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On the Adams Spectral Sequence for R-modules

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Abstract We discuss the Adams Spectral Sequence for R-modules based on commutative localized regular quotient ring spectra over a commutative S-algebra R in the sense of Elmendorf, Kriz, Mandell, May and Strickland. The formulation of this spectral sequence is similar to the classical case and the calculation of its E_2-term involves the cohomology of certain 'brave new Hopf algebroids' E^R_*E. In working out the details we resurrect Adams' original approach to Universal Coefficient Spectral Sequences for modules over an R ring spectrum.

We show that the Adams Spectral Sequence for S_R based on a commutative localized regular quotient R ring spectrum $E = R/I[X^{-1}]$ converges to the homotopy of the E-nilpotent completion

$$\pi_*\widehat{\mathrm{L}}^R_E S_R = R_*[X^{-1}]\widehat{_{I_*}}.$$

We also show that when the generating regular sequence of I_* is finite, $\widehat{\mathrm{L}}^R_E S_R$ is equivalent to $\mathrm{L}^R_E S_R$, the Bousfield localization of S_R with respect to E-theory. The spectral sequence here collapses at its E_2-term but it does not have a vanishing line because of the presence of polynomial generators of positive cohomological degree. Thus only one of Bousfield's two standard convergence criteria applies here even though we have this equivalence. The details involve the construction of an I-adic tower

$$R/I \longleftarrow R/I^2 \longleftarrow \cdots \longleftarrow R/I^s \longleftarrow R/I^{s+1} \longleftarrow \cdots$$

whose homotopy limit is $\widehat{\mathrm{L}}^R_E S_R$. We describe some examples for the motivating case $R = MU$.

AMS Classification 55P42, 55P43, 55T15; 55N20

Keywords S-algebra, R-module, R ring spectrum, Adams Spectral Sequence, regular quotient

Erratum

While this paper was in e-press, the authors discovered that the original versions of Theorems 6.3 and 6.4 were incorrect since they did not assume that the regular sequence u_j was finite. With the agreement of the Editors, we have revised this version to include the appropriate finiteness assumptions. We have also modified the Abstract and Introduction to reflect this and in Section 7 have replaced Bousfield localizations $\mathrm{L}^R_E X$ by E-nilpotent completions $\widehat{\mathrm{L}}^R_E X$. As far as we are aware, there are no further problems arising from this mistake.

Andrew Baker and Andrey Lazarev 9 May 2001

Introduction

We consider the Adams Spectral Sequence for R-modules based on localized regular quotient ring spectra over a commutative S-algebra R in the sense of [11, 16], making systematic use of ideas and notation from those two sources. This work grew out of a preprint [4] and the work of [6]; it is also related to ongoing collaboration with Alain Jeanneret on Bockstein operations in cohomology theories defined on R-modules [7].

One slightly surprising phenomenon we uncover concerns the convergence of the Adams Spectral Sequence based on $E = R/I[X^{-1}]$, a commutative localized regular quotient of a commutative S-algebra R. We show that the spectral sequence for $\pi_* S_R$ collapses at E_2, however for $r \geqslant 2$, E_r has no vanishing line because of the presence of polynomial generators of positive cohomological degree which are infinite cycles. Thus only one of Bousfield's two convergence criteria [10] (see Theorems 2.3 and 2.4 below) apply here. Despite this, when the generating regular sequence of I_* is finite, the spectral sequence converges to $\pi_* \mathrm{L}^R_E S_R$, where L^R_E is the Bousfield localization functor with respect to E-theory on the category of R-modules and

$$\pi_* \mathrm{L}^R_E S_R = R_*[X^{-1}]\widehat{\ }_{I_*},$$

the I_*-adic completion of $R_*[X^{-1}]$; we also show that in this case $\mathrm{L}^R_E S_R \simeq \widehat{\mathrm{L}}^R_E S_R$, the E-nilpotent completion of S_R. In the final section we describe some examples for the important case of $R = MU$, leaving more delicate calculations for future work.

To date there seems to have been very little attention paid to the detailed homotopy theory associated with the category of R-modules, apart from general results on Bousfield localizations and Wolbert's work on K-theoretic localizations in [11, 19]. We hope this paper leads to further work in this area.

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Background assumptions, terminology and technology

We work in a setting based on a good category of spectra $\mathcal{S}$ such as the category of $\mathbb{L}$-spectra of [11]. Associated to this is the subcategory of S-modules $\mathcal{M}_S$ and its derived homotopy category $\mathcal{D}_S$.

Throughout, R will denote a commutative S-algebra in the sense of [11]. There is an associated subcategory $\mathcal{M}_R$ of $\mathcal{M}_S$ consisting of the R-modules, and its derived homotopy category $\mathcal{D}_R$ and our homotopy theoretic work is located in the latter. Because we are working in $\mathcal{D}_R$, we frequently make constructions using cell R-modules in place of non-cell modules (such as R itself).

For R-modules M and N, we set

$$M^R_* N = \pi_* M \underset{R}{\wedge} N, \quad N^*_R M = \mathcal{D}_R(M, N)^*,$$

where $\mathcal{D}_R(M,N)^n = \mathcal{D}_R(M, \Sigma^n N)$.

We will use the following terminology of Strickland [16]. If the homotopy ring $R_* = \pi_* R$ is concentrated in even degrees, a *localized quotient* of R will be an R ring spectrum of the form $R/I[X^{-1}]$. A localized quotient is *commutative* if it is a commutative R ring spectrum. A localized quotient $R/I[X^{-1}]$ is *regular* if the ideal $I_* \triangleleft R_*$ is generated by a regular sequence $u_1, u_2, \ldots$ say. The ideal $I_* \triangleleft R_*$ extends to an ideal of $R_*[X^{-1}]$ which we will again denote by I_*; then as R-modules, $R/I[X^{-1}] \simeq R[X^{-1}]/I$.

We will make use of the language and ideas of algebraic derived categories of modules over a commutative ring, mildly extended to deal with evenly graded rings and their modules. In particular, this means that chain complexes are often bigraded (or even multigraded) objects with their first grading being homological and the second and higher ones being internal.

1 Brave new Hopf algebroids and their cohomology

If E is a commutative R-ring spectrum, the smash product $E\underset{R}{\wedge}E$ is also a commutative R-ring spectrum. More precisely, it is naturally an E-algebra spectrum in two ways induced from the left and right units

$$E \xrightarrow{\cong} E\underset{R}{\wedge}R \longrightarrow E\underset{R}{\wedge}E \longleftarrow E\underset{R}{\wedge}R \xleftarrow{\cong} E.$$

Theorem 1.1 *Let E^R_*E be flat as a left or equivalently right E_*-module. Then the following are true.* i) *(E_*, E^R_*E) is a Hopf algebroid over R_*.* ii) *for any R-module M, E^R_*M is a left E^R_*E-comodule.*

Proof This is proved using essentially the same argument as in [1, 15]. The natural map

$$E\underset{R}{\wedge}M \xrightarrow{\cong} E\underset{R}{\wedge}R\underset{R}{\wedge}M \longrightarrow E\underset{R}{\wedge}E\underset{R}{\wedge}M$$

induces the coaction

$$\psi\colon E^R_*M \longrightarrow \pi_*E\underset{R}{\wedge}E\underset{R}{\wedge}M \xrightarrow{\cong} E^R_*E\underset{E_*}{\otimes}E^R_*M,$$

which uses an isomorphism

$$\pi_*E\underset{R}{\wedge}E\underset{R}{\wedge}M \cong E^R_*E\underset{E_*}{\otimes}E^R_*M.$$

that follows from the flatness condition. □

For later use we record a general result on the Hopf algebroids associated with commutative regular quotients. A number of examples for the case $R = MU$ are discussed in Section 7.

Proposition 1.2 *Let $E = R/I$ be a commutative regular quotient where I_* is generated by the regular sequence $u_1, u_2, \ldots$. Then as an E_*-algebra,*

$$E^R_*E = \Lambda_{E_*}(\tau_i : i \geqslant 1),$$

where $\deg \tau_i = \deg u_i + 1$. Moreover, the generators τ_i are primitive with respect to the coaction, and $E^R_* E$ is a primitively generated Hopf algebra over E_*.

Dually, as an E_*-algebra,

$$E^*_R E = \widehat{\Lambda}_{E_*}(Q^i : i \geqslant 1),$$

where Q^i is the Bockstein operation dual to τ_i with $\deg Q^i = \deg u_i + 1$ and $\widehat{\Lambda}_{E_*}(\ )$ indicates the completed exterior algebra generated by the anti-commuting Q^i elements.

The proof requires the Künneth Spectral Sequence for R-modules of [11],

$$\mathrm{E}^2_{p,q} = \mathrm{Tor}^{R_*}_{p,q}(E_*, E_*) \Longrightarrow E^R_{p+q} E.$$

This spectral sequence is multiplicative, however there seems to be no published proof in the literature. At the suggestion of the referee, we indicate a proof of this due to M. Mandell and which originally appeared in a preprint version of [12].

Lemma 1.3 *If A and B are R ring spectra then the Künneth Spectral Sequence*

$$\mathrm{Tor}^{R_*}(A_*, B_*) \Longrightarrow A^R_* B = \pi_* A \underset{R}{\wedge} B$$

is a spectral sequence of differential graded R_*-algebras.

Sketch proof To deal with the multiplicative structure we need to modify the original construction given in Part IV section 5 of [11]. We remind the reader that we are working in the derived homotopy category $\mathcal{D}_R$.

Let

$$\cdots \longrightarrow F_{p,*} \xrightarrow{f_p} F_{p-1,*} \longrightarrow \cdots \xrightarrow{f_1} F_{0,*} \xrightarrow{f_0} A_* \to 0$$

be an free R_*-resolution of A_*. Using freeness, we can choose a map of complexes

$$\mu \colon F_{*,*} \underset{R_*}{\otimes} F_{*,*} \longrightarrow F_{*,*}$$

which lifts the multiplication on A_*.

For each $p \geqslant 0$ let $\mathbf{F}_p$ be a wedge of sphere R-modules satisfying $\pi_* \mathbf{F}_p = F_{p,*}$. Set $A'_0 = \mathbf{F}_0$ and choose a map $\varphi_0 \colon A'_0 \longrightarrow A$ inducing f_0 in homotopy. If $\mathbf{Q}_0$ is the homotopy fibre of φ_0 then

$$\pi_* \mathbf{Q}_0 = \ker f_0$$

and we can choose a map $\mathbf{F}_1 \longrightarrow \mathbf{Q}_0$ for which the composition $\varphi_1' \colon \mathbf{F}_1 \longrightarrow \mathbf{Q}_0 \longrightarrow \mathbf{F}_0$ induces f_1 in homotopy. Next take A_1' to be the cofibre of φ_1'. The map φ_0 has a canonical extension to a map $\varphi_1 \colon A_1' \longrightarrow A$. If $\mathbf{Q}_1$ is the homotopy fibre of φ_1 then

$$\pi_*\Sigma^{-1}\mathbf{Q}_1 = \ker f_1,$$

and we can find a map $\mathbf{F}_2 \longrightarrow \Sigma\mathbf{Q}_1$ for which the composite map $\varphi_2' \colon \mathbf{F}_2 \longrightarrow \mathbf{Q}_1 \longrightarrow \mathbf{F}_1$ induces f_2 in homotopy. We take A_2' to be the cofibre of φ_2' and find that there is a canonical extension of φ_1 to a map $\varphi_2 \colon A_2' \longrightarrow A$.

Continuing in this way we construct a directed system

$$A_0' \longrightarrow A_1' \longrightarrow \cdots \longrightarrow A_p' \longrightarrow \cdots \tag{1.1}$$

whose telescope A' is equivalent to A. Since we can assume that all consecutive maps are inclusions of cell subcomplexes, there is an associated filtration on A'. Smashing this with B we get a filtration on $A' \underset{R}{\wedge} B$ and an associated spectral sequence converging to A^R_*B. The identification of the E_2-term is routine.

Recall that A and therefore A' are R ring spectra. Smashing the directed system of (1.1) with itself we obtain a filtration on $A' \underset{R}{\wedge} A'$,

$$A_0' \underset{R}{\wedge} A_0' \longrightarrow \cdots \longrightarrow \bigcup_{i+j=k} A_i' \underset{R}{\wedge} A_j' \longrightarrow \bigcup_{i+j=k+1} A_i' \underset{R}{\wedge} A_j' \longrightarrow \cdots, \tag{1.2}$$

where the filtrations terms are unions of the subspectra $A_i' \underset{R}{\wedge} A_j'$. Proceeding by induction, we can realize the multiplication map $A' \underset{R}{\wedge} A' \longrightarrow A'$ as a map of filtered R-modules so that on the cofibres of the filtration terms of (1.2) it agrees with the pairing μ.

We have constructed a collection of maps $A_i' \underset{R}{\wedge} A_j' \longrightarrow A_{i+j}'$. Using these maps and the multiplication on B we can now construct maps

$$A_i' \underset{R}{\wedge} B \underset{R}{\wedge} A_j' \underset{R}{\wedge} B \longrightarrow A_{i+j} \underset{R}{\wedge} B$$

which induce the required pairing of spectral sequences. □

Proof of Proposition 1.2 As in the discussion preceding Proposition 5.1, making use of a Koszul resolution we obtain

$$\mathrm{E}^2_{*,*} = \Lambda_{E_*}(e_i : i \geqslant 1).$$

The generators have bidegree $\operatorname{bideg} e_i = (1, |u_i|)$, so the differentials

$$\mathrm{d}^r \colon \mathrm{E}^r_{p,q} \longrightarrow \mathrm{E}^r_{p-r,q+r-1}$$

are trivial on the generators e_i for dimensional reasons. Together with multiplicativity, this shows that spectral sequence collapses, giving

$$E^R_*E = \Lambda_{E_*}(\tau_i : i \geqslant 1),$$

where the generator τ_i has degree $\deg \tau_i = \deg u_i + 1$ and is represented by e_i.

For each i,

$$(R/u_i)^R_*(R/u_i) = \Lambda_{R_*/(u_i)}(\tau'_i)$$

with $\deg \tau'_i = |u_i| + 1$. Under the coproduct, τ'_i is primitive for degree reasons. By comparing the two Künneth Spectral Sequences we find that $\tau_i \in E^R_*E$ can be chosen to be the image of τ'_i under the evident ring homomorphism $(R/u_i)^R_*(R/u_i) \longrightarrow E^R_*E$, which is actually a morphism of Hopf algebroids over R_*. Hence τ_i is coaction primitive in E^R_*E.

For E^*_RE, we construct the Bockstein operation Q^i using the composition

$$R/u_i \longrightarrow \Sigma^{|u_i|+1}R \longrightarrow \Sigma^{|u_i|+1}R/u_i$$

to induce a map $E \longrightarrow \Sigma^{|u_i|+1}E$, then use the Koszul resolution to determine the Universal Coefficient Spectral sequence

$$\mathrm{E}_2^{p,q} = \mathrm{Ext}^{p,q}_{R_*}(E_*, E_*) \Longrightarrow E^{p+q}_R E$$

which collapses at its E_2-term. Further details on the construction of these operations appear in [16, 7]. $\square$

Corollary 1.4 i) *The natural map $E_* = E^R_*R \longrightarrow E^R_*E$ induced by the unit $R \longrightarrow R/I$ is a split monomorphism of E_*-modules.* ii) *E^R_*E is a free E_*-module.*

Proof An explicit splitting as in (i) is obtained using the multiplication map $E \underset{R}{\wedge} E \longrightarrow E$ which induces a homomorphism of E_*-modules $E^R_*E \longrightarrow E_*$. $\square$

We will use Coext to denote the cohomology of such Hopf algebroids rather than Ext since we will also make heavy use of Ext groups for modules over rings; more details of the definition and calculations can be found in [1, 15]. Recall that for E^R_*E-comodules L_* and M_* where L_* is E_*-projective, $\mathrm{Coext}^{s,t}_{E^R_*E}(L_*, M_*)$ can be calculated as follows. Consider a resolution

$$0 \to M_* \longrightarrow J_{0,*} \longrightarrow J_{1,*} \longrightarrow \cdots \longrightarrow J_{s,*} \longrightarrow \cdots$$

in which each $J_{s,*}$ is a summand of an extended comodule

$$E^R_*E \underset{E_*}{\square} N_{s,*},$$

for some E_*-module $N_{s,*}$. Then the complex

$$0 \to \mathrm{Hom}^*_{E^R_*E}(L_*, J_{0,*}) \longrightarrow \mathrm{Hom}^*_{E^R_*E}(L_*, J_{1,*}) \\ \longrightarrow \cdots \longrightarrow \mathrm{Hom}^*_{E^R_*E}(L_*, J_{s,*}) \longrightarrow \cdots$$

has cohomology

$$\mathrm{H}^s(\mathrm{Hom}^*_{E^R_*E}(L_*, J_{*,*})) = \mathrm{Coext}^{s,*}_{E^R_*E}(L_*, M_*).$$

The functors $\mathrm{Coext}^{s,*}_{E^R_*E}(L_*,\ )$ are the right derived functors of the left exact functor

$$M_* \rightsquigarrow \mathrm{Hom}^*_{E^R_*E}(L_*, M_*)$$

on the category of left E^R_*E-comodules. By analogy with [15], when $L_* = E_*$ we have

$$\mathrm{Coext}^{s,*}_{E^R_*E}(E_*, M_*) = \mathrm{Cotor}^{s,*}_{E^R_*E}(E_*, M_*).$$

2 The Adams Spectral Sequence for R-modules

We will describe the E-theory Adams Spectral Sequence in the homotopy category of R-module spectra. As in the classical case of sphere spectrum $R = S$, it turns out that the E_2-term is can be described in terms of the functor $\mathrm{Coext}_{E^R_*E}$.

Let L, M be R-modules and E a commutative R-ring spectrum with E^R_*E flat as a left (or right) E_*-module.

Theorem 2.1 *If E^R_*L is projective as an E_*-module, there is an Adams Spectral Sequence with*

$$\mathrm{E}^{s,t}_2(L, M) = \mathrm{Coext}^{s,t}_{E^R_*E}(E^R_*L, E^R_*M).$$

Proof Working throughout in the derived category $\mathcal{D}_R$, the proof follows that of Adams [1], with $S_R \simeq R$ replacing the sphere spectrum S. The canonical Adams resolution of M is built up in the usual way by splicing together the cofibre triangles in the following diagram.

$$\begin{tikzcd}
M \arrow[dr] & & \overline{E}\underset{R}{\wedge}M \arrow[ll] \arrow[dr] & & \overline{E}\underset{R}{\wedge}\overline{E}\underset{R}{\wedge}M \arrow[ll] \arrow[dr] & \cdots\cdots \arrow[l] \\
& E\underset{R}{\wedge}M \arrow[ur] & & E\underset{R}{\wedge}\overline{E}\underset{R}{\wedge}M \arrow[ur] & & {}
\end{tikzcd}$$

The algebraic identification of the E_2-term proceeds as in [1]. $\square$

In the rest of this paper we will have $L = S_R \simeq R$, and set

$$\mathrm{E}_2^{s,t}(M) = \mathrm{Coext}^{s,t}_{E^R_* E}(E_*, E^R_* M).$$

We will refer to this spectral sequence as the Adams Spectral Sequence based on E for the R-module M.

To understand convergence of such a spectral sequence we use a criterion of Bousfield [10, 14]. For an R-module M, let $D_s M$ $(s \geqslant 0)$ be the R-modules defined by $D_0 M = M$ and taking $D_s M$ to be the fibre of the natural map

$$D_{s-1}M \cong R \underset{R}{\wedge} D_{s-1}M \longrightarrow E \underset{R}{\wedge} D_{s-1}M.$$

Also for each $s \geqslant 0$ let $K_s M$ be the cofibre of the natural map $D_s M \longrightarrow M$. Then the *E-nilpotent completion* of M is the homotopy limit

$$\widehat{\mathrm{L}}^R_E M = \operatorname*{holim}_s K_s M.$$

Remark 2.2 It is easy to see that if $M \longrightarrow N$ is a map of R-modules which is an E-equivalence, then for each s, there is an equivalence $K_s M \longrightarrow K_s N$, hence

$$\widehat{\mathrm{L}}^R_E M \simeq \widehat{\mathrm{L}}^R_E N.$$

Theorem 2.3 *If for each pair (s,t) there is an r_0 for which $\mathrm{E}_r^{s,t}(M) = \mathrm{E}_\infty^{s,t}(M)$ whenever $r \geqslant r_0$, then the Adams Spectral Sequence for M based on E converges to $\pi_* \widehat{\mathrm{L}}^R_E M$.*

Although there is a natural map $\mathrm{L}^R_E M \longrightarrow \widehat{\mathrm{L}}^R_E M$, it is not in general a weak equivalence; this equivalence is guaranteed by another result of Bousfield [10].

Theorem 2.4 *Suppose that there is an r_1 such that for every R-module N there is an s_1 for which $\mathrm{E}_r^{s,t}(N) = 0$ whenever $r \geqslant r_1$ and $s \geqslant s_1$. Then for every R-module M the Adams Spectral Sequence for M based on E converges to $\pi_* \mathrm{L}^R_E M$ and*

$$\mathrm{L}^R_E M \simeq \widehat{\mathrm{L}}^R_E M.$$

3 The Universal Coefficient Spectral Sequence for regular quotients

Let R be a commutative S-algebra and $E = R/I$ a commutative regular quotient of R, where $u_1, u_2, \ldots$ is a regular sequence generating $I_* \lhd R_*$.

We will discuss the existence of the Universal Coefficient Spectral Sequence

$$\mathrm{E}^2_{r,s} = \mathrm{Ext}^{r,s}_{E_*}(E^R_* M, N_*) \Longrightarrow N^*_R M, \tag{3.1}$$

where M and N are R-modules and N is also an E-module spectrum in $\mathcal{M}_R$. The classical prototype of this was described by Adams [1] (who generalized a construction of Atiyah [2] for the Künneth Theorem in K-theory) and used in setting up the E-theory Adams Spectral Sequence. It is routine to verify that Adams' approach can be followed in $\mathcal{D}_R$. We remark that if E were a commutative R-algebra then the Universal Coefficient Spectral Sequence of [11] would be applicable but that condition does not hold in the generality we require.

The existence of such a spectral sequence depends on the following conditions being satisfied.

Conditions 3.1 E is a homotopy colimit of finite cell R-modules E_α whose R-Spanier Whitehead duals $\mathrm{D}_R E_\alpha = \mathcal{F}_R(E_\alpha, R)$ satisfy the two conditions (A) $E^R_* \mathrm{D}_R E_\alpha$ is E_*-projective; (B) the natural map

$$N^*_R M \longrightarrow \mathrm{Hom}_{E_*}(E^R_* M, N_*)$$

is an isomorphism.

Theorem 3.2 *For a commutative regular quotient $E = R/I$ of R, E can be expressed as a homotopy colimit of finite cell R-modules satisfying the conditions of* Condition 3.1. *In fact we can take $E^R_* \mathrm{D}_R E_\alpha$ to be E_*-free.*

The proof will use the following Lemma.

Lemma 3.3 *Let $u \in R_{2d}$ be non-zero divisor in R_*. Suppose that P is an R-module for which $E^R_* P$ is E_*-projective and for an E-module R-spectrum N,*

$$N^*_R P \cong \mathrm{Hom}_{E_*}(E^R_* P, N_*).$$

Then $E^R_* P \underset{R}{\wedge} R/u$ is E_*-projective and

$$N^*_R P \underset{R}{\wedge} R/u \cong \mathrm{Hom}_{E_*}(E^R_* P \underset{R}{\wedge} R/u, N_*).$$

Proof Smashing $E \underset{R}{\wedge} P$ with the cofibre sequence (3.2) and taking homotopy, we obtain an exact triangle

$$\begin{tikzcd} E^R_* P \arrow[rr, "u"] & & E^R_* P \arrow[dl] \\ & E^R_* P \underset{R}{\wedge} R/u \arrow[ul] & \end{tikzcd}$$

As multiplication by u induces the trivial map in E^R-homology, this is actually a short exact sequence of E_*-modules,

$$0 \to E^R_* P \longrightarrow E^R_* P \underset{R}{\wedge} R/u \longrightarrow E^R_* P \to 0$$

which clearly splits, so $E^R_* P \underset{R}{\wedge} R/u$ is E_*-projective.

In the evident diagram of exact triangles

$$\begin{tikzcd}
N^*_R P \arrow[rr] \arrow[dd] & & N^*_R P \arrow[dl] \arrow[dd] \\
& N^*_R P \underset{R}{\wedge} R/u \arrow[ul] \arrow[dd] & \\
\mathrm{Hom}_{E_*}(E^R_* P, N_*) \arrow[rr] & & \mathrm{Hom}_{E_*}(E^R_* P, N_*) \arrow[dl] \\
& \mathrm{Hom}_{E_*}(E^R_* P \underset{R}{\wedge} R/u, N_*) \arrow[ul] &
\end{tikzcd}$$

the map $N^*_R P \longrightarrow \mathrm{Hom}_{E_*}(E^R_* P, N_*)$ is an isomorphism, so

$$N^*_R P \underset{R}{\wedge} R/u \longrightarrow \mathrm{Hom}_{E_*}(E^R_* P \underset{R}{\wedge} R/u, N_*)$$

is also an isomorphism by the Five Lemma. $\square$

Proof of Theorem 3.2 Let $u_1, u_2, \ldots$ be a regular sequence generating $I_* \lhd R_*$. Using the notation $R/u = R/(u)$, we recall from [16] that

$$E = \underset{k}{\mathrm{hocolim}}\, R/u_1 \underset{R}{\wedge} R/u_2 \underset{R}{\wedge} \cdots \underset{R}{\wedge} R/u_k.$$

For $u \in R_{2d}$ a non-zero divisor, the R_*-free resolution

$$0 \to R_* \longrightarrow R_* \xrightarrow{u} R_*/(u) \to 0$$

corresponds to an R-cell structure on R/u with one cell in each of the dimensions 0 and $2d+1$. There is an associated cofibre sequence

$$\cdots \longrightarrow \Sigma^{2d} R \xrightarrow{u} R \longrightarrow R/u \longrightarrow \Sigma^{2d+1} R \longrightarrow \cdots, \tag{3.2}$$

for which the induced long exact sequence in E^R-homology shows that $E^R_* R/u$ is E_*-free. The dual $\mathrm{D}_R R/u$ is equivalent to $\Sigma^{-(2d+1)} R/u$, hence R/u is essentially self dual.

For an E-module spectrum N in $\mathcal{D}_R$, there are two exact triangles and morphisms between them,

$$\begin{tikzcd}
N^*_R R \arrow[rr] \arrow[dd] & & N^*_R R \arrow[dl] \arrow[dd] \\
& N^*_R R/u \arrow[ul] \arrow[dd] & \\
\mathrm{Hom}_{E_*}(E_*, N_*) \arrow[rr] & & \mathrm{Hom}_{E_*}(E_*, N_*) \arrow[dl] \\
& \mathrm{Hom}_{E_*}(E^R_* R/u, N_*) \arrow[ul] &
\end{tikzcd}$$

The identifications

$$N_* \cong N^*_R R \cong \mathrm{Hom}_{E_*}(E_*, N_*),$$

and the Five Lemma imply that

$$N^*_R R/u \cong \mathrm{Hom}_{E_*}(E^R_* R/u, N_*).$$

Lemma 3.3 now implies that each of the spectra $R/u_1 \underset{R}{\wedge} R/u_2 \underset{R}{\wedge} \cdots \underset{R}{\wedge} R/u_k$ satisfies conditions (A) and (B). $\square$

4 The Adams Spectral Sequence based on a regular quotient

For an R-module M, let $M^{(s)}$ denote the s-fold R-smash power of M,

$$M^{(s)} = M \underset{R}{\wedge} M \underset{R}{\wedge} \cdots \underset{R}{\wedge} M.$$

If M is an $R[X^{-1}]$-module, then

$$M^{(s)} = M \underset{R[X^{-1}]}{\wedge} M \underset{R[X^{-1}]}{\wedge} \cdots \underset{R[X^{-1}]}{\wedge} M.$$

Let $E = R/I[X^{-1}]$ be a localized regular quotient and $u_1, u_2, \ldots$ a regular sequence generating I_*. We will discuss the Adams Spectral Sequence based on E. By Remark 2.2, we can work in the category of $R[X^{-1}]$-modules and replace the Adams Spectral Sequence of S_R by that of $S_{R[X^{-1}]}$. To simplify notation, from now on we will replace R by $R[X^{-1}]$ and therefore assume that $E = R/I$ is a regular quotient of R.

First we identify the canonical Adams resolution giving rise to the Adams Spectral Sequence based on the regular quotient $E = R/I$. We will relate this to a tower described by the second author [12], but the reader should beware that his notation for $I^{(s)}$ is I^s which we will use for a different spectrum.

There is a fibre sequence $I \longrightarrow R \longrightarrow R/I$ and a tower of maps of R-modules

$$R \longleftarrow I \longleftarrow I^{(2)} \longleftarrow \cdots \longleftarrow I^{(s)} \longleftarrow I^{(s+1)} \longleftarrow \cdots$$

in which $I^{(s+1)} \longrightarrow I^{(s)}$ is the evident composite

$$I^{(s+1)} \longrightarrow R \underset{R}{\wedge} I^{(s)} = I^{(s)}.$$

Setting $R/I^{(s)} = \mathrm{cofibre}(I^{(s)} \longrightarrow R)$, we obtain a tower

$$R/I \longleftarrow R/I^{(2)} \longleftarrow \cdots \longleftarrow R/I^{(s)} \longleftarrow R/I^{(s+1)} \longleftarrow \cdots$$

which we will refer to as the *external I-adic tower*. The next result is immediate from the definitions.

Proposition 4.1 *We have*

$$D_0 S_R = R, \qquad D_s S_R = I^{(s)}, \quad (s \geqslant 1),$$

and

$$K_s S_R = R/I^{(s+1)} \quad (s \geqslant 0).$$

It is not immediately clear how to determine the limit

$$\widehat{\mathrm{L}}^R_E S_R = \operatorname*{holim}_s R/I^{(s)}.$$

Instead of doing this directly, we will adopt an approach suggested by Bousfield [10], making use of another E-nilpotent resolution, associated with the *internal I-adic tower* to be described below.

In order to carry this out, we first need to understand convergence. We will see that the condition of Theorem 2.3 is satisfied for a commutative regular quotient $E = R/I$.

Proposition 4.2 *The* E_2*-term of the E-theory Adams Spectral Sequence for* $\pi_* S_R$ *is*

$$\mathrm{E}_2^{s,t}(S_R) = \mathrm{Coext}^{s,t}_{E^R_* E}(E_*, E_*) = E_*[U_i : i \geqslant 1],$$

where $\mathrm{bideg}\, U_i = (1, |u_i| + 1)$*. Hence this spectral sequence collapses at its* E_2*-term*

$$\mathrm{E}_2^{*,*}(S_R) = \mathrm{E}_\infty^{*,*}(S_R)$$

and converges to $\pi_* \widehat{\mathrm{L}}^R_E S_R$.

Proof By Proposition 1.2,

$$E^R_*E = \Lambda_{R_*}(\tau_i : i \geqslant 1),$$

with generators τ_i which are primitive with respect to the coproduct of this Hopf algebroid. The determination of

$$\mathrm{Coext}^{*,*}_{E^R_*E}(E_*, E_*)$$

is now standard and the differentials are trivial for degree reasons. □

Induction on the number of cells now gives

Corollary 4.3 *For a finite cell R-module M, the E-theory Adams Spectral Sequence for π_*M converges to $\pi_*\widehat{\mathrm{L}}^R_E M$.*

5 The internal I-adic tower

Suppose that $I_* \lhd R_*$ is generated by a regular sequence $u_1, u_2, \ldots$. We will often indicate a monomial in the u_i by writing $u_{(i_1,\cdots,i_k)} = u_{i_1}\cdots u_{i_k}$. We will write $E = R/I$ and make use of algebraic results from [5] which we now recall in detail.

For $s \geqslant 0$, we define the R-module I^s/I^{s+1} to be the wedge of copies of E indexed on the distinct monomials of degree s in the generators u_i. For an explanation of this, see Corollary 5.4.

We will show that there is an (*internal*) I*-adic tower* of R-modules

$$R/I \longleftarrow R/I^2 \longleftarrow \cdots \longleftarrow R/I^s \longleftarrow R/I^{s+1} \longleftarrow \cdots$$

so that for each $s \geqslant 0$ the fibre sequence

$$R/I^s \longleftarrow R/I^{s+1} \longleftarrow I^s/I^{s+1}$$

corresponds to a certain element of

$$\mathrm{Ext}^1_{R_*}(R_*/I^s_*, I^s_*/I^{s+1}_*)$$

in E_2-term of the Universal Coefficient Spectral Sequence of [11] converging to $\mathcal{D}_R(R/I^s, I^s/I^{s+1})^*$. On setting $I^s = \mathrm{fibre}(R \longrightarrow R/I^s)$ we obtain another tower

$$R \longleftarrow I \longleftarrow I^2 \longleftarrow \cdots \longleftarrow I^s \longleftarrow I^{s+1} \longleftarrow \cdots$$

which is analogous to the external version of [12]. A related construction appeared in [3, 8] for the case of $R = \widehat{E(n)}$ (which was shown to admit a not necessarily commutative S-algebra structure) and $I = I_n$.

Underlying our work is the classical *Koszul resolution*

$$\mathbf{K}_{*,*} \longrightarrow R_*/I_* \to 0,$$

where

$$\mathbf{K}_{*,*} = \Lambda_{R_*}(e_i : i \geqslant 1),$$

which has grading given by $\deg e_i = |u_i| + 1$ and differential

$$\begin{aligned} \mathrm{d}\, e_i &= u_i, \\ \mathrm{d}(xy) &= (\mathrm{d}\, x)y + (-1)^r x \,\mathrm{d}\, y \quad (x \in \mathbf{K}_{r,*},\ y \in \mathbf{K}_{s,*}). \end{aligned}$$

Hence $(\mathbf{K}_{*,*}, \mathrm{d})$ is an R_*-free resolution of R_*/I_* which is a differential graded R_*-algebra. Tensoring with R_*/I_* and taking homology leads to a well known result.

Proposition 5.1 *As an R_*/I_*-algebra,*

$$\mathrm{Tor}^{R_*}_{*,*}(R_*/I_*, R_*/I_*) = \Lambda_{R_*/I_*}(e_i : i \geqslant 1).$$

Corollary 5.2 $\mathrm{Tor}^{R_*}_{*,*}(R_*/I_*, R_*/I_*)$ *is a free R_*/I_*-module.*

This is of course closely related to the topological result Proposition 1.2.

Now returning to our algebraic discussion, we recall the following standard result.

Lemma 5.3 ([13], Theorem 16.2) *For $s \geqslant 0$, I_*^s/I_*^{s+1} is a free R_*/I_*-module with a basis consisting of residue classes of the distinct monomials $u_{(i_1,\dots,i_s)}$ of degree s.*

Corollary 5.4 *For $s \geqslant 0$, there is an isomorphism of R_*-modules*

$$\pi_* I^s/I^{s+1} = I_*^s/I_*^{s+1}.$$

Hence $\pi_* I^s/I^{s+1}$ is a free R_*/I_*-module with a basis indexed on the distinct monomials $u_{(i_1,\dots,i_s)}$ of degree s.

Let $\mathrm{U}^{(s)}_*$ be the free R_*-module on a basis indexed on the distinct monomials of degree s in the u_i. For $s \geqslant 0$, set

$$\mathbf{Q}^{(s)}_{*,*} = \mathbf{K}_{*,*} \underset{R_*}{\otimes} \mathrm{U}^{(s)}_*, \quad \mathrm{d}^{(s)}_{\mathbf{Q}} = \mathrm{d} \otimes 1,$$

and also for $x \in \mathbf{K}_{*,*}$ write

$$x\widetilde{u}_{(i_1,\dots,i_s)} = x \otimes u_{(i_1,\dots,i_s)}.$$

There is an obvious augmentation

$$\mathbf{Q}^{(s)}_{0,*} \longrightarrow I^s_*/I^{s+1}_*.$$

Lemma 5.5 *For* $s \geqslant 1$,

$$\mathbf{Q}^{(s)}_{*,*} \xrightarrow{\varepsilon^{(s)}} I^s_*/I^{s+1}_* \to 0$$

is a resolution by free R_**-modules.*

Given a complex $(\mathbf{C}_{*,*}, \mathrm{d}_{\mathbf{C}})$, the k-shifted complex $(\mathbf{C}[k]_{*,*}, \mathrm{d}_{\mathbf{C}[k]})$ is defined by

$$\mathbf{C}[k]_{n,*} = \mathbf{C}_{n+k,*}, \quad \mathrm{d}_{\mathbf{C}[k]} = (-1)^k \, \mathrm{d}_{\mathbf{C}}\,.$$

There is a morphism of chain complexes

$$\partial^{(s+1)} \colon \mathbf{Q}^{(s)}_{*,*} \longrightarrow \mathbf{Q}^{(s+1)}[-1]_{*,*};$$

$$\partial^{(s+1)} e_{i_1} \cdots e_{i_r} \widetilde{u}_{(j_1,\dots,j_s)} = \sum_{k=1}^{r} (-1)^k e_{i_1} \cdots \widehat{e}_{i_k} \cdots e_{i_r} \widetilde{u}_{(j_1,\dots,j_s)}.$$

Using the identification $\mathbf{Q}^{(s+1)}[-1]_{n,*} = \mathbf{Q}^{(s+1)}_{n-1,*}$, we will often view $\partial^{(s+1)}$ as a homomorphism

$$\partial^{(s+1)} \colon \mathbf{Q}^{(s)}_{*,*} \longrightarrow \mathbf{Q}^{(s+1)}_{*,*}$$

of bigraded R_*-modules of degree -1.

There are also external pairings

$$\mathbf{Q}^{(r)}_{*,*} \underset{R_*}{\otimes} \mathbf{Q}^{(s)}_{*,*} \longrightarrow \mathbf{Q}^{(r+s)}_{*,*};$$

$$x\widetilde{u}_{(i_1,\dots,i_s)} \otimes y\widetilde{u}_{(j_1,\dots,j_s)} \longmapsto xy\widetilde{u}_{(i_1,\dots,i_s,j_1,\dots,j_s)} \quad (x, y \in \mathbf{K}_{*,*}).$$

In particular, each $\mathbf{Q}^{(r)}_{*,*}$ is a differential module over the differential graded R_*-algebra $\mathbf{K}^{(0)}_{*,*}$ and $\partial^{(s+1)}$ is a $\mathbf{K}^{(0)}_{*,*}$-derivation.

Theorem 5.6 *For $s \geqslant 1$, there is a resolution*

$$\mathbf{K}^{(s-1)}_{*,*} \xrightarrow{\varepsilon^{(s-1)}} R_*/I^s_* \to 0,$$

by free R_*-modules, where

$$\mathbf{K}^{(s-1)}_{*,*} = \mathbf{Q}^{(0)}_{*,*} \oplus \mathbf{Q}^{(1)}_{*,*} \oplus \cdots \oplus \mathbf{Q}^{(s-1)}_{*,*},$$

and the differential is

$$\mathrm{d}^{(s-1)} = (\mathrm{d}^{(0)}_{\mathbf{Q}}, \partial^{(1)} + \mathrm{d}^{(1)}_{\mathbf{Q}}, \partial^{(2)} + \mathrm{d}^{(2)}_{\mathbf{Q}}, \ldots, \partial^{(s-1)} + \mathrm{d}^{(s-1)}_{\mathbf{Q}}).$$

In fact $(\mathbf{K}^{(s-1)}_{*,*}, \mathrm{d}^{(s-1)})$ is a differential graded R_*-algebra which provides a multiplicative resolution of R_*/I^s, with the augmentation given by

$$\varepsilon^{(s-1)}(x_0, x_1\widetilde{u}_{\mathbf{i}_1}, \ldots, x_{s-1}\widetilde{u}_{\mathbf{i}_{s-1}}) = x_0 + x_1 u_{\mathbf{i}_1} + \cdots + x_{s-1}u_{\mathbf{i}_{s-1}}.$$

The algebraic extension of R_*-modules

$$0 \leftarrow R_*/I^s_* \longleftarrow R_*/I^{s+1}_* \longleftarrow I^s_*/I^{s+1}_* \leftarrow 0$$

is classified by an element of

$$\mathrm{Ext}^1_{R_*}(R_*/I^s_*, I^s_*/I^{s+1}_*) = \mathrm{Hom}_{\mathcal{D}_{R_*}}(R_*/I^s_*, I^s_*/I^{s+1}_*[-1]),$$

where $\mathrm{Hom}_{\mathcal{D}_{R_*}}$ denotes morphisms in the derived category $\mathcal{D}_{R_*}$ of the ring R_* [18]. This element is represented by the composite

$$\widetilde{\partial}^{(s)}_* : \mathbf{K}^{(s-1)}_{*,*} \xrightarrow{\mathrm{proj}} \mathbf{Q}^{(s-1)}_{*,*} \xrightarrow{\partial^{(s)}} \mathbf{Q}^{(s)}[-1]_{*,*}. \tag{5.1}$$

The analogue of the next result for ungraded rings was proved in [5]; the proof is easily adapted to the graded case.

Proposition 5.7 *For each $s \geqslant 2$, the following complex is exact:*

$$\mathrm{Tor}^{R_*}_{*,*}(R_*/I_*, R_*/I_*) \xrightarrow{\partial^{(1)}_*} \mathrm{Tor}^{R_*}_{*,*}(R_*/I_*, I_*/I^2_*)$$
$$\xrightarrow{\partial^{(2)}_*} \cdots \xrightarrow{\partial^{(s-1)}_*} \mathrm{Tor}^{R_*}_{*,*}(R_*/I_*, I^{s-1}_*/I^s_*).$$

Theorem 5.8 *For $s \geqslant 2$,*

$$\mathrm{Tor}^{R_*}_{*,*}(R_*/I_*, R_*/I^s_*) = R_*/I_* \oplus \operatorname{coker} \partial^{(s-1)}_*.$$

This is a free R_*/I_*-module and with its natural R_*/I_*-algebra structure, $\mathrm{Tor}^{R_*}_{*,*}(R_*/I_*, R_*/I^s_*)$ has trivial products.

Given this algebraic background, we can now construct the I-adic tower.

Theorem 5.9 *There is a tower of R-modules*

$$R/I \longleftarrow R/I^2 \longleftarrow \cdots \longleftarrow R/I^s \longleftarrow R/I^{s+1} \longleftarrow \cdots$$

whose maps define fibre sequences

$$R/I^s \longleftarrow R/I^{s+1} \longleftarrow I^s/I^{s+1}$$

which in homotopy realise the exact sequences of R_*-modules

$$0 \leftarrow R_*/I_*^s \longleftarrow R_*/I_*^{s+1} \longleftarrow I_*^s/I_*^{s+1} \leftarrow 0.$$

Furthermore, the following conditions are satisfied for each $s \geqslant 1$. (i) *$E_*^R R/I^s$ is a free E_*-module and the unit induces a splitting*

$$E_*^R R/I^s = E_* \oplus (\ker\colon E_*^R R/I^s \longrightarrow E_*);$$

(ii) *the projection map $R/I^{s+1} \longrightarrow R/I^s$ induces the zero map*

$$(\ker\colon E_*^R R/I^{s+1} \longrightarrow E_*) \longrightarrow (\ker\colon E_*^R R/I^s \longrightarrow E_*);$$

(iii) *the inclusion map $j_s\colon I^s/I^{s+1} \longrightarrow R/I^{s+1}$ induces an exact sequence*

$$E_*^R I^{s-1}/I^s \xrightarrow{\partial_*^{(s)}} E_*^R I^s/I^{s+1} \xrightarrow{j_{s*}} (\ker\colon E_*^R R/I^{s+1} \longrightarrow E_*) \to 0.$$

Proof The proof is by induction on s. Assuming that R/I^s exists with the asserted properties, we will define a suitable map $\delta_s\colon R/I^s \longrightarrow \Sigma I^s/I^{s+1}$ which induces a fibre sequence of the form

$$R/I^s \longleftarrow X^{(s+1)} \longleftarrow I^s/I^{s+1}, \tag{5.2}$$

for which $\pi_* X^{(s+1)} = R_*/I_*^{s+1}$ as an R_*-module.

If M is an R-module which is an E module spectrum, Theorem 3.2 provides a Universal Coefficient Spectral Sequence

$$\mathrm{E}_2^{*,*} = \mathrm{Ext}_{E_*}^{p,q}(E_*^R R/I^s, M_*) \Longrightarrow \mathcal{D}_R(R/I^s, M)^{p+q}.$$

Since $E_*^R R/I^s$ is E_*-free, this spectral sequence collapses to give

$$\mathcal{D}_R(R/I^s, M)^* = \mathrm{Hom}_{E_*}^*(E_*^R R/I^s, M_*).$$

In particular, for $M = I^s/I^{s+1}$,

$$\mathcal{D}_R(R/I^s, I^s/I^{s+1})^n = \mathrm{Hom}_{E_*}^n(E_*^R R/I^s, I_*^s/I_*^{s+1}).$$

By (5.1) and Theorem 5.6, there is an element

$$\widetilde{\partial}_*^{(s)} \in \mathrm{Hom}_{E_*}^0(E_*^R R/I^s, I_*^s/I_*^{s+1}[-1]) = \mathrm{Hom}_{E_*}^1(E_*^R R/I^s, I_*^s/I_*^{s+1}),$$

corresponding to an element $\delta_s\colon R/I^s \longrightarrow \Sigma I^s/I^{s+1}$ inducing a fibre sequence as in (5.2). It still remains to verify that $\pi_* X^{(s+1)} = R_*/I_*^{s+1}$ as an R_*-module.

For this, we will use the resolutions $\mathbf{K}_{*,*}^{(s-1)} \longrightarrow R_*/I_*^s \to 0$ and $\mathbf{K}_{*,*} \longrightarrow R_*/I_* \to 0$. These free resolutions give rise to cell R-module structures on R/I^s and E. By [11], the R-module $E \underset{R}{\wedge} R/I^s$ admits a cell structure with cells in one-one correspondence with the elements of the obvious tensor product basis of $\mathbf{K}_{*,*} \underset{R_*}{\otimes} \mathbf{K}_{*,*}^{(s-1)}$. Hence there is a resolution by free R_*-modules

$$\mathbf{K}_{*,*} \underset{R_*}{\otimes} \mathbf{K}_{*,*}^{(s-1)} \longrightarrow E_*^R R/I^s \to 0.$$

There are morphisms of chain complexes

$$\mathbf{K}_{*,*}^{(s-1)} \xrightarrow{\rho_s} \mathbf{K}_{*,*} \underset{R_*}{\otimes} \mathbf{K}_{*,*}^{(s-1)} \xrightarrow{\widetilde{\delta}_s} \mathbf{Q}_{*,*}^{(s)}[-1],$$

where ρ_s is the obvious inclusion and $\widetilde{\delta}_s$ is a chain map lifting $\widetilde{\partial}_*^{(s)}$ which can be chosen so that

$$\widetilde{\delta}_s(e_i \otimes x) = 0.$$

The effect of the composite $\widetilde{\delta}_s \rho_s$ on the generator $e_i \widetilde{u}_{(j_1,\dots,j_{s-1})} \in \mathbf{K}_{1,*}^{(s-1)}$ turns out to be

$$\widetilde{\partial}_*^{(s)} e_i \widetilde{u}_{(j_1,\dots,j_{s-1})} = \widetilde{u}_{(i,j_1,\dots,j_{s-1})},$$

while the elements of form $e_i \otimes \widetilde{u}_{(j_1,\dots,j_{k-1})}$ with $k < s$ are annihilated. The composite homomorphism

$$\mathbf{K}_{1,*}^{(s-1)} \xrightarrow{\widetilde{\delta}_s \rho_s} \mathbf{Q}_{0,*}^{(s)}[-1] \xrightarrow{\varepsilon_1} I_*^s/I_*^{s+1}[-1]$$

is a cocycle. There is a morphism of exact sequences

$$\begin{tikzcd}
0 & R_*/I_*^s \arrow[l] \arrow[d, equal] & \mathbf{K}_{0,*}^{(s-1)} \arrow[l] \arrow[d, "\alpha_0"'] & \mathbf{K}_{1,*}^{(s-1)} \arrow[l] \arrow[d, "\alpha_1"'] & \mathbf{K}_{2,*}^{(s-1)} \arrow[l] \arrow[d] \\
0 & R_*/I_*^s \arrow[l] & R_*/I_*^{s+1} \arrow[l] & I_*^s/I_*^{s+1} \arrow[l] & 0 \arrow[l]
\end{tikzcd}$$

where the cohomology class

$$[\alpha_1] \in \mathrm{Ext}_{R_*}^{1,*}(R_*/I_*^s, I_*^s/I_*^{s+1})$$

represents the extension of R_*-modules on the bottom row. It is easy to see that $[\alpha_1] = [\varepsilon_1 \widetilde{\delta}_s \rho_s]$, hence this class also represents the extension of R_*-modules

$$0 \leftarrow R_*/I_*^s \longleftarrow \pi_* X^{s+1} \longleftarrow I_*^s/I_*^{s+1} \leftarrow 0.$$

There is a diagram of cofibre triangles

$$\begin{tikzcd}
R/I^{s+1} \arrow[d] & I/I^{s+1} \arrow[l] \arrow[d] & {} \arrow[l] & \cdots & {} & I^{s-1}/I^{s+1} \arrow[l] \arrow[d] & I^s/I^{s+1} \arrow[l] \arrow[d, "="] \\
R/I \arrow[ur] & I/I^2 \arrow[ur] & & & {} \arrow[ur] & I^{s-1}/I^s \arrow[ur] & I^s/I^{s+1}
\end{tikzcd}$$

and applying $E^R_*(\ )$ we obtain a spectral sequence converging to $E^R_* R/I^{s+1}$ whose E_2-term is the homology of the complex

$$0 \to E^R_* R/I \xrightarrow{\partial^{(1)}_*} E^R_* I/I^2 \xrightarrow{\partial^{(2)}_*} E^R_* I^2/I^3 \longrightarrow \cdots \xrightarrow{\partial^{(s)}_*} E^R_* I^s/I^{s+1} \to 0,$$

where the $\partial^{(k)}_*$ are essentially the maps used to compute $\mathrm{Tor}^{R_*}_*(R_*/I_*, R_*/I^{s+1}_*)$ in [5]. By Proposition 5.7 and Theorem 5.8, this complex is exact except at the ends, where we have $\ker \partial^{(1)}_* = E_*$. As a result, this spectral sequence collapses at E_3 giving the desired form for $E^R_* R/I^{s+1}$. □

Corollary 5.10 *For any E-module spectrum N and $s \geqslant 1$,*

$$N^*_R R/I^s \cong \mathrm{Hom}_{E_*}(E^R_* R/I^s, N_*).$$

Proof This follows from Theorem 5.9(i). □

We will also use the following result.

Corollary 5.11 *For $s \geqslant 1$, the natural map*

$$E^R_* R/I^{s+1} \longrightarrow E^R_* R/I^s,$$

has image equal to $E_* = E^R_* R$.

Proof This follows from Theorem 5.9(ii). □

Corollary 5.12 *For any E-module spectrum N and $s \geqslant 1$,*

$$\operatorname*{colim}_s N^*_R R/I^s \cong N^*_R R \cong N_*.$$

Proof This is immediate from Corollaries 5.10 and 5.11 since

$$\operatorname*{colim}_s \mathrm{Hom}_{E_*}(E^R_* R/I^s, N_*) \cong \mathrm{Hom}_{E_*}(E_*, N_*).$$

□

6 The I-adic tower and Adams Spectral Sequence

Continuing with the notation of Section 5, the first substantial result of this section is

Theorem 6.1 *The I-adic tower*

$$R/I \longleftarrow R/I^2 \longleftarrow \cdots \longleftarrow R/I^s \longleftarrow R/I^{s+1} \longleftarrow \cdots$$

has homotopy limit

$$\operatorname*{holim}_s R/I^s \simeq \widehat{\mathrm{L}}^R_E S_R.$$

Our approach follows ideas of Bousfield [10] where it is shown that the following Lemma implies Theorem 6.1.

Lemma 6.2 *Let $E = R/I$. Then the following are true.* i) *Each R/I^s is E-nilpotent.* ii) *For each E-nilpotent R-module M,*

$$\operatorname*{colim}_s \mathcal{D}_R(R/I^s, M)^* = M_{-*}.$$

Proof (i) is proved by an easy induction on $s \geqslant 1$. (ii) is a consequence of Corollary 5.12. □

Since the maps $R_*/I_*^{s+1} \longrightarrow R_*/I_*^s$ are surjective, from the standard exact sequence for $\pi_*(\ )$ of a homotopy limit we have

$$\pi_*\widehat{\mathrm{L}}^R_E S_R = \lim_s R_*/I_*^s. \tag{6.1}$$

We can generalize this to the case where E is a commutative localized regular quotient.

Theorem 6.3 *Let $E = R/I[X^{-1}]$ be a commutative localized regular quotient of R. Then*

$$\pi_*\widehat{\mathrm{L}}^R_E S_R = R_*[X^{-1}]\widehat{{}_{I_*}} = \lim_s R_*[X^{-1}]/I_*^s.$$

If the regular sequence generating I_* is finite, then the natural map $S_R \longrightarrow \widehat{\mathrm{L}}^R_E S_R$ is an E-equivalence, hence

$$\mathrm{L}^R_E S_R \simeq \widehat{\mathrm{L}}^R_E S_R,$$
$$\pi_*\mathrm{L}^R_E S_R = R_*[X^{-1}]\widehat{{}_{I_*}}.$$

Proof The first statement is easy to verify.

By Remark 2.2, to simplify notation we may as well replace R by $R[X^{-1}]$ and so assume that $E = R/I$ is a commutative regular quotient of R.

Using the Koszul complex $(\Lambda_{R_*}(e_j : j), \mathrm{d})$, we see that $\mathrm{Tor}^{R_*}_{*,*}(E_*, (R_*)\widehat{_{I_*}})$ is the homology of the complex

$$\Lambda_{R_*}(e_j : j) \underset{R_*}{\otimes} (R_*)\widehat{_{I_*}} = \Lambda_{(R_*)\widehat{_{I_*}}}(e_j : j)$$

with differential $\mathrm{d}' = \mathrm{d} \otimes 1$. Since the sequence u_j remains regular in $(R_*)\widehat{_{I_*}}$, this complex provides a free resolution of $E_* = R_*/I_*$ as an $(R_*)\widehat{_{I_*}}$-module (this is *false* if the sequence u_j is not finite). Hence we have

$$\mathrm{Tor}^{R_*}_{*,*}(E_*, (R_*)\widehat{_{I_*}}) = \mathrm{Tor}^{(R_*)\widehat{_{I_*}}}_{*,*}(E_*, (R_*)\widehat{_{I_*}}) = E_*.$$

To calculate $E^R_* \widehat{\mathrm{L}}^R_E S_R$ we may use the Künneth Spectral Sequence of [11],

$$\mathrm{E}^{s,t}_2 = \mathrm{Tor}^{R_*}_{s,t}(E_*, \widehat{\mathrm{L}}^R_E S_R) \Longrightarrow E^R_{s+t} \widehat{\mathrm{L}}^R_E S_R.$$

By the first part, the E_2-term is

$$\mathrm{Tor}^{R_*}_{*,*}(E_*, (R_*)\widehat{_{I_*}}) = E_* = E^R_* R.$$

Hence the natural homomorphism

$$E^R_* S_R \longrightarrow E^R_* \widehat{\mathrm{L}}^R_E S_R$$

is an isomorphism. □

If the sequence u_j is infinite, the calculation of this proof shows that

$$E^R_* \widehat{\mathrm{L}}^R_E S_R = (R_*)\widehat{_{I_*}}/I_* \neq R_*/I_* = E_* S_R$$

and the Adams Spectral Sequence does not converge to the homotopy of the E-localization.

An induction on the number of cells of M proves a generalization of Theorem 6.3.

Theorem 6.4 *Let E be a commutative localized regular quotient of R and M a finite cell R-module. Then*

$$\pi_* \widehat{\mathrm{L}}^R_E M = M_*[X^{-1}]\widehat{_{I_*}} = R_*[X^{-1}]\widehat{_{I_*}} \underset{R_*}{\otimes} M_*.$$

If the regular sequence generating I_* is finite, then the natural map $M \longrightarrow \widehat{\mathrm{L}}^R_E M$ is an E-equivalence, hence

$$\mathrm{L}^R_E M \simeq \widehat{\mathrm{L}}^R_E M,$$
$$\pi_* \mathrm{L}^R_E M = M_*[X^{-1}]\widehat{_{I_*}} = R_*[X^{-1}]\widehat{_{I_*}} \underset{R_*}{\otimes} M_*.$$

The reader may wonder if the following conjecture is true, the algebraic issue being that it does not appear to be true that for a commutative ring A, the extension $A \longrightarrow A\widehat{_J}$ is always flat for an ideal $J \lhd A$, a Noetherian condition normally being required to establish such a result.

Conjecture 6.5 The conclusion of Theorem 6.4 holds when E is any commutative localized quotient of R.

7 Some examples associated with MU

An obvious source of commutative localized regular quotients is the commutative S-algebra $R = MU$ and we will describe some important examples. It would appear to be algebraically simpler to work with BP at a prime p in place of MU, but at the time of writing, it seems not to be known whether BP admits a commutative S-algebra structure.

Example A: $MU \longrightarrow H\mathbb{F}_p$.

Let p be a prime. By considering the Eilenberg-Mac Lane spectrum $H\mathbb{F}_p$ as a commutative MU-algebra [11], we can form $H\mathbb{F}_p \underset{MU}{\wedge} H\mathbb{F}_p$. The Künneth Spectral Sequence gives

$$\mathrm{E}^2_{s,t} = \mathrm{Tor}^{MU_*}_{s,t}(\mathbb{F}_p, \mathbb{F}_p) \Longrightarrow H\mathbb{F}_p\,{}^{MU}_{s+t} H\mathbb{F}_p.$$

Using a Koszul complex over MU_*, it is straightforward to see that

$$\mathrm{E}^2_{*,*} = \Lambda_{\mathbb{F}_p}(\tau_j : j \geqslant 0),$$

the exterior algebra over $\mathbb{F}_p$ with generators $\tau_j \in \mathrm{E}^2_{1,2j}$.

Taking $R = MU$ and $E = H\mathbb{F}_p$, we obtain a spectral sequence

$$\mathrm{E}_2^{s,t}(MU) = \mathrm{Coext}^{s,t}_{\Lambda_{\mathbb{F}_p}(\tau_j : j \geqslant 0)}(\mathbb{F}_p, \mathbb{F}_p) \Longrightarrow \pi_{s+t} \widehat{\mathrm{L}}^{MU}_{H\mathbb{F}_p} S_{MU},$$

where $I_\infty \lhd MU_*$ is generated by p together with all positive degree elements, so $MU_*/I_\infty = \mathbb{F}_p$. Also,

$$\pi_* \widehat{\mathrm{L}}^{MU}_{H\mathbb{F}_p} S_{MU} = (MU_*)\widehat{_{I_\infty}}.$$

More generally, for a finite cell MU-module M, the Adams Spectral Sequence has the form

$$\mathrm{E}_2^{s,t}(M) = \mathrm{Coext}^{s,t}_{\Lambda_{\mathbb{F}_p}(\tau_j : j \geqslant 0)}(\mathbb{F}_p, H\mathbb{F}_p{}^{MU}_* M) \Longrightarrow \pi_{s+t} \widehat{\mathrm{L}}^{MU}_{H\mathbb{F}_p} M,$$

where

$$\pi_* \widehat{\mathrm{L}}^{MU}_{H\mathbb{F}_p} M = (M_*)\widehat{_{I_\infty}}.$$

Example B: $MU \longrightarrow E(n)$**.**

By [11, 16], the Johnson-Wilson spectrum $E(n)$ at an *odd* prime p is a commutative MU-ring spectrum. According to proposition 2.10 of [16], at the prime 2 a certain modification of the usual construction also yields a commutative MU-ring spectrum which we will still denote by $E(n)$ rather than Strickland's $E(n)'$. In all cases we can form the commutative MU-ring spectrum $E(n) \underset{MU}{\wedge} E(n)$ and there is a Künneth Spectral Sequence

$$\mathrm{E}^2_{s,t} = \mathrm{Tor}^{MU_*}_{s,t}(E(n)_*, E(n)_*) \Longrightarrow E(n)^{MU}_{s+t}E(n).$$

By using a Koszul complex for $MU\langle n\rangle_*$ over MU_* and localizing at v_n, we find that

$$\mathrm{E}^2_{*,*} = \Lambda_{E(n)_*}(\tau_j : j \geqslant 1 \text{ and } j \neq p^k - 1 \text{ with } 1 \leqslant k \leqslant n),$$

where Λ denotes an exterior algebra and $\tau_j \in \mathrm{E}^2_{1,2j}$. So

$$E(n)^{MU}_* E(n) = \Lambda_{E(n)_*}(\tau_j : j \geqslant 1 \text{ and } j \neq p^k - 1 \text{ with } 1 \leqslant k \leqslant n)$$

as an $E(n)_*$-algebra.

When $R = MU$ and $E = E(n)$, we obtain a spectral sequence

$$\mathrm{E}^{s,t}_2(MU) = \mathrm{Coext}^{s,t}_{\Lambda_{E(n)_*}(\tau_j : j \geqslant n+1)}(E(n)_*, E(n)_*) \Longrightarrow \pi_{s+t}\widehat{\mathrm{L}}^{MU}_{E(n)}MU,$$

where

$$\pi_*\widehat{\mathrm{L}}^{MU}_{E(n)}MU = (MU_*)_{(p)}[v_n^{-1}]\widehat{}_{J_{n+1}}$$

and

$$J_{n+1} = (\ker\colon (MU_*)_{(p)}[v_n^{-1}] \longrightarrow E(n)_*) \triangleleft MU_*[v_n^{-1}].$$

In the E_2-term we have

$$\mathrm{E}^{s,t}_2(MU) = E(n)_*[U_j : 0 \leqslant j \neq p^k - 1 \text{ for } 0 \leqslant k \leqslant n],$$

with generator $U_j \in \mathrm{E}^{1,2j+1}_2(MU)$ corresponding to an exterior generator in $E(n)^{MU}_* E(n)$ associated with a polynomial generator of MU_* in degree $2j$ lying in $\ker MU_* \longrightarrow E(n)_*$.

More generally, for a finite cell MU-module M,

$$\mathrm{E}^{s,t}_2(M) = \mathrm{Coext}^{s,t}_{\Lambda_{E(n)_*}(\tau_j : j \geqslant n+1)}(E(n)_*, E(n)^{MU}_* M), \Longrightarrow \pi_{s+t}\widehat{\mathrm{L}}^{MU}_{E(n)}M,$$

where

$$\pi_*\widehat{\mathrm{L}}^{MU}_{E(n)}M = M\widehat{}_{J_{n+1}} = (MU_*)_{(p)}[v_n^{-1}]\widehat{}_{J_{n+1}} \underset{MU}{\otimes} M.$$

Example C: $MU \longrightarrow K(n)$**.**

We know from [11, 16] that for an odd prime p, the spectrum $K(n)$ representing the nth Morava K-theory $K(n)^*(\ )$ is a commutative MU ring spectrum. There is a Künneth Spectral Sequence

$$\mathrm{E}^2_{s,t} = \mathrm{Tor}^{MU_*}_{s,t}(K(n)_*, K(n)_*) \Longrightarrow K(n)^{MU}_{s+t}K(n),$$

and we have

$$\mathrm{E}^2_{*,*} = \Lambda_{K(n)_*}(\tau_j : 0 \leqslant j \neq p^n - 1).$$

Taking $R = MU$ and $E = K(n)$, we obtain a spectral sequence

$$\mathrm{E}^{s,t}_2(MU) = \mathrm{Coext}^{s,t}_{\Lambda_{K(n)_*}(\tau_j : 0 \leqslant j \neq n)}(K(n)_*, K(n)_*) \Longrightarrow \pi_{s+t}\widehat{\mathrm{L}}^{MU}_{K(n)}MU,$$

where

$$\pi_*\widehat{\mathrm{L}}^{MU}_{K(n)}MU = (MU_*)\widehat{_{I_{n,\infty}}}$$

with $I_{n,\infty} = \ker MU_* \longrightarrow K(n)_*$. In the E_2-term we have

$$\mathrm{E}^{s,t}_2(MU) = E(n)_*[U_j : 0 \leqslant j \neq p^n - 1],$$

with generator $U_j \in \mathrm{E}^{1,2j+1}_2(MU)$ corresponding to an exterior generator in $E(n)^{MU}_* E(n)$ associated with a polynomial generator of MU_* in degree $2k$ lying in $\ker MU_* \longrightarrow E(n)_*$ (or when $j = 0$, associated with p).

More generally, for a finite cell MU-module M,

$$\mathrm{E}^{s,t}_2(M) = \mathrm{Coext}^{s,t}_{\Lambda_{K(n)_*}(\tau_j : 0 \leqslant j \neq n)}(K(n)_*, K(n)^{MU}_* M) \Longrightarrow \pi_{s+t}\widehat{\mathrm{L}}^{MU}_{K(n)}M,$$

where

$$\pi_*\widehat{\mathrm{L}}^{MU}_{K(n)}M = (M_*)\widehat{_{I_{n,\infty}}} = (MU_*)\widehat{_{I_{n,\infty}}} \underset{MU_*}{\otimes} M_*.$$

Concluding remarks

There are several outstanding issues raised by our work.

Apart from the question of whether it is possible to weaken the assumptions from (commutative) regular quotients to a more general class, it seems reasonable to ask whether the internal I-adic tower is one of R ring spectra. Since $\mathrm{L}^R_E R = \operatorname{holim}_s R/I^s$ (at least when I_* is finitely generated), the localization theory of [11, 19] shows that this can be realized as a commutative R-algebra.

However, showing that each R/I^s is an R ring spectrum or even an R-algebra seem to involve far more intricate calculations. We expect that this will turn out to be true and even that the tower is one of R-algebras. This should involve techniques similar to those of [12, 6]. It is also worth noting that our proofs make no distinction between the cases where $I_* \triangleleft R_*$ is infinitely or finitely generated. There are a number of algebraic simplifications possible in the latter case, however we have avoided using them since the most interesting examples we know are associated with infinitely generated regular ideals in MU_*. The spectra E_n of Hopkins, Miller *et al.* have Noetherian homotopy rings and there are towers based on powers of their maximal ideals similar to those in the first author's previous work [3, 8].

We also hope that our preliminary exploration of Adams Spectral Sequences for R-modules will lead to further work on this topic, particularly in the case $R = MU$ and related examples. A more ambitious project would be to investigate the commutative S-algebra MSp from this point of view, perhaps reworking the results of Vershinin, Gorbounov and Botvinnik in the context of MSp-modules [9, 17].

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