \{0\}$, are decomposed into classes, such that each class contains exactly μ lines, which match together to give one fiber of $T(\mu,\nu)$. The image of $\{0\} \times [0,1]$ in $T(\mu,\nu)$ is also a fiber, called the 'middle fiber'. The pair (μ,ν) is an invariant of $T(\mu,\nu)$ if we normalize to $0 \leq \nu < \mu$. The following definition is Seifert's definition of a fibered space [Se1] adapted to the oriented case. *A Seifert manifold is a closed connected and oriented 3–manifold M, which can be decomposed into a collection of disjoint simple closed curves, called fibers, such that each fiber H has a neighborhood N, called a fiber neighborhood, which is homeomorphic to a fibered solid torus $T(\mu,\nu)$ by an orientation and fiber preserving homeomorphism mapping H to the middle fiber of $T(\mu,\nu)$.* By [Se2, Lemma 2], the numbers μ,ν are invariants of the fiber H, called the (oriented) *fiber invariants* of H. If $\mu > 1$, we call H an *exceptional fiber*; if $\mu = 1$, an *ordinary fiber*. In a fiber neighborhood of a fiber H all fibers except possibly H are ordinary fibers, so there are only finitely many (possibly zero) exceptional fibers in a Seifert manifold. For a Seifert manifold M, the *base* is the quotient space of M obtained by identifying each fiber to a point. The base is a closed connected surface, orientable or non-orientable. The genus of the non-orientable $\#^g\mathbb{RP}^2$ is g. Two Seifert manifolds are *equivalent* if there is a fiber and orientation preserving homeomorphism between them. We have the following classification result due to Seifert.

Theorem 2.1 [Se1] *An equivalence class of Seifert manifolds is determined by a system of invariants*

$$(\epsilon; g \,|\, b; (\alpha_1, \beta_1), \dots , (\alpha_r, \beta_r)) .$$

Here $\epsilon = \mathrm{o}$ if the base is orientable and $\epsilon = \mathrm{n}$ if not, and the non-negative integer g is the genus of the base. Moreover, $r \geq 0$ is the number of exceptional fibers, and (α_i, β_i) are the (oriented) Seifert invariants of the i'th exceptional fiber. The invariant b can take any value in $\mathbb{Z}$ ($-b$ is the Euler number of the locally trivial S^1–bundle $(\epsilon; g \,|\, b)$).

An oriented Seifert manifold M belonging to the class determined by the invariants $(\epsilon; g \,|\, b; (\alpha_1, \beta_1), \dots , (\alpha_r, \beta_r))$ belongs after reversing its orientation to the class determined by the invariants $(\epsilon; g \,|\, -r-b; (\alpha_1, \alpha_1-\beta_1), \dots , (\alpha_r, \alpha_r-\beta_r))$, $\epsilon = \mathrm{o}, \mathrm{n}$.

The Seifert invariants (α_i, β_i) of the i'th exceptional fiber are the unique integers such that $\alpha_i = \mu_i$, $\beta_i \nu_i \equiv 1 \pmod{\mu_i}$ and $0 < \beta_i < \alpha_i$, where μ_i, ν_i are the fiber invariants of that fiber. One can obtain $(\epsilon; g \,|\, b)$ from $(\epsilon; g \,|\, b; (\alpha_1, \beta_1), \dots , (\alpha_r, \beta_r))$ by cutting out fiber neighborhoods of the exceptional fibers and gluing in ordinary solid tori (i.e. $T(1,0)$'s) by certain fiber preserving homeomorphisms, see [Se2, Sect. 7], [M, Sect. 4.2] for details. The *Seifert Euler number* of the Seifert fibration $(\epsilon; g \,|\, b; (\alpha_1, \beta_1), \dots , (\alpha_r, \beta_r))$ is the rational number $e = -\left(b + \sum_{j=1}^{r} \beta_j/\alpha_j\right)$. (The reason for the choice of sign of e is the following. Let $\epsilon \in \{\mathrm{o}, \mathrm{n}\}$ and let X be a closed surface of genus g, orientable if $\epsilon = \mathrm{o}$ and non-orientable if $\epsilon = \mathrm{n}$. Then $(\epsilon; g \,|\, -\chi(X))$ is the unit tangent bundle of X, where $\chi(X)$ is the Euler characteristic of X. More generally, $(\epsilon; g \,|\, b)$ is a locally trivial S^1–bundle over the surface X. The number $-b$ is the Euler number of this bundle and is an obstruction to the existence of a section of $(\epsilon; g \,|\, b)$, see [M, Chap. 1]. The Seifert Euler number is a natural generalization of $-b$ when extending the above notions to orbifolds, see [T], [Sc], [M].)

Surgery presentations Any closed connected oriented 3–manifold can be obtained by Dehn-surgery on S^3 along a labelled link, the labels being the rational surgery coefficients, cf. [Li1], [Wa]. We use the standard convention for surgery coefficients, see e.g. [Ro1, Chap. 9], [Ro2]. In particular integer labelled links in S^3 can be identified with framed links with the framing indexes equal

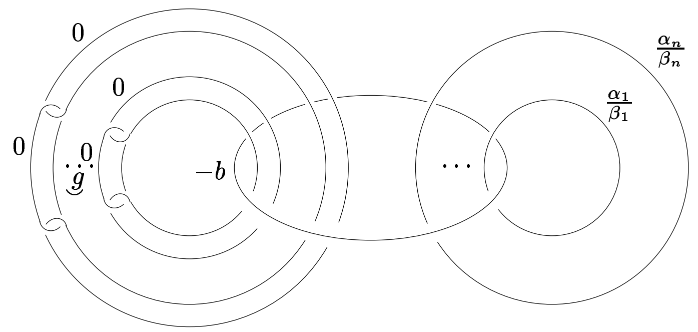

Figure 1: Surgery presentation of $(\mathrm{o}; g \mid b; (\alpha_1, \beta_1), \dots, (\alpha_n, \beta_n))$

to the labels. If M is a 3–manifold given by surgery on S^3 along a labelled link L we call L a surgery presentation of M. According to [M, Fig. 12 p. 146], the manifold $(\epsilon; g \mid b; (\alpha_1, \beta_1), \dots, (\alpha_n, \beta_n))$ has a surgery presentation as shown in Fig. 1 if $\epsilon = \mathrm{o}$ and as shown in Fig. 2 if $\epsilon = \mathrm{n}$. The $\underbrace{}_{g}$ indicate g repetitions.

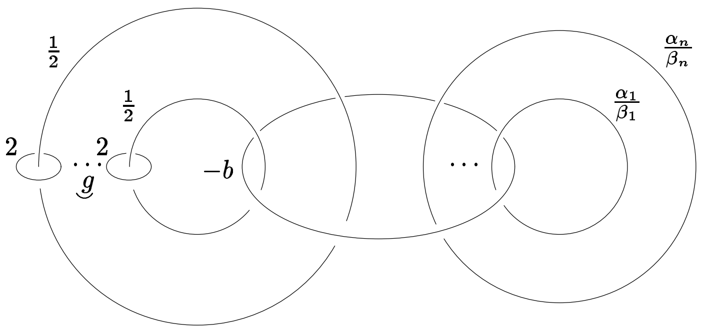

Figure 2: Surgery presentation of $(\mathrm{n}; g \mid b; (\alpha_1, \beta_1), \dots, (\alpha_n, \beta_n))$

Non-normalized Seifert invariants The so-called *non-normalized Seifert invariants*, see [Ne], [JN] or [NR], are sometimes more convenient to use in specific calculations. Let (α_j, β_j) be a pair of coprime integers with $\alpha_j > 0$, $j = 1, 2, \dots, n$. Then the Seifert manifold with non-normalized Seifert invariants $\{\epsilon; g; (\alpha_1, \beta_1), \dots, (\alpha_n, \beta_n)\}$ is given by a surgery presentation as shown in Fig. 1 with $b = 0$ if $\epsilon = \mathrm{o}$ and as shown in Fig. 2 with $b = 0$ if $\epsilon = \mathrm{n}$. It follows that these non-normalized invariants are not unique. In fact, by [JN, Theorem 1.5 and Theorem 1.8], the sets $\{\epsilon; g; (\alpha_1, \beta_1), \dots, (\alpha_n, \beta_n)\}$ and

$\{\epsilon'; g'; (\alpha'_1, \beta'_1), \dots, (\alpha'_m, \beta'_m)\}$ are two pairs of non-normalized Seifert invariants of the same Seifert manifold M if and only if $\epsilon = \epsilon'$, $g = g'$ (trivial), $\sum_{i=1}^n \beta_i/\alpha_i = \sum_{j=1}^m \beta'_j/\alpha'_j$, and disregarding any β_i/α_i and β'_j/α'_j which are integers, the remaining $\beta_i/\alpha_i \pmod 1$ are a permutation of the remaining $\beta'_j/\alpha'_j \pmod 1$. It follows that any Seifert manifold M has a unique set of non-normalized Seifert invariants (up to permutation of the indicis) of the form $\{\epsilon; g; (1, \beta_0), (\alpha_1, \beta_1), \dots, (\alpha_r, \beta_r)\}$ with $0 < \beta_i < \alpha_i$, $i = 1, \dots, r$, so $M = (\epsilon; g \mid \beta_0; (\alpha_1, \beta_1), \dots, (\alpha_r, \beta_r))$ in the terminology of Theorem 2.1. This implies that the Seifert Euler number of a Seifert manifold with non-normalized Seifert invariants $\{\epsilon; g; (\alpha_1, \beta_1), \dots, (\alpha_n, \beta_n)\}$ is given by $-\sum_{i=1}^n \beta_i/\alpha_i$.

Remark 2.2 [JN] operates with a generalization of oriented Seifert fibrations in which the pairs (α_j, β_j) are allowed to be equal to $(0, \pm 1)$. However, up to an orientation preserving homeomorphism, these generalized fibrations are Seifert manifolds as defined above or connected sums of the form $\#_{i=1}^k (S^1 \times S^2) \# \#_{i=1}^n L(p_i, q_i)$, cf. [JN, Theorem 5.1]. Since the RT–invariants behave nicely with respect to connected sums and since the lens spaces are (ordinary) Seifert manifolds, see the proof of Corollary 4.4, we will continue by only considering the Seifert manifolds in Theorem 2.1.

3 Modular categories and 3–manifold invariants

This is a preliminary section in which we recall concepts and notation from [Tu] used throughout in this paper. All monoidal categories in the following are assumed strict.

Ribbon categories and invariants of colored ribbon graphs A *ribbon category* $\mathcal{V}$ is a monoidal category with a braiding c and a twist θ and with a duality $(*, b, d)$ compatible with these structures. In $\mathcal{V}$ one has a well-defined trace $\mathrm{tr} = \mathrm{tr}_{\mathcal{V}}$ of morphisms and thereby a well-defined dimension $\dim = \dim_{\mathcal{V}}$ of objects. These take values in the commutative semigroup $K = K_{\mathcal{V}} = \mathrm{End}_{\mathcal{V}}(\mathbb{I})$, where $\mathbb{I}$ is the unit object (the multiplication being given by the composition of morphisms).

By a ($\mathcal{V}$–)*colored ribbon graph* we mean a ribbon graph Ω with an object of $\mathcal{V}$ attached to each band and annulus of Ω and with a compatible morphism of $\mathcal{V}$ attached to each coupon of Ω. We let $F = F_{\mathcal{V}}$ be the operator invariant of $\mathcal{V}$–colored ribbon graphs in $\mathbb{R}^3$ of Reshetikhin and Turaev, see [RT1], [RT2], [Tu, Chap. I].

We use the graphical calculus for morphisms of the ribbon category $\mathcal{V}$, see [Tu, Sect. I.1.6], [Ka, Chap. XIV]. In this calculus one represents a morphism f of $\mathcal{V}$ by a colored ribbon graph Ω mapped by F to f if such a ribbon graph exists. We then write $\Omega \doteq f$. We present ribbon graphs in figures according to the usual rules, cf. [Tu, Chap. I]. In particular we draw only the oriented cores of the annuli and bands, and we are careful to drawing all loops corresponding to twists in the ribbons. Analogous to the framing numbers in figures showing framed links we will sometimes indicate a certain number of twists in an annulus component of a ribbon graph by an integer instead of drawing the loops corresponding to these twists. In figures showing colored ribbon graphs these numbers will be put into parentheses to distinguish them from colors.

Modular categories A *monoidal Ab–category* is a monoidal category with all morphism sets equipped with an additive abelian group structure making the composition and tensor product bilinear (cf. [Ma]; Ab–categories are also called pre-abelian categories).

Let $\mathcal{V}$ be a *ribbon Ab–category*, i.e. a ribbon category such that the underlying monoidal category is a monoidal Ab–category. In particular, the semigroup $K = K_{\mathcal{V}}$ is a commutative unital ring, called the *ground ring* of $\mathcal{V}$. For any pair of objects V, W of $\mathcal{V}$, the abelian group $\mathrm{Hom}_{\mathcal{V}}(V,W)$ acquires the structure of a left K–module by $kf = k \otimes f$, $k \in K$, $f \in \mathrm{Hom}_{\mathcal{V}}(V,W)$, which makes composition and the tensor product of morphisms K–bilinear. An object V of $\mathcal{V}$ is called *simple* if $k \mapsto k\mathrm{id}_V$ is a bijection $K \to \mathrm{End}_{\mathcal{V}}(V)$. In particular the unit object $\mathbb{I}$ is simple. An object V of $\mathcal{V}$ is *dominated* by a family $\{V_i\}_{i\in I}$ if there exists a finite set of morphisms $\{f_r\colon V_{i(r)} \to V,\ g_r\colon V \to V_{i(r)}\}_r$ with $i(r) \in I$ such that $\mathrm{id}_V = \sum_r f_r g_r$.

A *modular category* is a tuple $(\mathcal{V}, \{V_i\}_{i\in I})$, where $\mathcal{V}$ is a ribbon Ab–category and $\{V_i\}_{i\in I}$ is a finite set of simple objects closed under duals (i.e. for any $i \in I$ there exists $i^* \in I$ such that V_{i^*} is isomorphic to the dual of V_i) and dominating all objects of $\mathcal{V}$, such that $V_0 = \mathbb{I}$ for a distinguished element $0 \in I$, and such that the so-called S–matrix $S = (S_{i,j})_{i,j\in I}$ is invertible over K. Here $S_{i,j} = \mathrm{tr}\left(c_{V_j,V_i} \circ c_{V_i,V_j}\right)$ is the invariant of the standard Hopf link with framing 0 and with one component colored by V_i and the other colored by V_j. The invertibility of S implies that $i \mapsto i^*$ is an involution in I.

Since V_i is a simple object, $\theta_{V_i}\colon V_i \to V_i$ is equal to $v_i \mathrm{id}_{V_i}$ for a $v_i \in K$, $i \in I$. The T–matrix $T = (T_{i,j})_{i,j\in I}$ is given by $T_{i,j} = \delta_{i,j} v_i$, where $\delta_{i,j}$ is the Kronecker delta equal to 1 if $i = j$ and to 0 otherwise. In Fig. 3 we give a graphical description of the entries of the $S-$ and $T-$matrices. In this and other figures we indicate the object V_i by i. Moreover, we put $\dim(i) = \dim(V_i)$, $i \in I$. We have used the identity $F(\bar{\Omega}) = \mathrm{tr}(F(\Omega))$, where $\bar{\Omega}$ is the closure of a colored ribbon graph Ω, cf. [Tu, Corollary I.2.7.2].

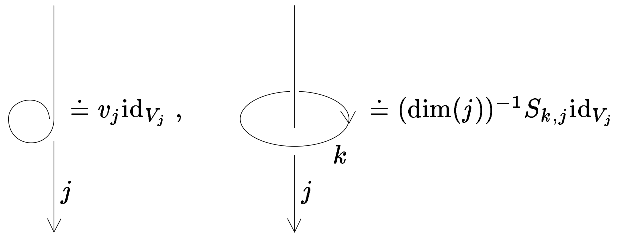

Figure 3

A *rank* of the modular category $(\mathcal{V}, \{V_i\}_{i\in I})$ is an element $\mathcal{D} = \mathcal{D}_{\mathcal{V}} \in K$ such that $\mathcal{D}^2 = \sum_{i\in I} (\dim(i))^2$. A modular category does not need to have a rank, but, as pointed out in [Tu, p. 76], we can always formally change $\mathcal{V}$ to a modular category with the same objects as $\mathcal{V}$ and with a rank. We let $\Delta = \Delta_{\mathcal{V}} = \sum_{i\in I} v_i^{-1} (\dim(i))^2$. For a modular category with a rank $\mathcal{D}$ we have

$$S^2 = \mathcal{D}^2 J \tag{1}$$

by [Tu, Formula (II.3.8.a)], where $J_{i,j} = \delta_{i^*,j}$, $i, j \in I$.

The RT–invariants of 3–manifolds We identify as usual an oriented framed link in $S^3 = \mathbb{R}^3 \cup \{\infty\}$ with a ribbon graph in S^3 (actually in $\mathbb{R}^3$) consisting solely of directed annuli, cf. [RT2], [Tu]. If L is a framed link in S^3 and B^4 is the closed 4–ball, oriented as the unit ball in $\mathbb{C}^2$, then we get a smooth closed connected oriented 4–manifold W_L by adding 2–handles to B^4 along the components of L in $S^3 = \partial B^4$ using the framing of L, see [Ki]. The manifold $M = M_L = \partial W_L$, oriented using the 'outward first' convention for boundaries, is the result of surgery on S^3 along L. Let Ω be a colored ribbon graph inside M and let $\Gamma(L,\lambda)$ be the colored ribbon graph obtained by fixing an orientation in L and coloring the i'th component of L by $V_{\lambda(L_i)}$. The RT–invariant of the pair (M,Ω) based on $(\mathcal{V}, \{V_i\}_{i\in I}, \mathcal{D})$ is given by

$$\begin{aligned} \tau_{(\mathcal{V},\mathcal{D})}(M,\Omega) &= \Delta^{\sigma(L)} \mathcal{D}^{-\sigma(L)-m-1} \\ &\quad \times \sum_{\lambda \in \mathrm{col}(L)} \left(\prod_{i=1}^{m} \dim(\lambda(L_i)) \right) F(\Gamma(L,\lambda) \cup \Omega), \end{aligned} \tag{2}$$

cf. [Tu, p. 82], where, as usual, we identify Ω with a colored ribbon graph in $S^3 \setminus L$. Here m is the number of components of L, $\sigma(L)$ is the signature of

W_L, i.e. the signature of the intersection form on $H_2(W_L;\mathbb{R})$, and $\mathrm{col}(L)$ is the set of mappings from the set of components of L to I. The signature $\sigma(L)$ is also equal to the signature of the linking matrix of L.

The mirror of a modular category The *mirror* of a modular category $(\mathcal{V},\{V_i\}_{i\in I})$ is a ribbon Ab–category $\overline{\mathcal{V}}$ with the same underlying monoidal Ab–category and the same duality as $\mathcal{V}$. If θ and c are the twist and braiding of $\mathcal{V}$, then the twist $\bar{\theta}$ and braiding $\bar{c}$ of $\overline{\mathcal{V}}$ are defined by $\bar{\theta}_V=(\theta_V)^{-1}$ and $\bar{c}_{V,W}=(c_{W,V})^{-1}$ for any objects V, W of $\mathcal{V}$, cf. [Tu, Sect. I.1.4]. By [Tu, Exercise II.1.9.2], $\left(\overline{\mathcal{V}},\{V_i\}_{i\in I}\right)$ is a modular category with S–matrix $\bar{S}=\left(S_{i^*,j}\right)_{i,j\in I}$, where $S=\left(S_{i,j}\right)_{i,j\in I}$ is the S–matrix of $\mathcal{V}$. Note that $\mathcal{D}$ is a rank of $\overline{\mathcal{V}}$ if and only if $\mathcal{D}$ is a rank of $\mathcal{V}$, since the dimensions of any object of $\mathcal{V}$ with respect to $\mathcal{V}$ and $\overline{\mathcal{V}}$ are equal, cf. [Tu, Corollary I.2.8.5]. By [Tu, Formula (II.2.4.a)] we have

$$\Delta_{\mathcal{V}}\Delta_{\overline{\mathcal{V}}}=\mathcal{D}^2. \tag{3}$$

We end this section by recalling the notion of a *unimodal modular category* also called a *unimodular category*, cf. [Tu, Sect. VI.2]. Moreover we give two small lemmas needed in the calculations of the RT–invariants of Seifert manifolds with non-orientable base.

Let $(\mathcal{V},\{V_i\}_{i\in I})$ be a modular category. An element $i\in I$ is called *self-dual* if $i=i^*$. For such an element we have a K–module isomorphism $\mathrm{Hom}_{\mathcal{V}}(V\otimes V,\mathbb{I})\cong K$, $V=V_i$. The map $x\mapsto x(\mathrm{id}_V\otimes\theta_V)c_{V,V}$ is a K–module endomorphism of $\mathrm{Hom}_{\mathcal{V}}(V\otimes V,\mathbb{I})$, so is a multiplication by a certain $\varepsilon_i\in K$. By the definition of the braiding and twist we have $(\varepsilon_i)^2=1$. In particular $\varepsilon_i\in\{\pm1\}$ if K is a field. The modular category $(\mathcal{V},\{V_i\}_{i\in I})$ is called *unimodal* if $\varepsilon_i=1$ for every self-dual $i\in I$. By copying a part of the proof of [Tu, Lemma VI.2.2] we get:

Lemma 3.1 *Let $(\mathcal{V},\{V_i\}_{i\in I})$ be a modular category and let $i\in I$ be self-dual. Moreover, let $V=V_i$ and let $\epsilon_i\in K$ be as above. Then*

$$d_V(\omega\otimes\mathrm{id}_V)=\varepsilon_i d_V^-(\mathrm{id}_V\otimes\omega) \tag{4}$$

for any isomorphism $\omega\colon V\to V^*$, where d_V^- is the operator invariant $F_{\mathcal{V}}$ of the left-oriented cap $\curvearrowleft$ colored with V. □

Let $(A,R,v,\{V_i\}_{i\in I})$ be a modular Hopf algebra over a commutative unital ring K, cf. [Tu, Chap. XI]. If we write the universal R–matrix as $R=\sum_j\alpha_j\otimes\beta_j\in$

$A^{\otimes 2}$, the element u is given by $u = \sum_j s(\beta_j)\alpha_j \in A$, where s is the antipode of the underlying Hopf algebra. Let $(\mathcal{V}, \{V_i\}_{i\in I})$ be the modular category induced by $(A, R, v, \{V_i\}_{i\in I})$, cf. [Tu, Chap. XI].

Lemma 3.2 *Let $i \in I$ be self-dual, let $V = V_i$ and let $\epsilon_i \in K_{\mathcal{V}} = K$ be as above. For any isomorphism $\omega\colon V \to V^*$, the composition*

$$V \xrightarrow{\omega} V^* \xrightarrow{(\omega^{-1})^*} V^{**} \xrightarrow{G} V$$

is given by multiplication with $\varepsilon_i uv$, where G^{-1} is the canonical K–module isomorphism between the finitely generated projective K–module V and its double dual V^{**}.

Proof Since $\mathcal{V}$ is a ribbon category, we have a canonical A–module isomorphism $\alpha_V\colon V \to V^{**}$ given by

$$\alpha_V = (d_V^- \otimes \mathrm{id}_{V^{**}})(\mathrm{id}_V \otimes b_{V^*}),$$

cf. [Tu, Corollary I.2.6.1]. Let $Q\colon V \to V$ be multiplication by uv. Then $\alpha_V = G^{-1} \circ Q$. To see this, write $b_{V^*}(1) = \sum_k g_k \otimes g^k \in V^* \otimes V^{**}$. This element is characterized by the following property: For any $\chi \in V^*$, $y \in V^{**}$ we have

$$y(\chi) = \sum_k y(g_k) g^k(\chi).$$

Now let $x \in V$ and get

$$\alpha_V(x) = \sum_k d_V^-(x \otimes g_k) \otimes g^k.$$

By using that $d_V^- = d_V c_{V,V^*}(\theta_V \otimes \mathrm{id}_{V^*})$ we get

$$\alpha_V(x) = \sum_k g_k(uv \cdot x) g^k \in V^{**}.$$

If $\chi \in V^*$ we therefore have

$$\alpha_V(x)(\chi) = \sum_k g_k(uv \cdot x) g^k(\chi) = G^{-1} \circ Q(x)(\chi).$$

If $f\colon U \to W$ is a morphism in $\mathcal{V}$, then the dual morphism $f^*\colon W^* \to U^*$ is given by $f^* = (d_W \otimes \mathrm{id}_{U^*})(\mathrm{id}_{W^*} \otimes f \otimes \mathrm{id}_{U^*})(\mathrm{id}_{W^*} \otimes b_U)$. By using the graphical calculus together with (4) one immediately gets that $(\omega^{-1})^* \circ \omega = \varepsilon_i \alpha_V$. □

4 The Reshetikhin–Turaev invariants of Seifert manifolds

In this section we calculate the RT–invariants of all oriented Seifert manifolds. Throughout, $(\mathcal{V}, \{V_i\}_{i\in I})$ is a fixed modular category with a fixed rank $\mathcal{D}$. We let $F = F_{\mathcal{V}}$, $\Delta = \Delta_{\mathcal{V}}$, $K = K_{\mathcal{V}}$, and $\tau = \tau_{(\mathcal{V},\mathcal{D})}$.

Notation For the next theorem and for later use we introduce some notation. Let $y(i,j) \in K$ be the scalar such that $F(T_{ij}) = y(i,j)\mathrm{id}_{V_j}$, where T_{ij} is the colored ribbon tangle in Fig. 4. That is, $y(i,j) = (\dim(j))^{-1}\mathrm{tr}(F(T_{ij}))$. We put

$$\kappa(j) = \sum_{i\in I} \dim(i) y(i,j), \qquad j \in I. \tag{5}$$

For every self-dual element $i \in I$, let $\varepsilon_i \in K$ be as in the last part of Sect. 3.

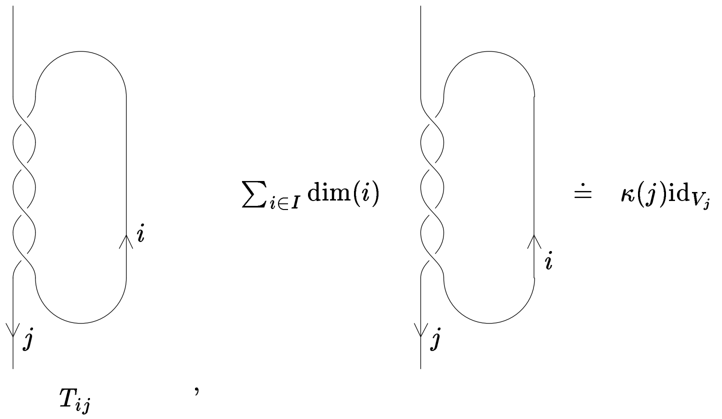

Figure 4

The group $SL(2,\mathbb{Z})$ is generated by two matrices

$$\Xi = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \qquad \Theta = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}. \tag{6}$$

For a tuple of integers $\mathcal{C} = (a_1, \ldots, a_n)$ we let

$$B_k^{\mathcal{C}} = \begin{pmatrix} \alpha_k^{\mathcal{C}} & \rho_k^{\mathcal{C}} \\ \beta_k^{\mathcal{C}} & \sigma_k^{\mathcal{C}} \end{pmatrix} = \Theta^{a_k}\Xi\Theta^{a_{k-1}}\Xi \ldots \Theta^{a_1}\Xi, \qquad k = 1, 2, \ldots, n \tag{7}$$

and let $B^{\mathcal{C}} = B_n^{\mathcal{C}}$. Moreover, we put

$$G^{\mathcal{C}} = T^{a_n} S T^{a_{n-1}} S \cdots S T^{a_1} S. \tag{8}$$

A continued fraction expansion

$$\frac{p}{q} = a_n - \cfrac{1}{a_{n-1} - \cfrac{1}{\cdots - \cfrac{1}{a_1}}}, \quad a_i \in \mathbb{Z},$$

$p, q \in \mathbb{Z}$ not both equal to zero, is abbreviated $(a_1, \ldots, a_n)$. Given pairs (α_j, β_j) of coprime integers we let $\mathcal{C}_j = (a_1^{(j)}, a_2^{(j)}, \ldots, a_{m_j}^{(j)})$ be a continued fraction expansion of α_j/β_j, $j = 1, 2, \ldots, n$.

Theorem 4.1 *The RT–invariant τ of $M = (\mathrm{o}; g \mid b; (\alpha_1, \beta_1), \ldots, (\alpha_n, \beta_n))$ is*

$$\tau(M) = (\Delta \mathcal{D}^{-1})^{\sigma_{\mathrm{o}}} \mathcal{D}^{2g-2-\sum_{j=1}^n m_j} \sum_{j \in I} v_j^{-b} \dim(j)^{2-n-2g} \left(\prod_{i=1}^n (S G^{\mathcal{C}_i})_{j,0} \right), \tag{9}$$

where

$$\sigma_{\mathrm{o}} = \mathrm{sign}(e) + \sum_{j=1}^n \sum_{l=1}^{m_j} \mathrm{sign}(\alpha_l^{\mathcal{C}_j} \beta_l^{\mathcal{C}_j}). \tag{10}$$

Here $e = -\left(b + \sum_{j=1}^n \frac{\beta_j}{\alpha_j}\right)$ is the Seifert Euler number.

The RT–invariant τ of the Seifert manifold M with non-normalized Seifert invariants $\{\mathrm{o}; g; (\alpha_1, \beta_1), \ldots, (\alpha_n, \beta_n)\}$ is given by the same expression with the exceptions, that the factor v_j^{-b} has to be removed and $e = -\sum_{j=1}^n \frac{\beta_j}{\alpha_j}$.

The RT–invariant τ of $M = (\mathrm{n}; g \mid b; (\alpha_1, \beta_1), \ldots, (\alpha_n, \beta_n))$ is

$$\begin{aligned} \tau(M) &= (\Delta \mathcal{D}^{-1})^{\sigma_{\mathrm{n}}} \mathcal{D}^{g-2-\sum_{j=1}^n m_j} \\ &\quad \times \sum_{j \in I} (\varepsilon_j)^g \, \delta_{j,j^*} v_j^{-b} \dim(j)^{2-n-g} \left(\prod_{i=1}^n (S G^{\mathcal{C}_i})_{j,0} \right), \end{aligned} \tag{11}$$

where $\delta_{j,k}$ is the Kronecker delta equal to 1 if $j = k$ and to 0 otherwise, and

$$\sigma_{\mathrm{n}} = \sum_{j=1}^n \sum_{l=1}^{m_j} \mathrm{sign}(\alpha_l^{\mathcal{C}_j} \beta_l^{\mathcal{C}_j}). \tag{12}$$

The RT–invariant τ of the Seifert manifold M with non-normalized Seifert invariants $\{\mathrm{n}; g; (\alpha_1, \beta_1), \ldots, (\alpha_n, \beta_n)\}$ is given by the same expression with the exception, that the factor v_j^{-b} has to be removed.

The theorem is also valid in case $n=0$. In this case one just has to put all sums $\sum_{j=1}^{n}$ equal to zero and all products $\prod_{i=1}^{n}$ equal to 1. Note that $\epsilon_j^g = 1$ if g is even and $\epsilon_j^g = \epsilon_j$ if g is odd since $\epsilon_j^2 = 1$.

Preliminaries Before giving the proof of Theorem 4.1 we make some preliminary remarks.

1) Let $\mathcal{C} = (a_1, \ldots, a_n) \in \mathbb{Z}^n$ and consider the matrices in (7). By [J, Proposition 2.5] we have that $(a_1, \ldots, a_k)$ is a continued fraction expansion of $\alpha_k^{\mathcal{C}}/\beta_k^{\mathcal{C}}$, $k = 1, 2, \ldots, n$, and that $\beta_k^{\mathcal{C}} = \alpha_{k-1}^{\mathcal{C}}$, $k = 2, 3, \ldots, n$. Note that $\alpha_1^{\mathcal{C}} = a_1$ and $\beta_1^{\mathcal{C}} = 1$.

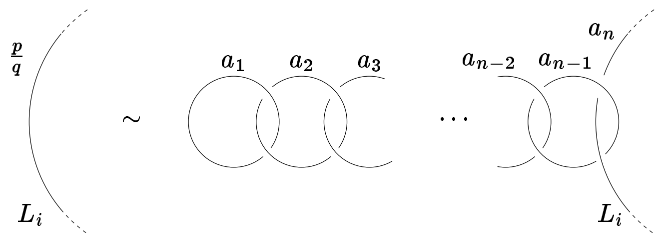

Figure 5

2) Two labelled links in S^3 are (surgery) equivalent if surgeries on S^3 along these labelled links result in 3–manifolds which are isomorphic as oriented 3–manifolds. Correspondingly we talk about equivalent surgery presentations. We have the following well-known fact [Ro1, p. 273]: Let $(a_1, \ldots, a_n)$ be a continued fraction expansion of $p/q \in \mathbb{Q}$ and let L be a labelled link with a component L_i with surgery coefficient p/q. Then this link is surgery equivalent to a link obtained from L by changing the surgery coefficient of L_i to a_n and shackling L_i with an integer labelled Hopf chain with $n-1$ components with labels $a_1, \ldots, a_{n-1}$ as shown in Fig. 5. For a proof of this, simply use standard surgery modifications, cf. [Ro1, Sect. 9.H], [Ro2]. (Begin by unknotting L_i in the presentation in the right-hand side of Fig. 5 and get rid of the Hopf chain, see the proof of [PS, Proposition 17.3]. Finally recover the original L_i by knotting.) Alternatively, see [KM2, Appendix].

3) The identity in Fig. 6 is due to Turaev, cf. [Tu, Exercise II.3.10.2]. For the sake of completeness we give a proof of it here.

Proof of the identity in Fig. 6 The axiom of domination for a modular category, see Sect. 3, implies that we for arbitrary $i, j \in I$ can write the identity

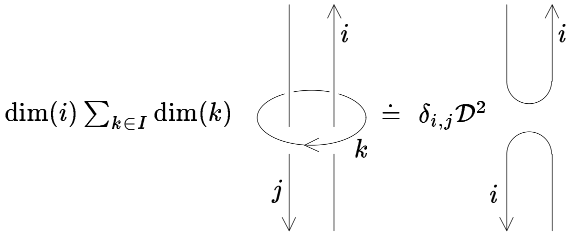

Figure 6

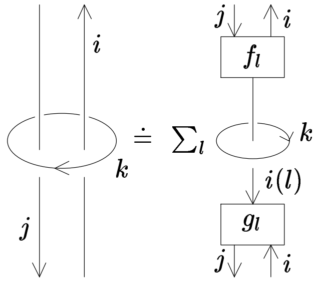

Figure 7

endomorphism of $V_j \otimes V_i^*$ as a finite sum

$$\mathrm{id}_{V_j \otimes V_i^*} = \sum_l f_l g_l,$$

where $f_l\colon V_{i(l)} \to V_j \otimes V_i^*$ and $g_l\colon V_j \otimes V_i^* \to V_{i(l)}$ are certain morphisms. By this we get the identity in Fig. 7. According to [Tu, Lemma II.3.2.3] we have $\dim(k) = d_0^{-1} d_k$, where the elements $d_i \in K$, $i \in I$, are defined by (50), cf. [Tu, p. 87]. By using this and [Tu, Lemma II.3.2.2 (i)] we get the identity shown in Fig. 8, where $x = \sum_{u\in I} d_u \dim(u)$. Since $V_i^* \cong V_{i^*}$ and $\mathrm{Hom}(\mathbb{I}, V_j \otimes V_{i^*}) = 0$ unless $i = j$, cf. [Tu, Lemma II.3.5], we get the result for $i \neq j$. Assume $i = j$. By [Tu, Lemma II.3.5], the K–module $\mathrm{Hom}(\mathbb{I}, V_i \otimes V_i^*)$ is generated by b_{V_i}, where b is part of the duality of the modular category. Similarly, $\mathrm{Hom}(V_i \otimes V_i^*, \mathbb{I})$ is generated by $d^-_{V_i}$, where $d^-_{V_i}\colon V_i \otimes V_i^* \to \mathbb{I}$ is the operator invariant of the left-oriented cap $\curvearrowleft$ colored with V_i. We can therefore write $\mathrm{id}_{V_i\otimes V_i^*} = fg + \sum_{l:i(l)\neq 0} f_l g_l$, where $f = ab_{V_i}$ and $g = a' d^-_{V_i}$, $a, a' \in K$. By this we get

$$d^-_{V_i} b_{V_i} = d^-_{V_i} \mathrm{id}_{V_i \otimes V_i^*} b_{V_i} = aa'(d^-_{V_i} b_{V_i})^2,$$

since $\mathrm{Hom}(V_r, V_s) = 0$ for any distinct $r, s \in I$, cf. [Tu, Lemma II.1.5]. Since $d^-_{V_i} b_{V_i} = \dim(i)$ we get $aa' = (\dim(i))^{-1}$. Combining this with the identity in

Fig. 8 and the fact that $xd_0^{-1} = \mathcal{D}^2$, cf. [Tu, p. 89], finally brings us to the identity in Fig. 6. □

The above proof does not use the existence of a rank. In case we don't have a rank, the identity in Fig. 6 is still valid if we replace $\mathcal{D}^2$ by $\sum_{u\in I}(\dim(u))^2$. One should also note that the orientation of the annulus component with color k does not play any role. This follows by the usual argument since we sum over all colors k.

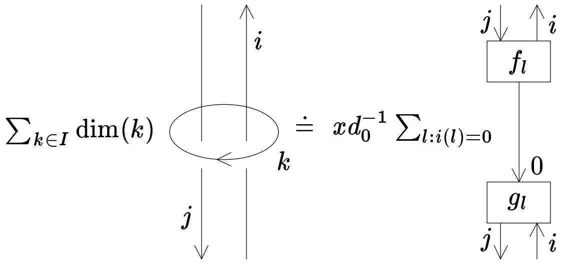

Figure 8

Proof of Theorem 4.1 Let $M = (\mathrm{o}; g\,|\,b; (\alpha_1,\beta_1),\dots,(\alpha_n,\beta_n))$ and let L be the link obtained from the link in Fig. 1 by replacing the component with surgery coefficient α_j/β_j by a chain according to $\mathcal{C}_j$ as in Fig. 5, $j = 1,\dots,n$. Note that L has $m = 2g+1+\sum_{j=1}^n m_j$ components. By (2) and the identities in Fig. 3 we have

$$\begin{aligned}\tau(M) &= \Delta^{\sigma(L)}\mathcal{D}^{-\sigma(L)-m-1}\sum_{j\in I}\dim(j)^{1-n}\left(\prod_{i=1}^n (SG^{\mathcal{C}_i})_{j,0}\right)\\ &\times \sum_{u_1,\dots,u_g,n_1,\dots,n_g\in I}\left(\prod_{l=1}^g \dim(u_l)\dim(n_l)\right)F(\Gamma(j,u_1,n_1,\dots,u_g,n_g)),\end{aligned}$$

where $\Gamma(j,u_1,n_1,\dots,u_g,n_g)$ is the colored ribbon graph shown in Fig. 9. If $g=0$ we have to replace the sum $\sum_{u_1,\dots,u_g,n_1,\dots,n_g\in I}$ by $v_j^{-b}\dim(j)$ here and can go directly to the calculation of $\sigma(L)$. Assume $g>0$. By using the identity in Fig. 6 with the component colored with u_j in Fig. 9 equal to the component colored with k in Fig. 6, $j=1,2,\dots,g$, we get

$$\sum_{u_1,\dots,u_g,n_1,\dots,n_g\in I}\left(\prod_{l=1}^{g}\dim(u_l)\dim(n_l)\right)F(\Gamma(j,u_1,n_1,\dots,u_g,n_g))$$
$$=\mathcal{D}^{2g}\sum_{n_1,\dots,n_g\in I}F(\Gamma(j,n_1,\dots,n_g)),$$

where $\Gamma(j,n_1,\dots,n_g)$ is the colored ribbon tangle shown in Fig. 10. The expression (9) now follows by the fact that $\sigma(L)=\sigma_{\text{o}}$, see below, and by

$$\sum_{n_1,\dots,n_g\in I}F(\Gamma(j,n_1,\dots,n_g))$$
$$=v_j^{-b}\sum_{n_1,\dots,n_g\in I}\left(\prod_{l=1}^{g}S_{n_l,j}S_{n_l^*,j}\dim(j)^{-2}\right)\dim(j)=\dim(j)^{1-2g}v_j^{-b}\mathcal{D}^{2g},$$

where the first equality follows by the identities in Fig. 3 and the last equality follows by (1) and the facts that S is symmetric and satisfies $S_{i,j}=S_{i^*,j^*}$, $i,j\in I$, cf. [Tu, Formula (II.3.3.a)].

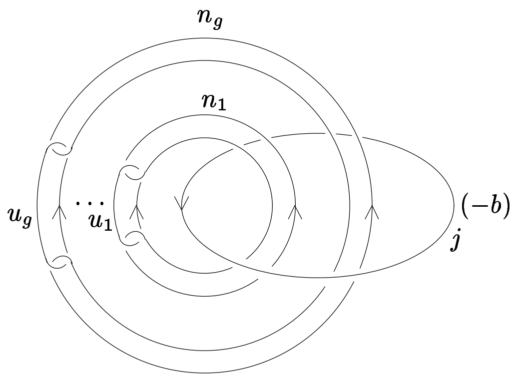

Figure 9: Oriented colored surgery presentation of $(\text{o};g\,|\,b)$

Let us show that $\sigma(L)=\sigma_{\text{o}}$. To this end let us use the notation

$$A(x_1,x_2,\dots,x_k)=\begin{pmatrix} x_1 & 1 & 0 & \cdots & 0\\ 1 & x_2 & 1 & \cdots & 0\\ 0 & 1 & x_3 & \cdots & 0\\ \cdots & \cdots & \cdots & \cdots & \cdots\\ 0 & 0 & 0 & \cdots & x_k\end{pmatrix}, \tag{13}$$

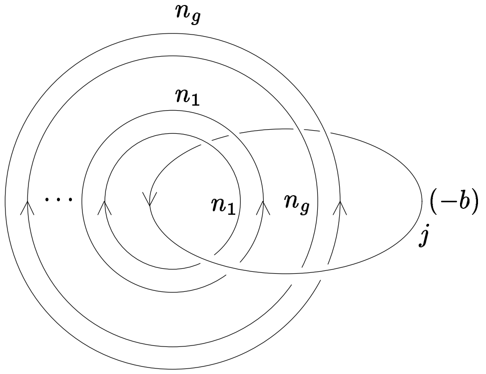

Figure 10: The colored ribbon tangle $\Gamma(j, n_1, \ldots, n_g)$

so $A(x_1, x_2, \ldots, x_k)_{ij}$ is x_i if $i = j$, 1 if $|i - j| = 1$, and 0 elsewhere, $i, j \in \{1, 2, \ldots, k\}$. The linking matrix of L is given by $\begin{pmatrix} 0 & 0 \\ 0 & A \end{pmatrix}$, where the zeroes refer to the first $2g$ rows and columns and

$$A = \begin{pmatrix} -b & e_1 & e_1 & \cdots & e_1 \\ e_1^t & A_1 & 0 & \cdots & 0 \\ e_1^t & 0 & A_2 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ e_1^t & 0 & 0 & \cdots & A_n \end{pmatrix}. \tag{14}$$

Here $A_j = A(a_{m_j}^{(j)}, a_{m_j-1}^{(j)}, \ldots, a_1^{(j)})$, i.e. the linking matrix of the j'th chain, and $e_1 = (1, 0, \cdots, 0)$. We write w^t for a vector w considered as a column vector. We will calculate the signature of A by reducing A using combined row and column operations. A main problem is to avoid dividing by zero. Let us consider A_1. Write $m = m_1$, $a_i = a_i^{(1)}$, $p_i = \alpha_i^{\mathcal{C}_1}$, and $q_i = \beta_i^{\mathcal{C}_1}$ to shorten notation. Assume first that $p_i \neq 0$ for all $i = 1, 2, \ldots, m$ (or equivalently that $q_i \neq 0$ for all $i = 1, 2, \ldots, m$ since $q_1 = 1$, $p_m = \pm\alpha_1 \neq 0$, and $q_i = p_{i-1}$, $i = 2, 3, \ldots, m$). Since $(a_1, a_2, \ldots, a_i)$ is a continued fraction expansion of p_i/q_i, $i = 1, 2, \ldots, m$, we can reduce A to

$$A' = \begin{pmatrix} -b - \frac{q_m}{p_m} & 0 & e_1 & \cdots & e_1 \\ 0 & A_1' & 0 & \cdots & 0 \\ e_1^t & 0 & A_2 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ e_1^t & 0 & 0 & \cdots & A_n \end{pmatrix}, \tag{15}$$

where $A_1' = \operatorname{diag}(p_m/q_m, p_{m-1}/q_{m-1}, \dots, p_1/q_1)$.

Next assume that $p_i = 0$ for at least one $i \in \{1, 2, \dots, m\}$. Let k be the smallest element in $\{1, 2, \dots, m\}$ such that $p_k = 0$. Choose a non-negative integer l such that $a_{k+1} = a_{k+2} = \cdots = a_{k+l} = 0$ and $a_{k+l+1} \neq 0$ or $k+l = m$. Let us first consider the case $k + l < m$ and let $a = a_{k+l+1}$. If $k > 1$ we reduce A to a matrix A' which is equal to A, except that A_1 is changed to $A_1' = \begin{pmatrix} C & 0 \\ 0 & D \end{pmatrix}$, where $D = \operatorname{diag}(p_{k-1}/q_{k-1}, p_{k-2}/q_{k-2}, \dots, p_1/q_1)$ and $C = A(a_m, a_{m-1}, \dots, a_{k+l+2}, a, 0, \dots, 0)$. If $k = 1$, let $A' = A$ and $A_1' = A_1$. Next reduce A' to a matrix A'' equal to A', except that A_1' is changed to

$$A_1'' = \begin{pmatrix} E & 0 & \cdots & 0 & 0 \\ 0 & G & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & G & 0 \\ 0 & 0 & \cdots & 0 & D \end{pmatrix}, \tag{16}$$

where $G = \operatorname{diag}(2, -1/2)$, and where $E = A(a_m, a_{m-1}, \dots, a_{k+l+2}, a, 0)$ if l is even and $E = A(a_m, a_{m-1}, \dots, a_{k+l+2}, a)$ if l is odd. (The row and column with D is not present if $k = 1$. Note that $A'' = A'$ if $l = 0$.) Assume that $k + l + 1 = m$. Then, if l is even, we reduce A'' futher to a matrix equal to the right-hand side of (15) with A_1' replaced by a matrix A_1''' equal to A_1'' with E replaced by $\operatorname{diag}(a, -1/a)$. If l is odd we let $A_1''' = A_1''$. Since $p_k = 0$ we have that $B_k^{\mathcal{C}_1} = \pm \begin{pmatrix} 0 & -1 \\ 1 & d \end{pmatrix} = \pm \Xi \Theta^d$ for a $d \in \mathbb{Z}$, so $B_{k+i}^{\mathcal{C}_1} = \pm \Xi^{i+1} \Theta^d$, $i = 1, 2, \dots, l$. Since $\Xi^2 = -1$ we therefore have $q_{k+i} = 0$ for i odd and $p_{k+i} = 0$ for i even, $i \in \{0, 1, \dots, l\}$. In particular $q_m = p_{k+l} = 0$ for l even. For l odd we have $p_m/q_m = a$. From this we also see that the signature of A_1''' is equal to $\sum_{j=1}^m \operatorname{sign}(p_j q_j)$. If $k + l + 1 < m$ we continue the diagonalization by reducing E in the same manner as we have reduced A_1 above. If l is odd, $p_{k+l+1}/q_{k+l+1} = a$ and the lower right block in A_1'', i.e. $\operatorname{diag}(G, \dots, G, D)$, has signature $\sum_{j=1}^{k+l} \operatorname{sign}(p_j q_j)$. If l is even, $p_{k+l} = 0$ and therefore $q_{k+l+1} = 0$ and $p_{k+l+2}/q_{k+l+2} = a_{k+l+2}$. In this case we begin by reducing A'' to a matrix equal to A'' with E replaced by $\begin{pmatrix} E' & 0 \\ 0 & F \end{pmatrix}$ in A_1'', where $E' = A(a_m, a_{m-1}, \dots, a_{k+l+2})$ and $F = \operatorname{diag}(a, -1/a)$. Note that the lower right block in the reduced A_1'', i.e. $\operatorname{diag}(F, G, \dots, G, D)$, has signature $\sum_{j=1}^{k+l+1} \operatorname{sign}(p_j q_j)$.

The only case left to consider is when $k + l = m$. In this case $B^{\mathcal{C}_1} = B_m^{\mathcal{C}_1} = \Xi^l B_k^{\mathcal{C}_1} = \pm \Xi^{l+1} \Theta^d$, so l is odd since $p_m = \pm \alpha_1 \neq 0$. But then $\beta_1 = \pm q_m = 0$, so this case is only relevant in case of non-normalized Seifert invariants. For letting the above calculation also work in this case, let us assume for the moment that $k+l=m$ and l is odd. We then reduce A to a matrix H equal to A, except that A_1 is replaced by a matrix H_1 equal to the right-hand side of (16) with E replaced by $J=A(0,0)$. Finally, we reduce H to a matrix equal to the right-hand side of (15) with A_1' replaced by a matrix H_2 equal to H_1 with J replaced by G. Note that the signature of H_2 is equal to $\sum_{j=1}^{m}\operatorname{sign}(p_jq_j)$.

This ends the reduction involving A_1. We can now continue as above reducing the parts in A involving A_j, $j=2,3,\dots,n$, and get the result.

The RT–invariant τ of the Seifert manifold with non-normalized Seifert invariants $\{\mathrm{o};g;(\alpha_1,\beta_1),\dots,(\alpha_n,\beta_n)\}$ is calculated as above by letting b be equal to zero everywhere, since this manifold has a surgery presentation as in Fig. 1 with $-b$ changed to 0.

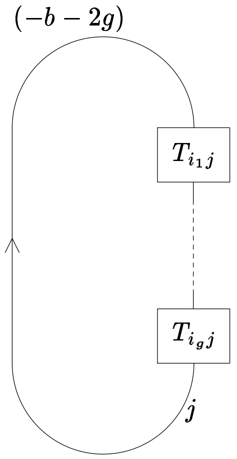

Figure 11: Oriented colored surgery presentation of $(\mathrm{n};g\,|\,b)$

Next let us calculate $\tau(M)$ for $M=(\mathrm{n};g\,|\,b;(\alpha_1,\beta_1),\dots,(\alpha_n,\beta_n))$. To obtain a surgery presentation of M with only integral surgery coefficients we make two left-handed twists about every component with surgery coefficient $1/2$ in the surgery presentation in Fig. 2. The components with surgery coefficients α_j/β_j are replaced by chains according to the continued fraction expansions $\mathcal{C}_j$, $j=1,\dots,n$, as before. Fig. 11 shows a colored oriented version of the new surgery diagram in the case where there are no exceptional fibers. The coupons T_{i_lj} represent colored ribbon tangles shown in Fig. 4. The resulting framed link

L has $m = g + 1 + \sum_{j=1}^{n} m_j$ components. By (2) and the identities in Fig. 3 we have

$$\tau(M) = \Delta^{\sigma(L)}\mathcal{D}^{-\sigma(L)-m-1}\sum_{j\in I}\dim(j)^{1-n}\left(\prod_{i=1}^{n}(SG^{\mathcal{C}_i})_{j,0}\right) \times \sum_{i_1,\dots,i_g\in I}\left(\prod_{l=1}^{g}\dim(i_l)\right)F(R(j,i_1,\dots,i_g)),$$

where $R(j,i_1,\dots,i_g)$ is the colored ribbon graph in Fig. 11. The expression (11) now follows by the fact that $\sigma(L) = \sigma_{\mathrm{n}}$, see below, and by Lemma 4.2 together with the identity

$$\sum_{i_1,\dots,i_g\in I}\left(\prod_{l=1}^{g}\dim(i_l)\right)F(R(j,i_1,\dots,i_g)) = v_j^{-b-2g}\dim(j)\kappa(j)^g,$$

which follows by combining Figures 3 and 4.

Let us show that $\sigma(L) = \sigma_{\mathrm{n}}$. The linking matrix of L is given by

$$A = \begin{pmatrix} 0 & w^t & 0 & 0 & 0 & 0 \\ w & -b-2g & e_1 & e_1 & \cdots & e_1 \\ 0 & e_1^t & A_1 & 0 & \cdots & 0 \\ 0 & e_1^t & 0 & A_2 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & e_1^t & 0 & 0 & \cdots & A_n \end{pmatrix},$$

where $w = (-2,-2,\dots,-2)$ is a vector of length g and A_j is given as in the case of oriented base, i.e. $A_j = A(a_{m_j}^{(j)}, a_{m_j-1}^{(j)},\dots,a_1^{(j)})$, see (13). By doing the same combined row and column operations as in the case of oriented base we reduce A to

$$A' = \begin{pmatrix} D & 0 & 0 & \cdots & 0 \\ 0 & A_1' & 0 & \cdots & 0 \\ 0 & 0 & A_2' & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & A_n' \end{pmatrix},$$

where A_j' is a diagonal matrix with signature $\sum_{k=1}^{m_j}\mathrm{sign}(\alpha_k^{\mathcal{C}_j}\beta_k^{\mathcal{C}_j})$, $j = 1,\dots,n$, and $D = \begin{pmatrix} 0 & w^t \\ w & e-2g \end{pmatrix}$, where, as usual, $e = -\left(b + \sum_{j=1}^{n}\beta_j/\alpha_j\right)$ is the Seifert Euler number. By this and the fact that the signature of D is zero it follows that $\sigma(L)$ is equal to σ_{n} in (12).

The RT–invariant τ of the Seifert manifold with non-normalized Seifert invariants $\{\mathrm{n}; g; (\alpha_1, \beta_1), \ldots, (\alpha_n, \beta_n)\}$ is calculated as above by letting b be equal to zero everywhere, since this manifold has a surgery presentation as in Fig. 2 with $-b$ changed to 0. □

The following lemma was first proved by the author in the case, where the modular categories are induced by the quantum groups associated to $\mathfrak{sl}_2(\mathbb{C})$. This was done by a rather long R–matrix calculation. After having presented the result to V. Turaev, he found the proof below using a geometric computation which works for an arbitrary modular category.

Lemma 4.2 *For all $j \in I$,*

$$\kappa(j) = \sum_{i \in I} \dim(i) y(i,j) = \varepsilon_j \mathcal{D}^2 v_j^2 \delta_{j,j^*} \left(\dim(j)\right)^{-1}.$$

Proof Let L_{ij} be the closure of the ribbon tangle T_{ij}. We have the isotopy shown in Fig. 12. Let $\omega_j \colon V_j \to (V_{j^*})^*$ be an isomorphism and use this and its inverse to reverse the orientation of one of the two strings passing through the component with color i in L'_{ij} (the link in the right-hand side of Fig. 12). This enables us to use the identity in Fig. 6, which gives us the identity in Fig. 13. The result now follows by applying Lemma 3.1 together with $F(L_{ij}) = \mathrm{tr}(F(T_{ij})) = y(i,j)\dim(j)$. □

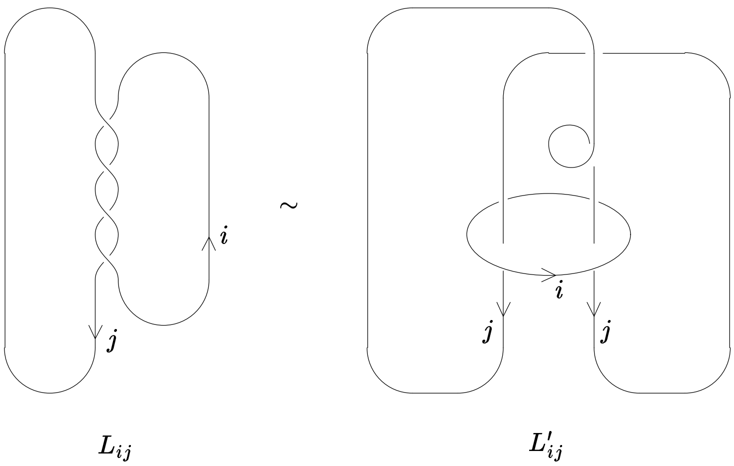

Figure 12: A fundamental isotopy

$$\sum_{i\in I}\dim(i)F(L_{ij}) \;\doteq\; v_j^2\left(\dim(j)\right)^{-1}\mathcal{D}^2\delta_{j,j^*}$$

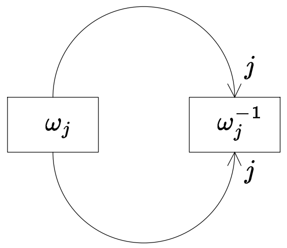

Figure 13

It follows that the lemma is also true in case we don't have a rank if one replaces $\mathcal{D}^2$ with $\sum_{u\in I}(\dim(u))^2$. The result in Lemma 4.2 is independent of how we direct the component with color i in T_{ij}. This follows by the usual argument since we sum over all colors i. If we reverse the direction of the component with color j we get $\kappa(j^*)$ instead of $\kappa(j)$, since the operator invariant F of a colored ribbon graph is unchanged by changing the direction of an annulus component if one at the same time changes the color of that component to the dual color. Observe however that $\kappa(j^*) = \kappa(j)$ since $j^{**} = j$.

Remark 4.3 In this remark we give some alternative expressions for the signatures (10) and (12). Similar formulas have been obtained in [FG] and [J] for the case $g = 0$ (so $\epsilon = \mathrm{o}$) in connection with calculations of framing corrections of Witten's 3–manifold invariants of lens spaces and other Seifert manifolds with base S^2. To this end we use the Rademacher Phi function Φ, which is defined on $PSL(2,\mathbb{Z}) = SL(2,\mathbb{Z})/\{\pm 1\}$ by

$$\Phi\left(\begin{bmatrix} p & r \\ q & s \end{bmatrix}\right) = \begin{cases} \frac{p+s}{q} - 12(\mathrm{sign}(q))\mathrm{s}(s,|q|) & ,q \neq 0 \\ \frac{r}{s} & ,q = 0. \end{cases} \tag{17}$$

Here, for $q > 0$, the Dedekind sum $\mathrm{s}(s,q)$ is given by

$$\mathrm{s}(s,q) = \frac{1}{4q}\sum_{j=1}^{q-1}\cot\frac{\pi j}{q}\cot\frac{\pi s j}{q} \tag{18}$$

for $q > 1$ and $\mathrm{s}(s,1) = 0$, $s \in \mathbb{Z}$. We refer to [RG] for a comprehensive description of this function and also to [KM2] for a detailed account of the presence of the Rademacher Phi function and the related Dedekind sums in topological settings. By [J, Formula (2.20)] we have

$$\sum_{l=1}^{m-1}\mathrm{sign}(\alpha_l^{\mathcal{C}}\beta_l^{\mathcal{C}}) = \frac{1}{3}\left(\sum_{l=1}^{m} a_l - \Phi(B^{\mathcal{C}})\right) \tag{19}$$

for any sequence of integers $\mathcal{C} = (a_1, \ldots, a_m)$. Formula (10) can therefore be changed to

$$\sigma_o = \operatorname{sign}(e) + \sum_{j=1}^{n} \operatorname{sign}(\alpha_j \beta_j) + \frac{1}{3} \sum_{j=1}^{n} \left(\sum_{l=1}^{m_j} a_l^{(j)} - \Phi(B^{\mathcal{C}_j}) \right), \tag{20}$$

where the second sum of course can be put equal to n if we work with normalized Seifert invariants ($\alpha_j > \beta_j > 0$). We can choose the $\mathcal{C}_j$ so that $|a_l^j| \geq 2$ for $l = 1, 2, \ldots, m_j - 1$ and $j = 1, 2, \ldots, n$. In this case we have that $\operatorname{sign}(\alpha_l^{\mathcal{C}_j} \beta_l^{\mathcal{C}_j}) = \operatorname{sign}(a_l^{(j)})$, $l = 1, 2, \ldots, m_j - 1$ and $j = 1, 2, \ldots, n$, so

$$\sigma_o = \operatorname{sign}(e) + \sum_{j=1}^{n} \operatorname{sign}(\alpha_j \beta_j) + \sum_{j=1}^{n} \sum_{l=1}^{m_j - 1} \operatorname{sign}(a_l^{(j)}). \tag{21}$$

The formula (21) generalizes [FG, Formula (2.7)]. The expressions (20) and (21) also hold for the signature σ_n in (12) if we remove the term $\operatorname{sign}(e)$.

We end this section by specializing to lens spaces. Let p, q be coprime integers. The lens space $L(p, q)$ is given by surgery on S^3 along the unknot with surgery coefficient $-p/q$. (Recall here that $L(p, -q)$ is diffeomorphic to $L(p, q)$ via an orientation reversing diffeomorphism.) From this surgery description we can directly calculate the RT–invariants of $L(p, q)$ by using a continued fraction expansion of $-p/q$ as in the proof of Theorem 4.1. We choose instead to calculate the invariants by identifying $L(p, q)$ with certain Seifert fibrations, see the proof below.

In the following corollary we include the possibilities $L(0, 1) = S^1 \times S^2$ and $L(1, q) = S^3$, $q \in \mathbb{Z}$. (Of course we immediately get from (2) that $\tau(S^3) = \mathcal{D}^{-1}$ and $\tau(S^1 \times S^2) = 1$, since S^3 and $S^1 \times S^2$ are given by surgeries on S^3 along the empty framed link and the unknot with framing 0 respectively.)

Corollary 4.4 *Let p, q be a pair of coprime integers and let $(a_1, \ldots, a_{m-1})$ be a continued fraction expansion of $-p/q$ if $q \neq 0$. If $q = 0$ we put $m = 3$ and $a_1 = a_2 = 0$. Then the RT–invariant τ of the lens space $L(p, q)$ is*

$$\tau(L(p, q)) = (\Delta \mathcal{D}^{-1})^{\sigma_o} \mathcal{D}^{-m} G_{0,0}^{\mathcal{C}}, \tag{22}$$

where $\mathcal{C} = (a_1, \ldots, a_{m-1}, 0)$ and

$$\sigma_o = \sum_{l=1}^{m-1} \operatorname{sign}(\alpha_l^{\mathcal{C}} \beta_l^{\mathcal{C}}) = \frac{1}{3} \left(\sum_{l=1}^{m-1} a_l - \Phi(B^{\mathcal{C}}) \right).$$

Proof Lens spaces are Seifert manifolds with base S^2 and zero, one or two exceptional fibers. In fact, let M be the Seifert manifold with non-normalized Seifert invariants $\{o;0;(\alpha_1,\beta_1),(\alpha_2,\beta_2)\}$, and let α_1',β_2' be integers such that $\alpha_2\beta_2'-\beta_2\alpha_2'=1$. By [JN, Theorem 4.4], M is isomorphic to $L(p,q)$ as oriented manifold, where $p=\alpha_1\beta_2+\alpha_2\beta_1$ and $q=\alpha_1\beta_2'+\alpha_2'\beta_1$. In particular $L(p,q)$ is isomorphic (as oriented manifold) to $\{o;0;(|q|,\operatorname{sign}(q)p),(1,0)\}=\{o;0;(|q|,\operatorname{sign}(q)p)\}$, see Remark 4.5 i). If $q=0$ we have $m=3$ and $a_1=a_2=0$ by assumption, so by (1) the right-hand side of (22) is equal to $\mathcal{D}^{-1}$ as it should be. If $q\neq 0$ we have

$$\begin{aligned}\tau(L(p,q)) &= (\Delta\mathcal{D}^{-1})^{\sigma_o}\mathcal{D}^{-m-2}\sum_{j\in I}\dim(j)(SG^{\mathcal{C}})_{j,0}\\ &= (\Delta\mathcal{D}^{-1})^{\sigma_o}\mathcal{D}^{-m-2}(S^2G^{\mathcal{C}})_{0,0}\end{aligned}$$

by Theorem 4.1, since $\mathcal{C}$ is a continued fraction expansion of q/p (also for $p=0$). The formula (22) then follows by (1). The formula for σ_o follows from (10) and (20). □

Remark 4.5 **i)** The manifold $\{o;0;(0,\pm1)\}$ is a Seifert fibration in the extended sense of [JN], see Remark 2.2. [JN, Theorem 4.4] is valid for these more general Seifert fibrations. Note also that it follows from this theorem, that a given lens space can have several distinct Seifert fibered structures.

ii) If $\mathcal{C}$ is chosen so that $|a_j|\geq 2$ for $j=1,\dots,m-1$ (which is possible if $|p/q|>1$ by [J, Lemma 3.1]), then $\sigma_o=\sum_{j=1}^{m-1}\operatorname{sign}(a_j)$ by (21).

iii) In the special case $(p,q)=(n,1)$, $|n|\geq 2$, the above coincides with the result obtained immediately by (2), cf. [Tu, p. 81] (by the conventions used here our $L(n,1)$ is equal to $-L(n,1)$ in [Tu]). The manifold $L(b,1)$, $b\in\mathbb{Z}$, is isomorphic (as oriented manifold) to the Seifert manifold with (normalized) Seifert invariants $(o;0\mid b)$.

5 A rational surgery formula for the Reshetikhin–Turaev invariant

In this section, as in the previous, $(\mathcal{V},\{V_i\}_{i\in I})$ is a fixed modular category with a fixed rank $\mathcal{D}$. We will use notation introduced above Theorem 4.1. Moreover, Φ is the Rademacher function, see (17).

The surgery formula, we are going to derive, concerns rational surgery along framed links in arbitrary closed oriented 3–manifolds. Before giving the result in the general case, let us first consider rational surgery along links in S^3. By using the surgery equivalence described in Fig. 5, the identities in Fig. 3, and the method used in the proof of Theorem 4.1 to calculate signatures we obtain:

Theorem 5.1 *Let L be a link in S^3 with m components and let M be the 3–manifold given by surgery on S^3 along L with surgery coefficient $p_i/q_i \in \mathbb{Q}$ attached to the i'th component, $i = 1,2,\dots,m$ (so we assume $q_i \neq 0$, $i = 1,2,\dots,m$, see the comments to (23)). Moreover, let Ω be a colored ribbon graph in M (also identified with a colored ribbon graph in $S^3 \setminus L$). Let L_0 be L considered as a framed link with all components given the framing 0. Finally, let $\mathcal{C}_i = (a_1^{(i)},\dots,a_{m_i}^{(i)})$ be a continued fraction expansion of p_i/q_i, $i = 1,2,\dots,m$. Then*

$$\begin{aligned}\tau(M,\Omega) &= (\Delta\mathcal{D}^{-1})^{\sigma+\sum_{i=1}^m c_i}\mathcal{D}^{-\sum_{i=1}^m m_i} \\ &\quad\times \sum_{\lambda\in\mathrm{col}(L)} \tau(S^3,\Gamma(L_0,\lambda)\cup\Omega)\left(\prod_{i=1}^m G^{\mathcal{C}_i}_{\lambda(L_i),0}\right),\end{aligned}$$

where $c_i = \frac{1}{3}\left(\sum_{j=1}^{m_i} a_j^{(i)} - \Phi(B^{\mathcal{C}_i})\right)$, $i = 1,\dots,m$, and σ is the signature of the linking matrix of L (with the surgery coefficients $p_1/q_1,\dots,p_m/q_m$ on the diagonal). □

We have used (19). Note that $\tau(S^3,\Gamma(L_0,\lambda)\cup\Omega) = \mathcal{D}^{-1}F(\Gamma(L_0,\lambda)\cup\Omega)$. Theorem 5.1 is a generalization of the defining surgery formula (2). This follows by the facts that if $a\in\mathbb{Z}$, then $\Phi(\Theta^a\Sigma) = a$ and $(T^aS)_{j,0} = v_j^a\dim(j)$.

In the case of surgery on arbitrary closed oriented 3–manifolds along framed links we do not have a preferred framing as above, i.e. we can not identify a framing of a link component with an integer in a canonical way, see Appendix B. Here, by a framed link in a closed oriented 3–manifold M, we mean a pair (L,Q), where $Q = \amalg_{i=1}^m Q_i\colon\ \amalg_{i=1}^m (B^2\times S^1)\to M$ is an embedding (or more precisely an isotopy class of such embeddings) and L is the image by Q of $\amalg_{i=1}^m(0\times S^1)$. For other definitions of framed links in 3–manifolds and how these relate to this definition we refer to Appendix B. To establish a surgery formula as above in this more general setting we will need the machinery of the TQFT of Reshetikhin and Turaev. We have to be precise with orientations because the TQFT–calculations are sensitive to these orientations. We will use the following conventions.

Conventions 5.2 The space $B^2\times S^1$ is the standard solid torus in $\mathbb{R}^3$ with the orientation induced by the standard right-handed orientation of $\mathbb{R}^3$. Here

S^1 is the standard unit circle in the xz–plane with centre 0 and oriented counterclockwise, i.e., e_3 is a positively oriented tangent vector in the tangent space $T_{e_1}S^1 \subseteq \mathbb{R}^3$, e_i being the i'th standard unit vector in $\mathbb{R}^3$, see Fig. 14. For a framed link (L,Q) as above we will always assume that each copy of $B^2 \times S^1$ is this oriented standard solid torus, and that Q is orientation preserving after giving the image of Q the orientation induced by that of M (we can always obtain this by composing some of the Q_i by $g \times \mathrm{id}_{S^1}$ if necessarily, where $g\colon B^2 \to B^2$ is an orientation reversing homeomorphism). Moreover, we orient L so that Q_i restricted to $S^1 \times \{0\}$ is orientation preserving for each i. The oriented meridian α and longitude β, see Fig. 14, represent a basis (over Λ) of $H_1(\Sigma_{(1;)};\Lambda) = \Lambda \oplus \Lambda$, $\Lambda = \mathbb{Z},\mathbb{R}$, $\Sigma_{(1;)} = S^1 \times S^1$. (For the notation $\Sigma_{(1;)}$, see Sect. 5.1.) We identify elements of $H_1(\Sigma_{(1;)};\Lambda)$ with 2–columns via $x[\alpha]+y[\beta] \longleftrightarrow \begin{pmatrix} x \\ y \end{pmatrix}$. The endomorphisms of $H_1(\Sigma_{(1;)};\Lambda)$ are identified with 2x2–matrices with entries in Λ acting on the 2–columns by multiplication on the left.

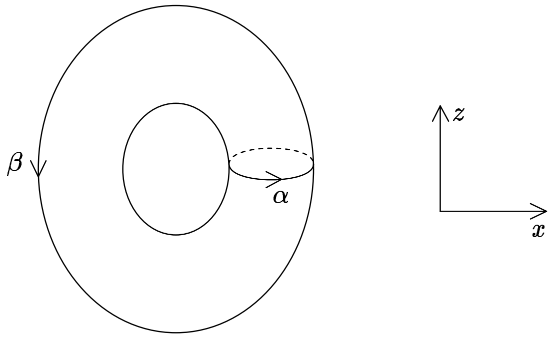

Figure 14

Let us recall the notion of rational surgery on M along (L,Q). Therefore, let $U_i = Q_i(B^2 \times S^1)$ and let $l_i = Q_i(e_1 \times S^1)$ oriented so that $[l_i] = [L_i]$ in $H_1(U_i;\mathbb{Z})$ where $L_i = Q_i(0 \times S^1)$. Moreover, let $\mu_i = Q_i(\partial B^2 \times 1)$ oriented so that $(\partial Q_i)_*([\alpha]) = [\mu_i]$ in $H_1(\partial U_i;\mathbb{Z})$, where ∂Q_i is the restriction of Q_i to $\partial B^2 \times S^1 = \Sigma_{(1;)}$. Let (p_i,q_i) be pairs of coprime integers, let $h_i\colon \partial U_i \to \partial U_i$ be homeomorphisms such that

$$(h_i)_*([\mu_i]) = \pm(p_i[\mu_i] + q_i[l_i]) \tag{23}$$

in $H_1(\partial U_i;\mathbb{Z})$, let h be the union of the h_i, and let $U = \amalg_{i=1}^m U_i$ be the image of Q. Then the 3–manifold $M' = (M \setminus \mathrm{int}(U)) \cup_h U$ is said to be the result of doing surgery on M along the framed link (L,Q) with surgery coefficients

$\{p_i/q_i\}_{i=1}^m$. If $q_i = 0$ so $p_i = \pm 1$ we just write ∞ for p_i/q_i. Such surgeries do not change the manifold (up to an orientation preserving homeomorphism). If, in (23), $p_i = 0$ and $q_i = \pm 1$ for all i, i.e. all surgery coefficients are 0, then we call M' the result of doing surgery on M along the framed link (L, Q). We equip M' with the unique orientation extending the orientation in $M \setminus \mathrm{int}(U)$. The above generalizes ordinary rational surgery along links in S^3, see Appendix B. We call a homeomorphism h satisfying (23) an *attaching map* for the surgery. We can and will always choose an orientation preserving attaching map. Up to an orientation preserving homeomorphism the result of doing surgery on M along the framed link (L, Q) with surgery coefficients $\{p_i/q_i\}_{i=1}^m$ is well defined, independent of the choices of representative Q and attaching map h.

For $\lambda \in \mathrm{col}(L)$ we let $\Gamma(L,\lambda) = \cup_{i=1}^m \Gamma(L_i, \lambda(L_i))$, where $\Gamma(L_i, j)$ is the colored ribbon graph equal to the directed annulus $Q_i(([-1/2, 1/2] \times 0) \times S^1)$ with oriented core L_i and color V_j, $j \in I$.

Theorem 5.3 *Let $\mathcal{C}_i = (a_1^{(i)}, \dots, a_{m_i}^{(i)}) \in \mathbb{Z}^{m_i}$ be a continued fraction expansion of p_i/q_i, $i = 1, \dots, m$. Moreover let Ω be a colored ribbon graph in M' (also identified with a colored ribbon graph in $M \setminus L$). Then*

$$\begin{aligned}\tau(M', \Omega) &= (\Delta \mathcal{D}^{-1})^{\mu + \sum_{i=1}^m c_i} \mathcal{D}^{-\sum_{i=1}^m m_i} \\ &\quad \times \sum_{\lambda \in \mathrm{col}(L)} \tau(M, \Gamma(L,\lambda) \cup \Omega) \left(\prod_{i=1}^m G^{\mathcal{C}_i}_{\lambda(L_i),0} \right),\end{aligned}$$

where μ is a sum of signs given by (33) and $c_i = \frac{1}{3}\left(\sum_{j=1}^{m_i} a_j^{(i)} - \Phi(B^{\mathcal{C}_i}) \right)$, $i = 1, \dots, m$.

This theorem obviously generalizes Theorem 5.1. Theorem 5.3 follows by Lemma 5.4 and Lemma 5.5 below. To prove these lemmas we use the machinery of the $2+1$–dimensional TQFT $(\tau, \mathcal{T})$ of Reshetikhin and Turaev, see [Tu, Chap. II and IV].

5.1 The TQFT $(\tau, \mathcal{T})$

The modular functor $\mathcal{T}$ for the TQFT $(\tau, \mathcal{T})$ is a functor from parametrized decorated surfaces (see below) to the category of finitely generated projective K–modules. Decorated surfaces will be denoted d-surfaces in the following. We begin by recalling the concepts and notation from [Tu] needed.

A *decorated type* or just a type is a tuple $t = (g; (W_1, \nu_1), \dots, (W_m, \nu_m))$, where g is a non-negative integer, $W_1, \dots, W_m$ are objects of $\mathcal{V}$, and $\nu_1, \dots, \nu_m \in$

$\{\pm 1\}$. The number m of pairs (W_j, ν_j) is allowed to be zero. For a type t as above we let

$$\Psi_t = \bigoplus_{i \in I^g} \mathrm{Hom}(\mathbb{I}, \Phi(t;i)), \tag{24}$$

where $\Phi(t;i) = W_1^{\nu_1} \otimes W_2^{\nu_2} \otimes \ldots \otimes W_m^{\nu_m} \otimes \bigotimes_{r=1}^{g}(V_{i_r} \otimes V_{i_r}^*)$ for every $i = (i_1, \ldots, i_g) \in I^g$. Here $W^{+1} = W$ and $W^{-1} = W^*$. Note that Ψ_t is a finitely generated projective K–module as a finite direct sum of such modules, cf. [Tu, Lemma II.4.2.1].

A connected *d-surface* is a connected closed oriented surface Σ of genus g with $m \geq 0$ distinguished ordered and oriented arcs $\gamma_1, \ldots, \gamma_m$, such that γ_j is marked with a pair (W_j, ν_j), where W_j is an object of $\mathcal{V}$ and $\nu_j \in \{\pm 1\}$, $j = 1, 2, \ldots, m$. The tuple $t(\Sigma) = (g; (W_1, \nu_1), \ldots, (W_m, \nu_m))$ is called the *type* of the d-surface. A non-connected closed oriented surface is said to be decorated if its connected components are decorated. A *d-homeomorphism of d-surfaces* is an orientation preserving homeomorphism of the underlying surfaces preserving the distinguished arcs together with their orientations, marks, and order (on each component).

For every type $t = (g; (W_1, \nu_1), \ldots, (W_m, \nu_m))$ there is a certain *standard d-surface* of type t, denoted Σ_t, which is the boundary of an oriented handlebody U_t of genus g with a certain partially colored ribbon graph R_t sitting inside, see [Tu, Sect. IV.1.2]. In particular, $\Sigma_{(1;)} = S^1 \times S^1$ is an ordinary oriented torus, see Fig. 15. The ribbon graph $R_{(1;)}$ lies in the interior of $U_{(1;)}$ and consists of an uncolored coupon with a cap-like uncolored, untwisted, and directed band attached to its top base. A *non-connected standard d-surface* is a disjoint union of a finite number of connected standard d-surfaces.

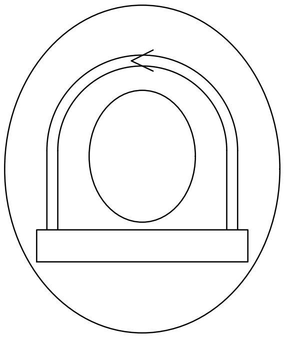

Figure 15: Projection of the standard handlebody $U_{(1;)}$

In the proof of Theorem 5.3 we will only need the standard surface $\Sigma_{(1;)}$. For this proof one can therefore ignore everything about the decoration with destinguished marked arcs. However, in Sect. 6 we will need such decorations. To avoid saying things twice, we continue by presenting the concepts using arbitrary types.

A connected *parametrized d-surface* is a connected d-surface Σ together with a d-homeomorphism $\Sigma_t \to \Sigma$ called the parametrization of Σ, where $t = t(\Sigma)$ is the type of Σ. A *non-connected parametrized d-surface* is defined similarly by using non-connected standard d-surfaces. A morphism in the category of parametrized d-surfaces, denoted a *d-morphism*, is a d-homeomorphism commuting with the parametrizations.

For a connected parametrized d-surface Σ of type t we have $\mathcal{T}(\Sigma) = \Psi_t$. If Σ is a non-connected parametrized d-surface with components $\Sigma_1, \dots, \Sigma_n$, then $\mathcal{T}(\Sigma)$ is equal to the non-ordered tensor product of the Ψ_{t_j}, $j = 1, \dots, n$, where t_j is the type of Σ_j. Moreover $\mathcal{T}(\emptyset) = K$. The modular functor $\mathcal{T}$ assigns the identity endomorphism to any d-morphism. By [Tu, Lemma IV.1.4.1], $\mathcal{T}$ is a modular functor in the sense of [Tu, Sect. III.1.2].

A *decorated* 3*–manifold* is a compact oriented 3–manifold M with parametrized decorated boundary ∂M and with an embedded colored ribbon graph Ω, which is compatible with the decoration of ∂M, see [Tu, p. 157]. In this paper we will only meet decorated 3–manifolds with empty boundary or boundary equal to a torus of type $(1;)$. In general, if ∂M contains no distinguished marked arcs, then Ω is a colored ribbon graph in the interior of M with all bases of bands lying on bases of coupons. A *d-homeomorphism of decorated* 3*–manifolds* is an orientation preserving homeomorphism of the underlying oriented 3–manifolds preserving all additional structure such as the decoration of the boundaries and the colored ribbon graphs. Such a d-homeomorphism restricts to a d-morphism of the boundaries.

A *decorated* 3*–cobordism* is a triple $(M, \partial_- M, \partial_+ M)$, where $\partial_- M$ and $\partial_+ M$ (denoted the bottom and top base respectively) are parametrized d-surfaces and M is a decorated 3–manifold with boundary $\partial M = (-\partial_- M) \amalg \partial_+ M$. Here $-N$ denotes the manifold N with the opposite orientation, where N is an oriented manifold. (To be precise $-$ is an involution in the space-structure of parametrized d-surfaces, see [Tu, Sect. IV.1.3 and Sect. III.1.1].) A *d-homeomorphism of decorated* 3*–cobordisms* is a d-homeomorphism of the underlying decorated 3–manifolds which preserves the bases.

A K–homomorphism $\tau(M) = \tau(M, \partial_- M, \partial_+ M)\colon\ \mathcal{T}(\partial_- M) \to \mathcal{T}(\partial_+ M)$ is constructed in [Tu, Sect. IV.1.8] making $(\tau, \mathcal{T})$ a topological quantum field theory

(TQFT) based on decorated 3–cobordisms and parametrized d-surfaces in the sense of [Tu, Sect. III.1.4], cf. [Tu, Theorem IV.1.9]. If $\partial_- M = \emptyset$, then $\tau(M)$ is determined by the element $\tau(M)(1_K) \in \mathcal{T}(\partial_+ M)$, and it is common practise in this case to identify $\tau(M)$ with this element. If M is closed with an embedded colored ribbon graph Ω, then $\tau(M)$ is the RT–invariant of the pair (M,Ω) as defined in (2). The map τ is called the *operator invariant* of decorated 3–cobordisms.

5.2 Gluing anomalies in the TQFT $(\tau, \mathcal{T})$

The TQFT $(\tau,\mathcal{T})$ has so-called (gluing) anomalies, see [Tu, Sect. III.1.4 and Sect. IV.4]. (There is a way to get rid of these anomalies by changing the TQFT sligthly, cf. [Tu, Sect. IV.9]. However from a computational point of view this 'killing' of anomalies does not make things easier.) To describe these anomalies we need some concepts from the theory of symplectic vector spaces.

If H_1 and H_2 are non-degenerate symplectic vector spaces, then a *Lagrangian relation* between H_1 and H_2 is a Lagrangian subspace of $(-H_1) \oplus H_2$. For a Lagrangian relation $N \subseteq (-H_1) \oplus H_2$ we write $N \colon H_1 \Longrightarrow H_2$. Let H_1, H_2 be non-degenerate symplectic vector spaces, and let $\Lambda(H_i)$ be the set of Lagrangian subspaces of H_i, $i = 1, 2$. A Lagrangian relation $N \colon H_1 \Longrightarrow H_2$ induces two mappings $N_* \colon \Lambda(H_1) \to \Lambda(H_2)$ and $N^* \colon \Lambda(H_2) \to \Lambda(H_1)$ given by

$$N_*(\lambda) = \{h_2 \in H_2 \mid \exists h_1 \in \lambda : (h_1,h_2) \in N\}$$

for $\lambda \in \Lambda(H_1)$ and

$$N^*(\lambda) = \{h_1 \in H_1 \mid \exists h_2 \in \lambda : (h_1,h_2) \in N\}$$

for $\lambda \in \Lambda(H_2)$. If $f \colon H_1 \to H_2$ is a symplectic isomorphism and $\lambda_i \in \Lambda(H_i)$, $i = 1, 2$, then $(N_f)_*(\lambda_1) = f(\lambda_1)$ and $(N_f)^*(\lambda_2) = f^{-1}(\lambda_2)$, where N_f is the graph of f.

For Lagrangian subspaces $\lambda_1, \lambda_2, \lambda_3$ of a symplectic vector space (H,ω), let $W = (\lambda_1 + \lambda_2) \cap \lambda_3$ and let $\langle .\,,.\rangle$ be the bilinear form on W defined by

$$\langle a,b\rangle = \omega(a_2,b) \tag{25}$$

for $a,b \in W$ with $a = a_1 + a_2$, $a_i \in \lambda_i$. This is a well-defined symmetric form, see e.g. [Tu, Sect. IV.3.5]. The *Maslov index* $\mu(\lambda_1,\lambda_2,\lambda_3) \in \mathbb{Z}$ is the signature of this bilinear form. It is invariant under cyclic permutations of the triple $(\lambda_1,\lambda_2,\lambda_3)$ and changes sign if we exchange λ_i and λ_j, $i \neq j$.

If Σ is a closed oriented surface, the real vector space $H_1(\Sigma;\mathbb{R})$ together with the intersection pairing

$$H_1(\Sigma;\mathbb{R}) \times H_1(\Sigma;\mathbb{R}) \to \mathbb{R} \tag{26}$$

is a non-degenerate symplectic vector space. For $\Sigma = \emptyset$ we let $H_1(\Sigma;\mathbb{R}) = 0$. For a parametrized d-surface Σ there is a certain Lagrangian subspace $\lambda(\Sigma) \subseteq H_1(\Sigma;\mathbb{R})$. For the standard d-surface Σ_t of type t, $\lambda_t = \lambda(\Sigma_t)$ is the kernel of the inclusion homomorphism $H_1(\Sigma_t;\mathbb{R}) \to H_1(U_t;\mathbb{R})$. For any connected parametrized d-surface Σ, $\lambda(\Sigma) = f_*(\lambda_t)$ where $f\colon \Sigma_t \to \Sigma$ is the parametrization. For a non-connected parametrized d-surface Σ, $\lambda(\Sigma)$ is the subspace of $H_1(\Sigma;\mathbb{R})$ generated by the Lagrangian subspaces of the connected components.

For any decorated 3–cobordism $(M, \partial_- M, \partial_+ M)$ we have

$$H_1(\partial M;\mathbb{R}) = (-H_1(\partial_- M;\mathbb{R})) \oplus H_1(\partial_+ M;\mathbb{R}),$$

and the kernel of the inclusion homomorphism $H_1(\partial M;\mathbb{R}) \to H_1(M;\mathbb{R})$ yields a Lagrangian relation $H_1(\partial_- M;\mathbb{R}) \Longrightarrow H_1(\partial_+ M;\mathbb{R})$ which is denoted $N(M)$. (Note that $N(M)$ does not depend on the parametrizations and marks of $\partial_\pm M$ and the colored ribbon graph in M.) We let $\lambda_-(M) = \lambda(\partial_- M)$ and $\lambda_+(M) = \lambda(\partial_+ M)$.

The anomalies of the TQFT $(\tau, \mathcal{T})$ are calculated in [Tu, Theorem IV.4.3]: Let $M = M_2 M_1$ be a decorated 3–cobordism obtained from decorated 3–cobordisms M_1 and M_2 by gluing along a d-morphism $p\colon \partial_+(M_1) \to \partial_-(M_2)$. Set

$$N_r = N(M_r)\colon H_1(\partial_-(M_r);\mathbb{R}) \Longrightarrow H_1(\partial_+(M_r);\mathbb{R})$$

for $r = 1, 2$. Then

$$\tau(M) = (\mathcal{D}\Delta^{-1})^m \tau(M_2)\tau(M_1) \tag{27}$$

with $m = \mu(p_*(N_1)_*(\lambda_-(M_1)), \lambda_-(M_2), N_2^*(\lambda_+(M_2)))$. If $\partial_- M_1 = \partial_+ M_2 = \emptyset$, then

$$m = \mu(p_*(N(M_1)), \lambda(-\partial M_2), N(M_2)), \tag{28}$$

a Maslov index for Lagrangian subspaces of $H_1(-\partial M_2;\mathbb{R}) = -H_1(\partial M_2;\mathbb{R})$. These anomalies do not depend on the colored ribbon graphs inside the decorated 3–cobordisms. By definition, $\mathcal{T}(p)\colon \mathcal{T}(\partial_+(M_1)) \to \mathcal{T}(\partial_-(M_2))$ is the identity and is therefore left out in (27).

5.3 The projective actions of the modular groups

We will need to know how the operator invariant τ of decorated 3–cobordisms changes when changing the parametrizations of the parametrized boundary d-surfaces. To this end we need the projective action of the modular group Mod_t, t a decorated type, cf. [Tu, Sect. IV.5]. Here Mod_t is the group of isotopy classes of d-homeomorphisms $\Sigma_t \to \Sigma_t$. For $t = (g;)$, $\mathrm{Mod}_t = \mathrm{Mod}_g$ is the usual modular group of genus g. In this paper we only consider the modular group Mod_1 of genus 1. The reader can therefore concentrate on this case if he/she prefers that. We will however state the following results using arbitrary types since it is not any longer. For a decorated type t, let $\Sigma = \Sigma_t$ and let $M(\mathrm{id}) = (\Sigma \times [0,1], \Sigma, \Sigma)$, where Σ is parametrized by the identity and $\Sigma \times [0,1]$ is the standard decorated cylinder over Σ, cf. [Tu, p. 158]. For $t = (1;)$ this is just an ordinary oriented cylinder cobordism without any ribbon graph inside (and without any marked arcs on the boundary tori, but with identity parametrizations attached to these tori). For an arbitrary d-homeomorphism $g\colon \Sigma \to \Sigma$, let $M(g)$ be as the decorated 3–cobordism $M(\mathrm{id})$ except that the bottom base is parametrized by g. Let

$$\epsilon(g) = \tau(M(g))\colon\ \Psi_t \to \Psi_t.$$

Then $g \mapsto \epsilon(g)$ is a projective linear action of Mod_t on $\mathcal{T}(\Sigma_t) = \Psi_t$. In fact we have

$$\epsilon(gh) = (\mathcal{D}\Delta^{-1})^{\mu(h_*(\lambda_t),\lambda_t,g_*^{-1}(\lambda_t))}\epsilon(g)\epsilon(h), \tag{29}$$

cf. [Tu, Formula (IV.5.1.a)]. By the axioms of a TQFT $\epsilon(\mathrm{id}) = \mathrm{id}$, so $\epsilon(g^{-1}) = (\epsilon(g))^{-1}$ by (29).

We use the action ϵ to describe the dependency of the operator invariant τ on the choice of parametrizations of bases. Let $(M, \partial_- M, \partial_+ M)$ be a decorated 3–cobordism with parametrizations $f_\pm\colon \Sigma_{t_\pm} \to \partial_\pm M$. Let $g_\pm\colon \Sigma_{t_\pm} \to \Sigma_{t_\pm}$ be d-homeomorphisms. Provide $\partial_- M$ and $\partial_+ M$ with the structure of parametrized d-surfaces via $f'_- = f_-(g_-)^{-1}\colon \Sigma_{t_-} \to \partial_- M$ and $f'_+ = f_+(g_+)^{-1}\colon \Sigma_{t_+} \to \partial_+ M$. Denote the resulting parametrized d-surfaces by $\partial'_- M$ and $\partial'_+ M$ respectively. These are the same oriented surfaces as $\partial_- M$, $\partial_+ M$ with the same (sets of totally ordered) distinguished marked arcs but with different parametrizations. The cobordism M with the newly parametrized bases is a decorated 3–cobordism, say M', between $\partial_-(M') = \partial'_- M$ and $\partial_+(M') = \partial'_+ M$. By definition of the modular functor $\mathcal{T}$ we have $\mathcal{T}(\partial_\pm M) = \mathcal{T}(\Sigma_{t_\pm}) = \mathcal{T}(\partial_\pm M')$, and by [Tu, Formula (IV.5.3.b)] we have

$$\tau(M')\epsilon(g_-) = (\mathcal{D}\Delta^{-1})^{\mu_+ - \mu_-}\epsilon(g_+)\tau(M)\colon\ \mathcal{T}(\Sigma_{t_-}) \to \mathcal{T}(\Sigma_{t_+}), \tag{30}$$

where, with $N = N(M) = N(M')$,

$$\begin{aligned}\mu_+ &= \mu(N_*(\lambda_-(M)), \lambda_+(M), \lambda_+(M')),\\ \mu_- &= \mu(\lambda_-(M), \lambda_-(M'), N^*(\lambda_+(M'))).\end{aligned}$$

5.4 The proof of Theorem 5.3

The standard handlebody $U_{(1;)}$ is a solid torus with a uncolored ribbon graph $R_{(1;)}$ inside consisting of one coupon and one band, see Fig. 15. By coloring the band with V_i and the coupon with $b_i = b_{V_i}$, $i \in I$, where b is part of the duality of $\mathcal{V}$, we get a decorated 3–manifold which is the oriented standard solid torus $B^2 \times S^1$ with a directed untwisted annulus with oriented core $0 \times S^1$ and color V_i. Let Y_i be this decorated 3–manifold considered as a decorated 3–cobordism between the empty surface and $\partial U_{(1;)} = \Sigma_{(1;)}$, where $\Sigma_{(1;)}$ is parametrized by the identity. By [Tu, Lemma IV.2.1.3] we have

$$\tau(Y_i) = b_i. \tag{31}$$

Let Ω be a colored ribbon graph in S^3 containing an annulus component or a band of color $\mathbb{I}$, and let Ω' be the colored ribbon graph obtained from Ω by eliminating this annulus (resp. band). Then it is a well-known fact that $F(\Omega') = F(\Omega)$, cf. [Tu, Exercise I.2.9.2]. Now let Y be the decorated 3–cobordism $(B^2 \times S^1, \emptyset, \Sigma_{(1;)})$ with the empty ribbon graph inside and with $\Sigma_{(1;)}$ parametrized by the identity. Then

$$\tau(Y) = \tau(Y_0) = b_0. \tag{32}$$

The first equality follows by the just mentioned fact about F together with the technique of presenting 3–cobordisms by ribbon graphs in $\mathbb{R}^3$, see [Tu, Sect. IV.2] and in particular [Tu, Formula (IV.2.3.a)]. (Alternatively, (32) follows directly from the definition of $\tau(Y)$, cf. [Tu, p. 160].)

By (24) and the definition of $\mathcal{T}$,

$$\mathcal{T}(\Sigma_{(1;)}) = \Psi_{(1;)} = \bigoplus_{i\in I} \mathrm{Hom}(\mathbb{I}, V_i \otimes V_i^*)$$

is a free K–module of rank card(I) with basis $\{b_i \colon \mathbb{I} \to V_i \otimes V_i^*\}_{i\in I}$ ($\mathrm{Hom}(\mathbb{I}, V_i \otimes V_i^*) \cong \mathrm{Hom}(V_i, V_i) \cong K$ since V_i is a simple object).

Lemma 5.4 *Let the situation be as in* Theorem 5.3. *Let $g_i \colon \Sigma_{(1;)} \to \Sigma_{(1;)}$ be a homeomorphism such that $(g_i)_* \colon H_1(\Sigma_{(1;)}, \mathbb{Z}) \to H_1(\Sigma_{(1;)}, \mathbb{Z})$ has matrix $B^{\mathcal{C}_i}$*

with respect to the basis $\{[\alpha],[\beta]\}$, $i=1,\dots,m$. *Then*

$$\tau(M',\Omega)=(\Delta\mathcal{D}^{-1})^{\mu}\sum_{\lambda\in\mathrm{col}(L)}\tau(M,\Gamma(L,\lambda)\cup\Omega)\left(\prod_{i=1}^{m}A^{(i)}_{\lambda(L_i),0}\right),$$

where $A^{(i)}=\left(A^{(i)}_{k,l}\right)_{k,l\in I}$ *is the matrix of* $\epsilon(g_i)\colon\ \Psi_{(1;)}\to\Psi_{(1;)}$ *with respect to the basis* $\{b_i\}_{i\in I}$. *The integer* μ *is given by the sum of Maslov indices*

$$\mu=\sum_{i=1}^{m}\mu((\partial Q_i)_*(\lambda_{(1;)}),(\partial Q_i\circ g_i)_*(\lambda_{(1;)}),N(X_i)),\tag{33}$$

where $X_i=(M_{i-1}\setminus\mathrm{int}(U_i),\partial U_i,\emptyset)$. *Here* M_i *is the manifold obtained by doing surgery on* M *along* $\left(\amalg_{j=1}^{i}L_j,\amalg_{j=1}^{i}Q_j\right)$ *with surgery coefficients* $\{p_j/q_j\}_{j=1}^{i}$, $i=1,2,\dots,m$, *and* $M_0=M$.

The $N(X_i)$ are here subspaces of the $H_1(\partial U_i;\mathbb{R})$. The integer μ in (33) does not depend on the colored ribbon graph Ω. Moreover μ is independent of the choice of the g_i since $(g_i)_*(\lambda_{(1;)})=\mathrm{Span}_{\mathbb{R}}\{p_i[\alpha]+q_i[\beta]\}$.

Proof Let $h_i:\partial U_i\to\partial U_i$ be the orientation preserving homeomorphisms determined by the commutative diagrams

$$\begin{tikzcd}
\Sigma_{(1;)} \arrow[r, "\partial Q_i"] \arrow[d, "g_i"'] & \partial U_i \arrow[d, "h_i"] \\
\Sigma_{(1;)} \arrow[r, "\partial Q_i"] & \partial U_i.
\end{tikzcd}$$

The disjoint union h of the h_i is an attaching map for the surgery considered in Theorem 5.3. According to the axioms for a TQFT (actually a cobordism theory), see [Tu, Sect. III.1.3], we can perform this surgery by consecutive gluings of the U_i to the corresponding boundary components in $M\setminus\mathrm{int}(U)$ along $h_i\colon\ \partial U_i\to\partial U_i\subseteq M\setminus\mathrm{int}(U)$, and we see that the general result follows from the case $m=1$. Therefore, assume $m=1$ and let $g=g_1$. Denote by X the decorated 3–cobordism $(M\setminus\mathrm{int}(U),\partial U,\emptyset)$, where $\partial U=-\partial(M\setminus\mathrm{int}(U))$ is parametrized by ∂Q. Denote by X' the decorated 3–cobordism equal to X, except that the base is parametrized by $\partial Q\circ g$. We identify U with the decorated 3–cobordism $(U,\emptyset,\partial U)$ with the empty ribbon graph, where ∂U is parametrized by ∂Q. Then $h\colon\ \partial_+U\to\partial_-X'$ is a d-morphism of parametrized d-surfaces and

$$\tau(M',\Omega)=k^{m_1}\tau(X')\tau(U)$$

by (27), where $k = \mathcal{D}\Delta^{-1}$ and m_1 is determined by (28). By (30) we get $\tau(X')\epsilon(g^{-1}) = k^{-\mu_-}\tau(X)$, and by the remarks following (29) we have $\epsilon(g^{-1}) = \epsilon(g)^{-1}$, so

$$\tau(M',\Omega) = k^{m_1-\mu_-}\tau(X)\epsilon(g)\tau(U).$$

Let Y be the decorated 3–cobordism in (32). By Conventions 5.2, $Q\colon Y \to U$ is a d-homeomorphism, so $\tau(U) = \tau(Y)$ by the axioms for a TQFT. Since $\tau(X)$ is K–linear we therefore have (use also (32))

$$\tau(M',\Omega) = k^{m_1-\mu_-}\tau(X)\epsilon(g)b_0 = k^{m_1-\mu_-}\sum_{j\in I} A_{j,0}\tau(X)b_j,$$

where $A = (A_{i,j})_{i,j\in I}$ is the matrix of $\epsilon(g)$ with respect to the basis $\{b_j\}_{j\in I}$. The set $\mathrm{col}(L)$ is identified with I since L has only one component. For $j \in I$ we let $U_j = ((U,\Gamma(L,j)),\emptyset,\partial U)$ be the decorated 3–cobordism identical with U, except that U_j has the colored ribbon graph $\Gamma(L,j)$ sitting inside. The pair $(M,\Gamma(L,j)\cup\Omega)$ can be obtained by gluing of U_j to X along $\mathrm{id}_{\partial U}\colon \partial_+U_j = \partial U \to \partial U = \partial_- X$ (which is a d-morphism). By (27) we therefore get

$$\tau(M,\Gamma(L,j)\cup\Omega) = k^{m_2}\tau(X)\tau(U_j),$$

where the integer $m_2 = \mu((N(U_j)),\lambda(-\partial X),N(X))$ by (28). Here $N(U_j) = N(U)$ since the Lagrangian relation of a decorated 3–cobordism does not depend on the colored ribbon graph sitting inside. By the commutative diagram

$$\begin{tikzcd}
\Sigma_{(1;)} \arrow[r, "i"] \arrow[d, "\partial Q"'] & U_{(1;)} \arrow[d, "Q"] \\
\partial U \arrow[r, "j"] & U,
\end{tikzcd}$$

i and j being inclusions, we get $N(U) = \ker(j_* : H_1(\partial U;\mathbb{R}) \to H_1(U;\mathbb{R})) = \partial Q_*(\lambda)$, where $\lambda = \lambda_{(1;)}$. Moreover, $\lambda(-\partial X) = \lambda(\partial U) = \partial Q_*(\lambda)$, so $m_2 = 0$. We have $\tau(U_j) = \tau(Y_j)$ since $Q\colon Y_j \to U_j$ is a d-homeomorphism of decorated 3–cobordisms by Conventions 5.2. By (31) we therefore get

$$\tau(M',\Omega) = k^{m_1-\mu_-}\sum_{j\in I}\tau(M,\Gamma(L,j)\cup\Omega)A_{j,0}.$$

From (30) we have $\mu_- = \mu(\lambda_-(X),\lambda_-(X'),N(X))$ since $\lambda_+(X') = 0$. Here $\lambda_-(X) = \lambda(\partial U) = \partial Q_*(\lambda)$ and $\lambda_-(X') = (\partial Q\circ g)_*(\lambda)$, so

$$\mu_- = \mu(\partial Q_*(\lambda),(\partial Q\circ g)_*(\lambda),N(X)).$$

By (28), $m_1 = \mu(h_*(N(U)),\lambda(-\partial X'),N(X'))$. Here $N(U) = \partial Q_*(\lambda)$, see above, so $h_*(N(U)) = (h\circ\partial Q)_*(\lambda) = (\partial Q\circ g)_*(\lambda)$. Moreover, $\lambda(-\partial X') = (\partial Q\circ g)_*(\lambda)$, so $m_1 = 0$. $\square$

To express the matrices $A^{(i)}$ in terms of the $S-$ and $T-$matrices we use the description of $\epsilon\colon \Psi_{(1;)} \to \Psi_{(1;)}$ given in [Tu, Sect. IV.5.4]. We have an isomorphism $[f] \mapsto M(f)\colon \mathrm{Mod}_1 \to SL(2,\mathbb{Z})$, where $[f]$ is the isotopy class represented by $f\colon \Sigma_{(1;)} \to \Sigma_{(1;)}$ and $M(f)$ is the matrix of the induced automorphism on 1–homologies $f_*\colon H_1(\Sigma_{(1;)};\mathbb{Z}) \to H_1(\Sigma_{(1;)};\mathbb{Z})$ with respect to the basis $\{[\alpha],[\beta]\}$, see Conventions 5.2. Let $[f_A]$ be the element in Mod_1 corresponding to the matrix $A \in SL(2,\mathbb{Z})$ under this isomorphism and let Θ and Ξ be the generators of $SL(2,\mathbb{Z})$ given in (6). By [Tu, pp. 193-195], the matrices of the K–module automorphisms $\epsilon(f_\Xi), \epsilon(f_\Theta)\colon \Psi_{(1;)} \to \Psi_{(1;)}$ with respect to the basis $\{b_i\}_{i\in I}$ are given by $\mathcal{D}S^{-1}$ and T respectively. (Note here that Ξ and Θ correspond to respectively s and t in [Tu]. Moreover our basis $\{[\alpha],[\beta]\}$ corresponds to $\{-[\alpha],[\beta]\}$ in [Tu, Fig. IV.5.1].)

Lemma 5.5 *Let $\mathcal{C} = (a_1,\dots,a_n) \in \mathbb{Z}^n$ and let $g = f_{B^\mathcal{C}} = f_\Theta^{a_n} f_\Xi f_\Theta^{a_{n-1}} f_\Xi \cdots f_\Theta^{a_1} f_\Xi$. The matrix of $\epsilon(g) : \Psi_{(1;)} \to \Psi_{(1;)}$ with respect to the basis $\{b_i\}_{i\in I}$ is given by*

$$G = \mathcal{D}^{-n}(\Delta\mathcal{D}^{-1})^m G^\mathcal{C} S^{-1}\hat{S},$$

where $\hat{S} = S$ if n is even and $\hat{S} = \bar{S}$ if n is odd. Here $\bar{S}$ is the S–matrix for the mirror of $\mathcal{V}$. Moreover, $m = \sum_{i=1}^{n-1} \mathrm{sign}(\alpha_i^\mathcal{C}\beta_i^\mathcal{C}) = \frac{1}{3}\left(\sum_{i=1}^n a_i - \Phi(B^\mathcal{C})\right)$.

Proof Let $h\colon \Sigma_{(1;)} \to \Sigma_{(1;)}$ be an arbitrary orientation preserving diffeomorphism, and let $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z})$ be the matrix of the induced automorphism $h_*\colon H_1(\Sigma_{(1;)};\mathbb{Z}) \to H_1(\Sigma_{(1;)};\mathbb{Z})$ with respect to the basis $\{[\alpha],[\beta]\}$. We have $\lambda_1 = \lambda_{(1;)} = \mathrm{Span}_\mathbb{R}\{[\alpha]\}$. Therefore $(f_\Theta)_*(\lambda_1) = \lambda_1$, and we get directly from (29) that

$$\epsilon(hf_\Theta^m) = \epsilon(h)\epsilon(f_\Theta^m), \qquad \epsilon(f_\Theta^m h) = \epsilon(f_\Theta^m)\epsilon(h)$$

and moreover $\epsilon(f_\Theta^m) = (\epsilon(f_\Theta))^m$ for all $m \in \mathbb{Z}$. Next consider composition with f_Ξ. We have that $(f_\Xi)_*(\lambda_1) = \lambda_2$, where $\lambda_2 = \mathrm{Span}_\mathbb{R}\{[\beta]\}$. Let $\lambda_3 = h_*(\lambda_1) = \mathrm{Span}_\mathbb{R}\{a[\alpha] + c[\beta]\}$, and let ω be the intersection pairing (26) with $\Sigma = \Sigma_{(1;)}$. Let $W = (\lambda_1+\lambda_2)\cap\lambda_3 = \lambda_3$ and let $\langle .,. \rangle$ be the bilinear form on W defined in (25). For $x = a[\alpha]+c[\beta]$ we have $\langle x,x\rangle = \omega(c[\beta], a[\alpha]+c[\beta]) = ac\omega([\beta],[\alpha])$. By definition, the Maslov index $\mu(h_*(\lambda_1),\lambda_1,(f_\Xi)_*^{-1}(\lambda_1)) = \mu(\lambda_1,\lambda_2,\lambda_3)$ is equal to the signature of $\langle .,. \rangle$ which again is equal to $-\mathrm{sign}(ac)$ since $\omega([\beta],[\alpha]) = -\omega([\alpha],[\beta]) = -1$. Therefore

$$\epsilon(f_\Xi h) = (\Delta\mathcal{D}^{-1})^{\mathrm{sign}(ac)}\epsilon(f_\Xi)\epsilon(h).$$

Let $g_i = f_\Theta^{a_i} f_\Xi f_\Theta^{a_{i-1}} \cdots f_\Theta^{a_1} f_\Xi$. In particular $g = g_n$, and $(g_i)_* : H_1(\Sigma_{(1;)};\mathbb{Z}) \to H_1(\Sigma_{(1;)};\mathbb{Z})$ has the matrix $B_i^\mathcal{C}$ with respect to the basis $\{[\alpha],[\beta]\}$. For $i \geq 1$

we have

$$\epsilon(g_{i+1}) = \epsilon(f_\Theta^{a_{i+1}} f_\Xi g_i) = (\Delta\mathcal{D}^{-1})^{\mathrm{sign}(\alpha_i^{\mathcal{C}}\beta_i^{\mathcal{C}})}(\epsilon(f_\Theta))^{a_{i+1}}\epsilon(f_\Xi)\epsilon(g_i).$$

Also note that $\epsilon(g_1) = (\epsilon(f_\Theta))^{a_1}\epsilon(f_\Xi)$. We therefore get

$$G = (\Delta\mathcal{D}^{-1})^m T^{a_n}(\mathcal{D}S^{-1})T^{a_{n-1}}(\mathcal{D}S^{-1})\cdots T^{a_1}(\mathcal{D}S^{-1}),$$

where $m = \sum_{i=1}^{n-1}\mathrm{sign}(\alpha_i^{\mathcal{C}}\beta_i^{\mathcal{C}})$. We also have $m = \frac{1}{3}\left(\sum_{i=1}^{n} a_i - \Phi(B^{\mathcal{C}})\right)$ by (19). By (1) we have that $S^{-1} = \mathcal{D}^{-2}\bar{S}$. Recall here that $\bar{S}_{i,j} = S_{i^*,j}$. The result now follows by using that $v_{i^*} = v_i$ and $S_{i^*,j^*} = S_{i,j}$ for all $i,j \in I$, see [Tu, Formulas (II.3.3.a-b)], and by using that $i \mapsto i^*$ is an involution in I. □

Note that $(G^{\mathcal{C}}S^{-1}\hat{S})_{j,0} = G^{\mathcal{C}}_{j,0}$ for all $j \in I$ since $0^* = 0$.

Remark 5.6 We have chosen in this paper to work with the generators Ξ and Θ for $SL(2,\mathbb{Z})$ since it seems to be the standard. However, the above result suggests that in the above setting it is more natural to work with $\Xi^{-1} = -\Xi$ and Θ. If we do this we will not need the strange factor $S^{-1}\hat{S}$ in the formula for G in Lemma 5.5. Note also that the use of Ξ^{-1} instead of Ξ causes no difficulties with respect to the matrices $B_k^{\mathcal{C}}$ in (7) since we actually only need these as elements of $PSL(2,\mathbb{Z})$ in any case. Another (more radical) way to avoid a factor such as $S^{-1}\hat{S}$ is to use $\bar{S}$ as the S–matrix for a modular category instead of S, see [Kir], [BK].

6 A second proof of formula (9)

In this section we use the surgery formula in Theorem 5.3 to calculate the invariant of $M = (\mathrm{o}; g \mid b;\ (\alpha_1,\beta_1),\dots,(\alpha_n,\beta_n))$. First assume that $b \neq 0$. Let $\alpha_{n+1} = 1$ and $\beta_{n+1} = b$, let Σ_g be a closed oriented surface of genus g, let $D_1,\dots,D_{n+1}$ be disjoint closed disks in Σ_g, and let $Q_i'\colon D_i \times S^1 \hookrightarrow \Sigma_g \times S^1$ be the inclusion, $i = 1,\dots,n+1$. Let $B^2 \times S^1$ be the oriented standard solid torus in $\mathbb{R}^3$, see Conventions 5.2, and let $k_i\colon B^2 \to D_i$ be orientation preserving homeomorphisms, $i = 1,\dots,n+1$. Moreover let $Q_i = Q_i' \circ (k_i \times id_{S^1})\colon B^2 \times S^1 \to \Sigma_g \times S^1$ and $L_i = Q_i(0 \times S^1)$. The manifold M is given by surgery on $\Sigma_g \times S^1$ along the link $L = \amalg_{i=1}^{n+1} L_i$ with framing $Q = \amalg_{i=1}^{n+1} Q_i$ and surgery coefficients $\{\alpha_i/\beta_i\}_{i=1}^{n+1}$. (The orientation of $\Sigma_g \times S^1$ is given by the orientation of Σ_g followed by the orientation of S^1, where S^1 is oriented as in the oriented standard solid torus $B^2 \times S^1$.) Let $\mathcal{C}_i$ be as above Theorem 4.1,

$i = 1, \dots, n$, let $\mathcal{C}_{n+1} = (-b, 0)$ (a continued fraction expansion of α_{n+1}/β_{n+1}), and let $m_{n+1} = 2$. By Theorem 5.3 we have

$$\begin{aligned}\tau(M) &= (\Delta\mathcal{D}^{-1})^{\mu+\sum_{i=1}^{n+1} c_i}\mathcal{D}^{-\sum_{i=1}^{n+1} m_i} \\ &\quad \times \sum_{\lambda\in\mathrm{col}(L)} \tau(\Sigma_g \times S^1, \Gamma(L,\lambda)) \left(\prod_{i=1}^{n+1} G^{\mathcal{C}_i}_{\lambda(L_i),0}\right).\end{aligned}$$

Here $(\Sigma_g \times S^1, \Gamma(L,\lambda)) = \Sigma_t \times S^1$ with $t = (g; (V_{\lambda(L_1)}, 1), \dots, (V_{\lambda(L_{n+1})}, 1))$. For an arbitrary type t we have

$$\tau(\Sigma_t \times S^1) = \mathrm{Dim}(\Psi_t),$$

where Ψ_t is the projective K–module given in (24) and Dim is the dimension in the (ribbon) category of finitely generated projective K–modules, see [Tu, Sect. I.1.7.1 and Appendix I]. This follows by [Tu, Theorem IV.7.2.1], the remarks following this theorem, and [Tu, Sect. IV.6.7]. The dimension $\mathrm{Dim}(\Psi_t)$ is calculated for an arbitrary type in [Tu, Sect. IV.12]. (The formula for this dimension is a generalization of Verlinde's well-known formula [V] to the setting of modular categories.) For $t = (g; (V_{i_1}, 1), \dots, (V_{i_m}, 1))$, $i_1, \dots, i_m \in I$, we have

$$\mathrm{Dim}(\Psi_t) = \mathcal{D}^{2g-2} \sum_{j\in I} (\dim(j))^{2-2g-m} \left(\prod_{k=1}^{m} S_{i_k,j}\right) \tag{34}$$

by [Tu, Theorem IV.12.1.1]. Putting the above together we get

$$\begin{aligned}\tau(M) &= (\Delta\mathcal{D}^{-1})^{\mu+\sum_{i=1}^{n+1} c_i}\mathcal{D}^{2g-2-\sum_{i=1}^{n+1} m_i} \\ &\quad \times \sum_{j\in I} (\dim(j))^{2-2g-n-1} \left(\prod_{i=1}^{n+1} (SG^{\mathcal{C}_i})_{j,0}\right).\end{aligned}$$

Here $SG^{\mathcal{C}_{n+1}} = S^2T^{-b}S$, so by (1) we get

$$(SG^{\mathcal{C}_{n+1}})_{j,0} = \mathcal{D}^2 \sum_{k,l\in I} \delta_{j^*,k}\delta_{k,l} v_k^{-b} S_{l,0} = \mathcal{D}^2 v_j^{-b} \dim(j),$$

where we use that $v_{j^*} = v_j$, cf. [Tu, p. 90], and that $\dim(j^*) = \dim(j)$ by [Tu, Corollary I.2.8.2] and the definition of a modular category. Therefore

$$\begin{aligned}\tau(M) &= (\Delta\mathcal{D}^{-1})^{\mu+\sum_{i=1}^{n+1} c_i}\mathcal{D}^{2g-2-\sum_{i=1}^{n} m_i} \\ &\quad \times \sum_{j\in I} v_j^{-b} (\dim(j))^{2-2g-n} \left(\prod_{i=1}^{n} (SG^{\mathcal{C}_i})_{j,0}\right).\end{aligned}$$

This expression is identical with (9) and (20) if

$$\mu + c_{n+1} = n + \mathrm{sign}(e). \tag{35}$$

Since $\mathcal{C}_{n+1} = (-b, 0)$ and $\Theta^{-b}\Xi = \begin{pmatrix} b & 1 \\ -1 & 0 \end{pmatrix}$ we immediately get $c_{n+1} = -\mathrm{sign}(b)$ by Lemma 5.5. By notation from Lemma 5.4 we have $M_0 = \Sigma_g \times S^1$ and

$$M_i = (\mathrm{o}; g \,|\, 0; (\alpha_1, \beta_1), \ldots, (\alpha_i, \beta_i)), \qquad i = 1, 2, \ldots, n.$$

Moreover $X_i = M_{i-1} \backslash \mathrm{int}(U_i)$ is obtained from $Y_i = (\Sigma_g \backslash \mathrm{int}(D_1 \cup \ldots \cup D_i)) \times S^1$ by pasting in $i-1$ solid tori $U_1, \ldots, U_{i-1}$ as explained above leaving one torus shaped cave. We have

$$\begin{aligned} \pi_1(Y_i) &= < a_1, b_1, \ldots, a_g, b_g, q_1, \ldots, q_i, h \mid \prod_{j=1}^{i} q_j \prod_{j=1}^{g} [a_j, b_j] = 1, \\ &\qquad [h, a_k] = [h, b_k] = [h, q_l] = 1,\ k = 1, \ldots, g,\ l = 1, \ldots, i >, \end{aligned}$$

cf. [JN, Sect. 6 pp. 34–35], [Se2, Sect. 10]. Here q_j corresponds to the 'partial cross-section' $\partial D_j \times \{1\}$, and h is a fiber. The generators $a_1, b_1, \ldots, a_g, b_g$ are induced by the usual generators of $\pi_1(\Sigma_g) =< a_1, b_1, \ldots, a_g, b_g \mid \prod_{j=1}^{g} [a_j, b_j] = 1 >$. By the theorem of Seifert and Van Kampen, gluing in the torus U_j adds a new generator t and two new relations $q_j^{\alpha_j} h^{\beta_j} = 1$ and $q_j^{\rho_j} h^{\sigma_j} = t$. The generator t and the last relation can be deleted by a Tietze transformation, so we get

$$\begin{aligned} \pi_1(X_i) &= < a_1, b_1, \ldots, a_g, b_g, q_1, \ldots, q_i, h \mid \prod_{j=1}^{i} q_j \prod_{j=1}^{g} [a_j, b_j] = 1, \\ &\qquad [h, a_k] = [h, b_k] = [h, q_l] = q_s^{\alpha_s} h^{\beta_s} = 1, \\ &\qquad k = 1, \ldots, g,\ l = 1, \ldots, i,\ s = 1, \ldots, i-1 > . \end{aligned}$$

By abelianizing we see that $H_1(X_i; \mathbb{Z}) = \mathbb{Z}^{2g} \oplus T$, where

$$T =< q_1, \ldots, q_i, h \mid \sum_{j=1}^{i} q_j = \alpha_s q_s + \beta_s h = 0,\ s = 1, \ldots, i-1 >,$$

and by the universal coefficient theorem we have $H_1(X_i; \mathbb{R}) = \mathbb{R}^{2g} \oplus (T \otimes_{\mathbb{Z}} \mathbb{R})$. Let

$$\begin{aligned} \lambda_1^{(i)} &= (\partial Q_i)_*(\lambda_{(1;)}) = \mathrm{Span}_{\mathbb{R}}\{q_i\}, \\ \lambda_2^{(i)} &= (\partial Q_i)_*(\mathrm{Span}_{\mathbb{R}}\{\alpha_i[\alpha] + \beta_i[\beta]\}) = \mathrm{Span}_{\mathbb{R}}\{\alpha_i q_i + \beta_i h\}, \\ \lambda_3^{(i)} &= \ker\left(i_*\colon\ H_1(\partial U_i; \mathbb{R}) \to H_1(X_i; \mathbb{R})\right), \end{aligned}$$

$i = 1, 2, \ldots, n+1$. Here $H_1(\partial U_i;\mathbb{R}) = \mathrm{Span}_{\mathbb{R}}\{q_i, h\}$. A small calculation shows that $\lambda_3^{(i)} = \mathrm{Span}_{\mathbb{R}}\{y_i\}$, where $y_1 = q_1$ and $y_i = q_i - \left(\sum_{j=1}^{i-1} \frac{\beta_j}{\alpha_j}\right) h$ for $i = 2, \ldots, n+1$. Let $\langle\ ,\ \rangle_i$ be the bilinear form on $(\lambda_1^{(i)} + \lambda_2^{(i)}) \cap \lambda_3^{(i)} = \lambda_3^{(i)}$ defined by (25) with $H = H_1(\partial U_i;\mathbb{R})$, ω equal to the intersection pairing ω_i on $H_1(\partial U_i;\mathbb{R})$, and $\lambda_j = \lambda_j^{(i)}$, $j = 1, 2, 3$. Moreover, let $\mu_i = \mu(\lambda_1^{(i)}, \lambda_2^{(i)}, \lambda_3^{(i)})$ be the Maslov index equal to the signature of $\langle .,. \rangle_i$. We get immediately that $\mu_1 = 0$. Let $i \in \{2, \ldots, n+1\}$ and let $x_i = \alpha_i q_i + \beta_i h$ and $t_i = \sum_{j=1}^{i-1} \frac{\beta_j}{\alpha_j}$. Then $y_i = \left(1 + \frac{\alpha_i}{\beta_i} t_i\right) q_i - \frac{t_i}{\beta_i} x_i$. Therefore

$$\langle y_i, y_i \rangle_i = \omega_i(-\frac{t_i}{\beta_i} x_i, q_i - t_i h) = \frac{\alpha_i}{\beta_i} t_{i+1} t_i,$$

where $t_{n+2} = -e$. Here $t_i > 0$, and for $i \le n$ we have $\alpha_i/\beta_i > 0$ and $t_{i+1} > 0$. Therefore $\mu_i = 1$ for $i = 2, \ldots, n$. Finally, $\mu_{n+1} = -\mathrm{sign}(b)\mathrm{sign}(e)$, so $\mu = \sum_{i=1}^{n+1} \mu_i = n - 1 - \mathrm{sign}(b)\mathrm{sign}(e)$. The identity (35) is therefore equivalent with the identity $\mathrm{sign}(e) + \mathrm{sign}(b) + \mathrm{sign}(b)\mathrm{sign}(e) + 1 = 0$ which is true. Note that the above also holds in case $n = 0$ (no exceptional fibers). In this case $e = -b$. In case $b = 0$ we ignore everything concerning the surgery along the component L_{n+1}. If $n > 0$ we have to show that $\mu = n + \mathrm{sign}(e)$, where $\mu = \sum_{i=1}^{n} \mu_i$ and $e = -\sum_{i=1}^{n} \frac{\beta_i}{\alpha_i} < 0$. Since $\mu = \sum_{i=1}^{n} \mu_i = n - 1$ this identity is true. If $n = 0$ the surgery formula is of no use. In this case $\tau(M) = \tau(\Sigma_g \times S^1) = \mathcal{D}^{2g-2} \sum_{j \in I} (\dim(j))^{2-2g}$ by (34) in accordance with Theorem 4.1.

7 A third proof of formula (9)

In this section we will use the formula in [Tu, Theorem X.9.3.1] for the RT–invariant of graph manifolds to derive (9). This formula is valid for unimodular categories with a rank, see Sect. 3.

Let $M = (\mathrm{o}; g \,|\, b; (\alpha_1, \beta_1), \ldots, (\alpha_n, \beta_n))$. It turns out to be an advantage to work with $-M$ instead of M. According to Theorem 2.1, $-M = (\mathrm{o}; g \,|\, -b - n; (\alpha_1, \alpha_1 - \beta_1), \ldots, (\alpha_n, \alpha_n - \beta_n))$. Let $\mathcal{C}_j = (a_1^{(j)}, \ldots, a_{m_j}^{(j)})$ be a continued fraction expansion of α_j/β_j with $a_l^j \ge 2$ and let A be the $m \times m$–matrix in (14), $m = 1 + \sum_{j=1}^{n} m_j$. By [O, Corollary 5 p. 30] and [Tu, Sect. X.9.2], the 3–manifold $-M$ is the 3–dimensional graph manifold determined by the matrix $-A$ and the integers $g_1, g_2, \ldots, g_m$, where $g_1 = g$ and $g_2 = \ldots = g_m = 0$.

Let $(\mathcal{V}, \{V_i\}_{i \in I})$ be a unimodular category with a fixed rank $\mathcal{D}$. By [Tu, Theorem X.9.3.1], the RT–invariant of the graph manifold N, determined by the symmetric square matrix $B=(a_{p,q})_{p,q=1}^m$ over $\mathbb{Z}$ and a sequence of non-negative integers $g_1,g_2,\dots,g_m$, is given by

$$\tau_{(\mathcal{V},\mathcal{D})}(N)=\Delta^{\sigma(B)}\mathcal{D}^b \times\sum_{\varphi\in I^m}\left(\prod_{p=1}^m v_{\varphi(p)}^{a_{p,p}}(\dim(\varphi(p)))^{2-2g_p-a_p}\prod_{p<q}(s^{\varphi}_{p,q})^{|a_{p,q}|}\right), \tag{36}$$

where $b=b_1(N)-b_0(N)-m-\mathrm{null}(B)-\sigma(B)$, where $\mathrm{null}(B)$ and $\sigma(B)$ are the nullity and signature of B respectively, and $a_p=\sum_{q\neq p}|a_{p,q}|$. Moreover, $s^{\varphi}_{p,q}=S_{\varphi(p),\varphi(q)}$ if $a_{p,q}\geq 0$ and $s^{\varphi}_{p,q}=S_{\varphi(p)^*,\varphi(q)}$ if $a_{p,q}<0$.

We have $b_0(-M)=1$ and $\sigma(-A)=-\sigma_o$, where σ_o is given by (10), (20). By [JN, Corollary 6.2], $b_1(-M)=b_1(M)=2g+\delta_{e,0}$, where $\delta_{e,0}=1$ if $e=0$ and 0 otherwise, and by the proof of (10) we have $\mathrm{null}(-A)=\mathrm{null}(A)=\delta_{e,0}$. If we write $-A=(a_{p,q})_{p,q=1}^m$ then

$$a_p=\begin{cases} n & ,p=1\\ 2 & ,p\in\{1,2,\dots,m\}\setminus\{1,m_1+1,m_1+m_2+1,\dots,m\}\\ 1 & ,p\in\{1,m_1+1,m_1+m_2+1,\dots,m\}.\end{cases}$$

According to [Tu, Exercise II.2.5] we have

$$\tau_{(\mathcal{V},\mathcal{D})}(M)=\tau_{(\overline{\mathcal{V}},\mathcal{D})}(-M),$$

where $\overline{\mathcal{V}}$ is the mirror of $\mathcal{V}$, see Sect. 3. The modular category $\left(\overline{\mathcal{V}},\{V_i\}_{i\in I}\right)$ is also unimodular, cf. [Tu, Exercise VI.2.3.1]. By (36) we get

$$\begin{aligned}\tau_{(\overline{\mathcal{V}},\mathcal{D})}(-M) &= (\Delta_{\overline{\mathcal{V}}}\mathcal{D}^{-1})^{-\sigma_o}\mathcal{D}^{2g-2-\sum_{j=1}^n m_j}\\ &\quad\times\sum_{j,k_1^1,\dots,k_{m_1}^1,k_1^2,\dots,k_{m_n}^n\in I}\bar{v}_j^b(\dim(j))^{2-2g-n}\\ &\quad\times\prod_{j=1}^n\prod_{l=1}^{m_j-1}\bar{v}_{k_l^j}^{-a^{(j)}_{m_j+1-l}}\bar{S}_{(k_l^j)^*,k_{l+1}^j}\prod_{j=1}^n\bar{v}_{k_{m_j}^j}^{-a_1^{(j)}}\dim(k_{m_j}^j)\bar{S}_{j^*,k_1^j}.\end{aligned}$$

Here $\bar{v}_i=v_i^{-1}$ and $\bar{S}_{j^*,k}=S_{j,k}$. By (3) we then get

$$\begin{aligned}\tau_{(\mathcal{V},\mathcal{D})}(M) &= (\Delta\mathcal{D}^{-1})^{\sigma_o}\mathcal{D}^{2g-2-\sum_{j=1}^n m_j}\\ &\quad\times\sum_{j,k_1^1,\dots,k_{m_1}^1,k_1^2,\dots,k_{m_n}^n\in I}v_j^{-b}(\dim(j))^{2-2g-n}\\ &\quad\times\prod_{j=1}^n\prod_{l=1}^{m_j-1}v_{k_l^j}^{a^{(j)}_{m_j+1-l}}S_{k_l^j,k_{l+1}^j}\prod_{j=1}^n v_{k_{m_j}^j}^{a_1^{(j)}}\dim(k_{m_j}^j)S_{j,k_1^j}\end{aligned}$$

$$= (\Delta\mathcal{D}^{-1})^{\sigma_0}\mathcal{D}^{2g-2-\sum_{j=1}^{n} m_j} \\ \times \sum_{j\in I} v_j^{-b}(\dim(j))^{2-2g-n}\left(\prod_{i=1}^{n}(SG^{\mathcal{C}_i})_{j,0}\right).$$

This expression is identical with (9).

8 The case of $\mathfrak{sl}_2(\mathbb{C})$

Let $t=\exp(i\pi/(2r))$, where r is an integer ≥ 2, and let U_t be the Hopf algebra considered in [RT2, Sect. 8], see below for details. With notation from [RT2] we have that $\left(U_t, R, v^{-1}, \{V_i\}_{i\in I}\right)$ is a modular Hopf algebra as defined in [Tu, Chap. XI]. Let $(\mathcal{V}_t, \{V_i\}_{i\in I})$ be the modular category induced by this modular Hopf algebra, cf. [Tu, Chap. XI].

Let us recall some notation and results from [RT2]. The quantum group $U_q(\mathfrak{sl}_2)$, $q=t^4$, is the $\mathbb{Q}(t)$–algebra with generators K, K^{-1}, X, Y subject to the relations

$$XY - YX = \frac{K^2-K^{-2}}{t^2-t^{-2}},$$
$$XK = t^{-2}KX, \quad YK = t^2KY, \quad KK^{-1} = K^{-1}K = 1.$$

The algebra U_t is given by the quotient of $U_q(\mathfrak{sl}_2)$ by the two-sided ideal generated by the elements $X^r, Y^r, K^{4r}-1$. In the following we will consider U_t as an algebra over $\mathbb{C}$.

To determine the RT–invariants of the Seifert manifolds with non-orientable base we need to determine the signs ε_i in Lemma 3.1 for all self-dual $i\in I=\{0,1,\ldots,r-2\}$. It is a well-known fact that all the simple objects V_i in $\mathcal{V}_t$ are self-dual. Let us provide some details. We let

$$[k] = \frac{t^{2k}-t^{-2k}}{t^2-t^{-2}} = \frac{\sin(\pi k/r)}{\sin(\pi/r)}$$

for an integer k. Let $\alpha\in\{-\sqrt{-1},\sqrt{-1},-1,1\}$. Then we have irreducible U_t–modules $\{V^i(\alpha)\}_{i\in I}$ with a basis (over $\mathbb{C}$) of weight vectors $\{e_n^i(\alpha)\}_{n=0}^i$ such that

$$\begin{aligned} Ke_n^i(\alpha) &= \alpha t^{i-2n}e_n^i(\alpha), \\ Xe_n^i(\alpha) &= \alpha^2[n][i+1-n]e_{n-1}^i(\alpha), \\ Ye_n^i(\alpha) &= e_{n+1}^i(\alpha) \end{aligned}$$

for $n = 0, 1, \ldots, i$, where $e^i_{-1}(\alpha) = e^i_{i+1}(\alpha) = 0$. Let $\{f^i_n(\alpha)\}^i_{n=0}$ be the basis of $V^i(\alpha)^*$ dual to $\{e^i_n(\alpha)\}^i_{n=0}$ as in [RT2, Sect. 8]. By using the antipode γ of U_t determined by $\gamma(K) = K^{-1}$, $\gamma(X) = -t^2X$ and $\gamma(Y) = -t^{-2}Y$, see [RT2, (8.1.4)], one gets

$$\begin{aligned} Kf^i_n(\alpha) &= \alpha^{-1}t^{2n-i}f^i_n(\alpha), \\ Xf^i_n(\alpha) &= -\alpha^2t^2[n+1][i-n]f^i_{n+1}(\alpha), \\ Yf^i_n(\alpha) &= -t^{-2}f^i_{n-1}(\alpha) \end{aligned}$$

for $n = 0, 1, \ldots, i$, where $f^i_{-1}(\alpha) = f^i_{i+1}(\alpha) = 0$. We have $V_i = V^i(1)$, $i \in I$. The following lemma follows by a straightforward computation using the above U_t–module structures.

Lemma 8.1 *Let $\alpha, \beta \in \{-\sqrt{-1}, \sqrt{1}, -1, 1\}$. A $\mathbb{C}$–linear map $h\colon V^i(\alpha)^* \to V^i(\beta)$ is a U_t–module isomorphism if and only if $\beta = \alpha^{-1}$ and*

$$h(f^i_n(\alpha)) = \delta_i(-1)^n t^{-2n} e^i_{i-n}(\alpha^{-1}) \tag{37}$$

for a $\delta_i \in \mathbb{C} \setminus \{0\}$. □

The lemma shows, as claimed above, that the module V_i is self-dual for all $i \in I$.

Lemma 8.2 *We have $\varepsilon_i = (-1)^i$ for all $i \in I$. In particular the modular category $(\mathcal{V}_t, \{V_i\}_{i\in I})$ is not unimodal.*

Proof We use Lemma 8.1 together with Lemma 3.2 to determine the signs ε_i, $i \in I$. To this end note that $v = uK^{-2}$, where u is the element u in Lemma 3.2. As indicated in the beginning of this section we shall use v^{-1} as the element v in Lemma 3.2. (This is due to different conventions in [Tu] and [RT2].) We see that $uv^{-1} = K^2$ (use that v^{-1} is central). Fix $i \in I$ and let $\omega = h^{-1}$, where $h\colon V_i^* \to V_i$ is given by (37) (with $\alpha = 1$). Moreover, let $e_n = e^i_n(1)$, $f_n = f^i_n(1)$. Then

$$\omega(e_n) = \delta_i^{-1}(-1)^{i-n}t^{2(i-n)}f_{i-n}.$$

Let $z = G \circ (\omega^{-1})^*(f_{i-n}) \in V_i$, where $G\colon V_i^{**} \to V_i$ is the canonical isomorphism as in Lemma 3.2. Then

$$G^{-1}(z)(f_m) = f_{i-n}(\omega^{-1}(f_m)) = \delta_i(-1)^m t^{-2m} f_{i-n}(e_{i-m}) = \delta_i(-1)^n t^{-2n}\delta_{n,m}.$$

On the other hand $G^{-1}(z)(f_m) = f_m(z)$, so we see that $z = \delta_i(-1)^n t^{-2n} e_n$. But then

$$G \circ (\omega^{-1})^* \circ \omega(e_n) = \delta_i^{-1}(-1)^{i-n}t^{2(i-n)}z = (-1)^i t^{2i-4n} e_n = (-1)^i K^2 e_n. \qquad \square$$

Let $\kappa(i)$, $i \in I$, be the quantity in (5). The R–matrix calculation of $\kappa(i)$ gives the result $\kappa(i) = (-1)^i \Lambda v_i^2 \left(\dim(i)\right)^{-1}$, $\Lambda = \sum_{u \in I} \left(\dim(u)\right)^2$, so gives together with Lemma 4.2 another proof of Lemma 8.2 and the fact that all the simple modules V_i, $i \in I$, are self-dual.

Let $\bar{v}_i$ be equal to the v_i in [RT2], i.e. $\bar{v}_i \mathrm{id}_{V_i}$ is equal to the map $V_i \to V_i$ given by multiplication with v, $i \in I$. Moreover, let $\bar{d}_i$ be equal to d_i in [RT2], i.e.

$$\sum_{i \in I} \bar{d}_i \bar{v}_i S_{i,j} = \bar{v}_j^{-1} \dim(j), \qquad j \in I.$$

Since $i = i^*$ here, this can also be written

$$\sum_{i \in I} \bar{d}_i \bar{v}_i S_{i^*,j} = \bar{v}_j^{-1} \dim(j), \qquad j \in I.$$

It follows that the $\bar{v}_i$ and $\bar{d}_i$ are equal to the v_i and d_i associated to $\overline{\mathcal{V}}_t$ in [Tu, Sect. II.3], where $\overline{\mathcal{V}}_t$ is the mirror of $\mathcal{V}_t$, see Sect. 3. By [RT2, Sect. 8.3],

$$\begin{aligned} \bar{v}_i &= t^{-i(i+2)}, \\ \bar{d}_i &= \sqrt{\frac{2}{r}} \sin(\pi/r) C_0 \dim(i) \end{aligned}$$

for $i \in I$. Here $C_0 = \exp(\sqrt{-1}d)$ is a square root of $C = \sum_{i \in I} \bar{v}_i^{-1} \dim(i) \bar{d}_i = \exp(2\sqrt{-1}d)$, where $d = \frac{3\pi(r-2)}{4r}$ and $\dim(i) = [i+1]$. In particular $\bar{d}_0 = \sqrt{\frac{2}{r}} \sin(\pi/r) \exp(\sqrt{-1}d)$. According to [Tu, pp. 88–89] we have that $\bar{d}_0 = \Delta_{\overline{\mathcal{V}}_t} \Lambda^{-1}$ and $C = \Delta_{\overline{\mathcal{V}}_t} \bar{d}_0$. (Here we use that the dimensions of any object of $\mathcal{V}_t$ with respect to $\mathcal{V}_t$ and $\overline{\mathcal{V}}_t$ are equal, cf. [Tu, Corollary I.2.8.5], so Λ is the same element in these two categories. By the same reason $\mathcal{D}$ is a rank of $\overline{\mathcal{V}}_t$ if and only if $\mathcal{D}$ is a rank of $\mathcal{V}_t$.) We see that $\Lambda = C\bar{d}_0^{-2} = \frac{r}{2} \frac{1}{\sin^2(\pi/r)}$. As a rank we choose

$$\mathcal{D} = \sqrt{\frac{r}{2}} \frac{1}{\sin(\pi/r)}. \tag{38}$$

Let $\{v_i\}_{i \in I}$ be the v_i associated to $\mathcal{V}_t$. Then

$$v_i = \bar{v}_i^{-1} = t^{i(i+2)} = t^{-1} t^{(i+1)^2}, \qquad i \in I. \tag{39}$$

By (3) we get $\Delta = \Delta_{\mathcal{V}_t} = \mathcal{D}^2 \Delta_{\overline{\mathcal{V}}_t}^{-1} = \mathcal{D}^2 \bar{d}_0 C^{-1}$, so

$$\Delta \mathcal{D}^{-1} = \mathcal{D} \bar{d}_0 C^{-1} = \exp(-\sqrt{-1}d) = \exp\left(\frac{\sqrt{-1}\pi}{4} \frac{3(2-r)}{r} \right). \tag{40}$$

The S–matrix of $\mathcal{V}_t$ is given by

$$S_{i,j} = \frac{\sin(\pi(i+1)(j+1)/r)}{\sin(\pi/r)}, \qquad i,j \in I, \tag{41}$$

cf. [RT2, Sect. 8].

Inspired by the Chern–Simons path integral invariant of Witten, see [Wi] and the introduction, the RT–invariant $\tau_{(\mathcal{V}_t,\mathcal{D})}$ is also called the quantum $SU(2)$–invariant at *level* $r-2$ and is denoted τ_r. We will take advantage of the fact that the projective action of $\mathrm{Mod}_1 = SL(2,\mathbb{Z})$ considered in (29) can be normalized to a linear action (in fact to a linear action of $PSL(2,\mathbb{Z})$) in the $\mathfrak{sl}_2(\mathbb{C})$–case. Let $\mathcal{R}\colon\ PSL(2,\mathbb{Z}) \to GL(r-1,\mathbb{C})$ be the unitary representation given by

$$\tilde{\Xi}_{jl} = \sqrt{\frac{2}{r}}\sin\left(\frac{jl\pi}{r}\right), \qquad \tilde{\Theta}_{jl} = e^{-\frac{i\pi}{4}}\exp\left(\frac{i\pi}{2r}j^2\right)\delta_{jl} \tag{42}$$

for $j,l \in I' = \{1,2,\dots,r-1\}$. Here we write $\tilde{M}$ for the matrix $\mathcal{R}(M)$. By changing the index set of the basis $\{b_i\}_{i\in I}$ to I' (so that the new V_j and b_j are equal to the old V_{j-1} and b_{j-1}, $j \in I'$) and comparing (42) with (38), (39) and (41) we get

$$\begin{aligned} S_{jl} &= \mathcal{D}\tilde{\Xi}_{jl}, \\ T_{jl} &= \exp\left(-\frac{i\pi}{2r}\right)\exp\left(\frac{i\pi}{2r}j^2\right)\delta_{jl} = e^{\frac{i\pi}{4}}\exp\left(-\frac{i\pi}{2r}\right)\tilde{\Theta}_{jl} \end{aligned} \tag{43}$$

for $j,l \in I'$. The representations $\mathcal{R}$ have been carefully studied by Jeffrey in [J] where she gives rather explicit formulas for $\tilde{M}$ in terms of the entries in $M \in SL(2,\mathbb{Z})$, see also the proof of Theorem 8.4 below. These representations are known from the study of affine Lie algebras, cf. [K].

The following corollary is an $\mathfrak{sl}_2(\mathbb{C})$–version of Theorem 5.3. It simply follows by choosing tuples of integers $\mathcal{C}_i$ such that $B_i = B^{\mathcal{C}_i}$, $i = 1,2,\dots,m$. By the first remark following Theorem 4.1, $\mathcal{C}_i$ is a continued fraction expansion of p_i/q_i. Besides note that $\Delta\mathcal{D}^{-1} = w^{-3}$ with $w = e^{\frac{i\pi}{4}}\exp\left(-\frac{i\pi}{2r}\right)$ by (40), and that $G^{\mathcal{C}} = w^{\sum_{k=1}^m a_k}\mathcal{D}^m\tilde{B}^{\mathcal{C}}$ for $\mathcal{C} = (a_1,\dots,a_m) \in \mathbb{Z}^m$ by (43), (7), and (8).

Corollary 8.3 *Let the situation be as in* Theorem 5.3 *and let* $B_i \in SL(2,\mathbb{Z})$ *with first column equal to* $\pm\begin{pmatrix} p_i \\ q_i \end{pmatrix}$, $i = 1,2,\dots,m$. *Then*

$$\begin{aligned} \tau_r(M',\Omega) &= \left(\mathrm{e}^{\frac{i\pi}{4}}\exp\left(-\frac{i\pi}{2r}\right)\right)^{\sum_{i=1}^m \Phi(B_i)-3\mu} \\ &\qquad\times \sum_{\lambda\in\mathrm{col}(L)} \tau_r(M,\Gamma(L,\lambda)\cup\Omega)\left(\prod_{i=1}^m (\tilde{B}_i)_{\lambda(L_i),1}\right), \end{aligned}$$

where μ is given by (33). □

The following theorem generalizes results in [Roz] to include the case of Seifert manifolds with non-orientable base. Specifically the expressions (44) and (45) are equivalent to [Roz, Formulas (2.7) and (2.8)] for $\epsilon = \mathrm{o}$. (Rozansky uses non-normalized Seifert invariants.) To state the theorem we need some notation. Multi-indices are denoted by an underline (e.g. $\underline{m}$). For $\underline{k} = (k_1, \dots, k_n), \underline{l} = (l_1, \dots, l_n) \in \mathbb{Z}^n$, $\underline{k} < \underline{l}$ if and only if $k_j < l_j$ for all $j = 1, \dots, n$. We let $\underline{1} = (1, \dots, 1)$. For $\underline{k} = (k_1, \dots, k_n) \in \mathbb{Z}_+^n$ we write $\sum_{\underline{m}=\underline{0}}^{\underline{k}}$ for $\sum_{m_1=0}^{k_1} \cdots \sum_{m_n=0}^{k_n}$ etc. In all expressions below e denotes the Seifert Euler number (except in factors such as $\mathrm{e}^{\frac{i\pi}{4}}$). Let $a_{\mathrm{o}} = 2$ and $a_{\mathrm{n}} = 1$. For a pair of coprime integers α, β we let β^* be the invers of β in the group of (multiplicative) units in $\mathbb{Z}/\alpha\mathbb{Z}$.

Theorem 8.4 *The RT–invariant at level $r-2$ of the Seifert manifold M with (normalized) Seifert invariants $(\epsilon; g \,|\, b; (\alpha_1, \beta_1), \dots, (\alpha_n, \beta_n))$, $\epsilon \in \{\mathrm{o}, \mathrm{n}\}$, is*

$$\begin{aligned} \tau_r(M) &= \exp\left(\frac{i\pi}{2r} \left[3(a_\epsilon - 1)\mathrm{sign}(e) - e - 12 \sum_{j=1}^n \mathrm{s}(\beta_j, \alpha_j) \right] \right) \\ &\quad \times (-1)^{a_\epsilon g} \frac{i^n r^{a_\epsilon g/2 - 1}}{2^{n + a_\epsilon g/2 - 1}} \frac{1}{\sqrt{\mathcal{A}}} e^{i \frac{3\pi}{4}(1 - a_\epsilon)\mathrm{sign}(e)} Z_\epsilon(M; r), \end{aligned} \tag{44}$$

where $\mathrm{s}(\beta_j, \alpha_j)$ is given by (18), *$\mathcal{A} = \prod_{j=1}^n \alpha_j$, and*

$$\begin{aligned} Z_\epsilon(M; r) &= \sum_{\gamma=1}^{r-1} (-1)^{\gamma a_\epsilon g} \frac{\exp\left(\frac{i\pi}{2r} e \gamma^2\right)}{\sin^{n + a_\epsilon g - 2}\left(\frac{\pi}{r}\gamma\right)} \sum_{\underline{\mu} \in \{\pm 1\}^n} \left(\prod_{j=1}^n \mu_j \right) \\ &\quad \times \sum_{\underline{m}=\underline{0}}^{\underline{\alpha}-\underline{1}} \exp\left(-\frac{i\pi}{r} \gamma \sum_{j=1}^n \frac{2rm_j + \mu_j}{\alpha_j} \right) \\ &\quad \times \exp\left(-2\pi i \sum_{j=1}^n \frac{\beta_j^*}{\alpha_j} [r m_j^2 + \mu_j m_j] \right). \end{aligned} \tag{45}$$

The RT–invariant at level $r-2$ of the Seifert manifold M with non-normalized Seifert invariants $\{\epsilon; g; (\alpha_1, \beta_1), \dots, (\alpha_n, \beta_n)\}$ is given by the same expression.

The theorem is also valid in case $n = 0$. In this case one just has to put $\mathcal{A} = 1$ and $\sum_{j=1}^n \mathrm{s}(\beta_j, \alpha_j) = 0$. Moreover, the sum $\sum_{\underline{\mu} \in \{\pm 1\}^n} \sum_{\underline{m}=\underline{0}}^{\underline{\alpha}-\underline{1}}$ in $Z_\epsilon(M; r)$ has to be put equal to 1, $\epsilon = \mathrm{o}, \mathrm{n}$.

Proof Let $M = (\epsilon; g \,|\, b; (\alpha_1, \beta_1), \dots, (\alpha_n, \beta_n))$, $\epsilon \in \{\mathrm{o}, \mathrm{n}\}$. Choose tuples of integers $\mathcal{C}_j = (a_1^{(j)}, \dots, a_{m_j}^{(j)})$ such that $B^{\mathcal{C}_j} = \begin{pmatrix} \alpha_j & \rho_j \\ \beta_j & \sigma_j \end{pmatrix}$ for some $\rho_j, \sigma_j \in \mathbb{Z}$, $j = 1, 2, \dots, n$. By Theorem 4.1, the first remark after Theorem 4.1, Lemma 8.1 and Lemma 8.2 we have

$$\begin{aligned} \tau_r(M) &= (\Delta \mathcal{D}^{-1})^{\sigma_\epsilon} \mathcal{D}^{a_\epsilon g - 2 - \sum_{j=1}^n m_j} \\ &\quad \times \sum_{j=1}^{r-1} (-1)^{(j-1)a_\epsilon g} v_j^{-b} \dim(j)^{2-n-a_\epsilon g} \left(\prod_{i=1}^n (SG^{\mathcal{C}_i})_{j,1} \right), \end{aligned}$$

where σ_ϵ is given by (10) if $\epsilon = \mathrm{o}$ and by (12) if $\epsilon = \mathrm{n}$. Here $\Delta\mathcal{D}^{-1} = w^{-3}$, where $w = e^{\frac{i\pi}{4}} \exp\left(-\frac{i\pi}{2r}\right)$, see (40). Moreover, $v_j = t^{-1} t^{j^2}$, see (39), $\dim(j) = [j] = \sqrt{\frac{2}{r}} \mathcal{D} \sin\left(\frac{\pi j}{r}\right)$, and $SG^{\mathcal{C}_i} = w^{\sum_{k=1}^{m_i} a_k^{(i)}} \mathcal{D}^{m_i+1} \tilde{N}_i$, where $N_i = \Xi B^{\mathcal{C}_i}$, so

$$\tau_r(M) = \alpha_\epsilon(r) \left(\frac{r}{2}\right)^{a_\epsilon g/2 + n/2 - 1} \sum_{j=1}^{r-1} (-1)^{j a_\epsilon g} \frac{t^{-bj^2} \prod_{i=1}^n (\tilde{N}_i)_{j,1}}{\sin^{n+a_\epsilon g-2}\left(\frac{\pi j}{r}\right)}, \tag{46}$$

where $\alpha_\epsilon(r) = (-1)^{a_\epsilon g} w^{\sum_{i=1}^n \sum_{k=1}^{m_i} a_k^{(i)} - 3\sigma_\epsilon} \exp\left(\frac{i\pi}{2r} b\right)$. By (20) and the equivalent expression for σ_{n} we get

$$\alpha_\epsilon(r) = (-1)^{a_\epsilon g} w^{\sum_{i=1}^n \Phi(B^{\mathcal{C}_i}) - 3\sum_{j=1}^n \mathrm{sign}(\alpha_j \beta_j) - 3(a_\epsilon - 1)\mathrm{sign}(e)} \exp\left(\frac{i\pi}{2r} b\right).$$

If $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{Z})$ with $c \neq 0$ we have by [J, Proposition 2.7 (a) and Proposition 2.8] that

$$\begin{aligned} \tilde{A}_{j,k} &= C \sum_{\mu = \pm 1} \sum_{\substack{\gamma \pmod{2rc} \\ \gamma = j \pmod{2r}}} \mu \exp\left(\frac{i\pi}{2rc}[a\gamma^2 - 2\mu\gamma k + dk^2]\right) \\ &= C \left\{ \sum_{n=m_1}^{|c|-1+m_1} \exp\left(\frac{i\pi}{2rc}[a(j+2rn)^2 - 2k(j+2rn) + dk^2]\right) \right. \\ &\qquad \left. - \sum_{n=m_2}^{|c|-1+m_2} \exp\left(\frac{i\pi}{2rc}[a(j+2rn)^2 + 2k(j+2rn) + dk^2]\right) \right\} \end{aligned}$$

for all $m_1, m_2 \in \mathbb{Z}$, where $C = i \frac{\mathrm{sign}(c)}{\sqrt{2r|c|}} e^{-\frac{i\pi}{4}\Phi(A)}$. For $m_1 = 0$ and $m_2 = -|c| + 1$ we get

$$\tilde{A}_{j,k} = C \sum_{\mu = \pm 1} \sum_{n=0}^{|c|-1} \mu \exp\left(\frac{i\pi}{2rc}[a(j + 2rn\mu)^2 - 2\mu k(j + 2rn\mu) + dk^2]\right).$$

If $A_i = \begin{pmatrix} a_i & b_i \\ c_i & d_i \end{pmatrix} \in SL(2,\mathbb{Z})$ such that $A_3 = A_1A_2$ we have

$$\Phi(A_3) = \Phi(A_1) + \Phi(A_2) - 3\mathrm{sign}(c_1c_2c_3). \tag{47}$$

Since the representation $\mathcal{R}$ is unitary we have $\mathcal{R}(A^{-1}) = \mathcal{R}(A)^*$, so $\tilde{A}_{j,k} = \overline{(\widetilde{A^{-1}})_{k,j}}$, where $\bar{\cdot}$ is complex conjugation. Here $A^{-1} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$, and (47) implies that $\Phi(A^{-1}) = -\Phi(A)$, so

$$\begin{aligned} \tilde{A}_{j,k} &= i\frac{\mathrm{sign}(c)}{\sqrt{2r|c|}} e^{-\frac{i\pi}{4}\Phi(A)} \\ &\quad \times \sum_{\mu=\pm1} \sum_{n=0}^{|c|-1} \mu \exp\left(\frac{i\pi}{2rc}[d(k+2rn\mu)^2 - 2\mu j(k+2rn\mu) + aj^2]\right). \end{aligned}$$

By this expression we get

$$\begin{aligned} (\tilde{N}_i)_{j,1} &= i\frac{e^{-\frac{i\pi}{4}\Phi(N_i)}}{\sqrt{2r\alpha_i}} \sum_{\mu=\pm1} \sum_{n=0}^{\alpha_i-1} \mu \\ &\quad \times \exp\left(\frac{i\pi}{2r\alpha_i}[-\beta_i j^2 - 2j(2rn+\mu) + \rho_i(2rn+\mu)^2]\right). \end{aligned}$$

By inserting this in (46) and using that $e = -b - \sum_{j=1}^n \frac{\beta_j}{\alpha_j}$ we get $\tau_r(M) = \kappa Z_\epsilon(M;r)$, where

$$\kappa = \frac{i^n r^{a_\epsilon g/2-1}}{2^{n+a_\epsilon g/2-1}} \frac{1}{\sqrt{\mathcal{A}}} \alpha_\epsilon(r) \exp\left(\frac{i\pi}{2r}\sum_{j=1}^n \frac{\rho_j}{\alpha_j}\right) \exp\left(-\frac{i\pi}{4}\sum_{j=1}^n \Phi(N_j)\right).$$

By (47) we have $\Phi(N_i) = \Phi(B^{\mathcal{C}_i}) - 3\mathrm{sign}(\alpha_i\beta_i)$ and get

$$\begin{aligned} \kappa &= (-1)^{a_\epsilon g} \frac{i^n r^{a_\epsilon g/2-1}}{2^{n+a_\epsilon g/2-1}} \frac{1}{\sqrt{\mathcal{A}}} \exp\left(i\frac{3\pi}{4}(1-a_\epsilon)\mathrm{sign}(e)\right) \\ &\quad \times \exp\left(\frac{i\pi}{2r}\left[3(a_\epsilon - 1)\mathrm{sign}(e) + b + \sum_{j=1}^n \frac{\rho_j}{\alpha_j} - \sum_{j=1}^n \Phi(N_j)\right]\right). \end{aligned}$$

The theorem now follows by using (17) together with the facts that $\mathrm{s}(a,b) = \mathrm{s}(a',b)$ if $a'a \equiv 1 \pmod b$ and $\mathrm{s}(-a,b) = -\mathrm{s}(a,b)$, cf. [RG, Chap. 3]. The case with non-normalized Seifert invariants follows as above by letting b be equal to zero everywhere. □

Let $M = (\epsilon; g \,|\, b; (\alpha_1,\beta_1),\dots,(\alpha_n,\beta_n))$. By (46) we have the following compact formula

$$\tau_r(M) = \alpha_\epsilon(r) \sum_{j=1}^{r-1} (-1)^{j a_\epsilon g} \frac{t^{-bj^2} \prod_{i=1}^n (\tilde{N}_i)_{j,1}}{\tilde{\Xi}_{j,1}^{n+a_\epsilon g-2}}, \tag{48}$$

where $\alpha_\epsilon(r) = (-1)^{a_\epsilon g} w^{\sum_{i=1}^n \Phi(N_i) - 3(a_\epsilon-1)\mathrm{sign}(e)} \exp\left(\frac{i\pi}{2r} b\right)$, $w = e^{\frac{i\pi}{4}} \exp\left(-\frac{i\pi}{2r}\right)$, and $N_i = \begin{pmatrix} -\beta_j & -\sigma_j \\ \alpha_j & \rho_j \end{pmatrix}$ for any integers ρ_j, σ_j such that $\alpha_j\sigma_j - \beta_j\rho_j = 1$.

Let us finally give a formula for $\tau_r(L(p,q))$. To this end let b,d be any integers such that $U = \begin{pmatrix} q & b \\ p & d \end{pmatrix} \in SL(2,\mathbb{Z})$. Assume $q \neq 0$, let $V = -\Xi U = \begin{pmatrix} p & d \\ -q & -b \end{pmatrix}$, and let $\mathcal{C}' = (a_1,a_2,\dots,a_{m-1}) \in \mathbb{Z}^{m-1}$ such that $B^{\mathcal{C}'} = V$. Then $\mathcal{C}'$ is a continued fraction expansion of $-p/q$ and $U = \Xi V = B^{\mathcal{C}}$ where $\mathcal{C} = (a_1,a_2,\dots,a_{m-1},0)$. By Corollary 4.4 and (43) we therefore get

$$\tau_r(L(p,q)) = \left(\mathrm{e}^{\frac{i\pi}{4}} \exp\left(-\frac{i\pi}{2r}\right)\right)^{\Phi(U)} \tilde{U}_{1,1}. \tag{49}$$

If $q=0$ we have $p=1$ and $L(p,q) = S^3$. In this case we have $U = \Xi\Theta^d$ and we immediately find from (43) that (49) is also true in this case. The identity (49) coincides with [J, Formula (3.7)].

Remark 8.5 It should not come as a surprise that we find the same result as Rozansky for the invariants of Seifert manifolds with orientable base. The calculation in [Roz] of these invariants follows the very same line as in the first part of Sect. 6. He uses a surgery formula [Roz, Formula (1.6)] which is identical with the surgery formula in Corollary 8.3 and a Verlinde formula [Roz, Formula (2.4)] which by (43) is identical with the Verlinde formula (34) of Turaev.

Remark 8.6 In more recent literature the symbol $U_q(\mathfrak{sl}_2(\mathbb{C}))$ normally refers to a Hopf algebra defined in a slightly different way than in the above text. It is well known [Kir], [BK], [Le] that Lusztig's version [Lu, Part V] of quantum deformations of simple complex finite dimensional Lie algebras at roots of unity is particular well suited to produce modular categories. Let us specialize to the $\mathfrak{sl}_2$–case. Let $\theta = \exp(i\pi/r)$, r an integer ≥ 2, and let $U_\theta(\mathfrak{sl}_2(\mathbb{C}))$ be Lusztig's version of the quantum group associated to θ and $\mathfrak{sl}_2(\mathbb{C})$. This is a Hopf algebra over $\mathbb{C}$, see the above references for the definition. (The root of unity θ is denoted q in [BK] and ε in [Kir].) The representation theory of $U_\theta(\mathfrak{sl}_2(\mathbb{C}))$ induces a modular category $(\mathcal{V}'_\theta, \{V'_i\}_{i\in I})$, $I = \{0,1,\dots,r-2\}$, with $S-$ and

T–matrices identical with the S– and T–matrices for the modular category $\mathcal{V}_t$, $t = \exp(i\pi/(2r))$, considered above, cf. [Kir, Theorem 3.9], [BK, Theorem 3.3.20]. (One should note different notation in [Kir] and [BK]. Note that the $\tilde{s}$–matrix in [BK] is Turaev's S–matrix of the mirror of $\mathcal{V}'_\theta$, i.e. $\tilde{s}_{i,j} = S_{i^*,j}$, $i,j \in I$, and that s in [Kir] is identical with $\tilde{s}$ in [BK] and vice versa.) The dimension of the simple object V'_i of $\mathcal{V}'_\theta$ is equal to the dimension of the simple object V_i of $\mathcal{V}_t$. The categories $\mathcal{V}_t$ and $\mathcal{V}'_\theta$ therefore also have the same ranks and the same Δ. Similar to the proofs of Lemma 8.1 and Lemma 8.2 we find that V'_i is self-dual with associated $\varepsilon_i = (-1)^i$, $i \in I$. We conclude that $\mathcal{V}'_\theta$ and $\mathcal{V}_t$ give the same invariants of the Seifert manifolds. Probably these two categories are equivalent giving the same invariants for all closed oriented 3–manifolds, but we will not check the details here.

In [Tu, Problems, question 8 p. 571] it is asked whether there exist unitary (or at least Hermitian) modular categories that are not unimodal. By combining the above with [Kir], [W] we can answer this question by a yes. The non-unimodal modular categories $(\mathcal{V}'_\theta, \{V'_i\}_{i\in I})$ provide such examples.

In [Tu, Chap. XII] Turaev constructs a unimodular category $(\mathcal{V}_p(a), \{W_i\}_{i\in I})$ for any primitive $4r$'th root of unity a using Kauffman's skein theoretical approach to the Jones polynomial together with the Jones–Wenzl idempotens. Here $I = \{0, 1, \dots, r-2\}$ as above and all the simple objects are self-dual. (In [Tu] W_i is denoted V_i.) In [Tu, Problems, question 24 p. 572] it is asked whether $\mathcal{V}_p(a)$ (with ground ring $\mathbb{C}$) is equivalent (as modular category) to the modular category $\left(\mathcal{V}''_q, \{V''_i\}_{i\in I}\right)$ induced by the representation theory of $U_q(\mathfrak{sl}_2(\mathbb{C}))$ for $q = -a^2$. Here $U_q(\mathfrak{sl}_2(\mathbb{C}))$ is given in [Ka, Sections VI.1 and VII.1] and differs slightly from the $U_q(\mathfrak{sl}_2(\mathbb{C}))$ given in this section and also from Lusztig's version, see above. For any primitive $2r$'th root of unity q the modular category $\mathcal{V}''_q$ is non-unimodal, so the answer to the above question is no. In fact, by using arguments similar to the proofs of Lemma 8.1 and Lemma 8.2, one finds that V''_i is self-dual with associated $\varepsilon_i = (-1)^i$, $i \in I$. One can construct a non-unimodal modular category $\mathcal{V}'_p(a)$ by changing the twist θ in $\mathcal{V}_p(a)$ slightly preserving all other structure. In fact one can construct a new twist θ' satisfying $\theta'_{W_i} = (-1)^i \theta_{W_i}$, $i \in I$. (One simply changes the twist θ_n, $n = 0, 1, 2, \dots$, in the skein category in [Tu, Sect. XII.2] into $(-1)^n \theta_n$.) By the definition of the elements ε_i, see Sect. 3 above Lemma 3.1, we immediately get $\varepsilon_i = (-1)^i$, $i \in I$, for $\mathcal{V}'_p(a)$. In his thesis [Th1] H. Thys shows that the modular category $\mathcal{V}'_p(a)$ is equivalent to the modular category $\mathcal{V}''_q$, if q is a primitive root of unity satisfying $q = a^2$, see also [Th2]. By using the twist θ' instead of θ in the last part of the proof of [Tu, Theorem XII.7.1] and in [Tu, Exercise XII.6.10 1)] one finds that the S– and T–matrices for $\mathcal{V}'_p(a)$ are identical to these matrices for $\mathcal{V}_a$. Moreover the dimension of the simple object W_i of $\mathcal{V}'_p(a)$ is equal to the dimension of the simple object V_i of $\mathcal{V}_a$, $i \in I$ (use [Tu, Sect. XII.6.8]). We conclude that these two modular categories give the same invariants of the Seifert manifolds. Probably these two categories are equivalent giving the same invariants for all closed oriented 3–manifolds, but we will not check the details here.

9 Appendices

A. Normalizations of the RT–invariants

As a convenience to the reader we compare in this appendix the normalizations of the RT–invariants used in the literature in particular the ones used in [RT2], [KM1], [TW1], [Le] and [Tu]. The invariants of 3–manifolds with embedded colored ribbon graphs constructed in [RT2], see also [TW1], are based on modular Hopf algebras. The definition of a modular Hopf algebra in [Tu, Chap. XI] is slightly simplified compared to [RT2], [TW1]. If $(A, R, v, \{V_i\}_{i\in I})$ is a modular Hopf algebra as defined in [RT2], [TW1], then $\left(A, R, v^{-1}, \{V_i\}_{i\in I}\right)$ is a modular Hopf algebra as defined in [Tu]. (The definition of the v_i's on p. 557 in [RT2] has to be changed according to [TW1]. That is, $v_i \mathrm{id}_{V_i}$ should be equal to the map $V_i \to V_i$ given by multiplication with v^{-1} instead of the map given by multiplication with v.) Let $(\mathcal{V}, \{V_i\}_{i\in I})$ be the modular category induced by the modular Hopf algebra $\left(A, R, v^{-1}, \{V_i\}_{i\in I}\right)$, cf. [Tu, Chap. XI], and let (M, Ω) be as in (2). The invariant of the pair (M, Ω) as defined in [RT2] is given by

$$\mathcal{F}(M,\Omega) = C^{-\sigma_-(L)} \sum_{\lambda\in \mathrm{col}(L)} \left(\prod_{i=1}^{m} d_{\lambda(L_i)} \right) F_{\mathcal{V}}(\Gamma(L,\lambda) \cup \Omega).$$

Here $C = \sum_{i\in I} v_i^{-1} \dim(i) d_i$, where $\{d_i\}_{i\in I}$ is the unique solution to

$$\sum_{i\in I} v_i S_{i,j} d_i = v_j^{-1} \dim(j) \quad , j \in I, \tag{50}$$

where S is the S–matrix of $\mathcal{V}$. Moreover $\sigma_-(L)$ is the number of negative eigenvalues of the intersection form of W_L. By comparing with [Tu, Sect. II.3] we see that $C = x d_0$, where $x = \sum_{i\in I} d_i \dim(i) = \Delta = \Delta_{\mathcal{V}}$, and $\mathcal{F}(M,\Omega) = \tau'_{\mathcal{V}}(M,\Omega) = (\Delta \mathcal{D}^{-1})^{b_1(M)} \mathcal{D} \tau_{(\mathcal{V},\mathcal{D})}(M,\Omega)$, where $\mathcal{D}$ is a rank of $\mathcal{V}$ and $b_1(M)$ is the first betti number of M. According to [Tu, p. 89] we also have $C = xd_0 = (\Delta\mathcal{D}^{-1})^2$. In [TW1] the invariant $\mathcal{F}(M,\Omega)$ is slightly changed to

$$\tau_A(M,\Omega) = C_0^{\sigma(L)-m} \sum_{\lambda\in\operatorname{col}(L)} \left(\prod_{i=1}^m d_{\lambda(L_i)}\right) F(\Gamma(L,\lambda)\cup\Omega),$$

where C_0 is a square root of C. For $C_0 = \Delta\mathcal{D}^{-1}$ we have

$$\tau_A(M,\Omega) = C_0^{-b_1(M)}\mathcal{F}(M,\Omega) = \mathcal{D}\tau_{(\mathcal{V},\mathcal{D})}(M,\Omega),$$

which follows by using that $\sigma_-(L) = (m - b_1(M) - \sigma(L))/2$. In case $A = U_t$, $t = \exp(i\pi/(2r))$, $r \geq 2$, see [RT2, Sect. 8] and the beginning of Sect. 8 in this paper, $\tau_A(M) = \tau_A(M,\emptyset)$ is equal to the invariant $\tau_r(M)$ in [KM1].

To compare with [Le] we use a more symmetric expression for $\tau_{(\mathcal{V},\mathcal{D})}$. To this end let $\Delta^- = \Delta$ and $\Delta^+ = \Delta_{\overline{\mathcal{V}}}$, so $\Delta^\pm = \sum_{i\in I} v_i^{\pm 1}(\dim(i))^2$. Moreover, let $\sigma_+(L)$ be the number of positive eigenvalues of the intersection form of W_L. Then, by using (3) and the above formula for $\sigma_-(L)$, one gets

$$\begin{aligned}\tau_{(\mathcal{V},\mathcal{D})}(M,\Omega) &= \mathcal{D}^{-b_1(M)-1}(\Delta^+)^{-\sigma_+(L)}(\Delta^-)^{-\sigma_-(L)} \\ &\quad \times \sum_{\lambda\in\operatorname{col}(L)} \left(\prod_{i=1}^m \dim(\lambda(L_i))\right) F(\Gamma(L,\lambda)\cup\Omega).\end{aligned}$$

The invariant $\mathcal{D}^{b_1(M)+1}\tau_{(\mathcal{V},\mathcal{D})}(M,\Omega)$ is the invariant considered in [Le] in case the modular categories are the ones induced by the quantum groups associated to simple finite dimensional complex Lie algebras.

B. Framed links in closed oriented 3–manifolds

In this appendix we discuss different ways of presenting a framing of a link L in an arbitrary closed oriented 3–manifold M. We will here explicitly work in the smooth category so we can use differential topological concepts. To simplify writing we restrict to the case of knots. The generalization to links will be obvious.

Three ways of defining a framing Let K be a knot in a closed orientable 3–manifold M, let $TM|_K$ be the restriction of the tangent bundle of M to K, and let $NK = TM|_K/TK$ be the normal bundle of K. Since M and K are orientable, NK is an orientable 2–dimensional real vector bundle over K. Isomorphism classes of oriented 2–dimensional real vector bundles over $K \cong S^1$ are in 1-1 correspondence with $\pi_1(BSO(2)) \cong \pi_0(SO(2)) = 0$, so NK is trivializable. We let $\mathcal{S}_{\mathrm{tr}}(K)$ be the set of isotopy classes of trivializations of NK. There is a 1-1 correspondence between $\mathcal{S}_{\mathrm{tr}}(K)$ and $\pi_1(GL(2,\mathbb{R})) \cong \pi_1(O(2)) \cong \mathbb{Z}$ through homotopy classes of transition functions. However, there is in general no canonical choise of this bijection. A normal vector field on K is a nowhere vanishing section in NK. Two normal vector fields on K are homotopic if they can be deformed into one another within the class of normal vector fields on K. We let $\mathcal{S}_{\mathrm{nvf}}(K)$ be the set of homotopy classes of normal vector fields on K. Finally let $\mathcal{S}_{\mathrm{emb}}(K)$ be the set of isotopy classes of embeddings $Q\colon B^2 \times S^1 \to M$ with $Q(0 \times S^1) = K$ (nothing about orientations here contrary to Conventions 5.2 in Sect. 5). A framing of K is an element in one of the sets $\mathcal{S}_{\mathrm{nvf}}(K)$, $\mathcal{S}_{\mathrm{tr}}(K)$, $\mathcal{S}_{\mathrm{emb}}(K)$.

Claim *We have a diagram of maps*

$$\begin{tikzcd} \mathcal{S}_{\mathrm{emb}}(K) \arrow[rr, shift left, "J"] \arrow[dr, "F"'] & & \mathcal{S}_{\mathrm{tr}}(K) \arrow[ll, shift left, "H_1{,}H_2"] \\ & \mathcal{S}_{\mathrm{nvf}}(K) \arrow[ur, "G_1{,}G_2"'] & \end{tikzcd}$$

with $J \circ H_\nu$ and $F \circ H_\nu \circ G_\nu$ the identity maps, $\nu = 1,2$. In particular H_ν and G_ν are injective, $\nu = 1,2$, and F and J surjective. The images of G_1 and G_2 have the same cardinality and they are disjoint with union $\mathcal{S}_{\mathrm{tr}}(K)$. The union of the images of H_1 and H_2 is $\mathcal{S}_{\mathrm{emb}}(K)$. Fix an orientation on K and let $-K$ be K with the opposite orientation. Then $H_1 = H_2$ if K and $-K$ are isotopic, so in particular this map is an isomorphism (with inverse J). If K and $-K$ are non-isotopic then H_1 and H_2 have disjoint images with the same cardinality.

Proof of claim Let Q be an embedding as above and let $\xi(x) = Q(0,x)$. Define a normal vector field on K by $X_Q(\xi(x)) = \Pi_K \circ T_{(0,x)}Q(e_1,0)$, where e_1 is the first standard unit vector in $\mathbb{R}^2$ and $\Pi_K\colon TM|_K \to NK$ is the projection. The map $Q \mapsto X_Q$ induces a map $F\colon \mathcal{S}_{\mathrm{emb}}(K) \to \mathcal{S}_{\mathrm{nvf}}(K)$. By $\sigma_Q(y,\xi(x)) = \Pi_K \circ T_{(0,x)}Q(y,0)$ we get a trivialization of NK. The map $Q \mapsto \sigma_Q$ induces a map $J\colon \mathcal{S}_{\mathrm{emb}}(K) \to \mathcal{S}_{\mathrm{tr}}(K)$. Let X be a normal vector field on K. Fix an orientation of NK and choose a normal vector field Y on K so $\{X,Y\}$ is a positively oriented frame for NK. Let σ_X be the corresponding trivialization of NK, i.e. $\sigma_X(ue_1 + ve_2, p) = uX(p) + vY(p)$. The map $X \mapsto \sigma_X$ induces a map $G_1\colon \mathcal{S}_{\mathrm{nvf}}(K) \to \mathcal{S}_{\mathrm{tr}}(K)$. Let G_2 be defined as G_1 but using the opposite orientation of NK. Finally for a trivialization σ of NK, a parametrisation $\xi\colon S^1 \to K$, and a tubular map $\tau\colon NK \to M$ we get an embedding $Q_\sigma\colon B^2 \times S^1 \to M$ by $Q_\sigma(y,x) = \tau(\sigma(y,\xi(x)))$. Here $\tau\colon NK \to M$ is an embedding, which on K is the inclusion $K \subset M$ and for which the differential induces the identity on the zero-section, cf. [BJ, p. 123]. The map $\sigma \mapsto Q_\sigma$ induces a map $H_1\colon \mathcal{S}_{\mathrm{tr}}(K) \to \mathcal{S}_{\mathrm{emb}}(K)$. Let H_2 be defined as H_1 but using $\xi \circ \kappa$ instead of ξ, where $\kappa\colon S^1 \to S^1$ is an orientation reversing diffeomorphism. By the property of the differential of τ, we get immediately the first claim (use that $\Pi_K \circ T_{(0,x)} Q_\sigma|_{T_0 B^2 \oplus 0} = \sigma(-,\xi(x))$.) Let $g\colon \mathbb{R}^2 \to \mathbb{R}^2$ be an orientation reversing diffeomorphism. For an embedding $Q\colon B^2 \times S^1 \to M$ we let $\bar{Q} = Q \circ (\mathrm{id}_{B^2} \times \kappa)$ and $\hat{Q} = Q \circ (g|_{B^2} \times \mathrm{id}_{S^1})$. Similarly for a trivialization σ of NK we let $\hat{\sigma} = \sigma \circ (g \times \mathrm{id}_K)$. Note that σ and $\hat{\sigma}$ are non-isotopic. The claims about G_1 and G_2 follows then basically by the observation $G_2([X]) = [\hat{\sigma}_X]$. Fix an orientation of K and let $-K$ be K with the opposite orientation. Then Q and $\bar{Q}$ are isotopic if and only if K and $-K$ are isotopic. The claims about H_1 and H_2 then basically follow from the observation $H_2([\sigma]) = \bar{Q}_\sigma$. (Use that the isotopy class of Q is completely determined by $Q|_{0 \times S^1}$ and $\Pi_K \circ T_{(0,x)} Q|_{T_0 B^2 \oplus 0}$.) □

Note that there are oriented knots K for which K and $-K$ are not isotopic, cf. [Tr]. Also note that for an embedding $Q\colon B^2 \times S^1 \to M$, F maps the isotopy classes of Q, $\hat{Q}$, $\bar{Q}$, and $\hat{\bar{Q}}$ to the same point. Here Q and $\hat{Q}$ are always non-isotopic.

Integral homology spheres Let us consider the case where $M = S^3$ or more generally where M is an integral homology sphere (meaning that $H_*(M;\mathbb{Z}) = H_*(S^3;\mathbb{Z})$). Then we have a well-defined linking number $\mathrm{lk}(.,.)$ between knots in M. If $Q\colon B^2 \times S^1 \to M$ is an embedding with $Q(0 \times S^1) = K$ we let K' be the knot $Q(e_1 \times S^1)$. Fix an orientation of K and give K' the induced orientation, i.e. $[K] = [K']$ in $H_1(U;\mathbb{Z})$, $U = Q(B^2 \times S^1)$. Note that we get the same parallel K' (up to isotopy) if we use $\hat{Q}$ or $\bar{Q}$ instead (see the above proof). We therefore have an identification $\mathcal{S}_{\mathrm{nvf}}(K) \cong \mathbb{Z}$ by $\mathrm{lk}(K,K')$. The framing corresponding to zero is sometimes called the *preferred framing*, cf. [Ro1, p. 31 and p. 136]. (We also have $\mathcal{S}_{\mathrm{nvf}}(K) \cong \mathbb{Z}$ in the general case, see the claim above, but we do not in general have a canonical choice of a zero.)

Notes on surgery Assume that M is an arbitrary closed oriented 3–manifold and that $Q\colon B^2 \times S^1 \to M$ is an orientation preserving embedding, where $B^2 \times S^1$ is the oriented standard solid torus and $U = Q(B^2 \times S^1)$ is given the orientation induced by that of M, see Conventions 5.2. Then $\hat{\bar{Q}}$ is also orientation preserving. However from a surgical point of view this causes no problems since rational surgery along (K,Q), $K = Q(0 \times S^1)$, with surgery coefficient p/q as defined in (23) is identical with rational surgery along $(K,\hat{\bar{Q}})$

with the same surgery coefficient p/q. If we change the orientation of M (M is connected) then by Conventions 5.2 we must use $\bar{Q}$ (or $\hat{Q}$) when doing surgery along K. This changes the signs of all surgery coefficients for a given surgery. If $M = S^3 = \mathbb{R}^3 \cup \{\infty\}$ given the standard right-handed orientation, then rational surgery on M along K as defined in (23), where K is given the preferred framing, is ordinary rational surgery on S^3 along K as defined in e.g. [Ro1, Sect. 9.F].

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