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Framed holonomic knots

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Abstract A holonomic knot is a knot in 3-space which arises as the 2-jet extension of a smooth function on the circle. A holonomic knot associated to a generic function is naturally framed by the blackboard framing of the knot diagram associated to the 1-jet extension of the function. There are two classical invariants of framed knot diagrams: the Whitney index (rotation number) W and the self linking number S.

For a framed holonomic knot we show that W is bounded above by the negative of the braid index of the knot, and that the sum of W and $|S|$ is bounded by the negative of the Euler characteristic of any Seifert surface of the knot.

The invariant S restricted to framed holonomic knots with $W = m$, is proved to split into n, where n is the largest natural number with $n \leq \frac{|m|}{2}$, integer invariants. Using this, the framed holonomic isotopy classification of framed holonomic knots is shown to be more refined than the regular isotopy classification of their diagrams.

AMS Classification 57M27; 58C25

Keywords Framing, holonomic knot, Legendrian knot, self-linking number, Whitney index

1 Introduction

1.A Holonomic knots and framing

Let $f\colon S^1 \to \mathbb{R}$ be a smooth function. The *holonomic plane curve* and *holonomic space curve* associated to f are the 1-jet extension c and the 2-jet extension C of f, respectively. That is, $c(t) = (f(t), f'(t))$, and $C(t) = (f(t), f'(t), f''(t))$ where $t \in S^1$, and where (x_0, x_1) and (x_0, x_1, x_2) are linear coordinates on $\mathbb{R}^2$ and $\mathbb{R}^3$, respectively.

A *holonomic knot* is a holonomic space curve which is an embedding. A *framed holonomic knot* is a holonomic knot with associated holonomic plane curve which is an immersion. If f is a function giving rise to a framed holonomic knot C then $(f'(t), f''(t)) \neq 0$, for all $t \in S^1$, and we consider the constant vector field ∂_{x_2} as a normal vector field along C.

In the space of smooth functions on the circle, the functions with associated holonomic space curve being a (framed) holonomic knot form an open and dense subset. A *(framed) holonomic isotopy* is a continuous 1-parameter family of (framed) holonomic knots, or equivalently a continuous path in the space of (framed) holonomic knots.

Vassiliev [5] introduced holonomic knots and proved that any knot class (topological isotopy class of knots) has a holonomic representative and also that there exists a natural isomorphism from finite type invariants of topological knots to finite type invariants of holonomic knots.

Birman and Wrinkle [2] showed that two holonomic knots which are topologically isotopic are in fact holonomically isotopic. From a combinatorial point of view this means that the holonomic isotopy classification of holonomic knots is identical to the isotopy classification of their diagrams. (A *knot diagram* is the image of a generic projection of a knot to a plane in $\mathbb{R}^3$, decorated with over and under crossing information at its double points. An *isotopy of a knot digram* is defined to be a sequence of planar isotopies and Reidemeister moves, see e.g. Kauffman [3].)

1.B Whitney index and self linking number

Following Kauffman [3], we say that two knot diagrams which can be deformed into each other by a sequence of planar isotopies and, second- and third Reidemeister moves (i.e. the moves the projections of which are self-tangency- and triple point instances) are called *regularly isotopic*. There are two simple invariants of regular isotopy:

Fix an orientation of the ambient $\mathbb{R}^3$. This orientation together with a fixed orientation of the projection direction associated to the knot diagram induce an orientation on the projection plane. The *Whitney index* W is the tangential degree of the knot diagram viewed as an oriented regular plane curve in the projection plane. The *self linking number* S is the linking number of a knot K which projects to the diagram and a copy of K shifted slightly in the projection direction, computed using the fixed orientation of the ambient $\mathbb{R}^3$.

The Whitney formula [6] expresses the Whitney index of a generic regular plane curve as follows. Let q be a point on C such that C lies on one side of the tangent line of C at q. Let $\mu(q) = \pm 1$ denote the winding number of C with respect to q', where q' is the point q shifted slightly into the half plane which contains the curve. The orientation of C and the point q induces an ordering of the preimages of a self intersection point p of C. Let $\epsilon_p = \pm 1$ be the sign of the orientation of the plane induced from the tangent vectors of the ordered branches of C intersecting at p. Then

$$W(C) = -\sum_p \epsilon(p) + \mu(q).$$

The self linking number is the sum of the crossing signs over all crossings of the diagram. Hence $W + S$ is an odd integer.

In our study of framed holonomic knots we use the orientation $dx_0 \wedge dx_1 \wedge dx_2 > 0$ and project along the x_2-axis oriented by $dx_2 > 0$ to define W and S for framed holonomic knots. The ranges of these invariants are easily found:

Proposition 1.1 *Let C be a framed holonomic knot. Then $W(C) < 0$, and if $W(C) = -1$ then $S(C) = 0$ and C represents the unknot. Let $m \leq -2$ and $n \in \mathbb{Z}$ be such that $m + n$ is odd. Then there exists a framed holonomic knot with $W(C) = m$ and $S(C) = n$.*

Proposition 1.1 is proved in Subsection 2.A.

It is more interesting to consider the ranges of W and S restricted to diagrams representing a fixed knot class K. It is easy to see that for any integers m and n such that $m + n$ is odd there exists a diagram D which represents K with $W(D) = m$ and $S(D) = n$. If the domains of W and S are restricted further to framed holonomic knots which represent K the situation changes drastically.

Theorem 1.2 *Let C be a framed holonomic knot representing the knot class K. Then*

$$W(C) \leq -\operatorname{braid}(K), \tag{1.1}$$

where $\operatorname{braid}(K)$ *is the braid index of K, and*

$$W(C) + |S(C)| \leq 2g(K) - 1, \tag{1.2}$$

where $g(K)$ denotes the genus of K.

Theorem 1.2 is proved in Subsection 3.C. The proof of (1.2) uses the Bennequin inequality [1] from the theory of Legendrian knots: if (x, y, z) are coordinates on $\mathbb{R}^3$ and $\mathbb{R}^3$ is oriented by $dx \wedge dy \wedge dz > 0$ then this inequality asserts that for the xy-diagram $\Gamma_{\mathbb{C}}$ of a knot Γ which is everywhere tangent to the field of hyperplanes $\ker(dz - ydx)$ and which represents the knot class K

$$S(\Gamma_{\mathbb{C}}) + |W(\Gamma_{\mathbb{C}})| \leq 2g(K) - 1. \tag{1.3}$$

It is a curious fact that the roles of S and W in (1.2) and (1.3) are reversed.

1.C New invariants of framed holonomic knots

Trace [4] showed that two knot diagrams D and D' are regularly isotopic if and only if they represent the same knot class, $W(D) = W(D')$, and $S(D) = S(D')$. The classification problem for framed holonomic knots resembles the problem of classifying knot diagrams up to regular isotopy in the following way. Regular isotopy is knot diagram isotopy without first Reidemeister moves (the move which projects to a cusp-instance) and framed holonomic isotopy is holonomic isotopy without the holonomic first Reidemeister move, see Figure 2.

Theorem 1.3 *On the space of framed holonomic knots with Whitney index equal to m, the invariant S splits. More precisely, to each framed holonomic knot C with $W(C) = m$ there is associated n, where n is the largest integer with $n \leq \frac{|m|}{2}$, integers $S_1(C), \dots, S_n(C)$, which are invariant under framed holonomic isotopy. Moreover,*

$$S(C) = \sum_{j=1}^{n} S_j(C). \tag{1.4}$$

The invariants S_j are defined in Definition 4.1 and Theorem 1.3 is proved in Subsection 4.A.

In Section 5 we give examples of framed holonomic knots representing the same knot class, with the same W and S but which are distinguished up to framed holonomic isotopy by the invariants S_j. This shows that the classification of framed holonomic knots up to framed holonomic isotopy is more refined than the regular isotopy classification of their diagrams. This result should be compared to the result of Birman and Wrinkle mentioned in Subsection 1.A.

1.D Holonomic regular homotopy

A *holonomic regular homotopy* is a continuous 1-parameter family of regular holonomic plane curves.

Proposition 1.4 *Two regular holonomic plane curves are holonomically regularly homotopic if and only if they have the same Whitney index.*

Proposition 1.4 is proved in Subsection 2.B. If the holonomic requirements in Proposition 1.4 are removed one obtains the classical Whitney-Graustein theorem [6]. The proof we present is independent of this theorem.

2 Diagrams of holonomic knots and Reidemeister moves

For the readers convenience, basic facts on the geometry of diagrams of holonomic knots are presented. For proofs of these facts, see [5], Proposition 1.

Let $f\colon S^1 \to \mathbb{R}$ be a generic function. Then the x_0x_1-diagram c of the framed holonomic knot C associated to f has the following properties:

- **P1** c is a regular curve and if p is a point on c in the upper (lower) half plane and v is the unit tangent of c at p then $\langle v, \partial_{x_0}\rangle > 0$ ($\langle v, \partial_{x_0}\rangle < 0$), where $\langle\, ,\rangle$ denotes the standard inner product on $\mathbb{R}^2$.
- **P2** c meets the x_0-axis at right angles at a finite number of points corresponding to the local extrema of f. The curvature of c at such a point p does not vanish and if p corresponds to maximum (minimum) of f the unit tangent of c at p equals $-\partial_{x_1}$ (∂_{x_1}).
- **P3** The only self intersection points of c are transverse double points which lie in the region $\{(x_0, x_1)\colon x_1 \neq 0\}$. The crossing number of a double point in the upper (lower) half plane is negative (positive) with respect to the orientation $dx_0 \wedge dx_1 \wedge dx_2 > 0$.

In generic 1-parameter families of *framed holonomic knots* the diagram changes by planar isotopy which preserve properties **P1**–**3** above except for a finite number of instances where one of the bifurcations in Figure 1 occur. Note that the Ω_2-moves always occur on the x_0-axis. The signs on the Ω_2-moves refer to the signs of the product of the second derivatives at the extrema meeting at the self-tangency moment of the function defining the holonomic knot. The Ω_3-move depicted occurs either in the upper- or lower half plane.

If the word *framed* above is omitted the corresponding result is: In generic 1-parameter families of *holonomic knots* the diagram changes by planar isotopy which preserve properties **P1**–**3** above except for a finite number of instances where one of the bifurcations in Figures 1 or 2 occur. The signs of the Ω_1-moves

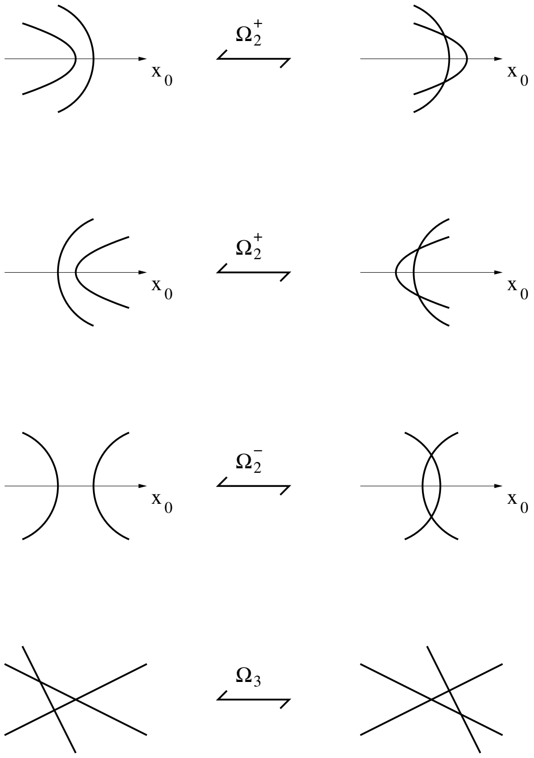


Figure 1: Framed holonomic Reidemeister moves

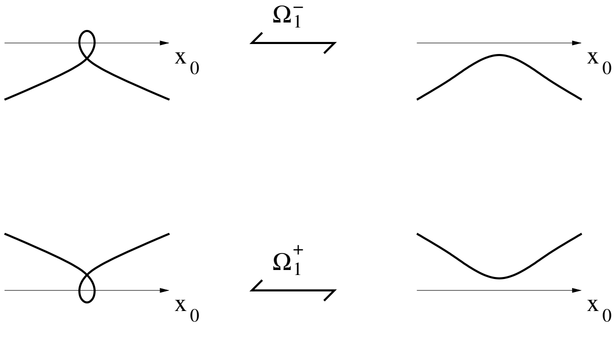


Figure 2: Holonomic versions of the first Reidemeister move

in Figure 2 refer to the sign of x_1 in the half plane where a double point is born or vanishes.

If we further omit the condition that the holonomic curve be an embedding then the list of diagram-bifurcations would be further extended and include also the move in Figure 3 (which might change the knot class of the holonomic curve). The signs on the Ω_0-moves in Figure 3 refer to the half plane where double

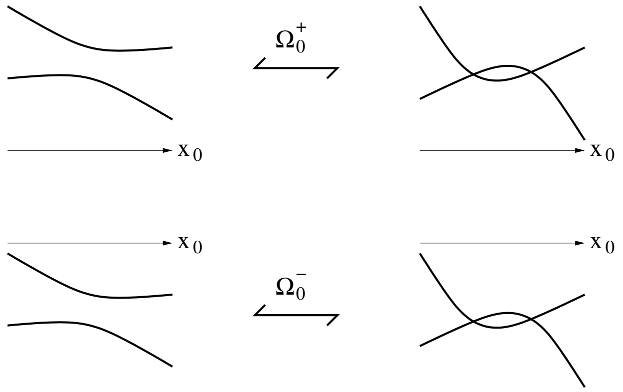


Figure 3: Framed holonomic crossing move

points are born or vanish.

2.A Proof of Proposition 1.1

Let $\pi(x_0, x_1, x_2) = (x_0, x_1)$. Let f be a generic function on the circle, let C be its associated holonomic knot, and let $c = \pi(C)$. To see that $W(C) < 0$ compute the Whitney index by looking at points p on c where the unit tangent equals ∂_{x_1}. These correspond to minima of f and all of them contribute -1 to W. The second statement is immediate.

To create a holonomic knot C with $W(C) = m$, $m \leq -2$ and $S(C) = n$, start from the holonomic unknot diagram (which is just the unit circle). If $n \geq 0$ ($n < 0$) apply Ω_1^+ (Ω_1^-) $m - 1$ times in such a way that the resulting diagram contains $m - 1$ consecutive loops along the x_0-axis. The resulting holonomic knot has $W = m$ and $S = -(m-1)$ if $n \geq 0$ ($S = m - 1$ if $n < 0$). Finally, if $n \geq 0$ apply Ω_0^- $\frac{n+m-1}{2}$ times and if $n < 0$ apply Ω_0^+ $\frac{n+m-1}{2}$ times to create new double points. The resulting holonomic knot C then has $W(C) = m$ and $S(C) = n$, as desired. $\square$

2.B Proof of Proposition 1.4

Let f be a function with associated holonomic plane curve c_f which is an immersion. If ϕ is a diffeomorphism of S^1 then also $f \circ \phi$ gives rise to a regular holonomic plane curve.

Let g be a function with regular plane holonomic curve c_g with $W(c_f) = W(c_g)$. Perturb f and g so that they become Morse functions. Then the proof of Proposition 1.1 implies that they have the same number of local extrema. Let ϕ_s, $0 \leq s \leq 1$ be a diffeotopy of S^1 which moves each critical point of f to a critical point of g of the same index. Then the critical sets of $\hat{g} = g \circ \phi_1$ and of f agree. Moreover, if t is local maximum (minimum) of f then it is a local maximum (minimum) of $g \circ \phi_1$. Let (s, r) be coordinates on the cylinder $S^1 \times \mathbb{R}$ and consider the vector field $V(s,r) = (f(s) - \hat{g}(s))\partial_r$. Let Φ^ρ be the flow of V. If $\hat{g}_\rho$ is the function with graph $\Phi^\rho(\Gamma_{\hat{g}})$, where $\Gamma_{\hat{g}}$ is the graph of $\hat{g}$, then $\hat{g}_\rho$ has a regular associated holonomic curve for each $\rho \leq 1$ and $\hat{g}_1 = f$.

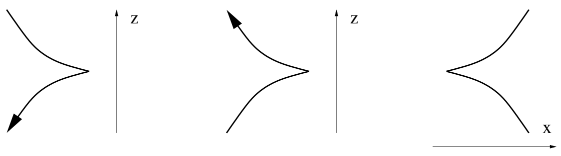
These two deformations together give the desired holonomic regular homotopy. □

3 Holonomic knots and front projections of Legendrian knots

3.A The front and complex projections of a Legendrian knot

Let Γ be a knot in $\mathbb{R}^3$ with coordinates (x, y, z) everywhere tangent to the plane field $\{\ker(dz - ydx)\}$. That is, Γ is a *Legendrian knot*. Assume moreover that Γ is generic among Legendrian knots, then the projection Γ_F of Γ to the xz-plane is a curve with transverse double points, isolated cusps, and without vertical tangencies. Moreover, given any curve in the xz-plane with these properties, there exists a unique Legendrian knot which projects to this curve. We associate the following numbers to Γ_F:

First we count cusps, let $\mathrm{Dcu}(\Gamma_F)$, $\mathrm{Ucu}(\Gamma_F)$, and $\mathrm{Lcu}(\Gamma_F)$ denote the number of down-cusps, up-cusps, and left-cusps respectively of Γ_F, see Figure 4.



(A) (B) (C)

Figure 4: (A) Down-cusp, (B) Up-cusp, and (C) left-cusp.

Second we count crossings, let $\mathrm{Ecr}(\Gamma_F)$ denote the number of crossing points where the tangent vectors has x-components of the same sign and $\mathrm{Ocr}(\Gamma_F)$ the number of crossing points where the tangent vectors has x-components of the opposite sign. See Figures 5 and 6.

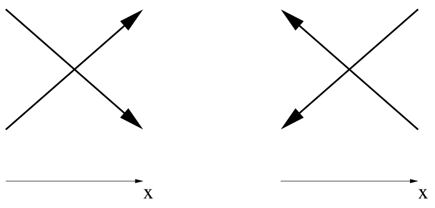


Figure 5: Crossing points with tangents with x-components of equal sign

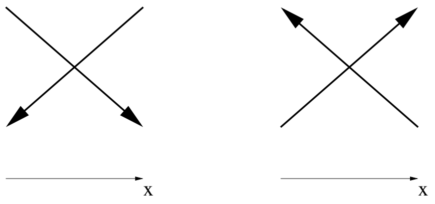


Figure 6: Crossing points with tangents with x-components of opposite signs

The projection $\Gamma_{\mathbb{C}}$ of Γ to the xy-plane is a generic knot diagram. It is straightforward to check that the invariants $W(\Gamma_{\mathbb{C}})$, where we use the orientation given by $dx \wedge dy$ in the xy-plane, and $S(\Gamma_{\mathbb{C}})$, where we use the orientation given by $dx \wedge dy \wedge dz$ in space, can be computed from data of Γ_F as follows,

$$W(\Gamma_{\mathbb{C}}) = \frac{1}{2}\left(\mathrm{Dcu}(\Gamma_F) - \mathrm{Ucu}(\Gamma_F)\right), \tag{3.1}$$

$$S(\Gamma_{\mathbb{C}}) = \mathrm{Ecr}(\Gamma_F) - \mathrm{Ocr}(\Gamma_F) - \mathrm{Lcu}(\Gamma_F). \tag{3.2}$$

3.B Legendrian knots associated to a holonomic one

Let C be a framed holonomic knot. We associate two Legendrian knots Γ^+ and Γ^-, everywhere tangent to $\ker(dx_1 - x_2 dx_0)$, to C by describing their front projections (in the x_0x_1-plane). The resulting Legendrian knots lie in $\mathbb{R}^3$ oriented by $dx_0 \wedge dx_2 \wedge dx_1 > 0$.

The first step in the construction of the fronts of Γ^+ and Γ^- is the same in both cases:

The points where the diagram of C has vertical tangents are all confined to the x_0-axis. Replace neighborhoods of such points in the diagram with cusped arcs as described in Figure 7. The second step however differs:

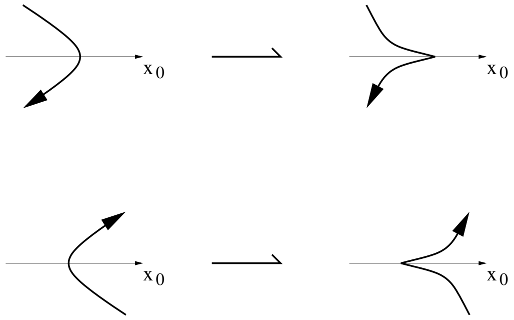


Figure 7: Replacing vertical tangencies with cusps

To obtain the front of Γ^+ we insert a zig-zag as in Figure 8 at all crossings in the lower half plane and keep the crossings in the upper half plane as they are.

To obtain the front of Γ^- we insert a zig-zag as in Figure 9 at all crossings in the upper half plane and keep the crossings in the lower half plane as they are.

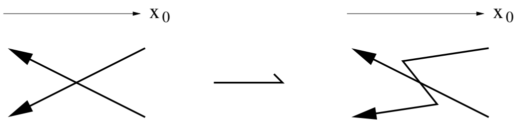


Figure 8: Inserting a zig-zag in the lower half plane

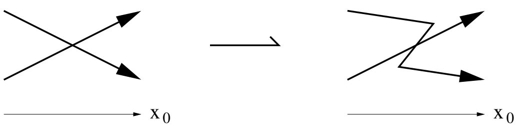


Figure 9: Inserting a zig-zag in the upper half plane

It is easy to check that Γ^- (Γ^+) is topologically isotopic to the knot C in $\mathbb{R}^3$ equipped with the orientation $dx_0 \wedge dx_1 \wedge dx_2 > 0$ ($dx_0 \wedge dx_1 \wedge dx_2 < 0$).

3.C Proof of Theorem 1.2

Using Ω_2-moves, we may obtain a closed braid representation of K. Equation (1.1) follows.

To prove (1.2), let $H_+(C)$ ($H_-(C)$) denote the number of intersection points of C in the upper (lower) half plane. Then $S(C) = H_-(C) - H_+(C)$. As noted before, $W(C)$ is the negative of the number of local minima of the function f giving rise to C.

Let Γ^+ and Γ^- be the Legendrian knots associated to C as in Subsection 3.B. Then

$$\begin{aligned} \mathrm{Dcu}(\Gamma^+_F) &= -W(C) + 2H_-(C), \\ \mathrm{Dcu}(\Gamma^-_F) &= -W(C) + 2H_+(C), \\ \mathrm{Ucu}(\Gamma^+_F) &= \mathrm{Ucu}(\Gamma^-_F) = -W(C), \end{aligned}$$

and hence

$$W(\Gamma^+_{\mathbb{C}}) = H_-(C), \tag{3.3}$$
$$W(\Gamma^-_{\mathbb{C}}) = H_+(C). \tag{3.4}$$

Also,

$$\begin{aligned} \mathrm{Lcu}(\Gamma^+_F) &= -W(C) + H_-(C), \\ \mathrm{Lcu}(\Gamma^-_F) &= -W(C) + H_+(C), \\ \mathrm{Ecr}(\Gamma^+_F) &= \mathrm{Ocr}(\Gamma^-_F) = H_+(C), \\ \mathrm{Ecr}(\Gamma^-_F) &= \mathrm{Ocr}(\Gamma^+_F) = H_-(C), \end{aligned}$$

and hence

$$S(\Gamma^+_{\mathbb{C}}) = H_+(C) - 2H_-(C) + W(C), \tag{3.5}$$
$$S(\Gamma^-_{\mathbb{C}}) = H_-(C) - 2H_+(C) + W(C). \tag{3.6}$$

Combining (3.3) and (3.5), respectively (3.4) and (3.6) with the Bennequin inequality (1.3) yields

$$\begin{aligned} -S(C) + W(C) &\leq 2g(K) - 1 \text{ and} \\ S(C) + W(C) &\leq 2g(K) - 1, \end{aligned}$$

since the genus does not depend on the orientation of the ambient space. The theorem follows. □

4 Splitting the self linking number

Consider the diagram of a framed holonomic knot C. The x_0-axis divides the diagram into cyclically ordered arcs (X_i, Y_i), $i = 1, \dots, m$, where the X_i lies in the upper half plane, the Y_i in the lower, and where $-m = W(C)$.

Let $(A_i, A_j) = (X_i, X_j)$ or $(A_i, A_j) = (Y_i, Y_j)$ where $i \neq j$. Define

$$\delta(A_i, A_j) = \begin{cases} 1 & \text{if } \partial A_i \text{ is contained in an unbounded component of } \mathbb{R} - \partial A_j, \\ 0 & \text{otherwise.} \end{cases}$$

Define

$$N(A_i, A_j) = |A_i \cap A_j| + \delta(A_i, A_j),$$

where $|S|$ denotes the number of points in the set S.

Let $\tilde{X}_i$ and $\tilde{Y}_i$ denote the preimages of X_i and Y_i, for $i = 1, \dots, m$. Let x_i and y_i denote the midpoints of $\tilde{X}_i$ and $\tilde{Y}_i$, respectively.

Consider two arcs X_i and X_j, $i \neq j$. Let $\gamma(x_i, x_j)$ denote the unique oriented arc connecting x_i to x_j with orientation agreeing with that of the circle. Define the *cyclic distance of* X_i *and* X_j as

$$d(X_i, X_j) = \min\Big\{ |\gamma(x_i, x_j) \cap \{y_1, \dots, y_m\}|, |\gamma(x_j, x_i) \cap \{y_1, \dots, y_m\}| \Big\}.$$

Define the cyclic distance of arcs Y_i and Y_j analogously.

Definition 4.1 Define

$$S_k(C) = \frac{1}{2} \left(\sum_{\{(Y_i, Y_j)\colon d(Y_i, Y_j) = k\}} N(Y_i, Y_j) - \sum_{\{(X_i, X_j)\colon d(X_i, X_j) = k\}} N(X_i, X_j) \right).$$

Remark 4.2 In terms of defining functions, the terms in the definition of S_k can be interpreted as follows. Let $f\colon S^1 \to \mathbb{R}$ be a function with associated framed holonomic knot C. Consider f as a periodic function with period T such that $f(0) = f(T)$ is the global minimum of f. Let $\Gamma_f \subset [0, T] \times \mathbb{R} \subset \mathbb{R}^2$ denote the graph of f. Then the arcs X_i (Y_i) are the holonomic curves corresponding to restrictions of f to subintervals of $[0, T]$, where f is increasing (decreasing). If (x, y) are coordinates on $\mathbb{R}^2$ then $|A_i \cap A_j|$ equals the number of lines $l_a = \{y = a\}$, $a \in \mathbb{R}$ which intersect A_i and A_j at equal angles, and $\delta(A_i, A_j) = 1$ if no l_a intersect both A_i and A_j, otherwise it is 0.

4.A Proof of Theorem 1.3

We check that S_k is invariant under framed holonomic Reidemeister moves. For Ω_3 this is immediate.

An Ω_2^+-move involving two arcs X and X' with distance $d(X, X') = k$ involves also two arcs Y and Y' with $d(Y, Y') = k$. At the move $\delta(X, X')$ and $\delta(Y, Y')$ are unchanged and the change in $|X \cap X'|$ and $|Y \cap Y'|$ are the same. Hence S_k remains constant.

At an Ω_2^--move involving arcs X and X', Y and Y' the change in $|X \cap X'|$ and $|Y \cap Y'|$ equals the change in $\delta(X, X')$ and $\delta(Y, Y')$, respectively. Hence S_k remains constant.

To prove (1.4) note that by using the Ω_2-moves we may move any framed holonomic knot diagram in such a way that its diagram is a closed braid with braid-axis parallel to the x_2-direction. (The linking number of this axis oriented in the positive x_2-direction and the holonomic knot with its natural orientation is negative.) Under such deformations both S and $S_1, \dots, S_n$, remain constant. Moreover for a diagram which is a closed braid $\delta(X_i, X_j) = \delta(Y_i, Y_j) = 0$ for all i, j. Hence both the left and right hand sides of (1.4) are equal to the difference of the number of double points in the lower and upper half planes. □

5 Examples

The framed holonomic knots K_1 in Figure 10 and K_2 in Figure 11 both represent the unknot. Since $S(K_1) = S(K_2) = -1$ and $W(K_1) = W(K_2) = -4$, K_1 and K_2 are regularly isotopic. Since K_1 is a closed braid $\delta(X_i, X_j) = 0 = \delta(Y_i, Y_j)$ for all i, j. Noting that all three intersection points in the diagram of K_1 in the upper half plane are intersections between arcs of cyclic distance 1, and that the two intersection points in the lower half plane are intersections of arcs of cyclic distance 1 respectively 2, we conclude that $S_1(K_1) = -2$ and $S_2(K_1) = 1$. A similar calculation gives $S_1(K_2) = 0$ and $S_2(K_2) = -1$. Hence K_1 and K_2 are *not* framed holonomically isotopic.

The framed holonomic knots K_3 in Figure 12 and K_4 in Figure 13 both represent the connected sum of the trefoil and its mirror image. Since $S(K_3) = S(K_4) = -1$ and $W(K_3) = W(K_4) = -4$, K_3 and K_4 are regularly isotopic. However, $S_1(K_3) = 0$ and $S_2(K_3) = -1$ but $S_1(K_4) = -4$ and $S_2(K_4) = 3$ so K_3 and K_4 are *not* framed holonomically isotopic.

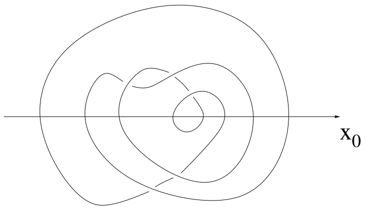


Figure 10: The framed holonomic knot K_1

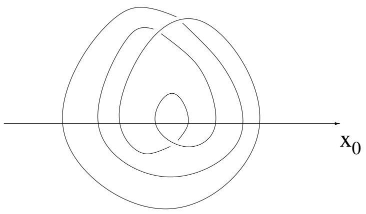


Figure 11: The framed holonomic knot K_2

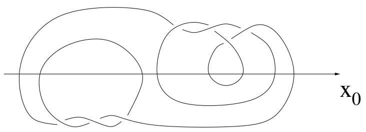


Figure 12: The framed holonomic knot K_3

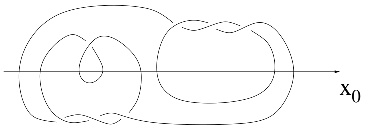


Figure 13: The framed holonomic knot K_4

References

[1] D. Bennequin, *Entrelacements et équations de Pfaff*, Soc. Math. de France, Astérisque **107–108** (1983) 87–161.

[2] J. S. Birman and N. C. Wrinkle, *Holonomic and Legendrian parametrizations of knots*, J. Knot Theory Ramifications **9** (2000), 293–309.

[3] L. Kauffman, *Knots and Physics*, World Scientific Publishing Co., Inc., River Edge, NJ (1991).

[4] B. Trace, *On the Reidemeister moves of a classical knot*, Proc. Amer. math. Soc. **89** (1983) 722–724.

[5] V. A. Vassiliev, *Holonomic links and Smale principles for multisingularities*, J. Knot Theory Ramifications **6** (1997) 115–123.

[6] H. Whitney, *On regular closed curves in the plane*, Composito Math. **4** (1936) 276–284.

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