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The slicing number of a knot

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Abstract An open question asks if every knot of 4–genus g_s can be changed into a slice knot by g_s crossing changes. A counterexample is given.

AMS Classification 57M25; 57N70

Keywords Slice genus, unknotting number

A question of Askitas, appearing in [O, Problem 12.1], asks the following: Can a knot of 4–genus g_s always be sliced (made into a slice knot) by g_s crossing changes? If we let $u_s(K)$ denote the *slicing number* of K, that is, the minimum number of crossing changes that are needed to convert K into a slice knot, one readily shows that $g_s(K) \leq u_s(K)$ for all knots, with equality if $g_s(K) = 0$. Hence, the problem can be restated as asking if $g_s(K) = u_s(K)$ for all K.

We will show that the knot 7_4 provides a counterexample; $g_s(7_4) = 1$ but no crossing change results in a slice knot: $u_s(7_4) = 2$. It is interesting to note that 7_4 already stands out as an important example. The proof that its unknotting number is 2, not 1, resisted early attempts [N]; ultimately, Lickorish [L] succeeded in proving that it cannot be unknotted with a single crossing change.

As noted by Stoimenow in [O], if one attempts to unknot a knot of 4–ball genus g_s instead of converting it into a slice knot, more than g_s crossing changes may be required. This is obviously the case with slice knots. For a more general example, let T denote the trefoil knot. One has $g_s(n(T\# -T)\#mT) = m$, but $u(n(T\# -T)\#mT) = m + 2n$, where u denotes the unknotting number.

The unknotting number, though itself mysterious, appears much simpler than the slicing number. Many of the three–dimensional tools that are available for studying the unknotting number do not apply to the study of the slicing number. As we will see, even for this low crossing knot, 7_4, the computation of its slicing number is far more complicated than its unknotting number.

In the last section of this paper we introduce a new slicing invariant, $U_s(K)$, that takes into account the sign of crossing changes used to convert a knot K

into a slice knot. This invariant is more closely related to the 4–genus and satisfies

$$g_s(K) \leq U_s(K) \leq u_s(K).$$

It seems likely that there are knots K for which $g_s(K) \neq U_s(K)$, and 7_4 seems a good candidate, but we have been unable to verify this.

A good reference for the knot theory used here, especially surgery descriptions of knots, crossing changes and branched coverings, is [R]. A reference for 4–dimensional aspects of knotting and also for the linking form of 3–manifolds is [G]. A careful analysis of the interplay between crossing changes and the linking form of the 2–fold branched cover of a knot appears in [L], which our work here generalizes. Different aspects of the relationship between crossing changes and 4–dimensional aspects of knotting appear in [CL]. A general discussion of slicing operations is contained in [A].

1 Background

Our goal is to prove that a single crossing change cannot change 7_4 into a slice knot. The key results concerning slice knots that we will be using are contained in the following theorem; details of the proof can be found in [CG, G, R].

Theorem 1.1 *If K is slice then:*

(1) *$\Delta_K(t) = \pm f(t)f(t^{-1})$ for some polynomial f, where $\Delta_K(t)$ is the Alexander polynomial.*

(2) *$|H_1(M(K), \mathbf{Z})| = n^2$ for some odd n, where $M(K)$ is the 2–fold branched cover of S^3 branched over K.*

(3) *There is a subgroup $H \subset H_1(M(K), \mathbf{Z})$ such that $|H|^2 = |H_1(M(K), \mathbf{Z})|$ and the $\mathbf{Q}/\mathbf{Z}$–valued linking form β defined on $H_1(M(K), \mathbf{Z})$ vanishes on H.*

Our analysis of 7_4 will focus on the 2–fold branched cover, $M(7_4)$, and its linking form. This is much as in Lickorish's unknotting number argument. However, in our case the necessary analysis of the 2–fold branched cover can only be achieved by a close examination of the infinite cyclic cover. In the next two sections we examine the 2–fold branched cover; in Section 4 we consider the infinite cyclic cover.

2 Crossing Changes and Surgery

If a knot K' is obtained from K by changing a crossing, surgery theory as described in [R] quickly gives that the 2-fold cover of K', $M(K')$, can be obtained from $M(K)$ by performing integral surgery on a pair curves, say S_1 and S_2, in S^3. It is also known [L, Mo] that $M(K')$ can be obtained from $M(K)$ by performing $p/2$ surgery on a single curve, say T, in S^3. Here it will be useful to observe that T can be taken to be S_1, as we next describe.

A crossing change is formally achieved as follows. Let D be a disk meeting K transversely in two points. A neighborhood of D is homeomorphic to a 3–ball, B, meeting K in two trivial arcs. In one view, a crossing change is accomplished by performing ± 1 surgery on the boundary of D, say S. Then S lifts to give the curves S_1 and S_2 in $M(K)$. In the other view, the crossing change is accomplished by removing B from S^3 and sewing it back in with one full twist. The 2–fold branched cover of B is a single solid torus, a regular neighborhood of its core T. A close examination shows the surgery coefficient in this case is $p/2$ for some odd p.

The lift of D to the 2-fold branched cover is an annulus with boundary the union of S_1 and S_2 and core T (the lift of an arc, τ, on D with endpoints the two points of intersection of D with K). Clearly T is isotopic to either S_i, as desired.

The following generalization of these observations will be useful. Rather than put a single full twist between the strands when replacing B, n full twists can be added. This is achieved by performing $\pm 1/n$ surgery on S and hence the 2–fold branched cover is obtained by performing p/n surgery on the S_i for some p, or, by a similar analysis, by performing $p'/2n$ surgery on T for some p'.

3 Results based on the 2–fold branched cover of 7_4

On the left in Figure 1 the knot 7_4 is illustrated. Basic facts about 7_4 include that it has 3–sphere genus 1 and that its Alexander polynomial is $\Delta_{7_4} = 4t^2 - 7t + 4$. Since the Alexander polynomial is irreducible, 7_4 is not slice, so we have $g_s(7_4) = 1$. Also, 7_4 is the 2–bridge $B(4, -4)$, and hence from the continued fraction expansion it has 2–fold branched cover the lens space, $L(15, 4)$.

The right diagram in Figure 1 represents a surgery diagram of 7_4. According to [R], surgery on the link $K \amalg K'$ with coefficient -1 and -2 yields S^3. Also

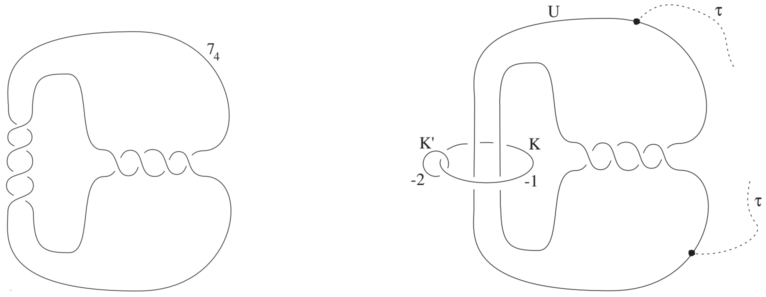

Figure 1: The knot 7_4

according to [R] the component K' could be ignored in the diagram if $-1/2$ surgery is performed on K instead. In both cases the effect is to put two full right handed twists in the two strands passing through K.

Notice that U is unknotted. After surgery is performed, U is converted into the knot 7_4.

If a knot J is obtained from 7_4 by a single crossing change, that change is achieved via a disk D meeting 7_4 in two points, marked schematically by the two dots in the right hand diagram. The path on D joining those two points is denoted τ, a portion of which is also indicated schematically. By sliding τ over K repeatedly it can be arranged that τ misses the small disk bounded by K' meeting K in one point. The boundary of D will be denoted S and one of its lift to the 2–fold branched cover of S^3 over U (this cover is again S^3 since U is unknotted) will be denoted S_1. Neither D nor S is drawn in the figure.

Since two full twists on the unknot U convert it into 7_4, the 2–fold branched cover of S^3 branched over 7_4 is, by our earlier discussion, obtained from S^3 by surgery on a single lift of K, say K_1, with surgery coefficient of the form $p/4$ for some p. Since we know that the cover is $L(15,4)$, we actually know that $p = 15$, though for the argument that follows, simply knowing that $p = \pm 15$ would be sufficient.

Theorem 3.1 *If the linking number of K_1 and S_1 in S^3 is divisible by 15 then J is not slice.*

Proof Suppose that the linking number is divisible by 15. Since $15/4$ surgery is performed on K_1, after repeatedly sliding S_1 over K_1 it can be arranged that the linking number of K_1 and S_1 is 0. The 2–fold cover of S^3 branched over J, that is $M(J)$, is obtained from S^3 by performing $15/4$ surgery on K_1 and $p/2$ surgery on S_1 for some odd p.

If J is slice, the order of the homology of $M(J)$ is an odd square and hence $p = \pm 5^{2k+1}3^{2j+1}q^2$, where q is relatively prime to 30.

We have that $H_1(M(J),\mathbf{Z}) = \mathbf{Z}_{15} \oplus \mathbf{Z}_{|p|}$ generated by the meridians of K_1 and S_1, denoted m_1 and m_2, respectively.

The $\mathbf{Q}/\mathbf{Z}$–valued linking form, β, on $H_1(M(J),\mathbf{Z})$ is orthogonal with respect to this direct sum decomposition since the linking number is now 0. Furthermore, from the surgery description we have that $\beta(m_1,m_1) = 4/15$ and $\beta(m_2,m_2) = 2/p$. The 5–torsion in $H_1(M(J),\mathbf{Z})$ is isomorphic to $\mathbf{Z}_5 \oplus \mathbf{Z}_{5^{2k+1}}$, generated by $n_1 = 3m_1$ and $n_2 = 3^{2j+1}q^2 m_2$. A quick calculation shows that $\beta(n_1,n_1) = 2/5$ and $\beta(m_2,m_2) = 2(3^{2j+1}q^2)^2/p = \pm 2(3^{2j+1}q^2)/5^{2k+1}$.

If J is slice, the linking form on the 5–torsion vanishes on a subgroup of order 5^{k+1}. Suppose that $n_1 + x5^l n_2$ has self–linking $0 \in \mathbf{Q}/\mathbf{Z}$, where x is relatively prime to 5. Then we would have

$$\frac{2}{5} \pm \frac{2x^2 5^{2l}(3^{2j+1}q^2)}{5^{2k+1}} = 0 \in \mathbf{Q}/\mathbf{Z}.$$

This implies that $l = k$, and hence that $2 \pm 2x^2 3^{2j+1}q^2 \equiv 0 \bmod 5$. Letting $q' = x3^j q$, this can be rewritten as $2 \pm 2(3{q'}^2) \equiv 0 \bmod 5$, or that $2 \equiv \mp {q'}^2 \bmod 5$. However, the only squares modulo 5 are ± 1, so this is impossible.

It follows from this that any element of self–linking 0 must be of the form $x5^l n_2$ for some l and x relatively prime to 5. One quickly computes that $l > k$, but such elements generate a subgroup of order 5^k, which is not large enough to satisfy the condition of Theorem 1.1, Statement 3. □

4 The Infinite Cyclic Cover of 7_4

The goal of this section is to prove the following result. It, along with Theorem 3.1, shows that 7_4 cannot be changed into a slice knot with a single crossing change.

Theorem 4.1 *If a crossing change converts 7_4 into a slice knot J, then the corresponding curve S_1 in $M(7_4)$ is null homologous in $H_1(L(15,4),\mathbf{Z})$.*

Before beginning the proof we need to set up notation and prove a lemma.

The infinite cyclic cover of J is built from the infinite cyclic cover of the unknot, U, by performing equivariant surgery on three families of curves: $\{\tilde{K}_i\}, \{\tilde{K}'_i\}$ and $\{\tilde{S}_i\}$, using the notation as before. (In each case, $i = -\infty, \ldots, \infty$.)

Following Rolfsen [R], one can draw that cover with the $\{\tilde{K}_i\}, \{\tilde{K}'_i\}$ drawn explicitly, and the $\{\tilde{S}_i\}$ unknown curves. From this one finds the presentation matrix of the infinite cyclic cover of J as a $\mathbf{Z}[t, t^{-1}]$ module, with respect to the basis given by the meridians of $\tilde{K}_0, \tilde{K}'_0$ and $\tilde{S}_0$, say k_0, k'_0, and s_0. The resulting presentation is given by the matrix

$$A = \begin{pmatrix} -2t + 3 - 2t^{-1} & 1 & g(t) \\ 1 & -2 & 0 \\ g(t^{-1}) & 0 & f(t) \end{pmatrix}.$$

Here $g(t)$ is an unknown polynomial describing the linking between the lifts of S and those of K. (Notice that the lifts of S do not link the lifts of K', since τ (and so S) misses the small disk bounded by K' and this disk lifts to a series of disjoint disks bounded by the $\tilde{K}'_i$ in the infinite cyclic cover.) Also, $f(t)$ is an unknown symmetric polynomial describing the self–linking of the lifts of S. (It might be helpful for the reader to note that if $g = 0$ and $f = 1$ then the determinant of the matrix is $4t - 7 + 4t^{-1}$, the Alexander polynomial of 7_4.)

Although g and f are unknown, two observations are possible. The first is that $f(1) = \pm 1$; this is because ± 1 surgery is being performed on S. The second is that $g(1) = 0$, or that $(t-1)$ divides g, which follows from the fact that S and K have 0 linking number, since S bounds the disk D in the complement of K.

Lemma 4.2 *If J is slice, then $4t - 7 + 4t^{-1} = \Delta_{7_4}$ divides g.*

Proof The determinant of A is given by

$$\Delta_J(t) = f(t)\Delta_{7_4}(t) + 2g(t)g(t^{-1}).$$

Since J is assumed to be slice we can rewrite this as

$$\pm H(t)H(t^{-1}) = f(t)\Delta_{7_4}(t) + 2g(t)g(t^{-1})$$

for some $H(t)$. Clearly, if H is divisible by Δ_{7_4} then $g(t)$ would also be and we would be done. So, assume that neither H or g has factor Δ_{7_4}.

Working modulo Δ_{7_4} we now have the equation:

$$(*) \qquad 2g(t)g(t^{-1}) = \pm H(t)H(t^{-1}) \ \in \ \mathbf{Z}[t, t^{-1}]/\langle 4t - 7 + 4t^{-1} \rangle.$$

There is an injection $\phi : \mathbf{Z}[t,t^{-1}]/\langle 4t-7+4t^{-1}\rangle \to \mathbf{Q}(\sqrt{-15})$ with $\phi(t^{\pm 1}) = (7 \pm \sqrt{-15})/4$. It follows that if equation $(*)$ holds then we could factor $2 = \pm(\frac{a}{c} + \frac{b}{c}\sqrt{-15})((\frac{a}{c} - \frac{b}{c}\sqrt{-15})$ with a, b, and c integers with $\gcd(a,b,c) = 1$. Simplifying we would have

$$\pm 2c^2 - a^2 - 15b^2 = 0.$$

Working modulo 5 and using that ± 2 is not a quadratic residue modulo 5, one sees immediately that a and c are both divisible by 5, which implies (working modulo 25) that b is divisible by 5 as well. Write $a = 5^s a'$, $b = 5^t b'$ and $c = 5^r c'$, with a', b', and c' relatively prime to 5. Hence:

$$\pm 2(5^{2s}c'^2) - 5^{2t}a'^2 - 3(5^{2r+1}b'^2) = 0.$$

If among the three exponents of 5 that appear in this equation there is a unique smallest exponent, then factoring out that power of 5 leaves an equation that clearly cannot hold modulo 5. Hence, there must be two exponents that are equal, and these must be the two even exponents. Factoring these out leaves the equation:

$$\pm 2c'^2 - a'^2 - 3(5^{2r'+1}b'^2) = 0.$$

Again using that ± 2 is not a quadratic residue modulo 5 gives a contradiction. □

We can now prove Theorem 4.1.

Proof of Theorem 4.1 The polynomial g determines the linking numbers of the lifts of K and S to the n-fold cyclic branched cover of S^3 branched over U as follows. Call the lifts $\bar{K}_i$ and $\bar{S}_i$ with i running from 0 to $n-1$. The linking numbers are given by equivariance and

$$\mathrm{lk}(\bar{K}_0, \bar{S}_i) = \bar{g}_i$$

where $\bar{g}_i$ is the coefficient of t^i in the reduction $\bar{g}$ of g to $\mathbf{Z}[t,t^{-1}]/\langle t^n - 1\rangle$.

In the case of the 2–fold cover we are hence interested in the even and odd index coefficients. For any integral polynomial $F(x) = \sum a_i t^i$ the sum of the even index coefficients is given by $(F(1)+F(-1))/2$ and the sum of the odd index coefficients is $(F(1)-F(-1))/2$. In our case we have seen that $g(t) = (t-1)(4t^2-7t+4)h(t)$ for some h. Hence, the sum of the even (or odd) coefficients is given by $\pm 15h(-1)$. In particular, the linking number is divisible by 15. Hence $\bar{S}_i = S_i$ is null homologous in the $L(15,4)$ obtained by surgery on K_1. □

5 Extensions

The proof that 7_4 has slicing number 2 clearly generalizes to other knots, though a general statement is somewhat technical. On the other hand, these methods seem not to apply effectively in addressing the next level of complexity—finding a knot K with $g_s(K) = 2$ but with slicing number 3.

Conjecture 5.1 *The difference $u_s(K) - g_s(K)$ can be arbitrarily large.*

In fact, this gap should be arbitrarily large even for knots with $g_s = 1$.

In retrospect, Askitas's question was optimistic. It is easily seen that if a knot can be converted into a slice knot by making n positive and n negative crossing changes, then $g_s(K) \leq n$. More generally, we have the following signed unknotting number.

Definition 5.2 For a knot K, let I denote the set of pairs of nonnegative integers (m, n) such that some collection of m positive crossing changes and n negative crossing changes converts K into a slice knot. Define the invariant $U_s(K)$ by

$$U_s(K) = \min_{(m,n)\in I} \{\max(m, n)\}$$

The following result has an elementary proof.

Theorem 5.3 *For all K, $g_s(K) \leq U_s(K)$.*

The only bounds that we know of relating to U_s are those arising from g_s, and so it is possible that $U_s(K) = g_s(K)$ for all K. However, a more likely conjecture is the following.

Conjecture 5.4 *The difference $U_s(K) - g_s(K)$ can be arbitrarily large.*

Even the following example is unknown.

Question Does $U_s(7_4) = 1$?

The example we describe below indicates that proving that $U_s(7_4) = 2$ may be quite difficult.

General Twisting One can think of performing a crossing change as grabbing two parallel strands of a knot with opposite orientation and given them one full twist. More generally, one can grab $2k$ parallel strands of K with k of the strands oriented in each direction and giving them one full twist. Call this a *generalized crossing change*. With a little care, the proof that 7_4 cannot be converted into a slice knot generalizes to show the following:

Theorem 5.5 *The knot 7_4 cannot be converted into a slice knot using a single generalized crossing change.*

On the other hand, consider Figure 2. The illustrated knot is slice since the dotted curve on the Seifert surface is unknotted and has framing 0. If a right–handed twist is put on the strands going through the circle labelled -1 and a left–handed twist is put on the strands going through the circle labelled $+1$, then the knot 7_4 results. Hence, 7_4 can be converted into a slice knot by performing one positive and one negative *generalized* crossing change.

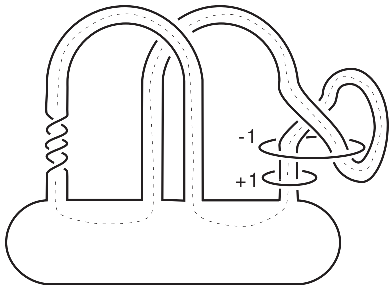

Figure 2: Twisting 7_4 to a slice knot

Since all the relevant techniques that we know of do not distinguish between crossing changes and generalized crossing changes, the difficulty associated to disproving showing that $U_s(7_4) = 2$ is now clear.

It is worth pointing out here that clearly 7_4 can be converted into a slice knot (actually the unknot) using two negative crossing changes, but an analysis of signatures and a minor generalization of the results of [CL] shows that it cannot be converted into a slice knot (or a knot with signature 0) using two positive generalized crossing changes.

Related to this discussion we have the follow result. Its proof is a bit technical to include here and will be described in detail elsewhere.

Theorem 5.6 *A knot K with 3–sphere genus $g(K)$ can be converted into the unknot using $2g(K)$ generalized crossing changes.*

Addendum (December 15, 2002) It has been pointed out to the author that results of Murakami and Yasuhara (*Four-genus and four-dimensional clasp number of a knot*, Proc. Amer. Math. Soc. 128 (2000), no. 12, 3693–3699) imply that $g_s(8_{16}) = 1$ but $u_s(8_{16}) = 2$. The methods used there are different from those of this paper.

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