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On the cohomology algebra of a fiber

Luc Menichi

Abstract Let $f : E \to B$ be a fibration of fiber F. Eilenberg and Moore have proved that there is a natural isomorphism of vector spaces between $H^*(F;\mathbb{F}_p)$ and $\mathrm{Tor}^{C^*(B)}(C^*(E),\mathbb{F}_p)$. Generalizing the rational case proved by Sullivan, Anick [2] proved that if X is a finite r-connected CW-complex of dimension $\leq rp$ then the algebra of singular cochains $C^*(X;\mathbb{F}_p)$ can be replaced by a commutative differential graded algebra $A(X)$ with the same cohomology. Therefore if we suppose that $f : E \hookrightarrow B$ is an inclusion of finite r-connected CW-complexes of dimension $\leq rp$, we obtain an isomorphism of vector spaces between the algebra $H^*(F;\mathbb{F}_p)$ and $\mathrm{Tor}^{A(B)}(A(E),\mathbb{F}_p)$ which has also a natural structure of algebra. Extending the rational case proved by Grivel-Thomas-Halperin [13, 15], we prove that this isomorphism is in fact an isomorphism of algebras. In particular, $H^*(F;\mathbb{F}_p)$ is a divided powers algebra and p^{th} powers vanish in the reduced cohomology $\tilde{H}^*(F;\mathbb{F}_p)$.

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1 Introduction

Let $f : E \twoheadrightarrow B$ be a fibration of fiber F with simply connected base B. A major problem in Algebraic Topology is to compute the homotopy type of F.

In 1966, S. Eilenberg and J. Moore [9] proved that the cohomology of F with coefficients in a field $\mathbb{k}$, denoted $H^*(F;\mathbb{k})$, is entirely determined, as graded vector spaces by the structure of $C^*(B;\mathbb{k})$-module induced on $C^*(E;\mathbb{k})$ through f.(Here $C^*(-;\mathbb{k})$ denotes the singular cochains.) More precisely, they generalize the classical notion of derived functor "Tor" to the differential case and obtain a natural isomorphism of graded vector spaces

$$H^*(F) \cong \mathrm{Tor}^{C^*(B)}(C^*(E),\mathbb{k}).$$

In the rational case, $C^*(X;\mathbb{Q})$ is equivalent to a commutative cochain algebra $A_{PL}(X)$ [23, 12] which carries the rational homotopy type of X. Moreover, the Eilenberg-Moore isomorphism is induced by a quasi-isomorphism between $A_{PL}(F)$ and a commutative cochain algebra A constructed from $A_{PL}(B)$ and $A_{PL}(E)$. In particular, the Eilenberg-Moore isomorphism is an isomorphism of graded algebras.

In general, the Eilenberg-Moore isomorphism does not give the multiplicative structure of $H^*(F;\Bbbk)$. However the main result of this paper asserts that for char $\Bbbk$ "sufficiently large ", the Eilenberg-Moore isomorphism is an isomorphism of graded algebras with respect to a natural multiplicative structure on Tor.

We now give the precise statement of our main result:

Over a field $\Bbbk$ of positive characteristic p, Anick [2, Proposition 8.7(a)] proved that if X is a finite r-connected CW-complex of dimension $\leq rp$, the algebra of singular cochains $C^*(X)$ is naturally linked to a commutative differential graded algebra $A(X)$ by morphisms of differential graded algebras inducing isomorphisms in homology. Therefore if we suppose that $f : E \hookrightarrow B$ is an inclusion of finite r-connected CW-complexes of dimension $\leq rp$, we obtain the isomorphism of graded vector spaces

$$\mathrm{Tor}^{C^*(B)}(C^*(E),\Bbbk) \cong \mathrm{Tor}^{A(B)}(A(E),\Bbbk).$$

Thus the Eilenberg-Moore isomorphism becomes

$$H^*(F;\Bbbk) \cong \mathrm{Tor}^{A(B)}(A(E),\Bbbk).$$

Now since $A(B)$ and $A(E)$ are commutative, $\mathrm{Tor}^{A(B)}(A(E),\Bbbk)$ has a natural structure of algebra. We prove

Theorem A *Assume the characteristic of the field $\Bbbk$ is an odd prime p and consider an inclusion $E \hookrightarrow B$ of finite r-connected CW-complexes ($r \geq 1$) of dimension $\leq rp$. Then the Eilenberg-Moore isomorphism*

$$H^*(F;\Bbbk) \cong Tor^{A(B)}(A(E),\Bbbk)$$

is an isomorphism of graded algebras.

As corollary of this main result, we obtain

Theorem B (8.7) *Let p be an odd prime. Consider the homotypy fiber F of an inclusion of finite r-connected CW-complexes of dimension $\leq rp$. Then the cohomology algebra $H^*(F;\mathbb{F}_p)$ is a divided powers algebra. In particular, p^{th} powers vanish in the reduced cohomology $\tilde{H}^*(F;\mathbb{F}_p)$.*

In fact, Theorem A is a consequence of the model theorem (Theorem 4.2) which establishes, for any fibration $F \hookrightarrow E \twoheadrightarrow B$, the existence of a coalgebra model up to homotopy of the ΩE-fibration

$$\Omega E \to \Omega B \to F.$$

Approaches concerning the general problem of computing the cohomology algebra of a fiber, different from our model theorem, are given in [8] and in [22].

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2 The two-sided bar construction

We use the terminology of [11]. In particular, a quasi-isomorphism is denoted $\xrightarrow{\simeq}$. Let A be an augmented differential graded algebra, M a right A-module, N a left A-module. Denote by d_1 the differential of the complex $M \otimes T(s\overline{A}) \otimes N$ obtained by tensorization, and denote by $s\overline{A}$ the suspension of the augmentation ideal $\overline{A}$, $(s\overline{A})_i = \overline{A}_{i-1}$. Let $|x|$ be the degree of an element x in any graded object. We denote the tensor product of the elements $m \in M$, $sa_1 \in s\overline{A}$, ... , $sa_k \in s\overline{A}$ and $n \in N$ by $m[sa_1|\cdots|sa_k]n$. Let d_2 be the differential on the graded vector space $M \otimes T(s\overline{A}) \otimes N$ defined by:

$$\begin{aligned} d_2 m[sa_1|\cdots|sa_k]n &= (-1)^{|m|} ma_1[sa_2|\cdots|sa_k]n \\ &\quad + \sum_{i=1}^{k-1} (-1)^{\varepsilon_i} m[sa_1|\cdots|sa_i a_{i+1}|\cdots|sa_k]n \\ &\quad - (-1)^{\varepsilon_{k-1}} m[sa_1|\cdots|sa_{k-1}]a_k n; \end{aligned}$$

Here $\varepsilon_i = |m| + |sa_1| + \cdots + |sa_i|$.

The *bar construction of A with coefficients in M and N*, denoted $B(M;A;N)$, is the complex $(M \otimes T(s\overline{A}) \otimes N, d_1 + d_2)$. We use mainly $B(M;A) = B(M;A;\mathbb{k})$. The *reduced bar construction of A*, denoted $B(A)$, is $B(\mathbb{k};A)$.

Let B be another augmented differential graded algebra, P a right B-module and Q a left B-module. Then we have the natural Alexander-Whitney morphism of complexes ([19, X.7.2] or [7, XI.6(3) computation of the $\vee$ product])

$$AW : B(M \otimes P; A \otimes B; N \otimes Q) \to B(M;A;N) \otimes B(P;B;Q)$$

where the image of a typical element $m \otimes p[s(a_1 \otimes b_1)|\cdots|s(a_k \otimes b_k)]n \otimes q$ is
$$\sum_{i=0}^{k}(-1)^{\zeta_i} m[sa_1|\cdots|sa_i]a_{i+1}\cdots a_k n \otimes pb_1\cdots b_i[sb_{i+1}|\cdots|sb_k]q.$$

$$\text{Here}\quad \zeta_i = \sum_{j=1}^{k}\left(|p| + \sum_{l=1}^{j-1}|b_l|\right)|a_j| + \left(|p| + \sum_{j=1}^{k}|b_j|\right)|n| + \sum_{j=i+1}^{k}(j-i)|a_j| + (k-i)|n| + |i||p| + \sum_{j=1}^{i-1}(i-j)|b_j|.$$

Property 2.1 *If there exist a morphism of augmented algebras $\Delta_A : A \to A \otimes A$ and morphisms of A-modules $\Delta_M : M \to M \otimes M$, $\varepsilon_M : M \twoheadrightarrow \mathbb{k}$, $\Delta_N : N \to N \otimes N$, $\varepsilon_N : N \twoheadrightarrow \mathbb{k}$ then $B(\varepsilon_M;\varepsilon_A;\varepsilon_N) : B(M;A;N) \twoheadrightarrow B(\mathbb{k};\mathbb{k};\mathbb{k}) = \mathbb{k}$ is an augmentation for $B(M;A;N)$ and the composite*

$$B(M;A;N) \xrightarrow{B(\Delta_M;\Delta_A;\Delta_N)} B(M\otimes M; A\otimes A; N\otimes N) \xrightarrow{AW} B(M;A;N)\otimes B(M;A;N)$$

is a morphism of complexes. In particular, if A is a differential graded Hopf algebra and if M and N are A-coalgebras then $B(M;A;N)$ is a differential graded coalgebra. This coalgebra structure on $B(M;A;N)$ is functorial.

Property 2.2 *Moreover, if M is A-semifree (in the sense of [11, §2]) then $B(M;A;N) \overset{\simeq}{\twoheadrightarrow} M \otimes_A N$ is a quasi-isomorphism of coalgebras.*

Theorem 2.3 ([11, 5.1] or [18, Theorem 3.9 and Corollary 3.10]) *Let $p : E \twoheadrightarrow B$ be a right G-fibration with B path connected. Then there is a natural quasi-isomorphism of coalgebras*

$$B(C_*(E);C_*(G)) \overset{\simeq}{\twoheadrightarrow} C_*(B).$$

Corollary 2.4 *Let $f : E \to B$ be a continuous pointed map with E and B path connected. If its homotopy fiber F is path connected, then there is a chain coalgebra $G(f)$ equipped with two natural isomorphisms of chain coalgebras*

$$C_*(F) \overset{\simeq}{\twoheadleftarrow} G(f) \overset{\simeq}{\twoheadrightarrow} B(C_*(\Omega B);C_*(\Omega E)).$$

This Corollary proves that the cohomology algebra $H^*(F)$ is determined by the Hopf algebra morphism $C_*(\Omega f) : C_*(\Omega E) \to C_*(\Omega B)$. This is the starting observation of our paper. In the next section, we extend Property 2.1 to Hopf algebras and coalgebras up to homotopy: i.e. we do not require strict coassociativity of the diagonals.

3 Hopf algebras and coalgebras up to homotopy

Let f, $g : A \to B$ be two morphisms of augmented differential graded algebras. A linear map $h : A \to B$ of (lower) degree $+1$ is a *homotopy of algebras* from f to g denoted $h : f \approx_a g$ if $\varepsilon_B h = 0$, $hd + dh = f - g$ and $h(xy) = h(x)g(y) + (-1)^{|x|} f(x)h(y)$ for $x, y \in A$. The symbol $\approx$ will be reserved to the usual notion of chain homotopy.

A *(cocommutative) coalgebra up to homotopy* is a complex C equipped with a morphism of complexes $\Delta : C \to C \otimes C$ and a morphism $\varepsilon : C \to \mathbb{k}$ such that $(\varepsilon \otimes 1) \circ \Delta = 1 = (1 \otimes \varepsilon) \circ \Delta$ (strict counitary), $(\Delta \otimes 1) \circ \Delta \approx (1 \otimes \Delta) \circ \Delta$ (homotopy coassociativity) and $\tau \circ \Delta \approx \Delta$ (homotopy cocommutativity). Here $\tau(x \otimes y) = (-1)^{|x||y|} y \otimes x$. Let C and C' be two coalgebras up to homotopy. A morphism of complexes $f : C \to C'$ is a *morphism of coalgebras up to homotopy* if $\Delta f \approx (f \otimes f)\Delta$ and $\varepsilon \circ f = \varepsilon$.

A *(cocommutative) Hopf algebra up to homotopy* is a differential graded algebra K equipped with two morphisms of algebras $\Delta : K \to K \otimes K$ and $\varepsilon : K \to \mathbb{k}$ such that $(\varepsilon \otimes 1) \circ \Delta = 1 = (1 \otimes \varepsilon) \circ \Delta$, $(\Delta \otimes 1) \circ \Delta \approx_a (1 \otimes \Delta) \circ \Delta$ and $\tau \circ \Delta \approx_a \Delta$. Let K, K' be two Hopf algebras up to homotopy. A morphism of augmented differential graded algebras $f : K \to K'$ is a *a morphism of Hopf algebras up to homotopy* if $\Delta f \approx_a (f \otimes f)\Delta$.

Lemma 3.1 *Suppose $\varphi \approx_a \varphi' : A \to A'$ and $\Psi \approx_a \Psi' : M \to M'$ via algebraic homotopies h and h', with φ, φ', Ψ, Ψ' morphisms of augmented chain algebras. Let $f : A \to M$ and $g : A' \to M'$ be two morphisms of augmented chain algebras such that $\Psi \circ f = g \circ \varphi$ and $\Psi' \circ f = g \circ \varphi'$. We summarize this situation by the "diagram"*

$$\begin{tikzcd} A \arrow[r, "h:\varphi\approx_a\varphi'"] \arrow[d, "f"'] & A' \arrow[d, "g"] \\ M \arrow[r, "h':\Psi\approx_a\Psi'"'] & M' \end{tikzcd}$$

If $h' \circ f = g \circ h$ (naturality of the homotopies) then the morphisms of augmented chain complexes $B(\Psi;\varphi)$ and $B(\Psi';\varphi')$ are chain homotopic.

Proof The explicit chain homotopy Θ between $B(\Psi;\varphi)$ and $B(\Psi';\varphi')$ is given by

$$\Theta(m[sa_1|\dots|sa_k]) = h'(m)[s\varphi'(a_1)|\dots|s\varphi'(a_k)]$$
$$-\sum_{i=1}^{k}(-1)^{\varepsilon_{i-1}}\Psi(m)[s\varphi(a_1)|\dots|s\varphi(a_{i-1})|sh(a_i)|s\varphi'(a_{i+1})|\dots|s\varphi'(a_k)]$$

Recall that $\varepsilon_{i-1} = |m|+|sa_1|+\cdots+|sa_{i-1}|$. Since Θ is just the chain homotopy obtained by tensorization,

$$d_1\Theta + \Theta d_1 = \Psi \otimes T(s\varphi) - \Psi' \otimes T(s\varphi').$$

It remains to check that $d_2\Theta + \Theta d_2 = 0$. □

From Lemma 3.1, one deduces:

Lemma 3.2 (i) *Let K (respectively C) be a Hopf algebra up to homotopy, coassociative up to a homotopy h_{assocK} (respectively h_{assocC}): $(\Delta\otimes 1)\circ\Delta \approx_a (1\otimes\Delta)\circ\Delta$ and cocommutative up a homotopy h_{comK} (respectively h_{comC}): $\tau\circ\Delta\approx_a\Delta$. Let $f : K \to C$ be a morphism of augmented algebras such that $\Delta_C f = (f\otimes f)\Delta_K$, $h_{assocC}f = (f\otimes f\otimes f)h_{assocK}$ and $h_{comC}f = (f\otimes f)h_{comK}$ (f commutes with the diagonals and the homotopies of coassociativity and cocommutativity). Then $B(C;K)$ with the diagonal*

$$B(C;K)\xrightarrow{B(\Delta_C;\Delta_K)} B(C\otimes C;K\otimes K)\xrightarrow{AW} B(C;K)\otimes B(C;K)$$

is a (cocommutative) coalgebra up to homotopy.

(ii) *Suppose given the following cube of augmented chain algebras*

$$\begin{tikzcd}
K \arrow[rr, "\varphi"] \arrow[dr, "\Delta_K"] \arrow[dd, "f"'] & & K' \arrow[dr, "\Delta_{K'}"] \arrow[dd, dotted, "g"] & \\
& K\otimes K \arrow[rr, "\varphi\otimes\varphi"] \arrow[dd, "\Psi"] & & K'\otimes K' \arrow[dd, "g\otimes g"] \\
C \arrow[rr, dotted, "f\otimes f"] \arrow[dr, "\Delta_C"'] & & C' \arrow[dr, dotted, "\Delta_{C'}"] & \\
& C\otimes C \arrow[rr, "\Psi\otimes\Psi"'] & & C'\otimes C'
\end{tikzcd}$$

where all the faces commute exactly except the top and the bottom ones. Suppose that the top face commutes up to a homotopy h_{top} : $(\varphi \otimes \varphi)\Delta_K \approx_a \Delta_{K'}\varphi$ and the bottom face commutes up to a homotopy $h_{bottom} : (\Psi \otimes \Psi)\Delta_C \approx_a \Delta_{C'}\Psi$ such that $h_{bottom} f = (g \otimes g)h_{top}$. Then the morphism of augmented chain complexes $B(\Psi;\varphi) : B(C;K) \to B(C';K')$ commutes with the diagonals up to chain homotopy.

4 The model Theorem

Let X be a graded vector space. We denote a free chain algebra (TX,∂) simply by TX except when the differential ∂ can be specified. In particular, a free chain algebra with zero differential is still denoted by $(TX,0)$.

Let $f : E \to B$ be a map between path connected pointed topological spaces with a path connected homotopy fiber F. Then there is a commutative diagram of augmented chain algebras as follows:

$$\begin{tikzcd} TX \arrow[r, "\simeq", "m_X"'] \arrow[d, "m(f)"'] & C_*(\Omega E) \arrow[d, "C_*(\Omega f)"] \\ TY \arrow[r, "\simeq", "m_Y"'] & C_*(\Omega B) \end{tikzcd} \tag{4.1}$$

where TX, TY are free chain algebras, m_X, m_Y are quasi-isomorphisms and $m(f) : TX \rightarrowtail TY$ is a free extension (in the sense of [11, §3]).

Theorem 4.2 *With the above:*

(1) *TX (respectively TY) can be endowed with an structure of Hopf algebras up to homotopy such that m_X (respectively m_Y) commutes with the diagonals up to a homotopy h_X (respectively h_Y) and such that the diagonal of TY extends the diagonal of TX, the homotopy of coassociativity of TY extends the homotopy of coassociativity of TY, the homotopy of cocommutativity of TY extends the homotopy of cocommutativity of TX and h_Y extends $(C_*(\Omega f) \otimes C_*(\Omega f))h_X$.*

(2) *$B(m_Y;m_X) : B(TY;TX) \overset{\simeq}{\rightrightarrows} B(C_*(\Omega B);C_*(\Omega E))$ is a morphism of coalgebras up to homotopy.*

(3) *The homology of the coalgebra up to homotopy $TY \otimes_{TX} \mathbb{k}$ is isomorphic to $H_*(F)$ as coalgebras.*

It is easy to see that the isomorphism of graded coalgebras between $H_*(TY \otimes_{TX} \Bbbk)$ and $H_*(F)$ fits into the commutative diagram of graded coalgebras:

$$\begin{tikzcd} H_*(TY) \arrow[r, "\cong", "H_*(m_Y)"'] \arrow[d, "H_*(q)"'] & H_*(\Omega B) \arrow[d, "H_*(\partial)"] \\ H_*(TY \otimes_{TX} \Bbbk) \arrow[r, "\cong"] & H_*(F) \end{tikzcd}$$

where $\partial : \Omega B \hookrightarrow F$ is the inclusion $\Omega B \times * \subset PB \times_B E$ and $q : TY \twoheadrightarrow TY \otimes_{TX} \Bbbk$ is the quotient map.

The exact commutativity of the diagram 4.1 is not important. If the diagram commutes only up to homotopy, since $m(f)$ is a free extension, using the lifting lemma [11, 3.6], we can replace m_Y by another m_Y which is homotopic to it, so that now the diagram strictly commutes. On the contrary, it is important that $m(f)$ is a free extension. We will show it in Section 5. Indeed, the general idea for the proof of part 1. is to keep control of the homotopies using the homotopy extension property of cofibrations.

Proof of Theorem 4.2 1. By the lifting lemma ([4, I.7 and II.1.11=II.2.11a)] or [11, 3.6]), we obtain a diagonal Δ_{TX} for TX such that the following diagram of augmented augmented chain algebras commutes up to a homotopy h_X:

$$\begin{tikzcd} TX \arrow[r, "\simeq", "m_X"'] \arrow[d, dotted, "\Delta_{TX}"'] & C_*(\Omega E) \arrow[d, "\Delta_{C_*(\Omega E)}"] \\ TX \otimes TX \arrow[r, "\simeq", "m_X \otimes m_X"'] & C_*(\Omega E) \otimes C_*(\Omega E) \end{tikzcd}$$

Moreover, since $C_*(\Omega E)$ is a differential graded Hopf algebra which is cocommutative up to homotopy [2, Proposition 7.1], by the unicity of the lifting ([4, II.1.11c)] or [11, 3.7]), Δ_{TX} is counitary, coassociative and cocommutative, all up to homotopy. The diagonal Δ_{TX} can be chosen to be stricly counitary [2, Lemma 5.4]. So TX is an Hopf algebra up to homotopy.

By the naturality of the lifting lemma with respect to the inclusion $m(f) : TX \rightarrowtail TY$ [11, 3.6], we may put a diagonal on TY, Δ_{TY} extending the diagonal on TX and there exists a homotopy h_Y between $(m_Y \otimes m_Y)\Delta_{TY}$ and $\Delta_{C_*(\Omega B)} m_Y$ extending $C_*(\Omega f)^{\otimes 2} h_X$. Again the diagonal on TY can be chosen to be counitary and so TY is also a Hopf algebra up to homotopy.

We give now a detailed proof that Δ_{TX} is cocommutative up to a homotopy h_{comX} and that Δ_{TY} is cocommutative up to a homotopy h_{comY} extending

h_{comX}. Since the diagonal on $C_*(\Omega E)$ is cocommutative up to a homotopy h_{comE}, by the unicity of the lifting ([4, II.1.11c)] or [11, 3.7]), Δ_{TX} is cocommutative up to a homotopy h_{comX}. More precisely, h_{comX} is such that in the diagram

$$\begin{tikzcd} TX \amalg TX \arrow[r, "(\Delta_{TX},\tau\Delta_{TX})"] \arrow[d, "(i_0,i_1)"'] & TX^{\otimes 2} \arrow[d, "m_X"', "\simeq"] \\ ITX \arrow[ur, dotted, "h_{comX}"] \arrow[r, "h_X - h_{comE}\circ m_X - \tau h_X"'] & C_*(\Omega E)^{\otimes 2} \end{tikzcd}$$

where ITX is the Baues-Lemaire cylinder ([11, 3.5] or [4, I.7.12]), the upper triangle commutes [4, II.1.11a)] and the lower triangle commutes up to a homotopy relative to $TX \amalg TX$ [4, II.1.11b)]. Now, since the homotopy of cocommutativity of $C_*(\Omega B)$ is natural [2, (23)] and the sums and negatives of homotopies are canonically defined [4, II.17.3], the homotopy $h_X - h_{comE} \circ m_X - \tau h_X$ is extended by $h_Y - h_{comB} \circ m_Y - \tau h_Y$. Therefore, by push out, we obtain a morphism $ITX \cup_{TX \amalg TX} (TY \amalg TY) \to TY^{\otimes 2}$ extending $m(f)^{\otimes 2} \circ h_{comX}$, Δ_{TY} and $\tau\Delta_{TY}$. The following square of unbroken arrows commutes up to homotopy:

$$\begin{tikzcd} ITX \cup_{TX\amalg TX} (TY \amalg TY) \arrow[r] \arrow[d, "(I(m(f)),i_o,i_1)"'] & TY^{\otimes 2} \arrow[d, "m_Y"', "\simeq"] \\ ITY \arrow[ur, dotted, "h_{comY}"'] \arrow[r, "h_Y - h_{comB}\circ m_Y - \tau h_Y"'] & C_*(\Omega B)^{\otimes 2} \end{tikzcd}$$

Using again the naturality of the lifting lemma [11, 3.6], we obtain the homotopy of cocommutativity of TY, h_{comY}. A similar proof shows that the homotopy of coassociativity on TY can be chosen to extend the homotopy of coassociativity on TX. So finally, the whole structure (homotopies included) of Hopf algebra up to homotopy on TY extends the structure on TX (Compare with the proof of Theorem 8.5(g)[2]).

2. Now Lemma 3.2 says exactly that part 1. implies part 2.

3. Since $TX \rightarrowtail TY$ is a semi-free extension of TX-modules (in the sense of [11, §2]) and by Property 2.2, the quasi-isomorphism of augmented chain complexes

$$B(TY;TX) \overset{\simeq}{\twoheadrightarrow} TY \otimes_{TX} \Bbbk$$

commutes exactly with the diagonals.

Since $B(TY;TX) \overset{\simeq}{\twoheadrightarrow} TY \otimes_{TX} \Bbbk$ is a diagonal preserving chain homotopy equivalence, the diagonal in $TY \otimes_{TX} \Bbbk$ is homotopy coassociative and homotopy cocommutative. By Corollary 2.4, $C_*(F)$ is weakly equivalent to $B(C_*\Omega B; C_*\Omega E)$

as coalgebras So part 2. implies that the coalgebra $H_*(TY \otimes_{TX} \mathbb{k})$ is isomorphic to $H_*(F)$. $\square$

5 The fiber of a suspended map

Let C be a coaugmented differential graded coalgebra. Consider the tensor algebra on $\overline{C}$, $T\overline{C}$, equipped with the differential obtained by tensorisation. The composite $C \stackrel{\Delta_C}{\to} C \otimes C \hookrightarrow T\overline{C} \otimes T\overline{C}$ extends to an unique morphism of augmented differential graded algebras

$$\Delta_{T\overline{C}} : T\overline{C} \to T\overline{C} \otimes T\overline{C}.$$

The tensor algebra $T\overline{C}$ equipped with this structure of differential graded Hopf algebra, is called the *Hopf algebra obtained by tensorization of the coalgebra C* and is denoted $\mathrm{TA}\overline{C}$ in this section.

Lemma 5.1 *Let X be a path connected space. Then there is a natural quasi-isomorphism of Hopf algebras $\mathrm{TA}\overline{C_*(X)} \stackrel{\simeq}{\to} C_*(\Omega\Sigma X)$.*

Proof The adjunction map ad induces a morphism of coaugmented coalgebras $C_*(ad) : C_*(X) \to C_*(\Omega\Sigma X)$. By universal property of $\mathrm{TA}\overline{C_*(X)}$, $C_*(ad)$ extends to a natural morphism of Hopf algebras. By the Bott-Samelson Theorem [17, appendix 2 Theorem 1.4], it is a quasi-isomorphism, since the functors H and T commute. $\square$

Theorem 5.2 *Let $f : E \to B$ be a continuous map between path connected spaces. Let F be the homotopy fiber of Σf. Then $C_*(F)$ is naturally weakly equivalent as coalgebras to $B(\mathrm{TA}\overline{C_*(B)}; \mathrm{TA}\overline{C_*(E)})$. In particular, the algebra $H^*(F)$ depends only of the morphism of coalgebras $C_*(f)$.*

Proof This is a direct consequence of Lemma 5.1, Corollary 2.4 and Property 2.1. $\square$

Consider the homotopy commutative diagram of chain algebras.

$$\begin{tikzcd}
(TH_+(E), 0) \arrow[r, "\simeq", "m_X"'] \arrow[d, "TH_+(f)"'] & \mathrm{TA}\overline{C_*(E)} \arrow[d, "\mathrm{TA}\overline{C_*(f)}"] \\
(TH_+(B), 0) \arrow[r, "\simeq", "m_Y"'] & \mathrm{TA}\overline{C_*(B)}
\end{tikzcd} \tag{5.3}$$

where m_X and m_Y induce the identity in homology.

When $H_*(f)$ is injective then $TH_+(f)$ is a free extension and we may choose m_Y so that the diagram (5.3) commutes. Thus Theorem 4.2 applies. The structures of Hopf algebra up to homotopy on $TH_+(E)$ and $TH_+(B)$ given by part 1. of Theorem 4.2 are the structures of Hopf algebra obtained by tensorization of the coalgebras $H_*(E)$ and $H_*(B)$. Part 3. of Theorem 4.2 claims that we have the isomorphism of graded coalgebras

$$\mathrm{TA}H_+(B) \otimes_{\mathrm{TA}H_+(E)} \mathbb{k} \cong H_*(F). \tag{5.4}$$

If $H_*(f)$ is not injective, this is not true in general: the algebra $H^*(F)$ does not depend only on the morphism of coalgebras $H_*(f)$. Indeed, over $\mathbb{F}_p$, take f to be a map from S^{2p-1} to $\mathbb{CP}^{p-1}$. Whatever is the map chosen, $H_*(f): H_*(S^{2p-1}) \to H_*(\mathbb{CP}^{p-1})$ is null. Let y_2 be a generator of $H^2(F)$. If f is the Hopf map, there is a map $\psi : \mathbb{CP}^p \to F$ such that the following diagram commutes

$$\begin{tikzcd}
S^{2p-1} \arrow[r, "f"] \arrow[d, "ad"'] & \mathbb{CP}^{p-1} \arrow[r] \arrow[d, "ad"'] & \mathbb{CP}^p \arrow[d, dotted, "\psi"] \\
\Omega\Sigma S^{2p-1} \arrow[r, "\Omega\Sigma f"'] & \Omega\Sigma\mathbb{CP}^{p-1} \arrow[r, "\partial"'] & F
\end{tikzcd}$$

Since $H^2(\psi)$ is an isomorphism, $y_2^p \neq 0$. On the contrary, if f is the constant map then $F \approx \Omega\Sigma\mathbb{CP}^{p-1} \times \Sigma S^{2p-1}$ and so $y_2^p = 0$.

Of course, the isomorphism of coalgebras (5.4) can be proved more easily with the Eilenberg-Moore spectral sequence applied to the $\Omega\Sigma E$-fibration

$$\Omega\Sigma E \xrightarrow{\Omega\Sigma f} \Omega\Sigma B \to F.$$

6 Proof of Theorem A

We recall first the natural structure of algebra on the torsion product of commutative algebras. Let $f : A \to M$, $g : A \to N$ be two morphisms of commutative differential graded algebras. The composite

$$\mathrm{Tor}^A(M,N)\otimes\mathrm{Tor}^A(M,N) \xrightarrow{\top} \mathrm{Tor}^{A\otimes A}(M\otimes M, N\otimes N) \xrightarrow{\mathrm{Tor}^{\mu_A}(\mu_M,\mu_N)} \mathrm{Tor}^A(M,N)$$

where $\top$ is the $\top$ product ([19, VIII.2.1] or [7, XI.Proposition 1.2.1]), defines a natural structure of commutative graded algebra on $\mathrm{Tor}^A(M,N)$ ([19, Theorem 2.2] or [7, XI.4 $\pitchfork$ product]).

Property 6.1 [16, A.3] *Suppose given a commutative diagram of augmented commutative differential graded algebras*

$$\begin{tikzcd} A \arrow[r, "\varphi", "\simeq"'] \arrow[d, "f"'] & A' \arrow[d, "g"] \\ M \arrow[r, "\Psi", "\simeq"'] & M' \end{tikzcd}$$

where φ and Ψ are quasi-isomorphisms. Then $Tor^{\varphi}(\Psi, \Bbbk) : Tor^{A}(M, \Bbbk) \longrightarrow Tor^{A'}(M', \Bbbk)$; is an isomorphism of graded commutative algebras.

Property 6.2 [19, VIII.2.3] *Consider a factorization $f = p \circ i$ where $i : A \rightarrowtail P$ is a morphism of commutative differential graded algebras such that P is an A-semifree module and $p : P \overset{\simeq}{\twoheadrightarrow} M$ is a quasi-isomorphism of commutative differential graded algebras. The homology of the commutative differential graded algebra $P \otimes_A N$, $H_*(P \otimes_A N)$, is the graded commutative algebra $Tor^{A}(M, N)$.*

Using this Property, Theorem A given in the Introduction derives from the following proposition:

Let $r \geq 1$ be a integer. Let p be the characteristic of the field $\Bbbk$ (except when the characteristic is 0: in this case, we set $p = +\infty$). We suppose now $p \neq 2$.

Definition 6.3 [16] A topological space X is (r,p)*-mild* or in the *Anick range* if it is r-connected and its homology with coefficient in $\Bbbk$ is concentrated in degrees $\leq rp$ and of finite type.

Proposition 6.4 *Let $f : E \to B$ be a continuous map between two topological spaces both (r,p)-mild with $H_{rp}(f)$ injective. Consider the homotopy fiber F and the induced fibration $p_0 : F \twoheadrightarrow E$. Then there are two morphisms of augmented commutative cochain algebras, denoted $A(f) : A(B) \to A(E)$ and $A(p_0) : A(E) \to A(F)$ such that:*

(1) *There is a commutative diagram of cochain complexes*

$$\begin{tikzcd}
C^*(B) \arrow[r, "C^*(f)"] \arrow[d, "\simeq"] & C^*(E) \arrow[r, "C^*(p_0)"] \arrow[d, "\simeq"] & C^*(F) \arrow[d, "\simeq"] \\
D_1(B) \arrow[r] & D_1(E) \arrow[r] & D_1(F) \\
D_2(B) \arrow[r] \arrow[u, "\simeq"'] \arrow[d, "\simeq"] & D_2(E) \arrow[r] \arrow[u, "\simeq"'] \arrow[d, "\simeq"] & D_2(F) \arrow[u, "\simeq"'] \arrow[d, "\Theta"', "\simeq"] \\
A(B) \arrow[r, "A(f)"] & A(E) \arrow[r, "A(p_0)"] & A(F)
\end{tikzcd}$$

where all the vertical maps are quasi-isomorphisms and where all the maps are morphisms of augmented cochain algebras except $\Theta : D_2(F) \overset{\simeq}{\rightarrow} A(F)$ who induces a morphism of graded algebras only in homology.

(2) *For any factorization $A(f) = \Phi \circ i$ where $i : A(B) \rightarrowtail C$ is a morphism of augmented commutative cochain algebras such that C is an $A(B)$-semifree module and where $\Phi : C \overset{\simeq}{\rightarrow} A(E)$ is a quasi-isomorphism of augmented commutative cochain algebras, there is a commutative diagram of augmented commutative cochain algebras*

$$\begin{tikzcd}
A(B) \arrow[r, "A(f)"] \arrow[dr, "i"'] & A(E) \arrow[r, "A(p_0)"] & A(F) \\
& C \arrow[u, "\Phi", "\simeq"'] \arrow[r] \arrow[dr] & D_3 \arrow[u, "\simeq"'] \arrow[d, "\simeq"'] \\
& & \Bbbk \otimes_{A(B)} C
\end{tikzcd}$$

In particular, the cohomology $H^*(F)$, is isomorphic as graded algebras to the cohomology $H^*(\Bbbk \otimes_{A(B)} C)$.

Over a field of characteristic zero, part (1) was proved by Sullivan [23] and part (2) is the Grivel-Thomas-Halperin theorem [1, 12].

The hypotheses of Theorem A are necessary: the space B must be (r,p)-mild. Indeed even for a path fibration $\Omega X \hookrightarrow PX \twoheadrightarrow X$, a commutative model of X does not determine the cohomology algebra of the loop space. The spaces $\Sigma\mathbb{CP}^p$ and $S^3\vee..\vee S^{2p+1}$ have the same commutative model but the cohomology algebras of their loop spaces are not isomorphic. The map $H_{rp}(f)$ must also be injective. Take the same example as in section 5: the suspension of the Hopf map $\Sigma f : \Sigma S^{2p-1} \to \Sigma\mathbb{CP}^{p-1}$.

Over a field of characteristic p, we can't improve Proposition 6.4, by "$\mathbb{k}\otimes_{A(B)}C$ is weakly equivalent as a cochain algebra to $C^*(F)$". For example, let $X = K(\mathbb{Z},4)_{2p+3}$ be the $2p+3$ skeleton of a $K(\mathbb{Z},4)$. The space X is $(3,p)$-mild and $C^*(\Omega X)$ is not weakly equivalent as a cochain algebra to any commutative cochain algebra. Indeed, there exist two CW-complexes denoted Y and $K(\mathbb{Z},3)$ with the same $2p+2$ skeleton, respectively homotopic to ΩX and $\Omega K(\mathbb{Z},4)$. The two morphisms of topological monoids

$$\Omega(Y_{2p+2}) \to \Omega Y \quad \text{and} \quad \Omega(K(\mathbb{Z},3)_{2p+2}) \to \Omega K(\mathbb{Z},3)$$

induce in homology two algebra morphisms which are isomorphisms in degree $\leq 2p$. Since $H_*(\Omega K(\mathbb{Z},3)) \cong \Gamma\alpha_2$ as algebras, ΩY is 1-connected, $H_2(\Omega Y) = \mathbb{F}_p\alpha_2$ and $\alpha_2^p = 0$. Suppose $C^*(Y)$ is weakly equivalent as a cochain algebra to a commutative cochain algebra A. We can suppose that A is of finite type. The dual of A, denoted $A^\vee$, is a cocommutative chain coalgebra. There is a quasi-isomorphism of chain algebras from the cobar construction of $A^\vee$, denoted $\Omega(A^\vee)$, to $C_*(\Omega Y)$. The Quillen construction on the coalgebra $A^\vee$ is a differential graded Lie algebra, denoted $\mathcal{L}_A$, such that $U\mathcal{L}_A := \Omega(A^\vee)$ [12, p. 307 and 315]. The homology of an universal enveloping algebra of a differential graded Lie algebra is isomorphic as graded Hopf algebras to the universal enveloping algebra of a graded Lie algebra [16, 8.3]: there a graded Lie algebra E equipped with the following isomorphism of graded algebras

$$H_*(\Omega Y) \cong H_*(U\mathcal{L}_A) \cong UE.$$

By the Poincaré-Birkoff-Witt Theorem [16, 1.2], $H_*(\Omega Y)$ admits a basis containing α_2^p. Thus α_2^p is non zero.

Proof of Proposition 6.4 By the naturality of Corollary 2.4 with respect to continuous maps, we have a commutative diagram of coalgebras:

$$\begin{tikzcd} C_*(F) \arrow[r, "C_*(p_0)"] & C_*(E) \arrow[r, "C_*(f)"] & C_*(B) \\ G(f) \arrow[r] \arrow[u, "\simeq"'] \arrow[d, "\simeq"] & G(E\to *) \arrow[r] \arrow[u, "\simeq"'] \arrow[d, "\simeq"] & G(B\to *) \arrow[u, "\simeq"'] \arrow[d, "\simeq"] \\ B(C_*(\Omega B);C_*(\Omega E)) \arrow[r] & BC_*(\Omega E) \arrow[r, "BC_*(\Omega f)"] & BC_*(\Omega B) \end{tikzcd} \tag{6.5}$$

There is also a commutative diagram of augmented chain algebras [10, Theorem I]

$$\begin{tikzcd} TX \arrow[r, "\simeq"] \arrow[d, "m(f)"'] & \Omega C_*(E) \arrow[r, "\simeq"] \arrow[d, "\Omega C_*(f)"] & C_*(\Omega E) \arrow[d, "C_*(\Omega f)"] \\ TY \arrow[r, "\simeq"] & \Omega C_*(B) \arrow[r, "\simeq"] & C_*(\Omega B) \end{tikzcd}$$

where Ω denotes the cobar construction, TX is a minimal (in the sense of [5, 2.1]) free chain algebra and $m(f): TX \rightarrowtail TY$ is a minimal free extension. Since the indecomposables functor Q preserves quasi-isomorphism between free chain algebras [5, 1.5],

$$X \cong s^{-1}H_+(E) \quad \text{and} \quad Y \cong s^{-1}H_+(E) \oplus s^{-1}\mathrm{coker} H_+(f) \oplus \ker H_+(f).$$

So X and Y are graded vector spaces of finite type concentrated in degree $\geq r$ and $\leq rp-1$. Denote by m_X the composite $TX \overset{\simeq}{\to} \Omega C_*(E) \overset{\simeq}{\to} C_*(\Omega E)$ and by m_Y the composite $TY \overset{\simeq}{\to} \Omega C_*(B) \overset{\simeq}{\to} C_*(\Omega B)$. By Theorem 4.2, $m(f): TX \rightarrowtail TY$ is an inclusion of Hopf algebras up to homotopy and $B(m_Y;m_X): B(TY;TX) \overset{\simeq}{\to} B(C_*(\Omega E);C_*(\Omega B))$ is a morphism of coalgebras up to homotopy.

By Anick's Theorem [2, 5.6], there exists a differential graded Lie algebra $L(E)$ and an isomorphism φ of Hopf algebras up to homotopy between the universal envelopping algebra of $L(E)$, $UL(E)$ and TX. By the naturality of Anick's Theorem with respect to Hopf algebras up to homotopy equipped with their homotopies ([20] D.33 and D.25, see also the proof of Theorem 8.5(g)[2]), there exists a differential graded Lie algebra morphism $L(f): L(E) \to L(B)$ and a commutative diagram of chain algebras

$$\begin{tikzcd} UL(E) \arrow[r, "\cong", "\varphi"'] \arrow[d, "UL(f)"'] & TX \arrow[d, "m(f)"] \\ UL(B) \arrow[r, "\cong", "\Psi"'] & TY \end{tikzcd}$$

where φ and Ψ are two algebra isomorphisms equipped with two homotopies of algebras

$$h_{top} : (\varphi \otimes \varphi)\Delta_{UL(E)} \approx_a \Delta_{TX}\varphi \quad \text{and} \quad h_{bottom} : (\Psi \otimes \Psi)\Delta_{UL(B)} \approx_a \Delta_{TY}\Psi$$

$$\text{such that} \quad h_{bottom}UL(f) = (m(f) \otimes m(f))h_{top}$$

(*the horizontal arrows commute with the diagonals up to natural homotopies*). By Lemma 3.2(ii), the isomorphism

$$B(\Psi;\varphi) : B(UL(B);UL(E)) \overset{\cong}{\to} B(TY;TX)$$

commutes up to chain homotopy with the diagonals. We give the Cartan-Chevalley-Eilenberg complex with coefficients [16, p. 242] $C_*(UL(B);L(E))$ the tensor product coalgebra structure of $UL(B) \otimes \Gamma sL(E)$. The injection $C_*(UL(B);L(E)) \overset{\simeq}{\to} B(UL(B);UL(E))$ is a quasi-isomorphism of coalgebras [11, 6.11]. By functoriality of the bar construction and of the Cartan-Chevalley-Eilenberg complex with coefficients, finally we get the commutative diagram of coalgebras up to homotopy

$$\begin{tikzcd} B(C_*(\Omega B);C_*(\Omega E)) \arrow[r] & BC_*(\Omega E) \arrow[r, "BC_*(\Omega f)"] & BC_*(\Omega B) \\ B(TY;TX) \arrow[r] \arrow[u, "B(m_Y;m_X)", "\simeq"'] & B(TX) \arrow[r, "B(m(f))"] \arrow[u, "B(m_X)", "\simeq"'] & B(TY) \arrow[u, "B(m_Y)", "\simeq"'] \\ B(UL(B);UL(E)) \arrow[r] \arrow[u, "B(\Psi;\varphi)", "\cong"'] & B(UL(E)) \arrow[r, "B(UL(f))"] \arrow[u, "B(\varphi)", "\cong"'] & B(UL(B)) \arrow[u, "B(\Psi)", "\cong"'] \\ C_*(UL(B);L(E)) \arrow[r] \arrow[u, "\simeq"'] & C_*L(E) \arrow[r, "C_*L(f)"] \arrow[u, "\simeq"'] & C_*L(B) \arrow[u, "\simeq"'] \end{tikzcd} \tag{6.6}$$

where all the coalgebras up to homotopy are counitary and coassociative exactly except $B(TY;TX)$, where all the morphisms commute exactly with the diagonals except $B(m_Y;m_X)$ and $B(\Psi;\varphi)$, and where all the vertical maps are quasi-isomorphisms. Define $A(f)$ to be $C^*L(f):C^*L(B)\to C^*L(E)$ and $A(p_0)$ to be the inclusion $C^*L(E)\hookrightarrow C^*(UL(B);L(E)$. By dualizing diagram 6.5 and diagram 6.6, we obtain the diagram of 1.

By the universal property of push out, there is a morphism of commutative cochain algebras

$$C^*(UL(B);L(B))\otimes_{C^*(L(B))}C^*(L(E))\xrightarrow{\cong}C^*(UL(B);L(E))$$

which is an isomorphism since $L(B)$ is of finite type. Recall that $C\overset{\simeq}{\twoheadrightarrow}C^*L(E)$ is a quasi-isomorphism of $C^*L(B)$-cochain algebras and that $C^*(UL(B);L(B))$ is $C^*(L(B))$-semifree. Set $D_3=C^*(UL(B);L(B))\otimes_{C^*L(B)}C$. Then we obtain the quasi-isomorphism $D_3\overset{\simeq}{\twoheadrightarrow}C^*(UL(B);L(B))\otimes_{C^*(L(B))}C^*(L(E))$. Symetrically, recall that $C^*(UL(B);L(B))\overset{\simeq}{\twoheadrightarrow}\Bbbk$ is a quasi-isomorphism of $C^*L(B)$-cochain algebras [11, 6.10] and that C is $C^*(L(B))$-semifree. Then we obtain the quasi-isomorphism $D_3\overset{\simeq}{\twoheadrightarrow}\Bbbk\otimes_{C^*L(B)}C$. □

7 Sullivan models mod p

We want to use our Theorem A for practical computations. Like in Rational Homotopy, we need two steps. First, we replace $A(f):A(B)\to A(E)$ by a morphism between Sullivan models. Second, we construct a factorization of this morphism between Sullivan models.

Contrary to the rational case [12, Proposition 14.6], modulo p, there is in general, no lifting lemma. Nevertheless, we have the following:

Corollary 7.1 • *Let $A(f):A(B)\to A(E)$ be a morphism of commutative cochain algebras as in Proposition 6.4. Let ΛY be a Sullivan model of $A(B)$, ΛX a Sullivan model of $A(E)$. Then there is an acyclic commutative cochain algebra U and a commutative diagram of commutative cochain algebras*

$$\begin{tikzcd} & \Lambda Y \arrow[r, "\simeq"] \arrow[dl, "\Psi"'] \arrow[d] & A(B) \arrow[d, "A(f)"] \\ \Lambda X & \Lambda X\otimes U \arrow[l, "\simeq"'] \arrow[r, "\simeq"] & A(E) \end{tikzcd}$$

• *Let $\Lambda Y \rightarrowtail C \overset{\simeq}{\twoheadrightarrow} \Lambda X$ be a factorization of $\Psi : \Lambda Y \to \Lambda X$ such that C is a ΛY-semifree module. Then the algebra $H^*(F)$ is isomorphic to $H^*(\Bbbk \otimes_{\Lambda Y} C)$. (This isomorphism identifies in homology $C^*(p_0) : C^*(E) \to C^*(F)$ and the quotient map $C \twoheadrightarrow \Bbbk \otimes_{\Lambda Y} C$.)*

Proof Since $A(E)$ is concentrated in degrees $\geq r+1$ and $H^{\geq (r+1)p}(E) = 0$, [16, Proposition 7.7 and Remark 7.8] gives the first part of this Corollary. For the second part, using Proposition 6.4, Property 6.1 twice and finally Property 6.2, we obtain the sequence of isomorphisms of graded algebras:

$$\begin{aligned} H^*(F) \cong \mathrm{Tor}^{A(B)}(A(E), \Bbbk) &\cong \mathrm{Tor}^{\Lambda Y}(\Lambda X \otimes U, \Bbbk) \\ &\cong \mathrm{Tor}^{\Lambda Y}(\Lambda X, \Bbbk) \cong H^*(\Bbbk \otimes_{\Lambda Y} C). \end{aligned}$$ □

As in the rational case, we can take a factorization of Ψ with relative Sullivan models. But mod p, since the p^{th} power of an element of even degree is always a cycle, our relative Sullivan model will have infinitely many generators. We'd rather use a free divided powers algebra ΓV where for $v \in V_{even}$, $v^p = 0$. But now arises the problem of constructing morphisms of commutative algebras from a free divided power algebra to any commutative algebra where the p^{th} powers are not zero. We give now an effective construction of a factorization of Ψ with divided powers algebras. Over $\mathbb{Q}$, this factorization will be just a factorization of Ψ through a minimal relative Sullivan model.

Let A be a commutative graded algebra, V and W two graded vector spaces. A Γ-*derivation* in $A \otimes \Gamma W$ is a derivation D such that $D\gamma^k(w) = D(w)\gamma^{k-1}(w)$, $k \geq 1$, $w \in W^{even}$. Any linear map $V \oplus W \to \Lambda V \otimes \Gamma W$ of degree k extends to a unique Γ-derivation over $\Lambda V \otimes \Gamma W$.

Lemma 7.2 *Let $\Psi : (\Lambda Y, d) \to (\Lambda X, d)$ be a morphism of commutative cochain algebras between two minimal Sullivan models such that X and Y are concentrated in degree ≥ 2. Then there is an explicit factorization of Ψ:*

$$(\Lambda Y, d) \overset{i}{\rightarrowtail} (\Lambda Y \otimes \Lambda coker\varphi \otimes \Gamma sker\varphi, D) \underset{p}{\overset{\simeq}{\twoheadrightarrow}} (\Lambda X, d)$$

where

- *φ is the composite $Y \hookrightarrow \Lambda Y \overset{\Psi}{\to} \Lambda X \twoheadrightarrow X$ and D is a Γ-derivation,*
- *i is an inclusion of augmented commutative cochain algebras such that $(\Lambda Y \otimes \Lambda coker\varphi \otimes \Gamma sker\varphi, D)$ is $(\Lambda Y, d)$-semifree, and*
- *p is a surjective quasi-isomorphism of commutative cochain algebras vanishing on $\Gamma sker\varphi$.*

Proof We will define
$$p := \varinjlim p_n.$$
We proceed by induction on $n \in \mathbb{N}^*$ to construct each p_n. Suppose we have constructed the factorization:
$$\big(\Lambda(Y^{\leq n}), d\big) \rightarrowtail \big(\Lambda(Y^{\leq n}) \otimes \Lambda(\mathrm{coker}\varphi^{\leq n}) \otimes \Gamma s(\ker\varphi^{\leq n}), D\big) \underset{p_n}{\overset{\simeq}{\twoheadrightarrow}} \big(\Lambda(X^{\leq n}), d\big)$$
We define now p_{n+1} extending Ψ and p_n.

Let $w \in \mathrm{coker}\varphi^{n+1}$. Define p_{n+1} in $\mathrm{coker}\varphi^{n+1}$ so that $p_{n+1}(w) \in X^{n+1}$ represents w. Then $dp_{n+1}(w)$ is a cycle of $\Lambda X^{\leq n}$. Since p_n is a surjective quasi-isomorphism, there is a cycle $z \in \Lambda(Y^{\leq n}) \otimes \Lambda(\mathrm{coker}\varphi^{\leq n}) \otimes \Gamma s(\ker\varphi^{\leq n})$ such that $p_n(z) = dp_{n+1}(w)$. Define $Dw = z$.

Let $v \in \ker\varphi^{n+1}$. Since $p_{n+1} : \Lambda(Y^{\leq n+1} \oplus \mathrm{coker}\varphi^{\leq n+1}) \twoheadrightarrow \Lambda(X^{\leq n+1})$ is a surjective morphism of graded algebras, there is $u \in \Lambda^{\geq 2}(Y^{\leq n} \oplus \mathrm{coker}\varphi^{\leq n})$ such that $p_{n+1}(v+u) = 0$. Since $D(v+u)$ is a cycle of $\Lambda(Y^{\leq n}) \otimes \Lambda(\mathrm{coker}\varphi^{\leq n}) \otimes \Gamma s(\ker\varphi^{\leq n})$ and p_n is a surjective quasi-isomorphism, there is $\alpha \in \Lambda(Y^{\leq n}) \otimes \Lambda(\mathrm{coker}\varphi^{\leq n}) \otimes \Gamma s(\ker\varphi^{\leq n})$ such that $p_n(\alpha) = 0$ and $D\alpha = D(v+u)$. Define $Dsv = v + u - \alpha$.

Now we have the commutative diagram of commutative cochain algebras:
$$\begin{tikzcd}
\Lambda(Y^{\leq n}) \otimes \Lambda(\mathrm{coker}\varphi^{\leq n}) \otimes \Gamma s(\ker\varphi^{\leq n}), D \arrow[r, "p_n", "\simeq"'] \arrow[d] & \Lambda(X^{\leq n}), d \arrow[d] \\
\Lambda(Y^{\leq n+1}) \otimes \Lambda(\mathrm{coker}\varphi^{\leq n+1}) \otimes \Gamma s(\ker\varphi^{\leq n+1}), D \arrow[r, "p_{n+1}"] \arrow[d] & \Lambda(X^{\leq n+1}), d \arrow[d] \\
\Lambda(Y^{n+1}) \otimes \Lambda(\mathrm{coker}\varphi^{n+1}) \otimes \Gamma s(\ker\varphi^{n+1}), \overline{D} \arrow[r, "\overline{p_{n+1}}", "\simeq"'] & \Lambda(X^{n+1}), 0
\end{tikzcd}$$
Since p_n and $\overline{p_{n+1}}$ are quasi-isomorphisms, by comparison of the E_2-term of the algebraic Serre spectral sequence associated to each column, p_{n+1} is a quasi-isomorphism. □

Example 7.3 Let $f : S^2 \hookrightarrow \mathbb{CP}^n$ be the inclusion of CW-complexes with $n \geq 2$. Applying Corollary 7.1, Ψ is $(\Lambda(x_2, y_{2n+1}), d) \to (\Lambda(x_2, z_3), d)$ with $dy_{2n+1} = x_2^{n+1}$ and $dz_3 = x_2^2$. Thus $\Psi y_{2n+1} = z_3 x_2^{n-1}$. By Lemma 7.2, Ψ factors through the commutative cochain algebra $(\Lambda(x_2, y_{2n+1}, z_3) \otimes \Gamma sy_{2n+1}, D)$ with $Dz_3 = x_2^2$ and $Dsy_{2n+1} = y_{2n+1} - z_3 x_2^{n-1}$. So $H^*(F) \cong \Lambda z_3 \otimes \Gamma sy_{2n+1}$ for $p \geq 2n$.

8 Proof of Theorem B

The key to the proof of Theorem A is to apply Anick's Theorem [2, 5.6]. One of the goals of Anick for developing this theorem was to prove a result suggested by McGibbon and Wilkerson [21, p. 699]: "If X is a finite simply-connected CW-complex then for large primes, p^{th} powers vanish in $\tilde{H}^*(\Omega X;\mathbb{F}_p)$." Anick [2, 9.1] proved precisely that "If X is (r,p)-mild then p^{th} powers vanish in $\tilde{H}^*(\Omega X;\mathbb{F}_p)$.". After Anick, Halperin proved in [16, Theorem 8.3 and Poincaré-Birkoff-Witt Theorem] that in fact:

Corollary 8.1 [16] *If X is (r,p)-mild then the algebra $H^*(\Omega X)$ is isomorphic to ΓsV where ΛV is a minimal Sullivan model of $A(X)$.*

Proof Apply Corollary 7.1 to $* \to X$. Consider the factorization of $(\Lambda V, d) \twoheadrightarrow (\mathbb{k}, 0)$,

$$(\Lambda V, d) \rightarrowtail (\Lambda V \otimes \Gamma sV, D) \overset{\simeq}{\twoheadrightarrow} \mathbb{k}$$

given by Lemma 7.2. See that the cofiber $(\mathbb{k},0)\otimes_{(\Lambda V,d)}(\Lambda V\otimes \Gamma sV, D)$ has a null differential [16, 2.6]. $\square$

Actually, we can show now that Anick's result on p^{th} powers and Halperin's result on a divided powers algebra structure remain valid if we consider the fiber of any fibration in the Anick range instead of just the loop fibration. But before we need some definitions concerning divided powers algebras with differential.

A *differential divided powers algebra* or Γ*-algebra* is a commutative cochain algebra A equipped with a system $(\gamma^k)_{k\in\mathbb{N}}$ of divided powers [6, page 124] such that $d\gamma^k(a) = d(a)\gamma^{k-1}(a)$. Let A, B be two Γ-algebras. A Γ*-morphism* $f : A \to B$ is a morphism of augmented commutative cochain algebras such that $f\gamma^k(a) = \gamma^k f(a)$.

A Γ*-free extension* is an inclusion of augmented commutative cochain algebras: $(A, d) \rightarrowtail (A\otimes\Gamma V, D)$ such that $V = \oplus_{k\in\mathbb{N}}V(k)$, $D : V(k) \to A\otimes\Gamma V(< k)$, $k \in \mathbb{N}$ and D is a Γ-derivation. In particular, if A is a Γ-algebra, than a Γ-free extension $(A, d) \rightarrowtail (A \otimes \Gamma V, D)$ is a Γ-morphism.

A commutative cochain algebra (respectively Γ-algebra) A is *admissible* (respectively Γ*-admissible*) if there is a surjective morphism of commutative cochain algebras (respectively Γ-morphism) $C \twoheadrightarrow A$ with C acyclic.

Property 8.2 [14, II.2.6] *Let $f : A \to B$ be a morphism of commutative cochain algebras (respectively a Γ-morphism). If f is surjective and A is admissible (respectively Γ-admissible) then so is B.*

Proposition 8.3 [14, II.2.7]

(i) *If $f : A \to B$ is a morphism of commutative cochain algebras with B admissible then we have the commutative diagram of commutative cochain algebras*

$$\begin{tikzcd} A \arrow[rr] \arrow[d] \arrow[drr] & & B \\ A \otimes \Gamma V' & & A \otimes \Lambda V \arrow[ll, "\simeq"'] \arrow[u, "\simeq"'] \end{tikzcd}$$

where $A \rightarrowtail A \otimes \Lambda V$ is a relative Sullivan model and $A \rightarrowtail A \otimes \Gamma V'$ is a Γ-free extension.

(ii) *In particular, if B is any admissible commutative cochain algebra, there are quasi-isomorphisms of commutative cochain algebras*

$$\Gamma V' \xleftarrow{\simeq} \Lambda V \xrightarrow{\simeq} B$$

where $\Gamma V'$ is a Γ-algebra.

The essential role of Γ-admissible algebras is that

Property 8.4 [3, 1.3] *If A is a Γ-admissible algebra then $H(A)$ is a divided powers algebra (not true if A was only a Γ-algebra!).*

Lemma 8.5 *Let A be a commutative cochain algebra. Assume that for some $r \geq 1$, A satisfies $A = \Bbbk \oplus \{A^i\}_{i \geq r}$.*

(i) *If $H^i(A) = 0$, $i \geq rp$, then A is admissible.*

(ii) *If A is a Γ-algebra and $H^i(A) = 0$, $i \geq rp+p-1$, then A is Γ-admissible.*

Proof (i) This lemma is just a slight improvement from [16, Lemma 7.6] and the proof is the same.

(ii) After replacing free commutative algebras by free divided powers algebras, the proof is the same as in (i). □

Lemma 8.6 *Let A and M be two commutative cochain algebras concentrated in degrees $\geq r+1$. Consider a morphism of algebras $A \to M$. If $H^{\geq rp+p}(A) = H^{\geq rp+p-1}(M) = 0$ then $\mathrm{Tor}^A(M, \Bbbk)$ is a divided powers algebra.*

Proof By Lemma 8.5 (i), A and M are admissible. By Proposition 8.3 (ii), there are quasi-isomorphisms of commutative cochain algebras

$$\Gamma X' \xleftarrow{\simeq} \Lambda X \xrightarrow{\simeq} A$$

where X and X' are concentrated in degrees $\geq r+1$. By Proposition 8.3 (i), we get the commutative diagram of commutative cochain algebras

$$\begin{tikzcd} A \arrow[r] & M \\ \Lambda X \arrow[u, "\simeq"] \arrow[r] \arrow[dr] & \Lambda X \otimes \Lambda Y \arrow[u, "\simeq"'] \arrow[d, "\simeq"] \\ & \Lambda X \otimes \Gamma Y' \end{tikzcd}$$

where Y and Y' are concentrated in degrees $\geq r$. Since $\Lambda X \rightarrowtail \Lambda X \otimes \Gamma Y'$ is a Γ-free extension, $\Lambda X \otimes \Gamma Y'$ is ΛX-semifree. Therefore, by push-out, we have the commutative diagram of commutative cochain algebras

$$\begin{tikzcd} \Lambda X \arrow[r] \arrow[d, "\simeq"'] & \Lambda X \otimes \Gamma Y' \arrow[d, "\simeq"] \\ \Gamma X' \arrow[r] & \Gamma X' \otimes \Gamma Y' \end{tikzcd}$$

where $\Lambda X \otimes \Gamma Y' \xrightarrow{\simeq} \Gamma X' \otimes \Gamma Y'$ is a quasi-isomorphism [11, 2.3(i)]. Since push-outs preserve Γ-free extension, $\Gamma X' \rightarrowtail \Gamma X' \otimes \Gamma Y'$ is a Γ-free extension. So $\Gamma X' \otimes \Gamma Y'$ is $\Gamma X'$-semifree, and by Property 6.1, the cohomology algebra of the cofiber $\Gamma Y'$ is $\mathrm{Tor}^A(M, \mathbb{k})$. Now since $\Gamma X'$ is a Γ-algebra, so is $\Gamma X' \otimes \Gamma Y'$. Since $\Gamma X' \otimes \Gamma Y'$ is concentrated in degrees $\geq r$ and its cohomology is null in degrees $\geq rp + p - 1$, by Lemma 8.5(ii), $\Gamma X' \otimes \Gamma Y'$ is Γ-admissible. Since $\Gamma X' \otimes \Gamma Y' \twoheadrightarrow \mathbb{k} \otimes_{\Gamma X'} (\Gamma X' \otimes \Gamma Y') = \Gamma Y'$ is a surjective Γ-morphism, by Property 8.2, $\Gamma Y'$ is a Γ-admissible. So by Property 8.4, $H(\Gamma Y')$ is a divided powers algebra. □

Theorem 8.7 *Let p be an odd prime and let $f : E \twoheadrightarrow B$ be a fibration of fiber F such that E and B are both (r,p)-mild with $H_{rp}(f)$ injective. Then the cohomology algebra $H^*(F; \mathbb{F}_p)$ is a (not necessarily free!) divided powers algebra. In particular, p^{th} powers vanish in the reduced cohomology $\tilde{H}^*(F; \mathbb{F}_p)$.*

Proof By Theorem A, $H^*(F; \mathbb{k}) \cong \mathrm{Tor}^{A(B)}(A(E), \mathbb{k})$. Since $A(B)$ and $A(E)$ are concentrated in degrees $\geq r+1$ and their cohomology is null in degrees $\geq rp$, by Lemma 8.6, $\mathrm{Tor}^{A(B)}(A(E), \mathbb{k})$ is a divided powers algebra. □

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Université d'Angers, Faculté des Sciences
2 Boulevard Lavoisier, 49045 Angers, FRANCE

Email: `Luc.Menichi@univ-angers.fr`

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