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On the CAT(0) dimension of 2-dimensional Bestvina-Brady groups

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Abstract Let K be a 2-dimensional finite flag complex. We study the CAT(0) dimension of the 'Bestvina-Brady group', or 'Artin kernel', Γ_K. We show that Γ_K has CAT(0) dimension 3 unless K admits a piecewise Euclidean metric of non-positive curvature. We give an example to show that this implication cannot be reversed. Different choices of K lead to examples where the CAT(0) dimension is 3, and either (i) the geometric dimension is 2, or (ii) the cohomological dimension is 2 and the geometric dimension is not known.

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Let Γ be a countable group. We denote the cohomological dimension of Γ by $\mathrm{cd}(\Gamma)$ and the geometric dimension by $\mathrm{gd}(\Gamma)$. See [7], pp 184–5, for definitions of these notions. It was shown by Eilenberg and Ganea [9] that if $\mathrm{cd}(\Gamma) \geq 3$ then $\mathrm{cd}(\Gamma) = \mathrm{gd}(\Gamma)$. The same is true if $\mathrm{cd}(\Gamma) = 0$ (Γ trivial), or $\mathrm{cd}(\Gamma) = 1$ (by the Stallings–Swan Theorem [11, 12]). There remains however the possibility that there exists a group Γ with $\mathrm{cd}(\Gamma) = 2$ but $\mathrm{gd}(\Gamma) = 3$, that is, a counter-example to the so-called Eilenberg-Ganea Conjecture. In the search for such an example, two promising families of groups have been proposed. Both constructions begin by choosing a finite flag 2-complex K which is acyclic but not contractible, such as, for example, any finite flag subdivision of the 2-spine of the Poincaré homology sphere. For the first family, due to Bestvina and Davis, one considers a torsion free finite index subgroup G_K of the right-angled Coxeter group associated to the 1-skeleton K^1. In the second case (Bestvina-Brady [1]) one considers the kernel Γ_K of the length homomorphism on the right-angled Artin group associated to K^1. These groups may be defined for an arbitrary finite flag complex K. In the case of interest, where K is a non-contractible, acyclic flag 2-complex, the groups G_K and Γ_K are known to have cohomological dimension 2 but their geometric dimensions are not known.

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Metric spaces of global nonpositive curvature, or CAT(0) spaces, provide a natural supply of contractible spaces. Moreover, in many cases the CAT(0) condition, and hence contractibility of a given space, may be easily verified in terms of local geometric information. This motivates our interest in CAT(0) spaces and group actions on these spaces, and leads us to the notion of the *CAT(0) dimension* of a group (we adopt the definition used by Bridson in [5]):

$$\dim_{ss}(\Gamma) = min(\mathcal{D} \cup \{\infty\}), \qquad \text{where}$$

$$\mathcal{D} = \{\dim(X) : X \text{ is a complete CAT(0) space on which } \Gamma \text{ acts properly by semi-simple isometries}\}.$$

Here $\dim(X)$ denotes the covering dimension of a metric space X (see [10]). Note that $\mathcal{D}$ may well be an empty set (i.e. no such actions exist) in which case $\dim_{ss}(\Gamma) = \infty$.

Let K be an arbitrary finite 2-dimensional flag complex. The main result of this paper (Theorem 2.1) states that, unless the 2-complex K is aspherical (in fact, unless it admits a piecewise Euclidean metric of nonpositive curvature), the group Γ_K does not act properly semi-simply on any 2-dimensional CAT(0) space, and hence has CAT(0) dimension 3. In particular, $\dim_{ss}(\Gamma_K) = 3$ in the case where K is acyclic but not aspherical, as for example when K is a flag decomposition of the 2-spine of the Poincaré homology sphere (Corollary 2.2). However, we do not rule out the possibility that $\dim_{ss}(\Gamma_K) = 2$ for some aspherical but non-contractible flag 2-complex K. We note also that this result is still a very long way from showing that some Γ_K with cohomological dimension 2 has geometric dimension 3. It merely indicates, as one might already expect, that for these examples a 2-dimensional Eilenberg-MacLane complex will not be so easy to find, if in fact such a complex exists.

Theorem 2.1 is a result of the fact that the group Γ_K contains many 'overlapping' $\mathbb{Z}\times\mathbb{Z}$ subgroups, which imply the existence of flat planes embedded in any CAT(0) space on which the group acts. Our argument proceeds by studying how these flat planes interact. The Bestvina-Davis examples, G_K, also contain $\mathbb{Z}\times\mathbb{Z}$ subgroups, however they arise in a less regular manner than in the Artin kernels, and the present techniques seem less applicable. It would be interesting to know whether similar results hold for the groups G_K.

It is already known that requiring that a group act properly semi-simply on a CAT(0) space imposes constraints both on the group and on the dimension of the space. For example, the Baumslag-Solitar groups $BS(n,m) = \langle x,t | t^{-1}x^n t = x^m\rangle$, for $1 \le n < m$, have geometric dimension 2 but do not admit any proper semi-simple actions on CAT(0) spaces (which may be seen by considering the translation length of x). There are also recent examples given of finitely presented groups Γ having $\mathrm{gd}(\Gamma) = 2$ but $\dim_{ss}(\Gamma) = 3$ (see [3, 5]). We note that Bridson's example [5] is actually an index two subgroup of a group with CAT(0) dimension 2. The examples of both [3] and [5] are all *CAT(0) groups*, that is they act properly and cocompactly on CAT(0) spaces.

In Corollary 2.3 we give a simple method for constructing infinitely many further examples of finitely presented groups with geometric dimension 2 but CAT(0) dimension 3. These are the groups Γ_K where K is any flag triangulation of a contractible 2-complex which does not admit a CAT(0) metric. Zeeman's dunce hat is such a complex. We note that the 3-dimensional CAT(0) actions which are known to exist for these groups are semi-simple, but not cocompact. It is not known whether any of the examples given by Corollary 2.3 are CAT(0) groups.

In the last section of the paper we give a refinement (Theorem 3.1) of the main result which, in the case that K is a simply-connected finite flag 2-complex, leads to an "if and only if" statement. This allows us to give an example of a group Γ_K where K admits a CAT(0) piecewise Euclidean metric, but where we still have $\mathrm{gd}(\Gamma_K) = 2$ and $\dim_{ss}(\Gamma_K) = 3$. Since submission of this paper, Noel Brady has informed me that this particular example has a cubic Dehn function [2], and so cannot be a CAT(0) group.

We close with the following remark: all the groups Γ considered in this paper have the property that every finite index subgroup of Γ contains a subgroup isomorphic to Γ. Thus all the examples given in this paper of 2-dimensional groups with CAT(0) dimension 3 have the further property that every finite index subgroup has CAT(0) dimension 3 as well. This distinguishes our examples from the example given by Bridson in [4] which has an index 2 subgroup with CAT(0) dimension 2. On the other hand, it is not known whether the CAT(0) dimension can drop when passing to finite index subgroups in any of the Artin group examples given in [2].

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1 Some CAT(0) geometry

We introduce just the basic concepts of CAT(0) geometry which we will need here. We refer to [6] for details.

Let (X,d) be a metric space. By a *geodesic segment* with endpoints x and y, usually denoted $[x,y]$, we mean the image of an isometric embedding of the closed interval $[0,d(x,y)]$ into X such that the endpoints are mapped to x and y. A *geodesic triangle* $\Delta(x,y,z)$ in X is simply a union of three geodesic segments $[x,y]\cup[y,z]\cup[z,x]$. Note that Δ need not be uniquely determined by the three points x,y and z. However, for every triple $x,y,z\in X$ there is (up to isometry) a unique *comparison triangle* $\Delta'(x,y,z)$ in the Euclidean plane $\mathbb{E}^2$ which has vertices x',y',z' such that $d_{\mathbb{E}^2}(x',y')=d(x,y)$, $d_{\mathbb{E}^2}(y',z')=d(y,z)$, and $d_{\mathbb{E}^2}(x',z')=d(x,z)$. In the case that x is distinct from both y and z, we write $\angle'_x(y,z)$ for the angle at x' between the sides $[x',y']$ and $[x',z']$ in the comparison triangle Δ'.

There are various equivalent formulations of global nonpositive curvature (for geodesic spaces) in terms of comparison triangles (see [6], Chapter II.1). We will make use of the following one:

Given distinct points $x,y,z\in X$ and geodesic segments $[x,y],[x,z]$ one defines the *(Alexandrov) angle* $\angle([x,y],[x,z])$ between the two geodesic segments to be

$$\lim_{\epsilon\to 0}\ \sup\{\angle'_x(p,q) : p\in[x,y], q\in[x,z] \text{ and } 0<d(p,x),d(q,x)<\epsilon\}\ .$$

A geodesic metric space X (one in which there is a geodesic segment joining any pair of points) is said to be a *CAT(0) space* if, for every geodesic triangle $\Delta(x,y,z)$ with distinct vertices, the Alexandrov angle between any two sides of Δ is no greater than the corresponding angle in the comparison triangle $\Delta'(x,y,z)$ in $\mathbb{E}^2$.

This definition leads quickly to the following:

Lemma 1.1 *Let X be a CAT(0) space, and $P=P(v_1,v_2,\dots,v_m)$ a geodesic polygon in X with distinct vertices $v_1,v_2,\dots,v_m$ $(m\geq 3)$ and sides $[v_i,v_{i+1}]$ for $1\leq i\leq m$ (indices mod m). Write $\phi_i=\angle([v_i,v_{i-1}],[v_i,v_{i+1}])$ for the angle of the polygon at v_i, for $i=1,\dots,m$. Then*

$$\text{angle sum of } P\ =\ \sum_{i=1}^{m}\phi_i\ \leq\ (m-2)\pi\ .$$

Proof If $m = 3$ then the result follows directly from the CAT(0) inequality. Otherwise, take the (unique) geodesic segment $[v_1, v_3]$ to form a triangle $\Delta(v_1, v_2, v_3)$ and an $(m-1)$-gon $P'(v_1, v_3, \ldots, v_m)$. Let ψ_1, $\psi_2 (= \phi_2)$ and ψ_3 be the angles of Δ, and $\phi'_1, \phi'_3, \phi_4, \ldots, \phi_m$ the angles of P'. Then, by induction on the number of sides of a polygon, we have

$$\psi_1 + \phi_2 + \psi_3 \leq \pi\,, \qquad \text{and} \tag{1}$$

$$\phi'_1 + \phi'_3 + \phi_4 + \cdots + \phi_m \leq (m-3)\pi \tag{2}$$

But $\phi_1 \leq \phi'_1 + \psi_1$ and $\phi_3 \leq \phi'_3 + \psi_3$, so that the result follows immediately by adding the two inequalities (1) and (2). $\square$

Let g be an isometry of a metric space (X, d). The *translation length of* g, denoted by $\ell(g)$, is defined to be

$$\ell(g) \;=\; \inf\{d(x, gx) \mid x \in X\}\,,$$

and the *minset of* g, denoted by $Min(g)$, is defined to be the possibly empty set

$$Min(g) \;=\; \{x \in X \mid d(x, gx) = \ell(g)\}\,.$$

An isometry of a CAT(0) space is called *semi-simple* if it has a nonempty minset. If g is semi-simple with nonzero translation length then we say that g is *hyperbolic.* In this case any g-invariant geodesic line in X shall be called an *axis of* g or g*-axis.* By [6] Proposition II.6.2, the g-axes all lie in $Min(g)$.

The following statements may be found in [6], Chapter II.6. Let g be a semi-simple isometry of a CAT(0) space X. Then $Min(g)$ is always a closed, convex subspace of X. If $\ell(g) \neq 0$ then $Min(g)$ is isometric to the metric product of $\mathbb{R}$ with a CAT(0) space Y, where each $\mathbb{R}$-fibre is an axis of g. The element g acts on $Min(g)$ by translating along the $\mathbb{R}$ factor and fixing the Y factor pointwise, and each isometry of X which commutes with g leaves $Min(g)$ invariant while respecting the product structure. Moreover, if g belongs to a group Γ of semi-simple isometries acting properly on X then $C_\Gamma(g)/\langle g \rangle$ acts properly by semi-simple isometries on Y. (c.f: Propositions II.6.9-10 of [6]). Here $C_\Gamma(g)$ denotes the centralizer of g in Γ.

We wish to restrict our attention now to group actions on 2-dimensional spaces. The following is a special case of Bridson's Flat Torus Theorem [4, 6]. We refer to [10] for the theory of 'covering dimension'. We note that the covering dimension of a simplicial complex agrees with the (simplicial) dimension, and that the covering dimension of a metric space is bounded below by the dimension of any subspace.

Proposition 1.2 *Let A be a free abelian group of rank 2 acting properly by semi-simple isometries on a CAT(0) space X of covering dimension 2. Then $Min(A) = \cap_{a \in A} Min(a)$ is an A-invariant isometrically embedded flat plane ($\cong \mathbb{E}^2$) and the group A acts by translations on $Min(A)$ with quotient a 2-torus.*

Proof By the Flat Torus Theorem, $Min(A)$ is non-empty and splits as a product $Y \times \mathbb{E}^2$, where each fibre $\{y\} \times \mathbb{E}^2$ has the desired properties. The dimension constraint ensures that Y consists of precisely one point. For if $p, q \in Y$ are distinct points then, by convexity, Y contains the geodesic segment $[p, q]$, and hence X contains a subspace $[p, q] \times \mathbb{E}^2$ of dimension 3, contradicting $\dim(X) = 2$. □

Notation Suppose that A is a free abelian group of rank 2 acting properly by semi-simple isometries on a CAT(0) space X of covering dimension 2. For any pair of nontrivial elements $x, y \in A$, we write $\theta(x, y)$ for the angle between any positively oriented x-axis and any positively oriented y-axis in $Min(A)$. More precisely

$$\theta(x, y) = \angle([p, x(p)], [p, y(p)]), \text{ for any } p \in Min(A).$$

Note that $0 < \theta(x, y) < \pi$ in the case that x and y generate A.

We now prove the following Lemma.

Lemma 1.3 *Suppose that Γ is a group acting properly by semi-simple isometries on a CAT(0) space X of covering dimension 2. Suppose that we have a cyclic sequence of $m \geq 4$ group elements $a_1, a_2, \ldots, a_m = a_0$ of Γ (indices taken mod m) such that, for each $i = 1, \ldots, m$, we have*

(i) $\langle a_{i-1}, a_{i+1} \rangle = F_2$ *(free group of rank 2), and*

(ii) $\langle a_i, a_{i+1} \rangle \cong \mathbb{Z} \times \mathbb{Z}$.

Then
$$\sum_{i=1}^{m} \theta(a_i, a_{i+1}) \geq 2\pi.$$

Proof For $i = 1, 2, \ldots, m$, write $\Pi(i, i+1)$ for the flat plane $Min(\langle a_i, a_{i+1} \rangle)$ in X (see Proposition 1.2). Let $C(i)$ denote the convex closure of $\Pi(i-1, i) \cup \Pi(i, i+1)$ in X.

Fix $i \in \{1, \ldots, m\}$. Note that $C(i)$ is contained in $Min(a_i)$ (by convexity of the minset), and that $Min(a_i)$ is isometric to $T \times \mathbb{R}$ where, since X is 2-dimensional, T must be an $\mathbb{R}$-tree (see Lemma 3.2 of [5] for more detail). With respect to this decomposition, $\Pi(i-1, i) \cong \tau \times \mathbb{R}$ and $\Pi(i, i+1) \cong \sigma \times \mathbb{R}$, where τ and σ are the axes in T for the elements a_{i-1} and a_{i+1} respectively, and $C(i) \cong H \times \mathbb{R}$ where H is the convex closure in T of $\tau \cup \sigma$.

By hypotheses (i) and (ii) the elements a_{i-1} and a_{i+1} generate a rank 2 free group which commutes with a_i and therefore acts properly semi-simply on T. It follows that their axes τ and σ in T are either disjoint, or intersect in a closed interval of finite length. We define the finite (possibly zero) length closed interval $I = I(i)$ in T as follows. If τ and σ are disjoint, then I is the unique shortest geodesic segment joining them. In this case $H = \tau \cup I \cup \sigma$. Otherwise, we set $I = \tau \cap \sigma$.

Now define the subspace $A(i) = I \times \mathbb{R}$ of $C(i)$, noting that $A(i)$ separates each of $\Pi(i-1, i)$ and $\Pi(i, i+1)$ into two $\langle a_i \rangle$-invariant components.

For each $i = 1, \ldots, m$, define $Q(i, i+1)$ to be the component of $\Pi(i, i+1) \setminus (A(i) \cup A(i+1))$ such that both $a_i(Q(i, i+1))$ and $a_{i+1}(Q(i, i+1))$ lie again in $Q(i, i+1)$. This is a sector of the plane with angle $\theta(a_i, a_{i+1})$ and bounded by positive semi-axes for a_i and a_{i+1} respectively.

Again fix $i \in \{1, \ldots, m\}$. Now, for some $p, q \in I(i)$ and $r, s \in \mathbb{R}$, we may write

$$\begin{aligned} Q(i-1, i) \cap A(i) &= \{p\} \times \{t \in \mathbb{R} : t \geq r\}, \text{ and} \\ Q(i, i+1) \cap A(i) &= \{q\} \times \{t \in \mathbb{R} : t \geq s\}. \end{aligned}$$

Let $t_0 = \max(r, s)$ and define

$$\begin{aligned} &B(i) = [p, q] \times \{t \in \mathbb{R} : t \geq t_0\}, \\ &\mu(i) = \{p\} \times \{t \in \mathbb{R} : t \geq t_0\}, \ b_i = (p, t_0) \in \mu(i), \text{ and} \\ &\widehat{Q}(i, i+1) = B(i) \cup Q(i, i+1). \end{aligned}$$

These definitions are illustrated in Figure 1 below. Note that the ray $\mu(i)$ is common to both $\widehat{Q}(i-1, i)$ and $\widehat{Q}(i, i+1)$. Also, $\widehat{Q}(i, i+1)$ is contained in the convex set $C(i)$.We now choose, for each $i = 1, \ldots, m$, a point v_i in $\mu(i)$, different from the basepoint b_i of the ray $\mu(i)$. These points may be chosen sufficiently far along their corresponding rays that the unique geodesic segment in X from v_i to v_{i+1} lies wholly in $\widehat{Q}(i, i+1)$. It is also easy to arrange that the v_i are mutually distinct. We now apply Lemma 1.1 to the polygon $P = P(v_1, v_2, \ldots, v_m)$. See Figure 1.

For each $i = 1, \ldots, m$, write

$$\alpha_i = \angle([v_i, v_{i-1}], [v_i, b_i]) \quad \text{and} \quad \beta_i = \angle([v_i, v_{i+1}], [v_i, b_i]).$$

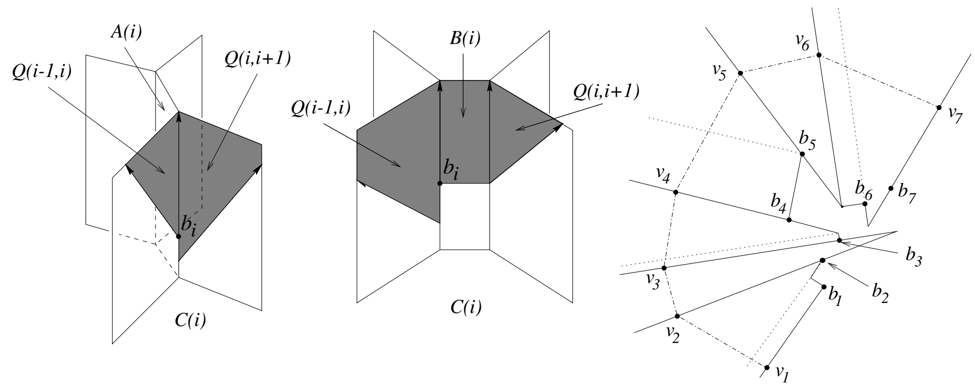

Figure 1: Examples of $C(i)$ and the construction of the polygon $P(v_1, .., v_m)$

From the Euclidean geometry of $\Pi(i, i+1)$ we have, for each i,

$$\beta_i + \alpha_{i+1} + \theta(i, i+1) = \pi\,. \tag{3}$$

Note that by choosing the points v_i equidistant from their corresponding base-points b_i and sufficiently far away, we may suppose that the angles α_i and β_i are all acute (in fact α_{i+1} and β_i approach one another as $d(v_i, b_i) = C \to \infty$, since for any fixed point b in the vicinity of either b_i or b_{i+1}, the triangle $\Delta(v_i, v_{i+1}, b)$ will tend towards an isosceles triangle).

As in Lemma 1.1, write $\phi_i = \angle([v_i, v_{i-1}], [v_i, v_{i+1}])$. By measuring in the convex subspace $C(i)$ of X, and supposing that α_i and β_i are acute, it is easily seen that

$$\phi_i = \alpha_i + \beta_i\,. \tag{4}$$

Combining (3) and (4), and applying Lemma 1.1, we now have

$$\sum_{i=1}^{m} \theta(i, i+1) \;=\; m\pi - \sum_{i=1}^{m} \phi_i \;\geq\; 2\pi\,,$$

completing the proof. □

Remark We have tried to make the proof of the above Lemma as elementary as possible. The argument may be simplified considerably by using the fact that, if X is a complete CAT(0) space, then its boundary ∂X (with the Tits, or angular, metric) is a complete CAT(1) space (see Theorem II.9.13 of [6]). Each flat sector $Q(i, i+1)$ determines a geodesic segment of length $\theta(i, i+1)$ between the points $a_i^{+\infty}$ and $a_{i+1}^{+\infty}$ in ∂X. The essence of the argument is to see that the union of these segments forms a locally geodesic circle in ∂X and hence must have total length at least 2π. (Local geodicity at $a_i^{+\infty}$ comes from the fact that, by the dimension constraint, the minset $Min(a_i)$ is isometric to $T\times\mathbb{R}$, where T is an $\mathbb{R}$-tree, and is convex in X). Note however that the more elementary proof makes no use at all of the hypothesis that X be complete.

2 Application to Artin kernels

A simplicial complex is called a *flag complex* if whenever $n+1$ vertices span a complete graph in the 1-skeleton they span an n-simplex in K. Let K be a finite connected 2-dimensional flag complex. Let $\mathcal{V}(K)$ denote the set of vertices of K and $\mathcal{E}(K)$ the set of ordered pairs $(u,v)\in\mathcal{V}(K)\times\mathcal{V}(K)$ such that u and v are distinct vertices joined by an edge in K. The elements $(u,v)\in\mathcal{E}(K)$ are thought of as directed edges of K.

The *right-angled Artin group* associated to the 1-skeleton of K is the group

$$A_K=\langle\mathcal{V}(K)\mid uv=vu \text{ if } (u,v)\in\mathcal{E}(K)\rangle\,.$$

The fact that K is a 2-dimensional flag complex ensures that mutually commuting subsets of the generating set $\mathcal{V}(K)$ contain at most three elements, and that whenever one sees three distinct mutually commuting generators they span a 2-simplex in K. In fact A_K acts freely on a 3-dimensional CAT(0) cube complex [1], so that

$$\mathrm{cd}(A_K)=\mathrm{gd}(A_K)=\dim_{ss}(A_K)=3\,.$$

Let $l:A_K\to\mathbb{Z}$ denote the 'length' homomorphism which takes every generator in $\mathcal{V}(K)$ to 1. We define the *Artin kernel*, or *Bestvina-Brady group*, to be the group

$$\Gamma_K=\ker(l:A_K\to\mathbb{Z})\,.$$

Since K is a connected complex, it follows (see [8]) that Γ_K is generated by the set

$$\mathcal{G}(K)=\{x_{(u,v)}=u^{-1}v:(u,v)\in\mathcal{E}(K)\}\,.$$

In fact, in [8], Dicks and Leary show that a complete presentation of Γ_K may be obtained by taking as relators all words $x^n_{(u_1,u_2)}x^n_{(u_2,u_3)}\cdots x^n_{(u_k,u_1)}$ where the sequence of edges $(u_1,u_2),(u_2,u_3),\ldots,(u_k,u_1)\in\mathcal{E}(K)$ forms a directed cycle in K, and where $n\in\mathbb{Z}$. In particular, we have the following relations in Γ_K:

$$\begin{aligned} x_{(v,u)}=x^{-1}_{(u,v)} &\quad \text{for all } (u,v)\in\mathcal{E}(K)\,,\\ x_{(u,v)}x_{(v,w)}=x_{(u,w)}=x_{(v,w)}x_{(u,v)} &\quad \text{whenever } u,v,w \text{ span a 2-simplex in } K\,. \end{aligned}$$

It is known, from [13], that if L is a full subcomplex of K, then the natural map $A_L \to A_K$ is injective. For $L < K$, we identify Γ_L and A_L with their images in Γ_K and A_K respectively, via this map. In particular, if σ is a 2-simplex in K, then Γ_σ is a free abelian group of rank 2 sitting inside $A_\sigma \cong \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}$.

Before stating the following Theorem we recall that a geodesic metric space is said to have *nonpositive curvature* if it is locally CAT(0), i.e: if every point has a convex open neighbourhood which is CAT(0) with the induced metric. If X is a complete geodesic metric space of nonpositive curvature then its universal cover $\widetilde{X}$ is a CAT(0) space (with the induced length metric). This is a generalisation of the Cartan-Hadamard Theorem (see [6] Chapter II.4 for a discussion). Any CAT(0) space is contractible ([6] II.1.5), so that in the above situation $\widetilde{X}$ is contractible and X is aspherical. We now have the following:

Theorem 2.1 *Let K be a finite connected 2-dimensional flag complex. If Γ_K acts properly by semi-simple isometries on a 2-dimensional CAT(0) space X, then K admits a complete piecewise Euclidean metric of nonpositive curvature, and in particular, K is aspherical.*

Proof We define a piecewise Euclidean metric on K as follows. Firstly define a Euclidean metric on each 2-simplex σ of K in such a way that the length of each edge (u,v) of σ is equal to the translation length of $x_{(u,v)}$ on X. (That this is always possible will become clear in the following paragraph). Now, as in [6] Chapter I.7, there is a complete geodesic metric on K defined by setting $d(x,y)$ to be the infimum of the lengths of all piecewise linear paths from x to y in K (where the length of such a path is just the sum of the lengths of its linear segments as measured inside the individual simplexes).

Suppose that u,v,w span a 2-simplex σ in K. The elements $x_{(u,v)}$, $x_{(u,w)}$ and $x_{(v,w)}$ generate a rank two abelian subgroup Γ_σ of Γ_K so that, by Proposition 1.2, their minsets intersect in a Γ_σ-invariant flat plane Π. Take any point $W \in \Pi$ and let Δ denote the triangle in Π with vertices W, $V = x_{(v,w)}(W)$ and $U = x_{(u,w)}(W)$. Then by the 'triangle relation' in $\Gamma(K)$ we have $x_{(u,v)}(V) = U$, and clearly the triangle $\Delta(U,V,W)$ is isometric to $\sigma(u,v,w)$. In particular, σ will have an angle at w precisely equal to $\theta(x_{(v,w)}, x_{(u,w)})$, and similarly for the other angles.

We recall ([6] Chapter II.5) that a 2-dimensional piecewise Euclidean metric complex K is nonpositively curved if and only if it satisfies the *link condition*, that every simple loop in $\mathrm{Lk}(v,K)$ has length at least 2π, for every vertex v in K. With the information given in the previous paragraph, it now follows immediately from Lemma 1.3 that the given piecewise Euclidean structure on K satisfies the link condition at every vertex, and hence is nonpositively curved. (Since K is a flag 2-complex, the link of any of its vertices is a graph in which all circuits have edge length at least 4. With regard to hypotheses (i) and (ii) of Lemma 1.3, it is not too hard to see that given distinct edges (u,v) and (u',v') the elements $x_{(u,v)}$ and $x_{(u',v')}$ always generate either $\mathbb{Z}\times\mathbb{Z}$ or a free subgroup $\mathbb{Z}\star\mathbb{Z}$ according to whether or not the two edges lie in a common simplex). □

Let K be a finite acyclic flag 2-complex with nontrivial fundamental group. It is known that $\mathrm{cd}(\Gamma_K)=2$ (Bestvina and Brady [1]), and that $\mathrm{gd}(\Gamma_K)\leq \mathrm{gd}(A_K)=3$, but it is unknown whether or not the geometric dimension of Γ_K agrees with the cohomological dimension. (Note, however, that if K is acyclic and has trivial fundamental group then it is contractible and the Bestvina-Brady argument shows that $\mathrm{cd}(\Gamma_K)=\mathrm{gd}(\Gamma_K)=2$.) There are examples of acyclic 2-complexes which are not aspherical. This happens, for instance, if the acyclic complex has nontrivial torsion in its fundamental group. One such example is the spine of the Poincaré homology sphere, namely the quotient of the dodecahedral tiling of S^2 in which opposite faces are identified with a $\frac{\pi}{5}$ twist. The fundamental group of this complex is finite. Theorem 2.1 now gives:

Corollary 2.2 *Let K be a finite acyclic flag 2-complex whose fundamental group has nontrivial torsion. Then $\dim_{ss}(\Gamma_K)=3$ while $cd(\Gamma_K)=2$. In these cases $gd(\Gamma_K)$ is not known.*

If K is a contractible complex then the Morse theory argument used in [1] shows that $\mathrm{gd}(\Gamma_K)=2$. In this case we have the following:

Corollary 2.3 *Let K be a finite contractible flag 2-complex which has no free edges. Then $\dim_{ss}(\Gamma_K)=3$ while $gd(\Gamma_K)=2$. In these cases Γ_K is finitely presented (see [1]; an explicit presentation is given in [8]) and FP_∞ (c.f. [1]).*

The hypotheses on K ensure that it cannot admit a piecewise Euclidean metric of nonpositive curvature. The "no free edges" condition implies that any such metric would have the geodesic extension property (see Proposition 5.10 of [6]), while contractibility implies that the metric would be (globally) CAT(0). Together these condition force the metric to be unbounded, contradicting K finite. Zeeman's dunce hat D (a 2-simplex $\Delta(a,b,c)$ with oriented edges (a,b), (a,c) and (b,c) all identified) is contractible with no free edges. Thus, for example, any flag triangulation of D will satisfy the hypotheses of Corollary 2.3.

3 Refinement of Theorem 2.1

As in the previous section, let K be a finite connected 2-dimensional flag complex. Suppose also that K is given a piecewise Euclidean metric. We associate to K a metric graph $L(K)$ as follows. The vertex set of $L(K)$ is defined to be the set $\mathcal{E}(K)$ of oriented edges of K. There is an edge of $L(K)$ between $(u,u'),(v,v') \in \mathcal{E}(K)$ if (u,u') and (v,v') are distinct edges of a common 2-simplex σ in K and either $u=v$ or $u'=v'$. The length of such an edge of $L(K)$ is defined as the angle in σ between the two sides in question. The situation of a single simplex is illustrated in Figure 2. Each simplex contributes to $L(K)$ a circle of length precisely 2π. These are identified pairwise along "great 0-circles" (pairs of antipodal points) according to the edge identifications between adjacent 2-simplexes in K. Note that $L(K)$ also contains, as a locally isometrically embedded subgraph, the link of each vertex in K. Thus Theorem 2.1 is a consequence of the following.

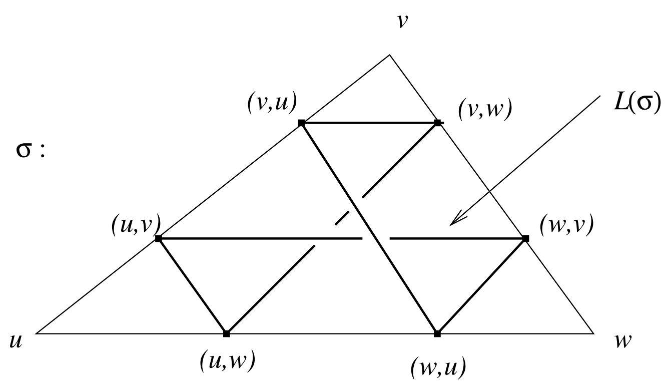

Figure 2: Defining $L(K)$ – the contribution from a 2-simplex σ

Theorem 3.1 *Let K be a finite connected 2-dimensional flag complex. If Γ_K acts properly by semi-simple isometries on a 2-dimensional (complete) CAT(0) space X, then K admits a piecewise Euclidean metric such that $L(K)$ is a CAT(1) metric graph (or equivalently $L(K)$ contains no simple closed circuit of length strictly less than 2π).*

Proof We will use the fact, as pointed out in the remark following Lemma 1.3, that if X is a complete CAT(0) metric space then its Tits boundary ∂X is CAT(1). In particular, any locally geodesic circle in ∂X has length at least 2π.

The metric on K is defined exactly as in the proof of Theorem 2.1. For any 2-simplex σ in K, let $\Pi(\sigma)$ denote the flat plane in X associated with Γ_σ. Then $L(\sigma)$ is isometric to $\partial\Pi(\sigma)$ by an isometry which associates a vertex (u,v) of $L(\sigma)$ with the boundary point $x^{+\infty}_{(u,v)}$. Since $\Pi(\sigma)$ is a convex subspace of X its boundary embeds canonically as a locally convex subspace of ∂X. Thus we have a locally isometric map $L(\sigma)\to\partial X$ for each 2-simplex σ. (In fact, this map is isometric since $L(\sigma)$ has diameter π). Combining these maps we have a map $\psi : L(K)\to\partial X$ which we claim is locally isometric. To check this, it suffices to look at a neighbourhood of each vertex (u,v) of $L(K)$. Such a neighbourhood may be chosen to lie inside the subgraph $L(N(u,v))$, where $N(u,v)$ denotes the union of all 2-simplices containing the edge (u,v). Now the $\Pi(\sigma)$ for $\sigma\in N(u,v)$ are mutually distinct planes all of which lie in $Min(x_{(u,v)})$. Since this minset is a convex subspace and, by the dimension constraint on X, has the structure of $T\times\mathbb{R}$ where T is an $\mathbb{R}$-tree, it follows that $L(N(u,v))$ now embeds locally isometrically in ∂X. This embedding is with respect to the intrinsic metric in the subgraph $L(N(u,v))$, not the metric induced from $L(K)$ (these metrics do not agree if the subgraph is not convex). However, locally these metrics do agree, showing that ψ is a local isometry.

Since the local isometry ψ maps simple closed circuits in $L(K)$ to locally geodesic circles of the same length in ∂X it now follows that all simple closed circuits in $L(K)$ have length at least 2π and hence that $L(K)$ is CAT(1).

[Note that the hypothesis that X be complete is actually artificial, since the same result may be obtained with a little more effort by reworking the original proof of Theorem 2.1. It is hypothesis (i) of Lemma 1.3 which needs refining so as to allow the case where $a_i = a_{i-1}a_{i+1}$, meaning that the three consecutive elements lie in the same $\mathbb{Z}\times\mathbb{Z}$ subgroup but in such a way that one will always have $\theta(a_{i-1},a_{i+1}) = \theta(a_{i-1},a_i)+\theta(a_i,a_{i+1})$.] □

Suppose now that K is a finite *simply-connected* piecewise Euclidean flag complex. We now define a piecewise Euclidean complex $T(K)$ as follows. For each 2-simplex σ in K let $T(\sigma)$ denote the the union of two isometric copies of σ glued along their edges as shown in Figure 3 so as to form a flat torus. Oriented edges of $T(\sigma)$ are labelled by elements of the generating set $\mathcal{G}(K)$ of Γ_K as indicated in the figure. We now construct $T(K)$ by taking the union of all $T(\sigma)$ for $\sigma\in K$ and identifying edges (by isometries) according to the labelling. Combinatorially, $T(K)$ is none other than the presentation complex associated to the finite presentation for Γ_K given in [8] (Corollary 3) for the case K simply-connected. Thus Γ_K acts freely, cocompactly and isometrically on the universal cover $\widetilde{T(K)}$. Moreover, observe that the link of every vertex in $\widetilde{T(K)}$, or rather the link of the unique vertex in $T(K)$, is isometric to the metric graph $L(K)$. If $L(K)$ is CAT(1) then $T(K)$ is non-positively curved and $\widetilde{T(K)}$ is CAT(0). We therefore have the following Corollary to Theorem 3.1.

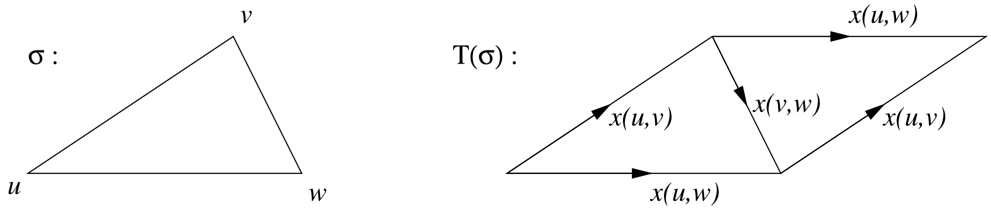

Figure 3: Building the complex $T(K)$ – the torus $T(\sigma)$ associated to a 2-simplex σ

Corollary 3.2 *Let K be a finite simply-connected 2-dimensional flag complex. Then the group Γ_K acts properly by semi-simple isometries (or even freely co-compactly) on a 2-dimensional (complete) CAT(0) space X if and only if K admits a piecewise Euclidean metric such that $L(K)$ is a CAT(1) metric graph.*

We finish with an example which shows that Theorem 3.1 really is stronger than Theorem 2.1. The 2-dimensional flag complex K_0 shown in Figure 4 clearly admits a CAT(0) piecewise Euclidean metric, however we now show that there is no piecewise Euclidean metric on K_0 such that $L_0 = L(K_0)$ is a CAT(1) graph. Suppose that K_0 did admit such a metric. In particular, K_0 would have to be locally CAT(0), and hence CAT(0), with respect to this metric. We now consider the lengths of the following circuits in L_0 (we describe a circuit by giving the sequence of vertices of L_0 – or rather, oriented edges of K_0 – through which it passes):

$$c_1 = (b, h, r, e, s, k, d, v, a, u, b)$$
$$c_2 = (b, x^{-1}, d, k, x, h, b)$$
$$c_3 = (b, x^{-1}, d, v, a, u, b)$$
$$c_4 = (h, r, e, s, k, x, h)$$

The circuit c_1 has length exactly 2π, since on the one hand it is a circuit in the CAT(1) link L_0, so at least 2π, while on the other hand its length is exactly the angle sum of the quadrilateral in K_0 with sides u, v, s, r, so at most 2π. The circuit c_2 has length 2π either by similar reasoning or by simply noting that its length is the total angle sum of two Euclidean triangles. But we also have

$$\ell(c_3) + \ell(c_4) = \ell(c_1) + \ell(c_2) - 2d_{L_0}(b,h) - 2d_{L_0}(d,k) < 4\pi\,.$$

Therefore at least one of the circuits c_3 or c_4 is strictly shorter than 2π, contradicting L_0 CAT(1).

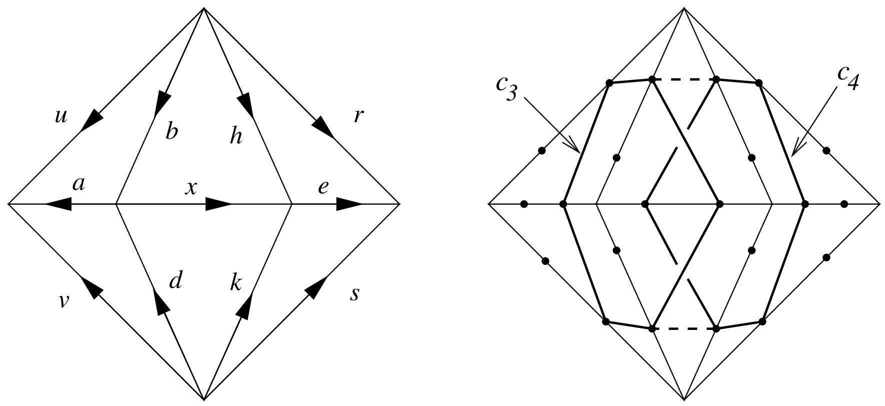

Figure 4: The flag complex K_0

By Theorem 3.1, we have that $\dim_{ss}(\Gamma_{K_0}) = 3$ while $\mathrm{gd}(\Gamma_{K_0}) = 2$.

Let L denote the subcomplex of K_0 which is the graph consisting of the three edges labelled a, x, e in Figure 4. Then K_0 is just the suspension of L. If $C(L)$ denotes the simplicial cone over L, then $\Gamma_{C(L)}$ is canonically isomorphic to A_L. Thus Γ_{K_0} is isomorphic to the double of A_L over the normal subgroup $\Gamma_L < A_L$:

$$\Gamma_{K_0} \cong A_L \underset{\Gamma_L}{\star} A_L\,.$$

Noel Brady [2] has a proof that Γ_{K_0} has a Dehn function which is at least cubic. It follows (see [6], for instance) that Γ_{K_0} cannot act properly cocompactly on any CAT(0) space, i.e: it is not a CAT(0) group. Without reproducing Brady's argument on the Dehn function we can still deduce that this example is not a CAT(0) group as follows. By the Exercises on page 499 of [6], the group Γ_{K_0} may be re-expressed as the trivial HNN-extension of A_L over Γ_L, and as such is a CAT(0) group only if Γ_L is quasi-isometrically embedded in A_L. However, Γ_L is sufficiently distorted in A_L for this not to be the case. Write $A_L = \langle u_1, u_2, u_3, u_4 | [u_1, u_2] = [u_2, u_3] = [u_3, u_4] = 1\rangle$, and observe that $\Gamma_L \cong F_3$, freely generated by the elements $a = u_2^{-1}u_1$, $x = u_2^{-1}u_3$ and $e = u_3^{-1}u_4$. For each $N \in \mathbb{N}$, the freely reduced word $w_N := (ax^N ex^{-N})^N$, of length $2(N^2+N)$, represents an element of the free group Γ_L which is equal in A_L to

$$\begin{aligned}((u_2^{-1}u_1)u_2^{-N}u_3^{N}(u_3^{-1}u_4)u_3^{-N}u_2^{N})^N &= (u_2^{-N}(u_2^{-1}u_1.u_3^{-1}u_4)u_2^{N})^N \\ &= u_2^{-N}(u_2^{-1}u_1.u_3^{-1}u_4)^N u_2^{N}\,.\end{aligned}$$

The latter word in the generators of A_L has length $6N$, implying that Γ_L is at least quadratically distorted in A_L, and so is not quasi-isometrically embedded.

References

[1] **M Bestvina**, **N Brady**, *Morse theory and finiteness properties of groups*, Invent. Math. 129 (1997) 445-470

[2] **N Brady**, Private communication (2002)

[3] **N Brady**, **J Crisp**, *Two-dimensional Artin groups with CAT(0) dimension 3*, Geometriae Dedicata (2002), to appear

[4] **M R Bridson**, *On the existence of flat planes in spaces of non-positive curvature*, Proc. A.M.S. 123 (1995) 223–235

[5] **M R Bridson**, *Length functions, curvature and the dimension of discrete groups*, Math. Res. Lett. 8 No.4 (2001) 557–567

[6] **M R Bridson**, **A Haefliger**, *Metric Spaces of Non-Positive Curvature*, Springer-Verlag (1999)

[7] **K S Brown**, *Cohomology of Groups*, Springer-Verlag (1982)

[8] **W Dicks**, **I J Leary**, *Presentations for subgroups of Artin groups*, Proc. Amer. Math. Soc. 127 No.2 (1999) 343–348

[9] **S Eilenberg**, **T Ganea**, *On the Lusternik-Schnirelmann category of abstract groups*, Ann. of Math. 65 (1957) 517–518

[10] **W Hurewicz**, **H Wallmann**, *Dimension Theory* Princeton Univ. Press (1941)

[11] **J R Stallings**, *On torsion-free groups with infinitely many ends*, Ann. of Math. 88 (1968) 312–334

[12] **R G Swan**, *Groups of cohomological dimension one*, J. Algebra 12 (1969) 585–601

[13] **H van der Lek**, *The Homotopy Type of Complex Hyperplane Complements*, Ph.D. thesis, University of Nijmegen (1983)

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