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Every orientable 3–manifold is a BΓ

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Abstract We show that every orientable 3–manifold is a classifying space BΓ where Γ is a groupoid of germs of homeomorphisms of $\mathbb{R}$. This follows by showing that every orientable 3–manifold M admits a codimension one foliation $\mathcal{F}$ such that the holonomy cover of every leaf is contractible. The $\mathcal{F}$ we construct can be taken to be C^1 but not C^2. The existence of such an $\mathcal{F}$ answers positively a question posed by Tsuboi (see [21]), but leaves open the question of whether $M = \mathrm{B}\Gamma$ for some C^∞ groupoid Γ.

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1 Acknowledgements

In August 2001, the P.U.C. in Rio de Janeiro held a conference on Foliations and Dynamics aimed at bringing together traditional foliators and 3–manifold topologists. At this conference, Takashi Tsuboi posed the question of the existence of typical foliations on 3–manifolds. I would very much like to thank Takashi for posing this question, for reading an early draft of this paper and catching numerous errors, and for introducting me to the beautiful subject of classical foliation theory. I'd also like to thank Curt McMullen for a useful conversation which helped clarify some analytic questions for me.

2 Typical foliations

Let M be a smooth n–manifold, and $\mathcal{F}$ a smooth codimension p foliation of M. The tangent bundle $T\mathcal{F}$ is a subbundle of TM, with orthogonal complement the normal bundle $\nu\mathcal{F}$. Choose some small $\epsilon > 0$, and let $E_\epsilon \subset \nu\mathcal{F}$ be the B^p bundle over M consisting of vectors of $\nu\mathcal{F}$ of length $\leq \epsilon$. For sufficiently small ϵ, for each leaf λ of $\mathcal{F}$, the restriction of E_ϵ to λ maps via the exponential map to a (typically immersed) tubular neighborhood of λ. The distribution $T\mathcal{F}$ can be pulled back via this exponential map to a (generally incomplete) connection on E_ϵ. For each point $p \in \lambda$, the ball $E_\epsilon(p)$ parameterizes the leaves of $\mathcal{F}$ near p. If $\gamma : I \to \lambda$ is a path, the holonomy of the connection on E_ϵ defines a diffeomorphism from a neighborhood of 0 in $E_\epsilon(\gamma(0))$ to a neighborhood of 0 in $E_\epsilon(\gamma(1))$. The germ of this map is called the *holonomy of $\mathcal{F}$ along γ*, and depends only on the homotopy class of γ rel. endpoints in λ. In particular, for any $p \in \lambda$, there is a homomorphism from $\pi_1(\lambda, p)$ to the group of germs at 0 of diffeomorphisms of $E_\epsilon(p)$ to itself which fix 0. The *holonomy cover* of λ is the covering space $\widetilde{\lambda}$ of λ corresponding to the kernel of this homomorphism.

If M and $\mathcal{F}$ are not necessarily smooth, there is still a well–defined holonomy homomorphism from $\pi_1(\lambda, p)$ to the group of germs at p of homeomorphisms of a transversal τ to $\mathcal{F}$ through p to itself fixing p. Therefore we can define the holonomy cover of a leaf in this case too.

Definition 2.1 A foliation $\mathcal{F}$ is *typical* if for each leaf λ, the holonomy cover $\widetilde{\lambda}$ of λ is contractible.

If we fix a finite collection $\{\tau_i\}$ of transversals to $\mathcal{F}$, let $\pi_1(\mathcal{F}, \{\tau_i\})$ denote the groupoid of homotopy classes of paths rel. endpoints tangent to $\mathcal{F}$, which start and end on some pair of transversals $\tau_i, \tau_j \in \{\tau_i\}$. This groupoid maps by the holonomy homomorphism to the groupoid of germs of homeomorphisms between subsets of the τ_i. If the union of the τ_i are a *total transversal* — i.e. if they intersect every leaf of $\mathcal{F}$ — we call the image of $\pi_1(\mathcal{F}, \{\tau_i\})$ the *holonomy groupoid of the foliation on* $\{\tau_i\}$. By embedding the τ_i disjointly in $\mathbb{R}^p$, we can think of the holonomy groupoid as a groupoid of germs of homeomorphisms of $\mathbb{R}^p$.

We are especially interested, throughout most of this article, with surface foliations of 3–manifolds. A surface foliation without sphere or projective plane leaves is typical iff the fundamental group $\pi_1(\lambda)$ of each leaf injects into the holonomy groupoid of $\mathcal{F}$. For a surface foliation of a 3–manifold, the holonomy groupoid Γ is a groupoid of germs of homeomorphisms of $\mathbb{R}$. If $\mathcal{F}$ is C^∞, the holonomy groupoid Γ is a groupoid of germs of diffeomorphisms of $\mathbb{R}$. For a more leisurely discussion of holonomy groupoids, see [9].

The significance of typical foliations is explained by the following theorem of Haefliger:

Theorem 2.2 (Haefliger) *Let $\mathcal{F}$ be a typical foliation of M. Then M is homotopic to* $\mathrm{B}\Gamma$ *where Γ is the holonomy groupoid of $\mathcal{F}$ (on any total transversal).*

Here BΓ is the *classifying space* of the groupoid Γ. For a construction of the classifying space of a groupoid, see [15]. Note that for Γ a groupoid of germs of homeomorphisms of $\mathbb{R}^p$, Γ is a *topological groupoid*, although the topology is not necessarily what one might expect. For $\phi \in \mathrm{Homeo}(\mathbb{R}^p)$ with $\phi(x) = y$, we denote the germ of ϕ at x by ϕ_x. We give $\mathrm{Homeo}(\mathbb{R}^p)$ the *discrete* topology and $\mathbb{R}^p$ the usual topology, and think of Γ as a quotient of the subspace

$$\{(\phi, x, y) \in \mathrm{Homeo}(\mathbb{R}^p) \times \mathbb{R}^p \times \mathbb{R}^p : \phi_x \in \Gamma, \phi(x) = y\}$$

by the equivalence relation:

$$(\phi, x, y) \sim (\psi, x, y) \text{ if } \phi_x = \psi_x$$

For X a topological space, the set of homotopy classes of Haefliger structures on X with coefficients in a groupoid Γ are classified by homotopy classes of maps $[X, \mathrm{B}\Gamma]$. If Γ is the groupoid of germs of C^r diffeomorphisms of $\mathbb{R}^p$ for some $r \in \{1, \ldots, \infty\}$, there is a natural "forgetful" map

$$\pi : \mathrm{B}\Gamma \to \mathrm{BO}(p, \mathbb{R})$$

which remembers only the linear part of each $\gamma \in \Gamma$.

Let ξ be the tautological bundle over $\mathrm{BO}(p, \mathbb{R})$. If M is a smooth manifold, Thurston showed ([18],[19]) that for every $\alpha : M \to \mathrm{B}\Gamma$ for which $(\pi \circ \alpha)^*\xi$ is isomorphic to a sub-bundle $\zeta \subset TM$, there is a C^r foliation $\mathcal{F}$ of M with $\nu\mathcal{F}$ homotopic to ζ, and with Haefliger structure homotopic to $\alpha^*\mathcal{H}$, for $\mathcal{H}$ the tautological Haefliger structure on BΓ. See [21] and [20] for the definition of a Haefliger structure and for more details.

Example If $\mathcal{F}$ is a p–dimensional foliation with every leaf homeomorphic to $\mathbb{R}^p$, then $\mathcal{F}$ is typical. Some familiar examples include:

(1) Kronecker foliations. Let π be a p–dimensional subspace of $\mathbb{R}^n$ which intersects the integer lattice $\mathbb{Z}^n$ only at the origin. Let $\widetilde{\mathcal{F}}$ be the foliation of $\mathbb{R}^n$ by planes parallel to π. Then $\widetilde{\mathcal{F}}$ descends to a foliation $\mathcal{F}$ of $T^n = \mathbb{R}^n/\mathbb{Z}^n$, all of whose leaves are p–planes.

(2) Hilbert modular surfaces. Let $\mathcal{O}$ denote the ring of integers in a real quadratic extension K of $\mathbb{Q}$, and let $e_1, e_2 : K \to \mathbb{R}$ denote the two real embeddings. Let Γ denote the group $PSL(2, \mathcal{O})$, and let $\widehat{\Gamma}$ denote a finite index torsion–free subgroup. Then e_i induces an embedding

$$\widehat{e_i} : \widehat{\Gamma} \to PSL(2, \mathbb{R})$$

for $i = 1, 2$, and we can let M be the quotient of $\mathbb{H}^2 \times \mathbb{H}^2$ by $\widehat{\Gamma}$, where

$$\gamma(x, y) = (\widehat{e_1}(\gamma)(x), \widehat{e_2}(\gamma)(y))$$

The foliation of $\mathbb{H}^2 \times \mathbb{H}^2$ by hyperbolic planes $\mathbb{H}^2 \times$ point descends to a foliation of M by hyperbolic planes.

(3) The Seifert conjecture. Let M be a 3–manifold, and X a nowhere zero vector field on M. Then Schweitzer ([14]) showed how to modify X to X' so that it possesses no closed integral curves; i.e. the integral foliation of X' has all leaves homeomorphic to $\mathbb{R}$.

Many 3–manifolds are already $B\Gamma$ for Γ a *group* of germs of homeomorphisms of $\mathbb{R}$ at some global fixed point.

Example The group $\text{Homeo}^+(\mathbb{R})$ of orientation preserving homeomorphisms of $\mathbb{R}$ is isomorphic by a suspension trick to a subgroup of the group of germs at 0 of elements of $\text{Homeo}^+(\mathbb{R})$ which fix 0. A finitely generated group Γ is isomorphic to a subgroup of $\text{Homeo}^+(\mathbb{R})$ iff Γ is *left orderable.* A closed 3–manifold M is a $K(\pi_1(M), 1)$ iff $\pi_1(M)$ is infinite and $\pi_2(M)$ is trivial; if $\pi_1(M)$ is also left orderable, then $\pi_1(M)$ is isomorphic to a group of germs of homeomorphisms of $\mathbb{R}$, and M is a classifying space for this group.

(1) If Γ is a finitely generated group which is *locally indicable* — i.e. every nontrivial finitely generated subgroup admits a surjective homomorphism to $\mathbb{Z}$ — then Γ is isomorphic to a subgroup of $\text{Homeo}^+(\mathbb{R})$. For Γ the fundamental group of an irreducible 3–manifold M, this happens when $H^1(M)$ is nontrivial. This observation is due to Boyer, Rolfsen and Wiest [1].

(2) If M is atoroidal and admits an orientable and co–orientable taut foliation $\mathcal{F}$, $\pi_1(M)$ admits a faithful representation in $\text{Homeo}^+(S^1)$. If the Euler class of $T\mathcal{F}$ is trivial, this representation lifts to a subgroup of $\text{Homeo}^+(\mathbb{R})$. If $H^1(M)$ is infinite, $\pi_1(M)$ is locally indicable as above. Otherwise, the Euler class of $T\mathcal{F}$ is torsion, and a finite index subgroup of $\pi_1(M)$ admits a faithful representation in $\text{Homeo}^+(\mathbb{R})$. See [2] for details.

Roberts, Shareshian and Stein showed ([12]) that there are examples of hyperbolic 3–manifold groups which are not left–orderable. More examples and a detailed analysis are given in [2].

On the other hand, these examples cannot in general be improved to C^1. If Γ is a finitely generated group of germs at 0 of C^1 homeomorphisms of $\mathbb{R}$

fixing 0, then $H^1(\Gamma)$ is nontrivial, by Thurston's stability theorem [17]. As the smoothness increases, our knowledge shrinks very rapidly. In fact, it is not even known if $\pi_1(\Sigma_g)$ admits a faithful representation in the group of germs at 0 of C^2 homeomorphisms of $\mathbb{R}$ fixing 0, for Σ_g the closed orientable surface of genus $g \geq 2$.

3 Open book decompositions

To build our foliations we will appeal to the following simple structure theorem.

Theorem 3.1 (Myers) *Let M be a closed orientable 3–manifold. Then there is a decomposition of M as $K \coprod N$ where K is a tubular neighborhood of a knot, and N is a surface bundle over S^1 with fiber the punctured surface Σ, such that the boundary curve $\partial\Sigma$ is not a meridian for K.*

Such a structure is called a *open book decomposition with connected binding.* Such a structure exists for any 3–manifold by Myers [10]. Note that Σ may be chosen to be a Seifert surface for K so that the boundary curve of $\partial\Sigma$ is not a meridian.

The existence of an open book decomposition for an orientable 3–manifold follows from structure theorems of Heegaard and Lickorish ([7]). First, M has a presentation as a union of two handlebodies — that is, it admits a Heegaard decomposition. Any two Heegaard decompositions of the same genus differ by cutting along the splitting surface and regluing by an automorphism of the surface. Lickorish showed that the automorphism group of the surface (up to isotopy) is generated by Dehn twists in a canonical set of curves; translating this into the 3–dimensional context, and starting from the standard genus g splitting of S^3, one sees that any 3–manifold M of Heegaard genus g is obtained by integer surgery on a link in S^3 of the form illustrated in figure 1, copied from [13].

M has the structure of an open book decomposition with one component for the binding for each component of the surgery presentation, and one component coming from the knot labelled J in figure 1. The fiber is a many–punctured disk. The number of components of the binding can be reduced at the cost of raising the genus of the fiber, until there is a single binding component.

In what follows, the fact that our open book decompositions have connected bindings is not an essential fact, but it simplifies the exposition somewhat.

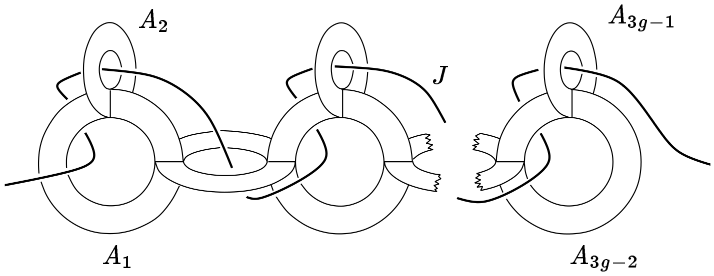

Figure 1: Any orientable 3–manifold of Heegaard genus g is obtained by integral surgery along a collection of components parallel to the cores of the annuli A_i in the figure. These links form a braid wrapping once around the curve J.

Since N is a surface bundle over a circle, it is determined by the class of the monodromy $\phi : \Sigma \to \Sigma$ up to homotopy. Such a class contains many smooth representatives.

4 Foliations of surface bundles

We inherit the notation M, N, K, Σ from the last section. Let I denote the closed unit interval.

Definition 4.1 A representation $\rho : \pi_1(\Sigma) \to \mathrm{Homeo}(I)$ is *generic* if for each nontrivial $\alpha \in \pi_1(\Sigma)$ the fixed points of $\rho(\alpha)$ in the interior of I are *isolated.* A generic representation whose image consists of C^∞ diffeomorphisms which are infinitely tangent to the identity at the two endpoints is called *well–tapered.*

The interior of I is homeomorphic to $\mathbb{R}$, and every self–homeomorphism of $\mathrm{int}(I)$ extends uniquely to a self–homeomorphism of I. Conversely, any self–homeomorphism of I is determined by its restriction to $\mathrm{int}(I)$. It follows that a choice of homeomorphism $a : \mathbb{R} \to \mathrm{int}(I)$ determines an *isomorphism*:

$$a_* : \mathrm{Homeo}(\mathbb{R}) \to \mathrm{Homeo}(I)$$

The key observation is that for $\rho : \pi_1(\Sigma) \to \mathrm{Homeo}(I)$ a generic representation, for each $\gamma \in \pi_1(\Sigma)$ and each $p \in \mathrm{fix}(\rho(\gamma))$ the germ of $\rho(\gamma)$ at p is nontrivial.

Lemma 4.2 *There exist generic representations.*

Proof Choose a faithful representation:

$$\sigma : \pi_1(\Sigma) \to PSL(2, \mathbb{R})$$

For example, σ could be a Fuchsian representation arising from a hyperbolic structure on Σ.

$PSL(2,\mathbb{R})$ can be naturally thought of as a subgroup of $\mathrm{Homeo}^+(S^1)$, the group of orientation–preserving homeomorphisms of the circle, by considering its action on the circle at infinity of the hyperbolic plane. For $\gamma \in PSL(2,\mathbb{R})$ nontrivial, γ has ≤ 2 fixed points in S^1. If we choose our representation σ corresponding to a hyperbolic structure on Σ where the boundary curves are homotopic to geodesic loops, then every nontrivial γ in the image of σ will be *hyperbolic*, and have exactly two fixed points in S^1. Moreover, $(\gamma - \mathrm{id})' \neq 0$ at either of these two fixed points.

Given a representation $\sigma : \pi_1(\Sigma) \to \mathrm{Homeo}^+(S^1)$, we can construct an oriented *foliated* circle bundle E over Σ by forming the quotient of

$$\widetilde{\Sigma} \times S^1$$

foliated by planes $\{\widetilde{\Sigma} \times \text{point}\}$ by the action of $\pi_1(\Sigma)$, where:

$$\gamma(s,\theta) = (\gamma(s), \sigma(\gamma)(\theta))$$

The Euler class of σ is then defined to be the Euler class of this topological circle bundle, i.e. the obstruction to trivializing this bundle over the 2-skeleton of Σ. Since Σ is a punctured surface, it is homotopic to a graph, and the Euler class of σ is 0. (See [8] for a discussion of Euler classes of representations and circle bundles.)

Thus, E is a *trivial* circle bundle $\Sigma \times S^1$, covered by $\widetilde{E} = \Sigma \times \mathbb{R}$. The foliation of E lifts to a foliation of $\widetilde{E}$ transverse to the $\mathbb{R}$ fibers, and determines a lift of σ to a representation:

$$\widetilde{\sigma} : \pi_1(\Sigma) \to \widetilde{PSL(2,\mathbb{R})}$$

Here $\widetilde{PSL(2,\mathbb{R})}$ is the universal cover of the group $PSL(2,\mathbb{R})$. It can be thought of as a group of homeomorphisms of $\mathbb{R}$ which are periodic with period 2π and which cover homeomorphisms of S^1 in $PSL(2,\mathbb{R})$. There is a short exact sequence:

$$0 \to \mathbb{Z} \to \widetilde{PSL(2,\mathbb{R})} \to PSL(2,\mathbb{R}) \to 0$$

Choose a homeomorphism $a : \mathbb{R} \to I$, and let $a_*\widetilde{\sigma}$ be the induced representation of $\pi_1(\Sigma)$ in $\mathrm{Homeo}(I)$. Then this representation is clearly generic. $\square$

Lemma 4.3 *Suppose $\rho : \pi_1(\Sigma) \to \mathrm{Homeo}(I)$ is a generic representation. Then there is a typical foliation $\mathcal{F}_\rho$ of N. If ρ is well–tapered, $\mathcal{F}_\rho$ can be chosen to be C^∞.*

Proof The representation ρ gives rise to a foliation of $\Sigma \times I$ by quotienting out the product foliation of $\widetilde{\Sigma} \times I$ by planes $\{\widetilde{\Sigma} \times \text{point}\}$ by the action of $\pi_1(\Sigma)$ in the usual way:

$$\gamma : (s, \theta) \to (\gamma(s), \rho(\gamma)(\theta))$$

This is basically the same construction that we used in lemma 4.2.

Then the top and bottom of $\Sigma \times I$ can be glued up by the monodromy ϕ of Σ to give a foliation $\mathcal{F}_\rho$ of N. By construction, π_1 of the closed leaf $\Sigma \times 0$ injects into the holonomy groupoid. Moreover, since for any nontrivial α the element $\rho(\alpha)$ has isolated fixed points in the interior of I, and since every essential closed loop α' in a leaf μ of $\mathcal{F}_\rho$ projects to some α by the projection $\Sigma \times I \to \Sigma$, it follows that the holonomy of α' is nontrivial on either side along μ. Thus $\mathcal{F}_\rho$ is typical.

Clearly, $\mathcal{F}_\rho$ is C^∞ in the interior of $\Sigma \times I$ if the representation is smooth. Moreover, if ρ is well–tapered, the holonomy on either side of a loop in the closed surface $\Sigma \times 0$ is infinitely tangent to the identity, and therefore $\mathcal{F}_\rho$ is smooth there too. □

Theorem 4.4 *Let M be a closed orientable 3–manifold. Then M admits a typical surface foliation $\mathcal{F}$.*

Proof We find an open–book decomposition of M into $K \coprod N$, by theorem 3.1. We foliate N with a typical foliation $\mathcal{F}_\rho$ by lemma 4.2 and lemma 4.3. We twist this foliation around ∂K and fill in K with a Reeb component to get a foliation $\mathcal{F}$. Since the meridian of ∂K is transverse to the circles $\partial\Sigma$, it can be chosen transverse to $\mathcal{F}_\rho|_{\partial N}$, and therefore the fundamental group of the torus ∂K injects into the holonomy groupoid of $\mathcal{F}$. All the leaves in the interior of K are planes, so their fundamental groups vacuously inject into the holonomy groupoid of $\mathcal{F}$. It follows that $\mathcal{F}$ is typical. □

Corollary 4.5 *Any closed orientable 3–manifold M is a classifying space $\mathrm{B}\Gamma$ for some groupoid Γ of germs of homeomorphisms of $\mathbb{R}$.*

Proof This follows from Haefliger's theorem 2.2. □

5 Finding well–tapered representations

In this section we will construct well–tapered representations of $\pi_1(\Sigma)$ for Σ a punctured surface of finite type. In fact, we will show that well–tapered representations are dense in the space of representations in C^∞ diffeomorphisms of I infinitely tangent to the identity at the boundary.

Definition 5.1 Let $C^\infty(I)$ denote the space of C^∞ real–valued functions on I. The C^r *topology* on $C^\infty(I)$ is the topology of uniform convergence of the first r derivatives. The C^∞ topology is the topology generated by the C^r topology for all r. That is, $f_i \to f$ in the C^∞ topology if the rth derivatives $f_i^{(r)} \to f^{(r)}$ uniformly, for each r.

Note that $C^\infty(I)$ with the C^∞ topology is *not* a Banach space. However, it is a Fréchet space — i.e. a metrizable and complete locally convex topological vector space. (see [3]).

Definition 5.2 Let $\mathcal{C}$ denote the group of C^∞ diffeomorphisms of I infinitely tangent to the identity on ∂I.

The subspace of $C^\infty(I)$ consisting of functions on I infinitely tangent to the identity on ∂I is an *affine* Fréchet space. The group $\mathcal{C}$ can be identified with an open subset of this space of functions, i.e. it is a Fréchet manifold. In particular it satisfies the *Baire property* — any countable intersection of open dense subsets of $\mathcal{C}$ is dense.

Remark Sergeraert showed in [16] that $\mathcal{C}$ is a perfect group.

Let Σ be a punctured surface of finite type. Fix a generating set $\alpha_1, \ldots, \alpha_n$ for $\pi_1(\Sigma)$. By hypothesis, the group $\pi_1(\Sigma)$ is free on these generators. The space of representations $\mathrm{Hom}(\pi_1(\Sigma), \mathcal{C})$ can therefore be identitifed with $\mathcal{C}^n$. This space also satisfies the Baire property.

Theorem 5.3 *Let Σ be a punctured surface of finite type. The set of well–tapered representations $\pi_1(\Sigma) \to \mathcal{C}$ is dense in the space Hom$(\pi_1(\Sigma), \mathcal{C}) = \mathcal{C}^n$ of all representations.*

Proof The strategy of the proof is as follows. Say an element $\gamma \in \mathcal{C}$ is *robustly generic* if the set of fixed points of γ in $\mathrm{int}(I)$ has no accumulation points, and if for each fixed point $p \in \mathrm{int}(I)$, γ is *not* infinitely tangent to the identity at p. For $J \subset \mathrm{int}(I)$ a *closed* subinterval, we say $\gamma \in \mathcal{C}$ is *robustly generic on* J if $\mathrm{fix}(\gamma) \cap J$ is finite, and for each $p \in \mathrm{fix}(\gamma) \cap J$, γ is not infinitely tangent to the identity at p. If

$$\cdots \subset J_i \subset J_{i+1} \subset \cdots$$

is an increasing sequence of closed subintervals of $\text{int}(I)$ whose union is $\text{int}(I)$, then an element $\gamma \in \mathcal{C}$ is robustly generic iff it is robustly generic on each J_i.

We will show that for each nontrivial $\alpha \in \pi_1(\Sigma)$ and each closed $J \subset \text{int}(I)$ the subset of $\rho \in \mathcal{C}^n$ for which $\rho(\alpha)$ is robustly generic on J is open and dense. If $\rho(\alpha)$ is robustly generic on J for every nontrivial α and every $J \subset \text{int}(I)$, then ρ is a well–tapered representation. It would follow from the Baire property for $\mathcal{C}^n$ that the set of well–tapered representations is dense — i.e. this would be sufficient to establish the theorem.

Firstly, we establish openness of the condition. Let $J \subset \text{int}(I)$ be closed, and suppose $\gamma \in \mathcal{C}$ is robustly generic on J. Then $\text{fix}(\gamma) \cap J$ is a finite set. For any $\epsilon > 0$, if ν is sufficiently close to γ, we can estimate that $\text{fix}(\nu) \cap J \subset N_\epsilon(\text{fix}(\gamma)) \cap J$, where $N_\epsilon(\cdot)$ denotes the ϵ–neighborhood of a set. Moreover, we can require that $\nu^{(n)}$ is arbitrarily close to $\gamma^{(n)}$ on this neighborhood, for any fixed n. Notice that the fixed points of ν are exactly the zeroes of $\nu - \text{id}$. By the robustness hypothesis, for each $p \in \text{fix}(\nu) \cap J$, and for some $\delta > 0$, there is a $\kappa > 0$ such that for some smallest n,

$$|(\nu - \text{id})^{(n)}(q)| > \delta$$

for all $q \in N_\kappa(p)$, and

$$(\nu - \text{id})^{(m)}(p) = 0$$

for all $m < n$. We change co–ordinates near p by setting $y = (x-p)^n$, and see that with respect to y:

$$\left| \frac{d}{dy}(\nu - \text{id})(0) \right| > 0$$

In particular, p is an *isolated* point of $\text{fix}(\nu)$, and therefore ν is robustly generic.

It remains to establish density. Pick a closed interval $J \subset \text{int}(I)$ and some $\alpha \in \pi_1(\Sigma)$, and suppose we have chosen some $\rho \in \mathcal{C}^n$ for which $\rho(\alpha)$ is not robustly generic on J. Fix a degree of smoothness m. For each $\epsilon > 0$, we can find a collection of closed intervals $J_i \subset \text{int}(I)$ for $1 \le i \le n$ such that for $\sigma \in \mathcal{C}^n$ chosen ϵ–close to ρ in the C^m norm, the restriction of $\sigma(\alpha)$ to J is determined by the restrictions of $\sigma(\alpha_i)$ to J_i. Moreover, for each δ, we can find an ϵ such that if σ is ϵ–close to ρ on the generating set $\{\alpha_i\}$, it is δ close to ρ on α, again in the C^m norm.

We will show how to modify $\rho(\alpha_i)$ on the subintervals J_i so that the modified $\rho(\alpha)$ is robustly generic on J. Think of the image of each generator α_i under ρ as a function from I to I:

$$\rho(\alpha_i)(x) = f_i(x)$$

Fix a positive integer $N \gg m$. Let $X_N(i)$ denote the space of invertible C^m functions $g_i : I \to I$ with the following properties:

(1) The function g_i agrees with f_i outside J_i.

(2) g_i is tangent to order m to f_i at ∂J_i.

(3) The restriction $g_i|_{J_i}$ is a real polynomial of degree $\leq N$.

The $X_N(i)$ determine a subspace of $\mathcal{C}^n$, which we denote by X_N. Clearly, X_N is a finite dimensional real analytic variety.

For any $\sigma \in X_N$, the restriction $\sigma(\alpha)|_J$ is a real analytic function, which extends to a holomorphic (complex–valued) function in a neighborhood of $J \subset \mathbb{C}$. In particular, if $\sigma(\alpha)$ is nontrivial on J, it is robustly generic there. Let $Y_N \subset X_N$ be the analytic subvariety defined by the condition that $\sigma(\alpha)|_J$ is trivial iff $\sigma \in Y_N$. For sufficiently large N, we claim $X_N \backslash Y_N$ is nonempty, and therefore is open and dense in X_N. Let $\tau : \pi_1(\Sigma) \to \text{Homeo}(I)$ denote a generic representation constructed in lemma 4.2. τ can be chosen to be real analytic on $\text{int}(I)$, by choosing some real analytic homeomorphism $a : \mathbb{R} \to \text{int}(I)$. We choose closed subintervals $K_i \subset \text{int}(J_i)$ and let $\sigma(\alpha_i)$ be C^m close to $\tau(\alpha_i)$ on K_i. If the K_i are large enough, they determine the value of $\sigma(\alpha)$ on some nonempty $K \subset J$. Since $\tau(\alpha)|_K$ is nontrivial (i.e. not equal to the identity there), it follows that $\sigma(\alpha)|_K$ is nontrivial. So $X_N \backslash Y_N$ is nonempty for large N and a suitable choice of J_i, and therefore it is open and dense in X_N. It follows that we can find $\sigma \in X_N$ which is C^m close to ρ for which $\sigma(\alpha)|_J$ is robustly generic.

If we C^N approximate σ by $\sigma' \in \mathcal{C}^n$ to make it C^∞, $\sigma'(\alpha)|_J$ is still robustly generic, by openness. It follows that we have established that the set of $\sigma \in \mathcal{C}^n$ for which $\sigma(\alpha)$ is robustly generic on J is dense.

Let J_i be an increasing sequence of closed subsets of $\text{int}(I)$ whose union is $\text{int}(I)$. Enumerate the nontrivial $\alpha \in \pi_1(\Sigma)$ in some order α_i, where we can assume if we like that this ordering agrees with the labels on the generating set. For each i, the set of $\sigma \in \mathcal{C}^n$ for which α_j is robustly generic on J_i for all $j \leq i$ is open and dense. The intersection of this countable family of open, dense sets is dense, and therefore there are a dense set of σ for which $\sigma(\alpha)$ is robustly generic for *every* nontrivial α. Such a σ is well–tapered. □

Notice that the proof actually tells us slightly more than the conclusion of the theorem. Given $\sigma \in \mathcal{C}^n$, we can find $\sigma' \in \mathcal{C}^n$ robustly generic which is obtained by successively perturbing $\sigma|_{J_i}$ on intervals $J_i \to \text{int}(I)$ by smaller and smaller amounts. So for any $\alpha \in \pi_1(\Sigma)$ we can choose these perturbations so that $\sigma'(\alpha)-\sigma(\alpha)$ tapers off as fast as desired as it approaches ∂I. Here our notation suggests that we are thinking of $\sigma(\alpha)$ and $\sigma'(\alpha)$ as functions from I to I. In particular, if $\sigma(\alpha)$ satisfies certain desirable properties — for instance, if $\text{int}(I)\cap\text{fix}(\sigma(\alpha))$ is empty — then this property may be inherited by σ' if we desire.

Definition 5.4 A function $f: I\to\mathbb{R}$ is *stretchable* if it satisfies the following conditions:

(1) f is C^∞ and f is positive on $\text{int}(I)$.

(2) f is infinitely tangent to 0 at the ends of I.
Furthermore, let X denote the vector field $f(t)\frac{d}{dt}$ on I, and $\phi_t: I\to I$ denote the diffeomorphism generated by time t flow along X. Define $\alpha: I\to I$ by the properties that $\alpha(\frac{1}{2})=\frac{1}{2}$ and $\alpha\phi_{2t}\alpha^{-1}=\phi_t$. Then

(3) $\alpha\in\mathcal{C}$.

We say in this case that ϕ_t is stretchable.

Stretchable functions certainly exist. By basic existence and uniqueness results for differential equations, α exists and is C^∞ on the interior of I; in order to ensure the correct boundary behavior of α, we just need f to taper off sufficiently fast as it approaches ∂I.

Corollary 5.5 *Let Σ be a hyperbolic punctured surface of finite type. There is a well–tapered representation $\rho:\pi_1(\Sigma)\to\mathcal{C}$ such that if γ denotes the image in $\pi_1(\Sigma)$ of a small loop around a puncture, the element $\rho(\gamma)$ fixes no points in the interior of I, and $\rho(\gamma)-id$ tapers off as fast as desired as it approaches ∂I.*

Proof Sergeraert's theorem implies that any element of $\mathcal{C}$ can be written as a product of commutators; however, this observation does not help us, since we have no bound on the number of commutators needed to express a given element of $\mathcal{C}$.

On the other hand, for f, X, ϕ_{2t}, α as above with ϕ_t stretchable, the commutator

$$[\phi_{2t},\alpha]=\phi_t$$

fixes no points in the interior of I. Since γ is a product of commutators, we can choose a representation σ taking the first commutator to ϕ_t and all the others to id. Then any ρ sufficiently close to σ will satisfy the conclustion of the corollary; such well–tapered ρ exist by theorem 5.3. □

6 Pixton actions

Theorem 6.1 *Let M be an orientable 3–manifold. Then M admits a C^1 typical surface foliation $\mathcal{F}$.*

Proof As before, we decompose $M = K \coprod N$. We foliate N by using a well–tapered representation of the fundamental group of a fiber, by theorem 5.3. Since the dynamics of the holonomy of $\partial\Sigma$ has no fixed points in the interior of I, the end of N has a single cylinder leaf, and plane leaves spiralling towards it, in a C^∞ fashion. This can be twisted around ∂K while keeping it C^1, where it will be eventually tangent to the suspension of a Pixton representation of $\mathbb{Z} \oplus \mathbb{Z}$ in the group of C^1 homeomorphisms of I; see [11]. With notation from corollary 5.5, as we spin the end of N around ∂K, the holonomy is generated by some $\beta : I \to I$ with no fixed points near 0 and the suspension α of $\rho(\gamma)$ by β. Explicitly: there is an inclusion $\varphi : I \to [p_1, p_0]$ where $0 < p_1 < p_0 < 1$. Choose a sequence $p_i \to 0$ where $\beta(p_i) = p_{i+1}$ and define α' by

$$\alpha'|_{[p_1,p_0]} = \varphi\rho(\gamma)\varphi^{-1}$$

and $\alpha' = \mathrm{id}$ outside $[p_1, p_0]$. Then define:

$$\alpha = \prod_{i=-\infty}^{\infty} \beta^i \alpha' \beta^{-i}$$

Such an α is the suspension of $\rho(\gamma)$ by β; clearly, the elements α and β commute. We would like to pick β and p_i such that α and β are both tangent to first order to the identity at 0. First of all, we should choose the p_i so that the ratios $[p_i, p_{i+1}]/[p_{i-1}, p_i] \to 1$ as $i \to \infty$, for instance by choosing $p_i = \frac{1}{i}$. The restriction of α to (p_i, p_{i+1}) is conjugate to a translation of $\mathbb{R}$; so we just need to choose a sequence of translations on each of the (p_i, p_{i+1}) which, after rescaling, converge geometrically in the C^1 topology to the identity. This requires β to expand the intervals where $(\alpha - \mathrm{id})'$ is very small, and contract the intervals where $(\alpha - \mathrm{id})'$ is larger. Let $\|f\|_\infty^i$ denote the L^∞ norm of f on the interval $[p_i, p_{i+1}]$. If $\rho(\gamma)$ decays sufficiently quickly to the identity near ∂I, we can find, for any $\delta_i > 0$, a restriction of β to $[p_i, p_{i+1}]$ for which $\|\beta'\|_\infty^i < \delta_i$ but which satisfies

$$\frac{\|(\alpha - \mathrm{id})'\|_\infty^{i+1}}{\|(\alpha - \mathrm{id})'\|_\infty^i} \le 1 - \epsilon_i$$

for some definite $\epsilon_i > 0$. Choose a sequence $\delta_i \to 0$ so that $\prod_i (1 - \epsilon_i) \to 0$. More details are found in [11]. $\square$

On the other hand, this foliation cannot be made C^2, by Kopell's lemma ([6]), since the holonomy representation of $\partial\Sigma$ on ∂N, thought of as a circle bundle over $\partial\Sigma$, has a fixed point. Thus, although the foliation of N is C^∞, this foliation cannot be smoothly twisted around the boundary torus and plugged in with a Reeb component.

Tantalisingly, the foliation $\mathcal{F}$ of N *can* be modified slightly so that it can be smoothly spun around ∂N:

Theorem 6.2 *The foliation $\mathcal{F}$ of N as above can be C^∞ approximated by a foliation $\mathcal{G}$ such that the holonomy of $\partial\Sigma$ has no fixed points.*

Proof Let γ be an essential simple arc in Σ. We cut open the foliation $\mathcal{F}$ along a small transverse foliated rectangle R, shift slightly and reglue. Since the holonomy along γ followed by an arc in $\partial\Sigma$ is nontrivial, this perturbation can be made to modify the holonomy of $\partial\Sigma$ to have no fixed points. Herman showed in [5] that for a subset $A \subset S^1$ of full measure, a C^∞ diffeomorphism from S^1 to itself with rotation number in A is C^∞ *conjugate* to a rotation. It follows that for suitable choice of generic shift along R, the rotation number of this boundary holonomy will lie in A, and therefore this element will be C^∞ conjugate to a rotation. Such a foliation can be spun *smoothly* around a Reeb component to give a C^∞ foliation of M. □

One might ask whether in this context, a "generic" perturbation of the resulting foliation, preserving the fixed–point free irrational dynamics on the boundary, can make the foliation typical, following the proof of theorem 5.3. Unfortunately, after shearing along R, the foliation of N is no longer necessarily complete; in fact, it will almost certainly blow up holonomy of some intervals in finite time. This is not to say that no such perturbation could exist, but new techniques would be required to find it.

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