ISSN 1472-2739 (on-line) 1472-2747 (printed)

Algebraic & Geometric Topology
Volume 2 (2002) 1–9
Published: 12 January 2002

Bihomogeneity of solenoids

Alex Clark
Robbert Fokkink

Abstract Solenoids are inverse limit spaces over regular covering maps of closed manifolds. M.C. McCord has shown that solenoids are topologically homogeneous and that they are principal bundles with a profinite structure group. We show that if a solenoid is bihomogeneous, then its structure group contains an open abelian subgroup. This leads to new examples of homogeneous continua that are not bihomogeneous.

AMS Classification 54F15; 55R10

Keywords Homogeneous continuum, covering space, profinite group, principal bundle

A topological space X is *homogeneous* if for every pair of points $x, y \in X$ there is a homeomorphism $h : X \to X$ satisfying $h(x) = y$. The space is *bihomogeneous* if for each such pair there is a homeomorphism satisfying $h(x) = y$ and $h(y) = x$. A compact and connected space is called a *continuum.* Knaster and Van Dantzig asked whether a homogeneous continuum is necessarily bihomogeneous. This was settled in the negative by Krystyna Kuperberg [5]. Subsequent counterexamples were given by Minc, Kawamura and Greg Kuperberg [8, 2, 4]. The counterexamples in [5, 4] are locally connected. Ungar [15] has studied stronger types of homogeneity conditions and showed that these conditions imply local connectivity.

A solenoid M_∞ is an inverse limit space over closed connected manifolds with bonding maps that are covering maps. We shall silently assume that the bonding maps are not $1-1$, so that M_∞ is not locally connected. McCord [7] has shown that solenoids are homogeneous provided that compositions of the bonding covering maps are regular. Minc [8] presented an example of a homogeneous but not bihomogeneous infinite-dimensional continuum similar to a solenoid, and Krystyna Kuperberg [6] observed that a similar construction could be used to construct a finite-dimensional solenoid which is homogeneous but not bihomogeneous. We shall show that M_∞ is bihomogeneous only if a certain condition related to commutativity (or lack thereof) of $\pi_1(M_i)$ is met. In case the solenoid is 2-dimensional, the condition is both necessary and sufficient.

1 Path-components of solenoids as left-cosets of the structure group

A *(strong) solenoid* M_∞ is an inverse-limit space of closed manifolds M_i with bonding maps $p_i\colon M_{i+1} \to M_i$ for $i \in \mathbb{N}$ which are covering maps, such that any composition $p_{i+k} \circ \ldots \circ p_i$ is regular. Solenoids are homogeneous spaces and they have dense path-components.

A G-bundle (E, B, p, F) is *principal* if the structure group G acts effectively on the fibers. As a consequence, the fiber F is homeomorphic to G, and G is naturally equivalent to the group of deck-transformations.

Theorem 1 (McCord, [7]) *Suppose that $M_\infty = \lim_{\leftarrow}(M_i, p_i)$ is a solenoid. Let $\pi_0\colon M_\infty \to M_0$ be the projection onto the first coordinate and let $\Gamma_0 = \pi_0^{-1}(m_0)$ be a fiber. Then $(M_\infty, M_0, \pi_0, \Gamma_0)$ is a principal-bundle.*

The projection π_0 is not to be confused with a homotopy group. Note that a solenoid $\lim_{\leftarrow}(M_i, f_i)$ is a principal bundle over any M_i and we have singled out M_0. The spaces M_i are called the *factor spaces* of the solenoid. We think of the fundamental groups $\pi_1(M_i)$ as (normal) subgroups of $\pi_1(M_0)$. The structure group Γ_0 is isomorphic to the profinite group $\lim_{\leftarrow} \pi_1(M_0)/\pi_1(M_i)$.

Choose base-points $m_i \in M_i$ such that $p_i(m_i) = m_{i-1}$, so $m_\infty = (m_i)$ is an element of M_∞. We identify the structure group Γ_0 with the fiber of m_0 and we identify m_∞ with the unit element of Γ_0. The fundamental group $\pi_1(M_0)$ acts on the base-point fiber Γ_0 by path lifting: for $g \in \Gamma_0$ and $\gamma \in \pi_1(M_0, m_0)$, define $g\circ\gamma$ as the end-point of the lifted path $\tilde{\gamma}$ starting from the initial-point g. One verifies that this right action of $\pi_1(M_0)$ commutes with left multiplication of Γ_0. More precisely, suppose that h is a deck-transformation and that $\tilde{\gamma}$ is a lifted path with initial-point g. Then $h(\tilde{\gamma})$ has initial point $h(g)$ and end-point $h(g \circ \gamma)$. Identify the structure group with the group of deck-transformations, so we get that $(hg) \circ \gamma = h(g \circ \gamma)$.

Definition 2 Suppose that $(M_\infty, M_0, \pi_0, \Gamma_0)$ is a solenoid. We shall call the $\pi_1(M_0)$-orbit of $e \in \Gamma_0$ the characteristic group and we shall denote it by γ_0. Let $K_\infty \subset \pi_1(M_0)$ be the intersection of all $\pi_1(M_i)$. Then γ_0 is isomorphic to $\pi_1(M_0)/K_\infty$ and we shall refer to K_∞ as the kernel of $\pi_1(M_0)$.

Our definition deviates from the common terminology, as in [14], where the equivalence class of γ_0 under inner automorphisms of Γ_0 is called the characteristic class. Note that γ_0 inherits a topology from Γ_0.

Lemma 3 *The path components of a solenoid are naturally equivalent to the left cosets* Γ_0/γ_0.

Proof Suppose that $x, y \in \Gamma_0$ are elements of the base-point fiber. Then $x \circ \gamma = y$ for some $\gamma \in \pi_1(M_0)$ if and only if there exists a path $\tilde{\gamma} \subset M_\infty$ that connects x to y. □

If we replace the base space M_0 by M_i for some index i, then we get a principal bundle $(M_\infty, M_i, \pi_i, \Gamma_i)$, where $\Gamma_i \subset \Gamma_0$ is the subgroup of transformations that leave M_i invariant. The topology of Γ_0 is induced by taking the Γ_i as an open neighborhood base of the identity. One verifies that the charateristic group of the bundle, denoted γ_i, is equal to $\gamma_0 \cap \Gamma_i$. Hence the γ_i are open subgroups of γ_0.

Lemma 4 *For* $j > i$ *the inclusion* $\Gamma_j \subset \Gamma_i$ *induces a natural isomorphism between* Γ_j/γ_j *and* Γ_i/γ_i.

Since path components are dense in M_∞, the characteristic subgroups γ_i are dense in Γ_i.

2 The permutation of path-components by self-homeomorphisms

A solenoid M_∞ can be represented as a subspace of $\prod M_i$, the Cartesian product of its factor spaces. We identify M_i with the subspace of $\prod M_i$ defined by:

$$M_i = \{(x_j) : x_j \in M_j,\ x_j = p^i_j(x_i) \text{ if } j \leq i,\ x_j = m_j \text{ if } j > i\}$$

where $p^i_j \colon M_i \to M_j$ is a composition of bonding maps. In this representation, the factor spaces M_i and M_∞ all have the same base-point.

A *morphism* between fiber bundles can be represented by a commutative diagram:

$$\begin{array}{ccc} E_1 & \xrightarrow{h} & E_2 \\ p_1 \downarrow & & \downarrow p_2 \\ B_1 & \xrightarrow{f} & B_2 \end{array}$$

We shall say that h is the *lifted map* and that f is the *base-map*. We say that morphisms are homotopic if their base-maps are. By the unique path-lifting property, a morphism between bundles with a totally disconnected fiber is determined by the base-map $f\colon B_1 \to B_2$ and the image under h of a single element of E_1. For pointed spaces, therefore, a bundle-morphism is determined by the base-map only. This implies that, for principal bundles with a totally disconnected fiber, bundle-morphisms commute with deck-transformations; i.e., for a lifted map h and a deck-transformation $\varphi\colon E_1 \to E_1$, we have that $h\circ\varphi = \psi \circ h$ for some deck-transformation $\psi\colon E_2 \to E_2$.

Lemma 5 *Suppose that $(E_i, B_i, p_i, \Gamma_i)$ are principal Γ_i-bundles with a totally disconnected fiber (for $i = 1, 2$). Then a base-point preserving bundle-morphism induces a homomorphism of the structure group. Furthermore, homotopic morphisms induce the same homomorphism.*

Proof First note that the lifted map h maps Γ_1 to Γ_2. Deck-transformations are (left) translations $x \to ax$ of the base-point fiber Γ_i $(i = 1, 2)$. Since a bundle-morphism commutes with deck-transformations, $h\colon \Gamma_1 \to \Gamma_2$ satisfies $h(ax) = f(a)h(x)$ for some $f\colon \Gamma_1 \to \Gamma_2$. Substitute $x = e$ to find that $h(ax) = h(a)h(x)$. Now homotopic bundle-morphisms give homotopic homomorphisms $h\colon \Gamma_1 \to \Gamma_2$. Since the groups are totally disconnected, the homomorphisms are necessarily the same. □

We shall say that a bundle morphism of a solenoid is an *automorphism* if the commutative diagram can be extended on the right-hand side

$$\begin{array}{ccccc}
M_\infty & \xrightarrow{h_1} & M_\infty & \xrightarrow{h_2} & M_\infty \\
\pi_j \downarrow & & \downarrow \pi_i & & \downarrow \pi_k \\
M_j & \xrightarrow{f_1} & M_i & \xrightarrow{f_2} & M_k
\end{array}$$

such that $f_2 \circ f_1$ is homotopic to p^j_k. We shall say that h_1 is the inverse of h_2. For instance, the covering projection $p^j_i\colon M_j \to M_i$ with lifted map id_{M_∞} yields an automorphism. We show that for every self-homeomorphism of a solenoid, there is an automorphism that acts in the same way on the space of path-components.

Theorem 6 *Suppose that h is a base-point preserving self-homeomorphism of a solenoid M_∞. Then h is homotopic to the lifted map of an automorphism of M_∞.*

Proof Since M_0 is an ANR, the composition $\pi_0 \circ h\colon M_\infty \to M_0$ extends to $H\colon U \to M_0$ for a neighborhood of $M_\infty \subset U$ in $\prod M_i$. The restriction $H\colon M_i \to M_0$ is well-defined for sufficiently large i. Note that H preserves the base-point of M_i. For sufficiently large i, the maps $H \circ \pi_i$ and $\pi_0 \circ h$ are homotopic. By the homotopy lifting property, $H \circ \pi_i$ can then be lifted to $\tilde{H}\colon M_\infty \to M_\infty$, which is homotopic to h.

Now apply the same argument to $\pi_i \circ h^{-1}$ to find a map $G\colon M_j \to M_i$ for sufficiently large j which can be lifted to $\tilde{G}\colon M_\infty \to M_\infty$. By choosing j and i sufficiently large, the composition $H \circ G\colon M_j \to M_0$ gets arbitrarily close to and hence homotopic to the covering map p_0^j. □

Theorem 6 and Lemma 5 describe how a self-homeomorphism acts on path-components of a solenoid (provided that it preserves the base-point).

Lemma 7 *Suppose that h is the lifted map of an automorphism of a solenoid M_∞. For some index i, h induces a monomorphism $\hat{h}\colon \Gamma_i \to \Gamma_0$ such that $\hat{h}^{-1}(\gamma_0) = \gamma_i$ and $\hat{h}(\Gamma_i)$ is an open subgroup of Γ_0.*

Proof By Lemma 5 we know that h induces a homomorphism $\hat{h}\colon \Gamma_i \to \Gamma_0$. Since homeomorphisms preserve path-components, Lemma 3 implies that h induces a homomorphism $\Gamma_i/\gamma_i \to \Gamma_0/\gamma_0$. Since h is the lifted map of an automorphism, it has an inverse g which induces a homomorphism $\hat{g}\colon \Gamma_j \to \Gamma_0$. The composition $\hat{g} \circ \hat{h}$, which is defined on an open subgroup, is equal to the identity. By Lemma 5, $\hat{g} \circ \hat{h}$ is equal to the homomorphism induced by p_i^j, which is the identity. □

3 An algebraic condition for bihomogeneity

Definition 8 Suppose that Γ_0 is the structure group of a solenoid with characteristic group γ_0. We define $\mathrm{Mon}(\Gamma_0, \gamma_0)$ as the set of monomorphisms $f\colon \Gamma_i \to \Gamma_0$, such that $f(\gamma_i) = \gamma_0 \cap f(\Gamma_i)$.

We say that an element of $\mathrm{Mon}(\Gamma_0, \gamma_0)$ is a *characteristic automorphism*. A self-homeomorphism H of M_∞ need not preserve the base-point. It can however be represented as a composition of a homeomorphism h that preserves the path-component of the base-point and a deck-transformation. Since h is homotopic to a base-point preserving homeomorphism, H permutes the path-components in the same way as a composition of a base-point preserving homeomorphism and a deck-transformation. In terms of Γ_0/γ_0, this is a composition of a characteristic automorphism φ and a left translation $z \to wz$ of Γ_0.

Definition 9 We say that a solenoid is *algebraically bihomogeneous* if it satisfies the following condition. For every $x, y \in \Gamma_0$ there are elements $w \in \Gamma_0$ and $\varphi \in \mathrm{Mon}(\Gamma_0, \gamma_0)$ such that $z \to w\varphi(z)$ switches the residue classes $x \bmod \gamma_0$ and $y \bmod \gamma_0$.

Obviously, bihomogeneity implies algebraic bihomogeneity. The condition of algebraic bihomogeneity may seem awkward, but fortunately there is a simpler characterization as we shall see below. We denote $x \sim y$ if x, y are in the same residue class of γ_0.

Lemma 10 *A solenoid M_∞ is algebraically bihomogeneous if and if only for every $z \in \Gamma_0$ there is a characteristic automorphism φ such that $\varphi(z) \sim z^{-1}$.*

Proof Suppose that M_∞ is algebraically bihomogeneous. For every $z \in \Gamma_0$, we can switch the cosets of z and e. More precisely, there exists a $w \in \Gamma_0$ and a $\varphi \in \mathrm{Mon}(\Gamma_0, \gamma_0)$ such that $zg = w\varphi(e)$ and $eg' = w\varphi(z)$ for $g, g' \in \gamma_0$. Since $\varphi(e) = e$, it follows that $w = zg$ and $\varphi(z) = g^{-1}z^{-1}g'$. Compose φ with the inner automorphism $x \to gxg^{-1}$ to obtain $\psi \in \mathrm{Mon}(\Gamma_0, \gamma_0)$ satisfying $\psi(z) \sim z^{-1}$.

If $\varphi(z) = z^{-1}g$ for some $g \in \gamma_0$, then compose φ with the inner automorphism $x \to gxg^{-1}$ to get $\psi \in \mathrm{Mon}(\Gamma_0, \gamma_0)$ satisfying $\psi(z) = gz^{-1}$. Then $x \to zg^{-1}\psi(x)$ switches the cosets of e and z. This implies algebraic bihomogeneity. □

Since $z \to z^{-1}$ is a homomorphism if and only if the group is abelian, we have the following corollary.

Corollary 11 *A solenoid with an abelian structure group Γ_i is algebraically bihomogeneous.*

This condition is automatically met if $\pi_1(M_i)$ is abelian.

Lemma 12 *Suppose that γ_0 is a characteristic group. Then* $\mathrm{Mon}(\Gamma_0, \gamma_0)$ *is countable.*

Proof There are only countably many subgroups γ_i and each of these is finitely generated. Hence, there are only finitely many homomorphisms $f\colon \gamma_i \to \gamma_0$. Since characteristic automorphisms are determined by their action on some γ_i, the result follows. $\square$

Theorem 13 *Let M_∞ be a bihomogeneous solenoid with structure group Γ_0. Then Γ_0 contains an open abelian subgroup.*

Proof Suppose that $\varphi\colon \Gamma_j \to \Gamma_0$ is a characteristic automorphism. For $g \in \gamma_0$ define the subset $V(\varphi, g) = \{z \in \Gamma_j \colon z\varphi(z) = g\} \subset \Gamma_0$. As φ ranges over $\mathrm{Mon}(\Gamma_0, \gamma_0)$ and g ranges over γ_0, the countable family of all $V(\varphi, g)$ covers Γ_0 by Lemma 10. Hence one of these sets, say $V(\varphi_0, g_0)$, is of second category in Γ_0. It follows that $K = \{z \in \Gamma_0 \colon z\varphi_0(z) = g_0\}$ is closed with non-empty interior in Γ_0. Since K has non-empty interior, there exist a $z_0 \in K$ and a neighborhood V of e such that $\varphi(z_0\delta) = \delta^{-1}z_0^{-1}g_0$ for all $\delta \in V$. It follows that $\varphi(\delta) = g_0^{-1}z_0\delta^{-1}z_0^{-1}g_0$. By composition with the inner automorphism $x \to z_0^{-1}g_0xg_0^{-1}z_0$, we get a homomorphism ψ such that $\psi(\delta) = \delta^{-1}$ for $\delta \in V$. The group generated by V is an open abelian subgroup of Γ_0. $\square$

For any neighborhood V of the identity, $\gamma_j \subset V$ for large enough j. Hence there exists an open abelian subgroup of Γ_0 if and only if γ_j is abelian for some j.

Corollary 14 *Let M_∞ be a solenoid and let K_∞ be the kernel of $\pi_1(M_0)$. Then M_∞ is algebraically bihomogeneous if and only if $\pi_1(M_j)/K_\infty$ is abelian for sufficiently large index j, or, equivalently, Γ_j is abelian for sufficiently large index j.*

4 An application

Our algebraic condition for (topological) bihomogeneity in Corollary 14 is necessary but not sufficient. For this, there should exist a homeomorphism $h\colon M_i \to M_i$ which induces an isomorphism $h_*\colon \pi_1(M_i) \to \pi_1(M_i)$ such that $h_*(x) = x^{-1}$ (modulo K_∞). The problem whether homomorphisms between fundamental groups are realized by continuous maps is known as the geometric realization problem. It is a classical result of Nielsen [9] that closed surfaces admit geometric realizations. This can be extended to certain three-dimensional manifolds [16]. The following result now follows from Nielsen's theorem.

Theorem 15 *A two-dimensional solenoid S_∞ with kernel $K_\infty \subset \pi_1(S_0)$ is bihomogeneous if and only if $\pi_1(S_i)/K_\infty$ is abelian for sufficiently large index i.*

One easily constructs two-dimensional solenoids that are not bihomogeneous, using results from geometric group theory. The fundamental group $\pi_1(S)$ of a closed surface is subgroup separable, see [13]; i.e., for every subgroup $H \subset \pi_1(S)$ there is a descending chain of subgroups of finite index with kernel H. Hence, there exists a solenoid with base-space S and kernel H. For a closed surface S of genus greater than 1, the fundamental group contains no abelian subgroup of finite index. Therefore, a solenoid with base-space S and kernel $\{e\}$ is a (simply-connected) continuum which is not bihomogeneous.

5 Final remarks

One-dimensional solenoids are indecomposable continua. It is not difficult to show that higher-dimensional solenoids are not. Rogers [10] has shown that a homogeneous, hereditarily indecomposable continuum is at most one-dimensional. His question whether there exists a homogeneous, indecomposable continuum of dimension greater than one remains open.

Our example of a non-bihomogeneous space is based on obstructions of the fundamental group, which seems to be characteristic for all examples so far. So it is natural to ask whether there exists a simply-connected Peano continuum that is homogeneous but not bihomogeneous. More generally, it is natural to ask whether there exists a continuum with trivial first Čech cohomology that is homogeneous but not bihomogeneous.

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University of North Texas, Department of Mathematics
Denton TX 76203-1430, U.S.A.
and
Technische Universiteit Delft, Faculty of Information Technology and Systems
Division Mediamatica, P.O. Box 5031, 2600 GA Delft, Netherlands

Email: `alexc@unt.edu, r.j.fokkink@its.tudelft.nl`

Received: 22 August 2001 Revised: 8 January 2002