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On Real-oriented Johnson-Wilson cohomology

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Abstract Answering a question of W. S. Wilson, I introduce a $\mathbb{Z}/2$-equivariant Atiyah-Real analogue of Johnson-Wilson cohomology theory $BP\langle n\rangle$, whose coefficient ring is the $\leq n$-chromatic part of Landweber's Real cobordism ring.

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1 Introduction

Recall Johnson-Wilson's spectrum $BP\langle n\rangle$, constructed in [11]. The complex cobordism spectrum MU, localized at 2, splits as a wedge-sum of suspensions of the Brown-Peterson spectrum BP [9]. We have $BP_* = \mathbb{Z}_{(2)}[v_1, v_2, \ldots]$, with $dim(v_i) = 2(2^i - 1)$. For each n, the Johnson-Wilson spectrum $BP\langle n\rangle$ comes with a map $BP \to BP\langle n\rangle$, and one has that

$$BP\langle n\rangle_* = \mathbb{Z}_{(2)}[v_1, v_2, \ldots, v_n]. \tag{1.1}$$

In particular, $BP\langle n\rangle_*$ is a quotient ring of BP_*. The fact that such $BP\langle n\rangle$ exists is, of course, today no longer surprising. In fact, one can construct BP almost formally by "killing a suitable regular ideal I_n in $MU_{(2)}$" (see [5]).

In connection with certain questions on Lie groups (which will not be discussed here), Steve Wilson recently asked if the spectrum $BP\langle n\rangle$ has a Landweber-Real analogue, i.e., if there exists a spectrum $BP\mathbb{R}\langle n\rangle$ whose coefficient ring is the quotient of Landweber's cobordism ring $M\mathbb{R}_\star$ ([2], [3], [8], [9]) by all elements "not related to $v_0, \ldots, v_n$". (Here, $M\mathbb{R}_\star$ denotes the $RO(\mathbb{Z}/2)$-graded coefficient ring, as opposed to the integer-graded coefficient ring.) This can be given an exact meaning, which I shall explain in the next section. First, however, I shall describe, in general terms, the main result of this paper, and its contribution to the present state of the subject.

In this paper, I completely answer Steve Wilson's question in the affirmative. The construction of the spectrum $BP\mathbb{R}\langle n\rangle$ is straightforward: analogously to MU-theory, general tools are now avaliable in $M\mathbb{R}$-theory. In particular, there is an embedding $MU_* \to M\mathbb{R}_\star$ (in an appropriate sense), and it is possible to "quotient out" $M\mathbb{R}$ by an ideal of MU_* using the tools of [5]. This method is described in detail in [8]. The construction of my spectrum $BP\mathbb{R}\langle n\rangle$ is that one simply "kills" the ideal I_n mentioned above, in the ring $M\mathbb{R}_\star$.

The contribution of this paper is in calculating the coefficient ring of $BP\mathbb{R}\langle n\rangle$. This is a non-trivial matter, since I_n is certainly not a regular ideal in $M\mathbb{R}_\star$. In fact, it is highly surprising that the spectrum $BP\mathbb{R}\langle n\rangle$ constructed in this "naive" way gives the coefficients that S. Wilson asked for.

To explain the issues involved, it should be mentioned at this point that we are dealing here with $RO(\mathbb{Z}/2)$-graded $\mathbb{Z}/2$-equivariant spectra [1], [10] ($M\mathbb{R}$ is $\mathbb{Z}/2$-equivariant), and that, therefore, questions of a "completion theorem" ([6]) arise. Indeed, Steve Wilson originally asked if a "homotopy fixed point spectrum" of $BP\langle n\rangle$ is the answer to his question. In this paper, we shall see that that is, in fact, false. The homotopy fixed point spectrum of $BP\langle n\rangle$ will be relevant to our calculations, but turns out not to have the right coefficients (they contain some spurious elements); the point is that the spectrum $BP\mathbb{R}\langle n\rangle$ constructed by killing the idea I_n in $M\mathbb{R}$ does not satisfy a "completion theorem" in the sense of [6].

This also makes our calculation new technically. In [8], where coefficients of numerous spectra obtained from $M\mathbb{R}$ by killing ideas are calculated, completion theorems for the relevant spectra always hold and are the bases of all the calculations. The present paper contains the first case where a calculation of coefficient of a "derived spectrum of $M\mathbb{R}$" is given where the spectrum does not satisfy a completion theorem (with the exception of $\mathbb{Z}/2$-equivariant constant Mackey functor spectra $H\mathbb{Z}/2$ and $H\mathbb{Z}$, which, in fact, could be called $BP\mathbb{R}\langle -1\rangle$ and $BP\mathbb{R}\langle 0\rangle$ from the point of view of this paper). I get my calculations by computing all the other terms of the "Tate diagram" of Greenlees-May [6]. It is somewhat amazing that the coefficients of $BP\mathbb{R}\langle n\rangle$ defined and calculated in this way are a quotient of $M\mathbb{R}_\star$, while the coefficients of the other terms of the Tate diagram, notably the Borel cohomology spectrum, are not.

In Section 2, I give a short review of Real cobordism theory and the main tools used in the paper, as well as the result of the calculations for $BP\mathbb{R}\langle n\rangle_\star$. In particular, for $n = 1$, we get that the fixed points spectrum $BP\mathbb{R}\langle 1\rangle^{\mathbb{Z}/2}$ is kO, the connective cover of orthogonal K-theory KO. However, this is not true in other twists, i.e., if we first suspend $BP\mathbb{R}\langle 1\rangle$ by copies of the sign representation of $\mathbb{Z}/2$ and then take its fixed points. In Section 3, I compute the coefficients of the Tate and Borel cohomology spectra of $BP\mathbb{R}\langle n\rangle$, which appear in the Tate diagram for $BP\mathbb{R}\langle n\rangle$. Finally, in Section 4, I calculate the geometric spectrum of $BP\mathbb{R}\langle n\rangle$ to fill in the Tate diagram, and use it to get the coefficients of $BP\mathbb{R}\langle n\rangle$ itself. It is interesting to note that the coefficients of the Tate and geometric spectra of $BP\mathbb{R}\langle n\rangle$ are small: in this sense, one might say that $BP\mathbb{R}\langle n\rangle$ "nearly has descent".

2 Review of $M\mathbb{R}$-theory and statement of the main result

I shall now describe some basic aspects of Landweber cobordism theory ([9], [2], [3], [8]). First, the infinite loop spaces making up $M\mathbb{R}$ are the same as the infinite loop spaces of the complex cobordism spectrum MU, but there is a $\mathbb{Z}/2$-action, and the dimensions are indexed differently. Namely, on the level of prespectra, MU is obtained from the sequence of Thom spaces of n-dimensional canonical complex bundles γ_n on $BU(n)$. Denoting the Thom space of γ_n by $BU(n)^{\gamma_n}$, we get structure maps

$$\Sigma^2 BU(n)^{\gamma_n} \to BU(n+1)^{\gamma_{n+1}}.$$

In the case of $M\mathbb{R}$, we use the same Thom spaces $BU(n)^{\gamma_n}$, but with the $\mathbb{Z}/2$-action by complex conjugation. The space $BU(n)^{\gamma_n}$ is placed in dimension $n(1+\alpha)$, where 1 and α denote the trivial and the sign representations of $\mathbb{Z}/2$, respectively. This is because γ_n is a Real bundle in the sense of Atiyah [4]. Hence, the structure maps are

$$\Sigma^{1+\alpha} BU(n)^{\gamma_n} \to BU(n+1)^{\gamma_{n+1}}.$$

(Note that the $\mathbb{Z}/2$-representation $1+\alpha$ is just $\mathbb{C}$ with $\mathbb{Z}/2$-action by complex conjugation.) Spectrification then makes $M\mathbb{R}$ a $\mathbb{Z}/2$-equivariant spectrum, indexed on $RO(\mathbb{Z}/2)$, i. e. in dimensions $k+l\alpha$, for all $k,l \in \mathbb{Z}$. In this paper, we will denote the $RO(\mathbb{Z}/2)$-grading by the subscript $\star$, to distinguish it from $\mathbb{Z}$-grading, which will be denoted by the subscript $*$ as usual. We work locally at the prime 2 in this paper. The Real Brown-Peterson spectrum is obtained from $M\mathbb{R}$ via the Real version of the Quillen idempotent, analogous to the way the Brown-Peterson spectrum BP is obtained from MU. In [8], we calculated the $RO(\mathbb{Z}/2)$-graded coefficient ring of $BP\mathbb{R}$. Namely, we have that

$$BP\mathbb{R}_\star = \mathbb{Z}_{(2)}[v_n\sigma^{l2^{n+1}}, a]/\sim . \tag{2.1}$$

The relations are

$$v_0 = 2 \tag{2.2}$$

$$(v_n\sigma^{l2^{n+1}})a^{2^{n+1}-1} = 0 \tag{2.3}$$

$$(v_n\sigma^{l2^{n+1}})(v_m\sigma^{k2^{m+1}}) = v_nv_m\sigma^{l2^{n+1}+k2^{m+1}} \text{ for } m \leq n. \tag{2.4}$$

Here, $n \geq 0$, and l ranges over all integers. The dimensions of elements are that v_n has dimension $(2^n-1)(1+\alpha)$, a has dimension $-\alpha$, and the operator σ has dimension $-1+\alpha$.

As described in [8], for each $n \geq 0$, the Real Johnson-Wilson spectrum $BP\mathbb{R}\langle n\rangle$ is obtained by killing the sequence of elements $v_{n+1}, v_{n+2}, \ldots$ in $BP\mathbb{R}$, in the manner of [5]. This is again a $\mathbb{Z}/2$-equivariant spectrum indexed on $RO(\mathbb{Z}/2)$. In particular, the infinite loop space of $BP\mathbb{R}\langle n\rangle$ in dimension $k+l\alpha$ is the same as the infinite loop space of $BP\langle n\rangle$ in dimension $k+l$, but with an additional action by $\mathbb{Z}/2$, which depends on k and l, not just their sum.

For a $\mathbb{Z}/2$-equivariant spectrum E, there are several kinds of "fixed points spectra" associated with E. What we usually consider as the fixed point spectrum is the Lewis-May fixed point spectrum $E^{\mathbb{Z}/2}$, obtained by first forgetting the $RO(\mathbb{Z}/2)$-graded spectrum to one graded on $\mathbb{Z}$, i.e., considering only the spaces in dimensions $k+0\alpha$, and then taking the fixed points spacewise [10]. This gives a nonequivariant spectrum. Similarly, for each $l \in \mathbb{Z}$, one also has $(\Sigma^{-l\alpha}E)^{\mathbb{Z}/2}$, called the fixed point spectrum twisted by l. This is obtained by first taking only the $\mathbb{Z}$-graded spectrum consisting of the spaces in dimensions $k+l\alpha$, and then taking fixed points spacewise. There are also the Borel homology and cohomology fixed point spectra of E. Recall that $E\mathbb{Z}/2$ is the universal contractible free $\mathbb{Z}/2$-space, which may be thought of as $S(\infty\alpha) = \mathrm{colim}_k S(k\alpha)$, where $S(k\alpha)$ is the unit sphere in the representation $k\alpha$. The Borel homology spectrum $E\mathbb{Z}/2_+ \wedge E$, and the Borel cohomology spectrum is $F(E\mathbb{Z}/2_+, E)$. The Borel homology and cohomology fixed points of E are obtained by taking the fixed points (in the above sense, with possible twist by l) of the Borel homology and cohomology spectra, respectively. In particular, the Borel cohomology fixed points

$$F(E\mathbb{Z}/2_+, E)^{\mathbb{Z}/2}$$

is $E^{h\mathbb{Z}/2}$, the homotopy fixed points spectrum of E. For the Borel homology, note that since $E\mathbb{Z}/2_+ \wedge E$ is a free spectrum indexed on $RO(\mathbb{Z}/2)$, its fixed points can be computed using the Adams isomorphism, which gives that

$$(E\mathbb{Z}/2_+ \wedge E)^{\mathbb{Z}/2} \simeq E\mathbb{Z}/2_+ \wedge_{\mathbb{Z}/2} E.$$

We also have the geometric fixed points spectrum of E. This is is a $\mathbb{Z}$-graded nonequivariant spectrum, whose infinite loopspace is

$$\mathrm{colim}_V \Omega^{V^{\mathbb{Z}/2}} E_V^{\mathbb{Z}/2}$$

where the colimit ranges over all finite-dimensional representations V of $\mathbb{Z}/2$, and E_V denotes the V-th space of E. The geometric fixed points can be calculated by first taking $S^{\infty\alpha} \wedge E$, then taking the fixed points of this spectrum in the sense above. Here, $S^{\infty\alpha}$ is the one-point compactification of the infinite-dimensional representaiton $\infty\alpha$.

The various spectra associated with a $\mathbb{Z}/2$-equivariant spectrum E are organized by the Tate diagram. We have the cofiber sequence

$$E\mathbb{Z}/2_+ \to S^0 \to \widetilde{E\mathbb{Z}/2}$$

where the cofiber is the unreduced suspension of $E\mathbb{Z}/2$. Hence, we have that $\widetilde{E\mathbb{Z}/2}$ is just $S^{\infty\alpha}$. Smashing with E and mapping into $F(E\mathbb{Z}/2_+, E)$ gives the Tate diagram

$$\begin{tikzcd}
E\mathbb{Z}/2_+ \wedge E \arrow[r] \arrow[d, "\simeq"'] & E \arrow[r] \arrow[d] & \widetilde{E\mathbb{Z}/2} \wedge E \arrow[d] \\
E\mathbb{Z}/2_+ \wedge F(E\mathbb{Z}/2_+, E) \arrow[r] & F(E\mathbb{Z}/2_+, E) \arrow[r] & \widetilde{E\mathbb{Z}/2} \wedge F(E\mathbb{Z}/2_+, E).
\end{tikzcd}$$

The rightmost term on the bottom row, $\widetilde{E\mathbb{Z}/2} \wedge F(E\mathbb{Z}/2_+, E)$, is the Tate cohomology of E, which we also denote by $t(E)$.

Taking $E = BP\mathbb{R}\langle n\rangle$, we get the Tate diagram for $BP\mathbb{R}\langle n\rangle$

$$\begin{tikzcd}
E\mathbb{Z}/2_+ \wedge BP\mathbb{R}\langle n\rangle \arrow[r] \arrow[d, "\simeq"'] & BP\mathbb{R}\langle n\rangle \arrow[r] \arrow[d] & \widetilde{E\mathbb{Z}/2} \wedge BP\mathbb{R}\langle n\rangle \arrow[d] \\
E\mathbb{Z}/2_+ \wedge F(E\mathbb{Z}/2_+, BP\mathbb{R}\langle n\rangle) \arrow[r] & F(E\mathbb{Z}/2_+, BP\mathbb{R}\langle n\rangle) \arrow[r] & t(BP\mathbb{R}\langle n\rangle).
\end{tikzcd}$$

Here, $t(BP\mathbb{R}\langle n\rangle) = \widetilde{E\mathbb{Z}/2} \wedge F(E\mathbb{Z}/2_+, BP\mathbb{R}\langle n\rangle))$. One sometimes also refers to the fixed points spectra obtained from the spectra in the Tate diagram by the same names as the corresponding equivariant spectra. Note that we can also take twisted fixed points, by first desuspending by $S^{l\alpha}$, and then taking fixed points. However, note that the rightmost column, i.e., the geometric and the Tate spectra, are α-periodic, and hence do not depend on the twist l. The middle column of the Tate diagram is an equivalence if and only if the rightmost column is an equivalence. We call an $RO(\mathbb{Z}/2)$-graded equivariant spectrum *complete* if this condition holds. More generally, a "completion theorem" holds if the middle vertical arrow of the Tate diagram is a completion in some suitable sense. For more information, see [6]. Unlike $BP\mathbb{R}$, the spectrum $BP\mathbb{R}\langle n\rangle$ is not complete, i.e., it is not equivalent to its Borel cohomology spectrum.

All these spectra help in computing the coefficients of $BP\mathbb{R}\langle n\rangle$. There are spectral sequences that compute the coefficients of the Borel homology, Borel cohomology, and Tate cohomology terms, while $\widetilde{E\mathbb{Z}/2}\wedge BP\mathbb{R}\langle n\rangle$ can be identified using geometric methods.

Theorem 2.1 (1) *The coefficients of the Tate spectrum of* $BP\mathbb{R}\langle n\rangle$ *are*

$$t(BP\mathbb{R}\langle n\rangle)_\star = \mathbb{Z}/2[\sigma^{2^{n+1}},\sigma^{-2^{n+1}},a,a^{-1}]$$

where σ *has dimension* $-1+\alpha$, *and* a *has dimension* $-\alpha$.

(2) *The coefficients of the Borel cohomology spectrum of* $BP\mathbb{R}\langle n\rangle$ *are*

$$F(E\mathbb{Z}/2_+,BP\mathbb{R}\langle n\rangle)_\star = (\mathbb{Z}_{(2)}[v_k\sigma^{l2^{k+1}},a]/\sim)\oplus\mathbb{Z}/2[\sigma^{2^{n+1}},\sigma^{-2^{n+1}},a].$$

Here, $0\le k\le n$, *and* l *ranges over all integers. The relations are*

$$\begin{aligned} v_0 &= 2\\ v_k a^{2^{k+1}-1} &= 0\\ (v_n\sigma^{l2^{n+1}})(v_m\sigma^{k2^{m+1}}) &= v_nv_m\sigma^{l2^{n+1}+k2^{m+1}} \text{ for } m\le n. \end{aligned}$$

For $BP\mathbb{R}\langle n\rangle$ itself, we have the following theorem.

Theorem 2.2 *The coefficient ring of* $BP\mathbb{R}\langle n\rangle$ *is*

$$BP\mathbb{R}\langle n\rangle_\star = (\mathbb{Z}_{(2)}[v_k\sigma^{l2^{k+1}},a]/\sim)\oplus\mathbb{Z}/2[\sigma^{-2^{n+1}},a].$$

with the same relations (2.2), (2.3) and (2.4) as in $BP\mathbb{R}_\star$.

For readers who prefer not to use the $RO(\mathbb{Z}/2)$-grading, the (untwisted or twisted) coefficients of $BP\mathbb{R}$ and $BP\mathbb{R}\langle n\rangle$ can be described using nonequivariant Milnor words with dimensional shifts. For an element x of dimension $k+l\alpha$, we say that the twist of x is l. Recalling the calculation of $BP\mathbb{R}_\star$, for a fixed twist l, we can describe the coefficients of $(\Sigma^{-l\alpha}BP\mathbb{R})^{\mathbb{Z}/2}$, the twist l fixed points of $BP\mathbb{R}$, in terms of just the Milnor generators v_n's, but with shifted dimensions. Namely, fix $l \in \mathbb{Z}$. For a sequence of nonnegative integers $R = (r_0, r_1, \ldots)$ of which all but finitely many are 0, we write the monomial $v_R = \prod_{i \geq 0} v_i^{r_i}$. Let $n = min(R)$ be the smallest number such that $i_n \neq 0$, and let $|v_R|$ denote the dimension of v_R in BP_*. The additive generators of $BP\mathbb{R}_\star$ as a $\mathbb{Z}_{(2)}$-module are the monomials v_R, with the following possibilities. If $|v_R| \leq l$, the v_R has dimension

$$|v_R| - l - k$$

where $0 \leq k \leq 2^{n+1} - 1$ is congruent to $|v_R|$ modulo 2^{n+1}. This is 0 if $k = 2^{n+1} - 1$, it generates a copy of $\mathbb{Z}_{(2)}$ if $k = 0$, and a copy of $\mathbb{Z}/2$ otherwise. If $l > |v_R|$, then v_R is in dimension

$$|v_R| - l - k'$$

where $0 \leq k' \leq 2^{n+1} - 1$ is congruent to $|v_R| - l$ modulo 2^{n+1}. Again, this is 0 if $k' = 2^{n+1} - 1$, it generates a copy of $\mathbb{Z}_{(2)}$ if $k' = 0$, and it generates a copy of $\mathbb{Z}/2$ else.

For each l, the elements of the homotopy groups of the twist l fixed points of $BP\mathbb{R}\langle n\rangle$ are the relevant ones from $BP\mathbb{R}$, and some extra elements.

Corollary 2.3 *Let $l \in \mathbb{Z}$. If $l \geq 0$, then the elements of $BP\mathbb{R}\langle n\rangle_\star$ in twist l are the same as the twist l elements of $BP\mathbb{R}_\star$ that do not contain v_s for any $s > n$. If $l < 0$, then the elements of $BP\mathbb{R}\langle n\rangle_\star$ in twist l are the twist l elements of $BP\mathbb{R}_\star$ not containing v_s for any $s > n$, as well as an extra copy of $\mathbb{Z}/2$ in dimension $k2^{n+1}$ for each k such that $0 > k2^{n+1} \geq l$. (In the notation of Theorem 2, this element corresponds to the generator by $\sigma^{-k2^{n+1}} a^{k2^{n+1}-l}$.)*

3 Tate and Borel cohomology calculations

The goal of this section is to prove Theorem 2.1. To compute the Tate cohomology of $BP\mathbb{R}\langle n\rangle$, we consider the Tate spectral sequence

$$E_2 = \widehat{H}^*(\mathbb{Z}/2, BP\langle n\rangle_*[\sigma, \sigma^{-1}]) \Rightarrow \widehat{BP\mathbb{R}\langle n\rangle}_\star. \tag{3.1}$$

We can compare this to the Tate spectral sequence for $BP\mathbb{R}$, which is that

$$E_2 = \widehat{H}^*(\mathbb{Z}/2, BP_*[\sigma, \sigma^{-1}]) \Rightarrow \widehat{BP\mathbb{R}}_\star. \tag{3.2}$$

(see [8, 7]). The E_1-term of (3.2) is $BP_\star[\sigma, \sigma^{-1}, a, a^{-1}]$, where $BP_\star$ is the same as BP_*, with the exception that the dimension of v_n is $(2^n - 1)(1 + \alpha)$ instead of $2(2^n - 1)$. Note that with a different choice of generators (multiplying v_n

by σ^{2^n-1}), this is in fact equal to $BP_*[\sigma,\sigma^{-1},a,a^{-1}]$. In (3.2), $\mathbb{Z}/2$ acts by $(-1)^{(|v_R|_{\mathbb{C}})/2+l}$ on the monomial $v_R\sigma^l$. Here, $|v_R|_{\mathbb{C}}$ denotes the dimension of a monomial v_R in BP_*. In [8], it was shown that (3.2) has the differentials

$$d_{2^{k+1}-1}(\sigma^{-2^k}) = v_k a^{2^{k+1}-1} \tag{3.3}$$

for $k \geq 1$. These differentials wipe out all elements except $\mathbb{Z}/2[a,a^{-1}]$. Namely, a typical element of the E_1-term of the spectral sequence (3.2) is $v_R\sigma^{2^s l}a^t$, where $l \in \mathbb{Z}$ is odd, $t \in \mathbb{Z}$, and $R = (r_0, r_1, \ldots)$ is a sequence of nonnegative integers, of which only finitely many are nonzero, with $v_R = \prod v_i^{r_i}$. The filtration degree of this element is t. The differential (3.3) gives that if $s \leq min(R)$, then

$$d_{2^{s+1}-1}(v_R\sigma^{2^s l}a^t) = v_R v_s \sigma^{2^s l+1} a^{t+2^{s+1}-1} \tag{3.4}$$

for all $l \neq 0$. So the element is the source of a differential if $s \leq min(R)$ or if $R = (0,0,\ldots)$ and $l \neq 0$, and it is the target of a differential if $s > min(R)$ or if $l = 0$ and $R \neq (0,0,\ldots)$. Note that every monomial $v_R\sigma^{2^s l}a^t$ in the E_1-term of appears either in the source or target of a differential (3.4), except when $l = 0$. (For complete details on this, see [7].) Thus, the only surviving elements are powers of a.

In (3.1), the E_1-term is now

$$BP\langle n\rangle_\star[\sigma,\sigma^{-1},a,a^{-1}].$$

Again, this is the same as $BP\langle n\rangle_*[\sigma,\sigma^{-1},a,a^{-1}]$, by replacing the generators v_i by $v_i\sigma^{2^i-1}$. The differentials are same as the ones as (3.3). An element of the E_1-term is $v_R\sigma^{2^s l}a^t$, but now $R = (v_0, v_1, \ldots, v_n)$. If $s \leq min(R)$, this is the source of a differential. If $s > min(R)$ or if $l = 0$ and $R \neq (0,0,\ldots,0)$, this is the target of a differential. However, suppose that $R = (0,\ldots,0)$ and $s > n$. In the spectral sequence (3.1), we get a differential

$$d_{2^{s+1}-1}(\sigma^{2^s l}a^t) = v_s\sigma^{2^s(l+1)}a^{t+2^{s+1}-1}. \tag{3.5}$$

The target of this differential is now 0 in the Tate spectral sequence for $BP\mathbb{R}\langle n\rangle$. Thus, the monomials in $\sigma^{2^{n+1}}, \sigma^{-2^{n+1}}, a, a^{-1}$ survive to the E_∞-term of the Tate spectral sequence (3.1). This proves the first part of Theorem 2.1.

For the Borel cohomology of $BP\mathbb{R}\langle n\rangle$, we use the Borel cohomology spectral sequence

$$E_2 = H^*(\mathbb{Z}/2, BP\langle n\rangle_*[\sigma,\sigma^{-1}]) \Rightarrow F(E\mathbb{Z}/2_+, BP\mathbb{R}\langle n\rangle)_\star. \tag{3.6}$$

We compare this to both the Tate spectral sequence (3.1), and to the Borel cohomology spectral sequence for $BP\mathbb{R}$

$$E_2 = H^*(\mathbb{Z}/2, BP_*[\sigma,\sigma^{-1}]) \Rightarrow F(E\mathbb{Z}/2_+, BP\mathbb{R})_\star. \tag{3.7}$$

The E_1-term of (3.6) is $BP\langle n\rangle_\star[\sigma, \sigma^{-1}, a]$, which is just the part of the E_1-term of the Tate spectral sequence (3.1) consisting of only the elements with nonnegative filtration degrees (i. e. nonnegative powers of a). The differentials are the same as in (3.1), i. e. $d_{2^{k+1}-1}$ for $0 \le k \le n$, except that now we only allow the differentials with sources and targets both having nonnegative filtration degrees. Thus, the monomials $v_R\sigma^{2^s l}a^t$ with $t \le 2^{min(R)+1} - 2$ and $s > min(R)$ will survive (3.6), since in (3.1), they are targets of differentials $d_{2^{s+1}-1}$ with sources having negative filtration degrees. Also, as before, the monomials $\sigma^{2^s l}a^t$ survives for any $s > n$ and $t \in \mathbb{Z}$. This gives the second part of Theorem 2.1.

4 The coefficients of $BP\mathbb{R}\langle n\rangle$

We prove Theorem 2.2 in this section. To this end, we will first compute the coefficients of the geometric spectrum $\widetilde{E\mathbb{Z}/2} \wedge BP\mathbb{R}\langle n\rangle$ by induction on n. The following lemma was shown in [8].

Lemma 4.1 *$\widetilde{E\mathbb{Z}/2} \wedge BP\mathbb{R}\langle 0\rangle$ is $H\mathbb{Z}/2_m$, the $\mathbb{Z}/2$-equivariant cohomology spectrum corresponding to the constant Mackey functor.*

Proposition 4.2 *For $n \ge 0$, the coefficients of the geometric spectrum $\widetilde{E\mathbb{Z}/2} \wedge BP\mathbb{R}\langle n\rangle$ are*

$$\mathbb{Z}/2[\sigma^{-2^{n+1}}, a, a^{-1}]$$

where the dimensions of σ and a are as above.

Proof We work by induction. As shown in [8], the coefficents of $\widetilde{E\mathbb{Z}/2} \wedge BP\mathbb{R}\langle 0\rangle$ is

$$(\widetilde{E\mathbb{Z}/2} \wedge BP\mathbb{R}\langle 0\rangle)_\star = (H\mathbb{Z}/2_m)_\star = \mathbb{Z}/2[\sigma^{-2}, a, a^{-1}].$$

Suppose that the statement is true for $\widetilde{E\mathbb{Z}/2} \wedge BP\mathbb{R}\langle n-1\rangle$. We filter $\widetilde{E\mathbb{Z}/2} \wedge BP\mathbb{R}\langle n\rangle$ by copies of $\widetilde{E\mathbb{Z}/2} \wedge BP\mathbb{R}\langle n-1\rangle$. Namely, consider the map

$$v_n : \Sigma^{(2^n-1)(1+\alpha)}(\widetilde{E\mathbb{Z}/2} \wedge BP\mathbb{R}\langle n\rangle) \to \widetilde{E\mathbb{Z}/2} \wedge BP\mathbb{R}\langle n\rangle.$$

The cofiber of this is a suspension of $\widetilde{E\mathbb{Z}/2} \wedge BP\mathbb{R}\langle n-1\rangle$. Iterating the map gives an exact couple, which in turn gives a spectral sequence

$$\begin{aligned} E_1 = (\widetilde{E\mathbb{Z}/2} \wedge BP\mathbb{R}\langle n-1\rangle)_\star[v_n] &= \mathbb{Z}/2[\sigma^{-2^n}, a, a^{-1}][v_n] \\ &\Rightarrow (\widetilde{E\mathbb{Z}/2} \wedge BP\mathbb{R}\langle n\rangle)_\star. \end{aligned} \tag{4.1}$$

By comparing with the other spectral sequences (3.1), (3.2) and (3.6), we get that the differentials of the spectral sequence are only

$$d_1(\sigma^{-2^n}) = v_n a^{2^{n+1}-1}$$

and its multiples by powers of σ^{-2^n}. Hence, by arguments similar to that for the Tate spectral sequence, σ^{-l2^n} is the source of a differential for all l odd, and all monomials containing v_n are targets of differentials. This gives that the E_∞-term of the spectral sequence (4.1) is just $\mathbb{Z}/2[\sigma^{-2^{n+1}}, a, a^{-1}]$ as claimed. □

Now in the bottom row of the Tate diagram for $BP\mathbb{R}\langle n\rangle$, we have the cofiber sequence

$$\begin{aligned} &E\mathbb{Z}/2_+ \wedge F(E\mathbb{Z}/2_+, BP\mathbb{R}\langle n\rangle)_\star \\ &\to (\mathbb{Z}_{(2)}[v_k\sigma^{l2^{n+1}}, a]/\sim) \oplus \mathbb{Z}/2[\sigma^{2^{n+1}}, \sigma^{-2^{n+1}}, a] \\ &\to \mathbb{Z}/2[\sigma^{2^{n+1}}, \sigma^{-2^{n+1}}, a, a^{-1}]. \end{aligned}$$

By comparison of the spectral sequences computing them, it is straightforward to see that the map from the Borel cohomology term to the Tate term is just the inclusion on the monomials containing only a and powers of σ, and kills all monomials containing any v_k. Thus, the coefficient of the fibers is

$$(\mathbb{Z}_{(2)}[v_k\sigma^{l2^{n+1}}, a]/\sim) \oplus \mathbb{Z}/2[a^{-1}].$$

Here, $\sim$ denotes the relations (2.2), (2.3) and (2.4). This is the Borel homology of $BP\mathbb{R}\langle n\rangle$. Hence, for the top row of the Tate diagram, we get the cofiber sequence

$$\begin{aligned} (\mathbb{Z}_{(2)}[v_k\sigma^{l2^{n+1}}, a]/\sim) \oplus \mathbb{Z}/2[a^{-1}] &\to BP\mathbb{R}\langle n\rangle_\star \\ &\to \mathbb{Z}/2[\sigma^{-2^{n+1}}, a, a^{-1}]. \end{aligned}$$

The connecting map is the identity on a^{-1} and kills $\sigma^{-2^{n+1}}$ and a. Therefore, the middle term gives

$$BP\mathbb{R}\langle n\rangle_\star = (\mathbb{Z}_{(2)}[v_k\sigma^{l2^{n+1}}, a]/\sim) \oplus \mathbb{Z}/2[\sigma^{-2^{n+1}}, a]$$

where $\sim$ denotes the relations (2.2), (2.3) and (2.4).

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