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The co-rank conjecture for 3-manifold groups

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Abstract In this paper we construct explicit examples of both closed and non-compact finite volume hyperbolic manifolds which provide counterexamples to the conjecture that the co-rank of a 3-manifold group (also known as the cut number) is bounded below by one-third the first Betti number.

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1 Introduction

The *co-rank* of a group G, which we denote by $c_1(G)$, is the maximal rank of a free group homomorphically surjected by G. Clearly, $c_1(G) \leq b_1(G) = \dim(G^{ab}\otimes\mathbb{Q})$, where $G^{ab} = (G/[G,G])$. If M is a manifold, $c_1(M) = c_1(\pi_1(M))$. If M is a compact 3-manifold, $c_1(M)$ is also called the *cut number* of M, and is equal to the maximal number of components of a surface F embedded in M for which $M \setminus F$ is connected.

The free Abelian groups $\mathbb{Z}^n$ show that, in general, there is no lower bound for $c_1(G)$ in terms of $b_1(G)$. For a genus g surface group it is well-known that $c_1(\Sigma_g) = g$ (as is proved below). In his talk [18], J Stallings discussed the following conjecture on a lower bound for $c_1(M)$ for M a compact 3-manifold, which has recently received some attention. According to A Sikora this conjecture has its origins in work of T Kerler connected to quantum invariants of 3-manifolds.

Conjecture 1.1 *If M is a 3-manifold, then $c_1(M) \geq \frac{b_1(M)}{3}$.*

Notice that as particular cases, if $b_1(M) = 4$ or 5, Conjecture 1.1 would imply $c_1(M) \geq 2$. In this note we construct explicit counterexamples to this conjecture. In particular, we prove:

Theorem 1.2 (1) *There exist closed hyperbolic 3-manifolds M such that $b_1(M) = 5$ and $c_1(M) = 1$.*

(2) *There exist compact 3-manifolds M with toroidal boundary so that $b_1(M) = 4$ and $c_1(M) = 1$.*

Remark A Sikora [17] has also recently proved the existence of counterexamples to this conjecture with $b_1 \geq 7$ and $c_1 \leq 2$. Moreover, S. Harvey [8] has recently constructed examples of closed hyperbolic 3-manifolds with b_1 arbitrarily large, yet $c_1 = 1$.

The rest of the paper is organized as follows. In section 2 we fix some notation and make a few elementary observations about co-rank and surface groups. In section 3 we discuss automorphisms of surfaces and when they extend over a handlebody. This is needed to prove Theorem 1.2. We discuss the contents of section 3 further in section 6. Parts (1) and (2) of Theorem 1.2 are proved in sections 4 and 5.

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2 Background and notation: Co-rank and surfaces

2.1 Notation

Throughout the rest of this paper F_k will denote a free group of rank k, Σ_g will denote a closed, oriented surface of genus g, and $\mathrm{Mod}(\Sigma_g)$ will denote its mapping class group. That is, $\mathrm{Mod}(\Sigma_g)$ is the group of isotopy classes of orientation preserving diffeomorphisms of Σ_g. We will refer to an element of $\mathrm{Mod}(\Sigma_g)$, or any representative of that element, as an *automorphism* of Σ_g.

The canonical homomorphism $\mathrm{Mod}(\Sigma_g) \to \mathrm{Aut}(H_1(\Sigma_g))$ has a non-trivial kernel which is called the *Torelli group*. We will denote it by

$$\mathcal{I}(\Sigma_g) \lhd \mathrm{Mod}(\Sigma_g).$$

An automorphism f is in $\mathcal{I}(\Sigma_g)$ if and only if $f_* : H_1(\Sigma_g) \to H_1(\Sigma_g)$ is the identity.

2.2 Surfaces

In this section we record some lemmas concerning epimorphisms from surface groups to free groups. Since we shall make use of it, we give the proof that $c_1(\Sigma_g) = g$, although this is well-known.

Lemma 2.1 $c_1(\Sigma_g) = g$.

Proof Let $\phi : \pi_1(\Sigma_g) \to F_k$ be a surjective homomorphism and let $f : \Sigma_g \to X_k$ be a map where X_k is a wedge of k circles with $\pi_1(X_k)$ identified with F_k so that $f_* = \phi$. Making f transverse to k points (one in each circle, each different than the wedge point), the preimage is a disjoint union of k 1-submanifolds $\gamma_1, ..., \gamma_k$. It is easy to see that $[\gamma_1], ..., [\gamma_k]$ must represent k linearly independent elements of $H_1(\Sigma_g; \mathbb{Z})$ (with a pull-back orientation). Since the curves $\gamma_1, ..., \gamma_k$ are pairwise disjoint, the intersection form on $H_1(\Sigma_g; \mathbb{Z})$ is trivial on the span $< [\gamma_1], ..., [\gamma_k] >$. This is a non-degenerate, skew-symmetric form, so that $k \leq \frac{1}{2} b_1(\Sigma_g) = g$.

Clearly $c_1(\Sigma_g) \geq g$, so $c_1(\Sigma_g) = g$. □

We will also need the following well-known fact, whose proof we sketch.

Lemma 2.2 *Suppose $\phi : \pi_1(\Sigma_g) \to F_g$ is an epimorphism. Then there exists a handlebody H with $\pi_1(H)$ identified to F_g, and a homeomorphism $\tau : \Sigma_g \to \partial H$, such that if $i : \partial H \to H$ is the inclusion map, then $(i \circ \tau)_* = \phi$.*

Proof Given $\phi : \pi_1(\Sigma_g) \to F_g$, let $f : \Sigma_g \to X_g$ and $\gamma_1, ..., \gamma_g$ be as in the previous proof. We may homotope f so that no component of γ_j is homotopically trivial for any $j = 1, ..., g$. We construct a compression body H by first thickening Σ_g to $\Sigma_g \times [-1, 1]$, then attaching 2-handles along the curves on $\Sigma_g \times \{-1\}$ corresponding to $\cup_{j=1}^{g} \gamma_j$ on Σ_g, and finally capping off all $2-$sphere boundary components with $3-$handles.

It must be that H is a handlebody. If this were not the case, one could find an (oriented) curve γ on Σ_g, disjoint from $\gamma_1, ..., \gamma_g$ with $[\gamma], [\gamma_1], ..., [\gamma_g]$ being linearly independent. As in the previous proof, this is impossible by dimension considerations.

Let $\tau : \Sigma_g \to \Sigma_g \times \{1\} = \partial H$ be the obvious homeomorphism. By construction, we can extend f to $\widetilde{f} : H \to X_g$ (that is, $\widetilde{f} \circ \tau = f$). Moreover, the induced homomorphism $\widetilde{f}_* : \pi_1(H) \to \pi_1(X_g) = F_g$ must then be surjective. It follows from the Hopfian property for free groups (see [14]) that $\widetilde{f}_*$ must be an isomorphism. Therefore $\widetilde{f}_*$ identifies $\pi_1(H)$ with F_k and $(i \circ \tau)_* = f_* = \phi$, where $i : \partial H \to H$ is the inclusion. □

3 Extending automorphisms of surfaces

Recall that an automorphism $f : \Sigma_g \to \Sigma_g$ extends over a handlebody H if there exists a homeomorphism $\tau : \Sigma_g \to \partial H$ and a homeomorphism $\tilde{f} : H \to H$ such that $\tau^{-1} \circ \tilde{f}|_{\partial H} \circ \tau = f$.

Lemma 3.1 *For $f \in \mathrm{Mod}(\Sigma_g)$, f extends over a handlebody H if and only if for every simple closed curve $\gamma \subset \Sigma_g$ for which $\tau(\gamma)$ bounds a disk in H, $\tau(f(\gamma))$ also bounds a disk in H.*

Proof If f extends over a handlebody, then clearly $\tau(\gamma)$ bounds a disk if and only if $\tau(f(\gamma))$ does.

Suppose that for every simple closed curve $\gamma \subset \Sigma_g$ for which $\tau(\gamma)$ bounds a disk in H, $\tau(f(\gamma))$ also bounds a disk. Choose a complete set of (pairwise disjoint) meridional disks $D_1, ..., D_g$ for H. The images $\tau \circ f \circ \tau^{-1}(\partial(D_1)), ..., \tau \circ f \circ \tau^{-1}(\partial(D_g))$ bound disks $D'_1, ..., D'_g$ which we may assume are pairwise disjoint.

We now extend f to $\tilde{f} : H \to H$. This is done by first extending over the disks D_j, mapping these to D'_j by any homeomorphism which extends $(\tau \circ f \circ \tau^{-1})|_{\partial D_j}$, and then extending over a regular neighborhood of $\partial H \cup \bigcup_{j=1}^{g} D_j$. What is left is a 3-ball, and we may extend over this in any fashion. □

The important point for us is existence of certain types of automorphisms that do not extend. The following theorem was proven independently by Johannson and Johnson [10] and Casson [3]. Neither of these works were ever published, so we include in section 6 a sketch of the proof (as given in [10]) for completeness.

Theorem 3.2 *For every $g \geq 2$, there exists $f \in \mathcal{I}(\Sigma_g)$ which does not extend over any handlebody. Furthermore, for every odd integer n, f^n does not extend.*

Indeed, using Thurston's classification of automorphisms of surfaces [20], the automorphism f can be chosen to be pseudo-Anosov. We state this below and prove this in section 6, since we make use of some of the notation developed in proving Theorem 3.2.

Theorem 3.3 *For any $g \geq 2$, there exists pseudo-Anosov mapping classes $f \in \mathcal{I}(\Sigma_g)$ so that no odd power of f extends over any handlebody.*

Remark In the genus 2 case, *every* automorphism $f \in \mathcal{I}(\Sigma_2)$ which does not extend over a handlebody is pseudo-Anosov. To see this, first note that by Thurston's classification of automorphisms of surfaces, and because $\mathcal{I}(\Sigma_2)$ is torsion free, f is either reducible or pseudo-Anosov. If $f \in \mathcal{I}(\Sigma_2)$ is reducible, we can extend f to $\widetilde{f} : H \to H$ where H is a compression body defined by the reducing curves of f with upper boundary, $\partial_+ H$, identified to Σ_2. The lower boundary of the compression body, $\partial_- H$, must be a (possibly empty) disjoint union of tori. Since the map induced by inclusion $i_* : H_1(\partial_- H) \to H_1(H)$ is injective and $\widetilde{f}$ acts trivially on $H_1(H)$, it follows that $(\widetilde{f}|_{\partial_- H})_* : H_1(\partial_- H) \to H_1(\partial_- H)$ must be the identity. An automorphism acting trivially on the homology of a torus is trivial, and so we can compress away the lower boundary completely and extend. It follows that if $f \in \mathcal{I}(\Sigma_2)$ does not extend over any handlebody, then it must be pseudo-Anosov.

4 Examples: Closed 3-manifolds

Let M_f be the mapping torus of a pseudo-Anosov automorphism $f \in \mathcal{I}(\Sigma_2)$ that does not extend over any handlebody (by Theorem 3.3, such an f exists).

Theorem 4.1 *M_f is hyperbolic, $b_1(M_f) = 5$, and $c_1(M_f) = 1$*

Proof Since f is pseudo-Anosov, M_f is hyperbolic by Thurston's Geometerization Theorem for Haken Manifolds [19]. An elementary calculation shows that $b_1(M_f) = 1 + \mathrm{rk}(\mathrm{fix}(f_*))$ where $\mathrm{fix}(f_*)$ is the fixed subgroup of $H_1(\Sigma_2; \mathbb{Z})$. So, $b_1(M_f) = 5$.

Suppose $c_1(M_f) > 1$. Then there exists an epimorphism $\phi : \pi_1(M_f) \to F_2$. Let $\Sigma_2 \subset M_f$ denote the fiber, and note that $\pi_1(\Sigma_2) \lhd \pi_1(M_f)$. This implies that $\phi(\pi_1(\Sigma_2)) \lhd F_2$ is a finitely generated normal subgroup, which must therefore be either trivial or of finite index. Now $\phi(\pi_1(\Sigma_2))$ cannot be trivial since this would imply F_2 is the image of $\pi_1(M_f)/\pi_1(\Sigma_2) \cong \mathbb{Z}$. It follows that $\phi(\pi_1(\Sigma_2))$ must have finite index. Note that $rk(\phi(\pi_1(\Sigma_2))) = 1 + [F_2 : \phi(\pi_1(\Sigma_2))]$. By Lemma 2.1, $c_1(\Sigma_2) = 2$, so that $[F_2 : \phi(\pi_1(\Sigma_2))] = 1$ and $\phi|_{\pi_1(\Sigma_2)}$ must be surjective.

By Lemma 2.2, there exists a handlebody H with $\Sigma_2 = \partial H$ such that the inclusion $i : \Sigma_2 \to H$ has $i_* = \phi|_{\pi_1(\Sigma_2)}$. Since f does not extend over any handlebody, Lemma 3.1 implies the existence of a simple closed curve $\gamma \subset \Sigma_2$ such that γ bounds a disk in H, but $f(\gamma)$ does not. By Dehn's Lemma and the Loop Theorem (see eg [16]), $f(\gamma)$ is homotopically non-trivial in H. Representing these curves by based loops of the same name in Σ_2 (with basepoint fixed by f), this says $i_*([\gamma]) = 1$, while $i_*(f_*([\gamma])) = i_*([f(\gamma)]) \neq 1$. That is, $\phi([\gamma]) = 1$ while $\phi(f_*([\gamma])) \neq 1$.

Now we note that $\pi_1(M_f)$ is an HNN extension of $\pi_1(\Sigma_2)$ with conjugation by the stable letter, t, acting by f_*. It follows that

$$1 = \phi(t)\phi(t^{-1}) = \phi(t)\phi([\gamma])\phi(t^{-1}) = \phi(t[\gamma]t^{-1}) = \phi(f_*[\gamma]) \neq 1.$$

This contradiction proves that $c_1(M_f) = 1$. □

Corollary 4.2 *There are infinitely many closed hyperbolic 3-manifolds M with $b_1(M) = 5$ and $c_1(M) = 1$.*

Proof Given M_f as above, with f as in Theorem 3.3, the cyclic covers M_{f^n}, for odd n, provide infinitely many such manifolds. □

Using the nature of the construction of the automorphism f in Theorem 4.1 we can extend Corollary 4.2 to the following; a proof is given in section 6.

Corollary 4.3 *There are infinitely many non-commensurable closed hyperbolic 3-manifolds M with $b_1(M) = 5$ and $c_1(M) = 1$.*

A similar argument can be made to work for genus 3 bundles. In this case if we consider an $f \in \mathcal{I}(\Sigma_3)$ as in Theorem 3.3, then $b_1(M_f) = 7$, so that Conjecture 1.1 would predict $c_1(M_f) = 3$.

Theorem 4.4 *There are infinitely many closed hyperbolic 3-manifolds M with $b_1(M) = 7$ and $c_1(M) \leq 2$.*

Proof Let $f \in \mathcal{I}(\Sigma_3)$ be as in Theorem 3.3. The proof of Theorem 4.1 applies verbatim to the bundle M_f. The only point to remark being that if $\pi_1(M_f)$ surjects a free group of rank 3, then the fiber group must also surject. □

Remark Notice that the argument breaks down for genus 4 bundles. In this case, $b_1(M_f) = 9$ so that a counterexample to Conjecture 1.1 requires $c_1(M_f) \leq 2$. The argument above only guarantees $c_1(M_f) \leq 3$.

5 Examples: Bounded 3-manifolds

Here we sketch the proof of the second part of Theorem 1.2.

Theorem 5.1 *There exists compact 3-manifolds M with toroidal boundary so that $b_1(M) = 4$ and $c_1(M) = 1$.*

Remark In fact, the manifolds we construct below can be shown to be hyperbolic using Thurston's Geometerization Theorem for Haken manifolds (see [15] for example), and a variant of [7] (see [4] for a proof that M is irreducible).

Sketch of proof Let $\alpha, \beta, \gamma, \delta,$ and ϵ be the simple closed curves shown on the surface Σ_2, and let $a, b, c,$ and d be the generators for $\pi_1(\Sigma_2)$ shown (see Figure 1).

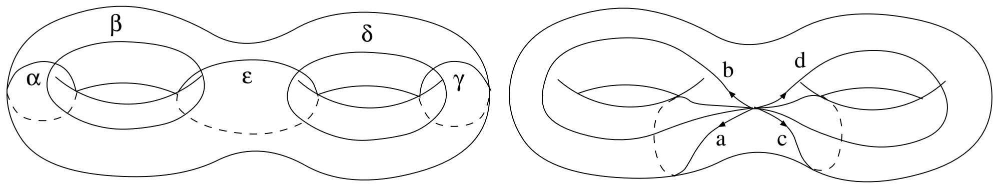

Figure 1

We denote the Dehn twist in $\alpha, \beta, \gamma, \delta,$ and ϵ by $T_\alpha, T_\beta, T_\gamma, T_\delta,$ and T_ϵ respectively. Let C_1 denote the separating simple closed curve on Σ_2 represented by $[a, b] = aba^{-1}b^{-1}$, and set

$$C_2 = T_\alpha^{-1} \circ (T_\alpha \circ T_\epsilon \circ T_\gamma^2 \circ T_\beta^{-1} \circ T_\delta^{-1})^3(C_1)$$

Now construct a 3-manifold M by attaching 2-handles to $\Sigma_2 \times [-1, 1]$ along C_1 in $\Sigma_2 \times \{-1\}$ and along C_2 in $\Sigma_2 \times \{1\}$. Since both C_1 and C_2 are separating curves, $b_1(M) = b_1(\Sigma_2) = 4$. We claim that $c_1(M) = 1$.

We begin by finding a presentation for $\pi_1(M)$. By considering the action of each of the Dehn twists above on $\pi_1(\Sigma)$, we can explicitly write down a word $w = w(a, b, c, d)$ representing C_2. It is given by

$$\begin{gathered} w = ac^{-1}abc^{-1}adc^{-3}ac^{-1}abc^{-1}abac^{-1}abc^{-1}adc^{-3}ac^{-1}ab \\ c^{-1}adc^{-3}adc^{-4}adc^{-4}adc^{-3}ac^{-1}abc^{-1}adc^{-3}ac^{-1}aba^{-1}b^{-1}a^{-1}cb^{-1}a^{-1}ca^{-1} \\ c^3d^{-1}a^{-1}cb^{-1}a^{-1}ca^{-1}b^{-1}a^{-1}cb^{-1}a^{-1}ca^{-1}c^3d^{-1}a^{-1}cb^{-1}a^{-1}ca^{-1} \\ c^3d^{-1}a^{-1}c^4d^{-1}a^{-1}c^4d^{-1}a^{-1}c^3d^{-1}a^{-1}cb^{-1}a^{-1}ca^{-1}c^3d^{-1}a^{-1}c. \end{gathered}$$

An application of Van Kampen's Theorem implies

$$\pi_1(M) = < a, b, c, d \mid [a, b], [c, d], w > .$$

To prove that $c_1(M) = 1$, we suppose there exists an epimorphism $\phi : \pi_1(M) \to F_2$, and find a contradiction. The idea of the proof is as follows. Since $[a, b] = 1$ and $[c, d] = 1$ and maximal Abelian subgroups of free groups are cyclic, there exists $g, h \in F_2$ such that $\phi(a), \phi(b) \in < g >$ and $\phi(c), \phi(d) \in < h >$. We let $m, n, j, k \in \mathbb{Z}$ be such that

$$\phi(a) = g^m\,,\ \phi(b) = g^n\,,\ \phi(c) = h^k \ \text{ and } \phi(d) = h^j.$$

The subgroup of F_2 generated by g and h contains the subgroup generated by $\phi(a), \phi(b), \phi(c)$, and $\phi(d)$, but ϕ is surjective, so g and h must generate all of F_2. By the Hopfian property for free groups, we see that g and h form a basis for F_2. Since ϕ is a homomorphism, we have

$$1 = \phi(w(a, b, c, d)) = w(\phi(a), \phi(b), \phi(c), \phi(d)) = w(g^m, g^n, h^k, h^j).$$

This imposes restrictions on the integers m, n, j, and k. There are then several cases to analyze, each of which results in a contradiction to the surjectivity of ϕ. It follows that there is no such epimorphism ϕ, and hence $c_1(M) = 1$. □

Remark Another proof of this theorem goes as follows. Choose two separating curves C_1 and C_2 on Σ_2 so that (a) $C_1 \cup C_2$ fills Σ_2, and (b) for every handlebody H with $\partial H = \Sigma_2$, at most one of C_1 and C_2 bounds a disk. If one constructs M as in the above proof, then M would provide an example proving the theorem. To see this, note that any epimorphism from $\pi_1(M)$ onto F_2 would induces a epimorphism from $\pi_1(\Sigma_2)$ onto F_2 in which C_1 and C_2 are both mapped to 1. By applying Dehn's Lemma and the Loop Theorem along with Lemma 2.2 above, we would have a contradiction.

Of course, the difficulty is in finding two curves satisfying (a) and (b). The two curves C_1 and C_2 in the given proof *do* satisfy (a) and (b)–condition (b) is essentially what is shown in the proof (which we only sketched). In fact, Lemma 2.2 of [5] describes an algorithm to decide if two curves can both bound disks in any handlebody, so it should possible to implement this to give another proof that C_1 and C_2 satisfy (b). This algorithm is based on analyzing the intersections of the pair of curves, and it seems that likely that this could be computationally more difficult than the given proof (the geometric intersection number of C_1 and C_2 is 72).

6 The proof of Theorem 3.2

In this section we sketch the proof of Theorem 3.2 following [10]. We include this sketch since [10] did not appear, and although examples as in Theorem 3.2 appear to be well-known, no explicit example appears to be recorded in the literature.

We begin by considering a Heegaard embedding $h : \Sigma_g \to S^3$ (ie, an embedding such that $h(\Sigma_g)$ bounds handlebodies on both sides). We perturb this Heegaard splitting using an automorphism $f \in \mathcal{I}(\Sigma_g)$ to give a Heegaard splitting of a new manifold $M_{h,f}$ as follows.

We let H_+ and H_- be the handlebodies on the positive and negative sides of $h(\Sigma_g)$ respectively (so the positive unit normal to $h(\Sigma_g)$ points into H_+). The manifold $M_{h,f}$ is constructed by gluing ∂H_- to ∂H_+ by

$$h \circ f \circ h^{-1} : \partial H_- \to \partial H_+.$$

If $h \circ f \circ h^{-1}$ extended over the handlebody H_+, then one can check that $M_{h,f} \cong S^3$. In particular, suppose f extends over *some* handlebody $\widetilde{f} : H \to H$. Then if H_+ and H_- are the positive and negative handlebodies in a genus g Heegaard splitting of S^3, any diffeomorphism $\widetilde{h} : H \to H_+$ restricts to a Heegaard embedding $h : \Sigma_g \to S^3$, and $h \circ f \circ h^{-1}$ extends over H_+ by $\widetilde{h} \circ \widetilde{f} \circ \widetilde{h}^{-1}$. So to prove that an automorphism f does not extend over any handlebody, it suffices to show that for any Heegaard embedding $h : \Sigma_g \to S^3$, the manifold $M_{h,f}$ is not diffeomorphic to S^3.

Next we note that since $f \in \mathcal{I}(\Sigma_g)$, $M_{h,f}$ will be an integral homology 3-sphere. We let $\mu(M_{h,f}) \in \mathbb{Z}/2\mathbb{Z}$ denote the Rohlin invariant of $M_{h,f}$ (see eg [6]). Since $\mu(S^3) = 0$, we wish to find $f \in \mathcal{I}(\Sigma_g)$ such that for every Heegaard embedding h, $\mu(M_{h,f}) = 1$. In [11], Johnson studies the *Birman-Craggs homomorphisms* (see [1]) and gives a very effective way of finding such f.

In what follows, the main ideas and relevant theorems of [11] necessary for our purpose are stated without proofs (see [11] for the proofs and complete references).

A *symplectic quadratic form* (Sp-form for short) on $H_1(\Sigma_g;\mathbb{Z}/2\mathbb{Z})$ is a function

$$\omega : H_1(\Sigma_g;\mathbb{Z}/2\mathbb{Z}) \to \mathbb{Z}/2\mathbb{Z}$$

such that

$$\omega(a+b) = \omega(a) + \omega(b) + a \cdot b$$

where $a \cdot b$ is the symplectic bilinear form given by the mod 2 intersection number of a and b. We denote the set of Sp-forms on $H_1(\Sigma_g;\mathbb{Z}/2\mathbb{Z})$ by $\Omega = \Omega(\Sigma_g)$.

Given an element $a \in H_1(\Sigma_g;\mathbb{Z}/2\mathbb{Z})$, we obtain a function $\overline{a} : \Omega \to \mathbb{Z}/2\mathbb{Z}$ defined by $\overline{a}(\omega) = \omega(a)$. If $\{a_1, b_1, ..., a_g, b_g\}$ is a symplectic basis for $H_1(\Sigma_g;\mathbb{Z}/2\mathbb{Z})$ (with respect to $\cdot$), then the *Arf-invariant* of a form $\omega \in \Omega$ is defined to be $Arf(\omega) = \left(\sum_{i=1}^{g} \overline{a_i}\overline{b_i}\right)(\omega)$. We denote the set of Sp-forms with zero Arf-invariant by $\Psi = \Psi(\Sigma_g) \subset \Omega(\Sigma_g)$.

Now let $h : \Sigma_g \to S^3$ be a Heegaard embedding. Seifert's linking form defines an Sp-form by setting

$$\omega_h(a) = \lambda(h(\gamma_a), h(\gamma_a)^+)$$

where γ_a is a simple closed curve representing the class $a \in H_1(\Sigma_g;\mathbb{Z}/2\mathbb{Z})$ and $\lambda(h(\gamma_a), h(\gamma_a)^+)$ is the mod 2 linking number of $h(\gamma_a)$ and its push off in the positive normal direction, $h(\gamma_a)^+$. The form ω_h lies in Ψ, and furthermore, any $\omega \in \Psi$ can be realized by some Heegaard embedding of Σ_g.

The following facts from [11] will essentially complete the proof (labellings below are those of [11]).

- **Corollary 1 to Theorem 1** $\mu(M_{h,f})$ depends only on ω_h and f.
- **Lemma 11** If we denote the Abelian group of functions from Ψ into $\mathbb{Z}/2\mathbb{Z}$ by $\mathbb{Z}/2\mathbb{Z}^{\Psi}$, then there is a homomorphism
$$\sigma : \mathcal{I}(\Sigma_g) \to \mathbb{Z}/2\mathbb{Z}^{\Psi}$$
such that if we denote the image of f under σ by σ_f, then $\sigma_f(\omega_h) = \mu(M_{h,f})$.
- **Consequence of Theorem 4** The constant function 1 is in the image of σ for every $g \geq 2$.

It now follows that if we let $f \in \mathcal{I}(\Sigma_g)$ be such that $\sigma_f = 1$, then for any Heegaard embedding h, $\mu(M_{h,k}) = 1$, and hence f cannot extend over any handlebody. This completes the proof of the first statement of Theorem 3.2

For the second, note the following consequence of the proof of the first statement in Theorem 3.2, which gives the second statement of Theorem 3.2.

Scholium 6.1 *If $f \in \mathcal{I}(\Sigma_g)$ satisfies $\sigma_f = 1$, then for any odd integer n, f^n does not extend over any handlebody.*

Proof Note that for any odd n,

$$\sigma_{f^n} = n\sigma_f = \sigma_f = 1$$

so f^n cannot extend over any handlebody. □

6.1

Here we extend Theorem 3.2 to obtain pseudo-Anosov maps for all genera ≥ 2. In the notation developed above, Theorem 3.3 follows from Scholium 6.1 and

Theorem 6.2 *For any $g \geq 2$, there exists pseudo-Anosov mapping classes $f \in \mathcal{I}(\Sigma_g)$ for which $\sigma_f = 1$.*

Proof We use the notation of the proof of Theorem 3.2. Let $h \in \mathcal{I}(\Sigma_g)$ be such that $\sigma_h = 1$, and let $\phi \in ker(\sigma)$ be a pseudo-Anosov mapping class with stable and unstable laminations $[\lambda_s], [\lambda_u] \in \mathcal{PML}(\Sigma_g)$. By choosing a different h if necessary, we may assume that $h([\lambda_s]) = [\mu] \neq [\lambda_u]$.

Let n be a positive integer and V_s, V_u, and V_μ be neighborhoods in $\mathcal{PML}(\Sigma_g)$ of $[\lambda_s], [\lambda_u]$, and $[\mu]$, respectively, so that

- For all $[\nu] \in \overline{V}_\mu$ and $[\eta] \in \overline{V}_u$, $\nu \pitchfork \eta$ and $\nu \cup \eta$ fills Σ_g (so, $\overline{V}_\mu \cap \overline{V}_u = \emptyset$).
- $\phi^n(\mathcal{PML}(\Sigma_g) \setminus V_u) \subset \overline{V}_s$, $\phi^{-n}(\mathcal{PML}(\Sigma_g) \setminus V_s) \subset \overline{V}_u$, and $\overline{V}_s \cap \overline{V}_u = \emptyset$
- $h(\overline{V}_s) = \overline{V}_\mu$
- $\overline{V}_s \cong \overline{V}_u \cong \overline{V}_\mu \cong B^n$

The first property is possible to arrange since $\mu \pitchfork \lambda_u$ and $\mu \cup \lambda_u$ fills Σ_g implies the existence of disjoint neighborhoods with the same property. The second follows from standard properties of the dynamical behavior of the action of ϕ on $\mathcal{PML}(\Sigma_g)$ (see for example [9], Chapter 8). The third is possible because $h([\lambda_s]) = [\mu]$ and h acts by a homeomorphism on $\mathcal{PML}(\Sigma_g)$. The last property is possible because $\mathcal{PML}(\Sigma_g)$ is a manifold.

We let $f = h\phi^n \in \mathcal{I}(\Sigma_g)$ and note that

$$f(\mathcal{PML}(\Sigma_g) \setminus V_u) = h(\phi^n(\mathcal{PML}(\Sigma_g) \setminus V_u)) \subset h(\overline{V}_s) = \overline{V}_\mu$$

and

$$f^{-1}(\mathcal{PML}(\Sigma_g) \setminus V_\mu) = \phi^{-n}(h^{-1}(\mathcal{PML}(\Sigma_g) \setminus V_\mu)) = \phi^{-n}(\mathcal{PML}(\Sigma_g) \setminus V_s) \subset \overline{V}_u$$

In particular, $f(\overline{V}_\mu) \subset \overline{V}_\mu$ and $f^{-1}(\overline{V}_u) \subset \overline{V}_u$. By the Brouwer Fixed Point Theorem, f has fixed points $[\nu_s] \in \overline{V}_\mu$ and $[\nu_u] \in \overline{V}_u$. Since f is clearly not periodic, and because $\nu_s \pitchfork \nu_u$ and $\nu_s \cup \nu_u$ fill Σ_g, it follows that f must be pseudo-Anosov with stable and unstable laminations $[\nu_s]$ and $[\nu_u]$ respectively. Since σ is a homomorphism, $\sigma_f = 1$. ◻

6.2

Following [10], we construct some explicit examples of $f \in \mathcal{I}(\Sigma_2)$ which do not extend over any handlebody.

Example 6.3 If $a, b \in H_1(\Sigma_2; \mathbb{Z}/2\mathbb{Z})$ satisfy $a \cdot b = 1$, we can find a pair of transversely intersecting simple closed curves α and β representing a and b respectively such that $\alpha \cap \beta$ is exactly 1 point. The regular neighborhood, $N(\alpha \cup \beta)$ is homeomorphic to a torus-minus-disk embedded in Σ_2 and $\gamma = \partial N(\alpha \cup \beta)$ is a separating essential simple closed curve. If we denote a Dehn twist about γ by T_γ (note that $T_\gamma \in \mathcal{I}(\Sigma_2)$) then according to [11] (Lemma 12a)

$$\sigma_{T_\gamma} = \overline{a}\overline{b}.$$

Now fix a symplectic basis a_1, b_1, a_2, b_2 for $H_1(\Sigma_2; \mathbb{Z}/2\mathbb{Z})$, and let $\gamma_1, ..., \gamma_{10}$ be separating essential simple closed curves associated, as above, to the following 10 pairs of elements $a, b \in H_1(\Sigma_2; \mathbb{Z}/2\mathbb{Z})$ (with $a \cdot b = 1$).

$$\begin{array}{lll}
a_1, b_1 \rightsquigarrow \gamma_1\,, & a_1, b_1 + a_2 \rightsquigarrow \gamma_2\,, & a_1, b_1 + b_2 \rightsquigarrow \gamma_3\,, \\
b_1, a_1 + a_2 \rightsquigarrow \gamma_4\,, & a_1, b_1 + a_2 + b_2 \rightsquigarrow \gamma_5\,, & b_1, a_1 + b_2 \rightsquigarrow \gamma_6\,, \\
b_1, a_1 + a_2 + b_2 \rightsquigarrow \gamma_7\,, & a_1 + b_1, a_1 + a_2 \rightsquigarrow \gamma_8\,, & a_1 + b_1, a_1 + b_2 \rightsquigarrow \gamma_9\,, \\
a_1 + b_1, a_1 + a_2 + b_2 \rightsquigarrow \gamma_{10}. & &
\end{array}$$

To compute $\sigma_{T_{\gamma_j}}$ for each $j = 1, ..., 10$, we note first that the defining characteristic of Sp-forms implies $\overline{a+b} = \overline{a} + \overline{b} + a \cdot b$. We also note that if $\phi : \Psi \to \mathbb{Z}/2\mathbb{Z}$, then $\phi^2 = \phi$. It then follows that

$$\begin{array}{ll}
\sigma_{T_{\gamma_1}} = \overline{a}_1\overline{b}_1\,, & \sigma_{T_{\gamma_2}} = \overline{a}_1\overline{b}_1 + \overline{a}_1\overline{a}_2\,, \\
\sigma_{T_{\gamma_3}} = \overline{a}_1\overline{b}_1 + \overline{a}_1\overline{b}_2\,, & \sigma_{T_{\gamma_4}} = \overline{b}_1\overline{a}_1 + \overline{b}_1\overline{a}_2\,, \\
\sigma_{T_{\gamma_5}} = \overline{a}_1\overline{b}_1 + \overline{a}_1\overline{a}_2 + \overline{a}_1\overline{b}_2 + \overline{a}_1\,, & \sigma_{T_{\gamma_6}} = \overline{b}_1\overline{a}_1 + \overline{b}_1\overline{b}_2\,, \\
\sigma_{T_{\gamma_7}} = \overline{b}_1\overline{a}_1 + \overline{b}_1\overline{a}_2 + \overline{b}_1\overline{b}_2 + \overline{b}_1\,, & \sigma_{T_{\gamma_8}} = \overline{a}_1\overline{b}_1 + \overline{a}_1\overline{a}_2 + \overline{b}_1\overline{a}_2 + \overline{a}_2\,, \\
\sigma_{T_{\gamma_9}} = \overline{a}_1\overline{b}_1 + \overline{a}_1\overline{b}_2 + \overline{b}_1\overline{b}_2 + \overline{b}_2\,, & \sigma_{T_{\gamma_{10}}} = \overline{a}_1\overline{b}_1 + \overline{a}_1\overline{a}_2 + \overline{a}_1\overline{b}_2 + \overline{b}_1\overline{a}_2 + \overline{b}_1\overline{b}_2 \\
 & \quad + \overline{a}_1 + \overline{b}_1 + \overline{a}_2 + \overline{b}_2 + 1.
\end{array}$$

Any word in $T_{\gamma_1}, ..., T_{\gamma_{10}}$ such that the total exponent of each T_{γ_j} is odd provides an automorphism in $\mathcal{I}(\Sigma_2)$ which does not extend over any handlebody.

6.3 Proof of Corollary 4.3

Let θ be any word in $T_{\gamma_1}, ..., T_{\gamma_{10}}$ such that the total exponent of each T_{γ_j} is odd. Let $n \geq 1$ be an integer, define $\theta_n = T_{\gamma_1}^{2n}\theta$, and let M_n denote the mapping torus of θ_n. By the remarks in section 6.2, θ_n does not extend over any handlebody. Note that by the Remark at the end of section 3, θ_n is pseudo-Anosov, however we require the following description to gain extra control of commensurability. By Lemma 1.1 of [12] the manifolds M_n can be described as surgeries on a 1 cusped hyperbolic 3-manifold. Since the degree of the invariant trace-field gets arbitrarily large on such a sequence of surgeries (see [13]) and the invariant trace-field is an invariant of the commensurability class, by subsequencing if necessary we obtain the set of non-commensurable manifolds. □

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