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Common subbundles and intersections of divisors

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Abstract Let V_0 and V_1 be complex vector bundles over a space X. We use the theory of divisors on formal groups to give obstructions in generalised cohomology that vanish when V_0 and V_1 can be embedded in a bundle U in such a way that $V_0 \cap V_1$ has dimension at least k everywhere. We study various algebraic universal examples related to this question, and show that they arise from the generalised cohomology of corresponding topological universal examples. This extends and reinterprets earlier work on degeneracy classes in ordinary cohomology or intersection theory.

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Keywords Vector bundle, divisor, degeneracy, Thom-Porteous, formal group

1 Introduction

There are a number of different motivations for the theory developed here, but perhaps the most obvious is as follows. Suppose we have a space X with vector bundles V_0 and V_1. (Throughout this paper, the term "vector space" refers to finite-dimensional complex vector spaces equipped with Hermitian inner products, and similarly for "vector bundle".) We define the *intersection index* of V_0 and V_1 to be the largest k such that V_0 and V_1 can be embedded isometrically in some bundle U in such a way that $\dim(V_{0x} \cap V_{1x}) \geq k$ for all $x \in X$. We write $\mathrm{int}(V_0, V_1)$ for this intersection index. Our aim is to use invariants from generalised cohomology theory to estimate $\mathrm{int}(V_0, V_1)$, and to investigate the topology of certain universal examples related to this question.

We will show in Proposition 5.3 that $\mathrm{int}(V_0, V_1)$ is also the largest k such that there is a linear map $V_0 \to V_1$ of rank at least k everywhere. This creates a link with the theory of degeneracy loci and the corresponding classes in the cohomology of manifolds or Chow rings of varieties, which are given by the determinantal formula of Thom and Porteous. The paper [9] by Pragacz is a convenient reference for comparison with the present work. The relevant theory is based strongly on Schubert calculus, and could presumably be transferred to complex cobordism (and thus to other complex-orientable theories) by the methods of Bressler and Evens [1].

However, our approach will be different in a number of ways. Firstly, we use the language of formal groups, as discussed in [10] (for example). We fix an even periodic cohomology theory E with a complex orientation $x \in \widetilde{E}^0\mathbb{C}P^\infty$. For any space X we have a formal scheme $X_E = \mathrm{spf}(E^0X)$, the basic examples being $S := (\mathrm{point})_E$ and $\mathbb{G} := \mathbb{C}P^\infty_E = \mathrm{spf}(E^0[\![x]\!])$, which is a formal group over S. If V is a complex vector bundle over X, we write PV for the associated bundle of projective spaces. It is standard that $E^0(PV) = E^0(X)[\![x]\!]/f_V(x)$, where $f_V(x) = \sum_{i+j=\dim(V)} c_i x^j$, where c_i is the i'th Chern class of V. In geometric terms, this means that the formal scheme $D(V) := (PV)_E$ is naturally embedded as a divisor in $\mathbb{G} \times_S X_E$. Most of our algebraic constructions will have a very natural interpretation in terms of such divisors. We will also consider the bundle $U(V) = \coprod_{x \in X} U(V_x)$ of unitary groups associated to V. A key point is that $E^*U(V)$ is the exterior algebra over E^*X generated by $E^{*-1}PV$. This provides a very natural link with exterior algebra, and could be regarded as the "real reason" for the appearance of determinantal formulae, which seem rather accidental in other approaches. Our divisorial approach also leads to descriptions of various cohomology rings that are manifestly independent of the choice of complex orientation, and depend functorially on $\mathbb{G}$. This functorality implicitly encodes the action of stable cohomology operations and thus gives a tighter link with the underlying homotopy theory.

We were also influenced by work of Kitchloo [5], in which he investigates the cohomological effect of Miller's stable splitting of $U(n)$, and draws a link with the theory of Schur functions.

In Section 3 we use the theory of Fitting ideals to define an intersection index $\mathrm{int}(D_0, D_1)$, where D_0 and D_1 are divisors on $\mathbb{G}$. In Section 4 we identify $E^*U(V)$ with the exterior algebra generated by $E^{*-1}PV$, and show that this identification is an isomorphism of Hopf algebras. In Section 5 we use this to prove our first main theorem, that $\mathrm{int}(V_0, V_1) \leq \mathrm{int}(D(V_0), D(V_1))$; this implicitly gives all the relations among Chern classes that are universally satisfied when $\mathrm{int}(V_0, V_1) \geq k$ for some given integer k. Next, in Section 6 we study the universal examples of our various algebraic questions, focusing on the scheme $\mathrm{Int}_r(d_0, d_1)$ which classifies pairs (D_0, D_1) of divisors of degrees d_0 and d_1 such that $\mathrm{int}(D_0, D_1) \geq k$. Our next task is to construct spaces whose associated schemes are these algebraic universal examples. In Section 7 we warm up by giving a divisorial account of the generalised cohomology of Grassmannians and flag spaces, and then in Section 8 we show that the space

$$I'_r(d_0, d_1) := \{(V_0, V_1) \in G_{d_0}(\mathbb{C}^\infty) \times G_{d_1}(\mathbb{C}^\infty) \mid \dim(V_0 \cap V_1) \geq k\}$$

satisfies $I'_r(d_0, d_1)_E = \mathrm{Int}_r(d_0, d_1)$. (The origin of the present work is that the author needed to compute the cohomology of certain spaces similar to $I'_r(d_0, d_1)$ as input to another project; it would take us too far afield to discuss the background.) This completes the main work of the paper, but we have added three more sections exploring the isomorphism $E^*U(V) \simeq \lambda^* E^{*-1}PV$ in more detail. Section 9 treats some purely algebraic questions related to this situation, and in Sections 10 and 11 we translate all the algebra into homotopy theory. In particular, this gives a divisorial interpretation of the work of Mitchell, Richter and others on filtrations of $\Omega U(n)$: the scheme associated to the k'th stage in the filtration of $\Omega_X U(V)$ is $D(V)^k/\Sigma_k$, and the scheme associated to $\Omega_X U(V)$ is the free formal group over X_E generated by $D(V)$.

Appendix A gives a brief treatment of the functional calculus for normal operators, which is used in a number of places in the text.

Remark 1.1 There is a theory of degeneracy loci for morphisms with symmetries, where the formulae involve Pfaffians instead of determinants. It would clearly be a natural project to reexamine this theory from the point of view of the present paper, but so far we have nothing to say about this.

2 Notation and conventions

2.1 Spheres

We take $\mathbb{R}^n \cup \{\infty\}$ as our definition of S^n, with ∞ as the basepoint; we distinguish S^1 from the homeomorphic space $U(1) := \{z \in \mathbb{C} \mid |z| = 1\}$. Where necessary, we use the homeomorphism $\gamma\colon U(1) \to S^1$ given by

$$\gamma(z) = (z+1)(z-1)^{-1}/i$$
$$\gamma^{-1}(t) = (it+1)/(it-1).$$

One checks that $\gamma(e^{i\theta}) = \cot(-\theta/2)$, which is a strictly increasing function of θ for $0 < \theta < 2\pi$.

2.2 Fibrewise spaces

We will use various elementary concepts from fibrewise topology; the book of Crabb and James [3] is a convenient reference. Very few topological technicalities arise, as our fibrewise spaces are all fibre bundles, and the fibres are usually finite complexes.

In particular, given spaces U and V over a space X, we write $U \times_X V$ for the fibre product, and U^n_X for the fibre power $U \times_X \ldots \times_X U$. If U is pointed (in other words, it has a specified section $s\colon X \to U$) and E is any cohomology theory we write $\widetilde{E}^*_X U = E^*(U, sX)$. We also write $\Sigma_X U$ for the fibrewise suspension of U, which is the quotient of $S^1 \times U$ in which $\{\infty\} \times U \cup S^1 \times sX$ is collapsed to a copy of X. This satisfies $\widetilde{E}^*_X \Sigma_X U = \widetilde{E}^{*-1}_X U$. We also write $\Omega_X U$ for the fibrewise loop space of U, which is the space of maps $\omega\colon S^1 \to U$ such that the composite $S^1 \to U \to X$ is constant and $\omega(\infty) \in sX$. If V is another pointed space over X, we write $U \wedge_X V$ for the fibrewise smash product. If W is an unpointed space over X then we write $W_{+X} = W \amalg X$, which is a pointed space over X in an obvious way.

2.3 Tensor products over schemes

If T is a scheme and M, N are modules over the ring $\mathcal{O}_T$, we will write $M \otimes_T N$ for $M \otimes_{\mathcal{O}_T} N$. Similarly, we write $\lambda^k_T M$ for $\lambda^k_{\mathcal{O}_T} M$, the k'th exterior power of M over $\mathcal{O}_T$.

2.4 Free modules

Given a ring R and a set T, we write $R\{T\}$ for the free R-module generated by T.

3 Intersections of divisors

Let $\mathbb{G}$ be a commutative, one-dimensional formal group over a scheme S. Choose a coordinate x so that $\mathcal{O}_{\mathbb{G}} = \mathcal{O}_S[\![x]\!]$. Let D_0 and D_1 be divisors on $\mathbb{G}$ defined over S, with degrees d_0 and d_1 respectively. This means that $\mathcal{O}_{D_i} = \mathcal{O}_{\mathbb{G}}/f_i = \mathcal{O}_S[\![x]\!]/f_i(x)$ for some monic polynomial $f_i(x)$ of degree d_i such that $f_i(x) = x^{d_i}$ modulo nilpotents. It follows that $\mathcal{O}_{D_i}$ is a free module of rank d_i over $\mathcal{O}_S$, with basis $\{x^j \mid 0 \leq j < d_i\}$.

As D_0 and D_1 are closed subschemes of $\mathbb{G}$ we can form their intersection, so that

$$\mathcal{O}_{D_0\cap D_1} = \mathcal{O}_{\mathbb{G}}/(f_0, f_1) = \mathcal{O}_S[\![x]\!]/(f_0(x), f_1(x)).$$

Typically this will not be a projective module over $\mathcal{O}_S$, so some thought is required to give a useful notion of its size. We will use a measure coming from the theory of Fitting ideals, which we now recall briefly.

Let R be a commutative Noetherian ring, and let M be a finitely generated R-module. We can then find a presentation $P_1 \xrightarrow{\phi_1} P_0 \xrightarrow{\phi_0} M$, where P_0 and P_1 are finitely generated projective modules of ranks p_0 and p_1 say, and $M = \operatorname{cok}(\phi_1)$. The exterior powers $\lambda^j P_i$ are again finitely generated projective modules. We define $I_j(\phi_1)$ to be the smallest ideal in R modulo which we have $\lambda^j(\phi_1) = 0$. More concretely, if P_0 and P_1 are free then ϕ_1 can be represented by a matrix A and $I_j(\phi_1)$ is generated by the determinants of all $j\times j$ submatrices of A. We then define $I_j(M) = I_{p_0-j}(\phi_1)$; this is called the j'th Fitting ideal of M. It is a fundamental fact that this is well-defined; this was already known to Fitting (see [8, Chapter 3], for example), but we give a proof for the convenience of the reader.

Proposition 3.1 *The ideal $I_j(M)$ is independent of the choice of presentation of M.*

Proof We temporarily write $I_j(M, P_*, \phi_*)$ for the ideal called $I_j(M)$ above.

Put $N = \ker(\phi_0)$ and let $\beta\colon N \to P_0$ be the inclusion. Then ϕ_1 factors as $P_1 \xrightarrow{\alpha} N \xrightarrow{\beta} P_0$, where α is surjective. For any ideal $J \leq R$ we see that $\lambda^k\alpha$ is surjective mod J, so $\lambda^k\phi_1$ is zero mod J iff $\lambda^k\beta$ is zero mod J. This condition depends only on the map $\phi_0\colon P_0 \to M$, so we can legitimately define $I_j(M, P_0, \phi_0) := I_j(M, P_*, \phi_*)$.

Now suppose we have another presentation $Q_1 \xrightarrow{\psi_1} Q_0 \xrightarrow{\psi_0} M$, where Q_i has rank q_i. Define $\chi_0\colon P_0 \oplus Q_0 \to M$ by $(u, v) \mapsto \phi_0(u) + \psi_0(v)$. It will suffice to prove that

$$I_j(M, P_0, \phi_0) = I_j(M, P_0 \oplus Q_0, \chi_0) = I_j(M, Q_0, \psi_0),$$

and by symmetry we need only check the first of these. By projectivity we can choose a map $\theta\colon Q_0 \to P_0$ with $\phi_0\theta = \psi_0$, and define $\chi_1\colon P_1\oplus Q_0 \to P_0\oplus Q_0$ by $(u, v) \mapsto (\phi_1(u) - \theta(v), v)$. It is easy to check that this gives another presentation

$$P_1 \oplus Q_0 \xrightarrow{\chi_1} P_0 \oplus Q_0 \xrightarrow{\chi_0} M.$$

If $k \leq q_0$ then $\lambda^k\chi_1$ is certainly nonzero, because the composite

$$\lambda^k Q_0 \to \lambda^k(P_1 \oplus Q_0) \xrightarrow{\lambda^k\chi} \lambda^k(P_0 \oplus Q_0) \to \lambda^k Q_0$$

is the identity, and $\lambda^k Q_0 \neq 0$. If $k > q_0$ and $\lambda^k\chi_1 = 0$ then (by restricting to $\lambda^{k-q_0}P_1 \otimes \lambda^{q_0}Q_0$) we see that $\lambda^{k-q_0}\phi_1 = 0$.

For the converse, notice that $\lambda^* N$ is a graded ring for any module N, and that $\lambda^*\alpha$ is a ring map for any homomorphism α of R-modules. One can check that $\lambda^{j+q_0}(P_1 \oplus Q_0)$ is contained in the ideal in $\lambda^*(P_1 \oplus Q_0)$ generated by $\lambda^j P_1$. It follows that if $\lambda^j\phi_1 = 0$ then $\lambda^{j+q_0}\chi_1 = 0$.

This shows that $I_r(\phi_1) = I_{r+q_0}(\chi_1)$, and thus that $I_r(M, P_0, \phi_0) = I_r(M, P_0 \oplus Q_0, \chi_0)$, as required. □

It is clear that

$$I_0(M) \leq \ldots \leq I_m(M) = R,$$

and we define

$$\operatorname{rank}(M) = \operatorname{rank}_R(M) = \min\{r \mid I_r(M) \neq 0\}.$$

We call $\operatorname{rank}(M)$ the *Fitting rank* of M. For example, if R is a principal ideal domain with fraction field K, one can check that $\operatorname{rank}(M) = \dim_K(K \otimes_R M)$ for all M. However, we will mostly be interested in rings R with many nilpotents, for which there is no such simple formula.

The following lemma is easily checked from the definitions.

Lemma 3.2 (a) *The Fitting rank is the same as the ordinary rank for projective modules.*

(b) *If N is a quotient of M then* $\operatorname{rank}(N) \leq \operatorname{rank}(M)$.

(c) *If there is a presentation $P \to Q \to M$ then* $\operatorname{rank}(Q) - \operatorname{rank}(P) \leq \operatorname{rank}(M) \leq \operatorname{rank}(Q)$. □

(It is not true, however, that $\operatorname{rank}(M \oplus N) = \operatorname{rank}(M) + \operatorname{rank}(N)$; indeed, if $a \neq 0$ and $a^2 = 0$ then $\operatorname{rank}(R/a) = 0$ but $\operatorname{rank}(R/a \oplus R/a) = 1$.)

Definition 3.3 The *intersection multiplicity* of D_0 and D_1 is the integer

$$\operatorname{int}(D_0, D_1) := \operatorname{rank}_{\mathcal{O}_S}(\mathcal{O}_{D_0 \cap D_1}).$$

We also put

$$\operatorname{Int}_r(D_0, D_1) = \operatorname{spec}(\mathcal{O}_S / I_{r-1}(\mathcal{O}_{D_0 \cap D_1})),$$

which is the largest subscheme of S over which we have $\operatorname{int}(D_0, D_1) \geq r$.

Remark 3.4 Let S' be a scheme over S, so that $\mathbb{G}' := \mathbb{G} \times_S S'$ is a formal group over S'. We refer to divisors on $\mathbb{G}'$ as divisors on $\mathbb{G}$ over S'. Given two such divisors D_0 and D_1, we get a closed subscheme $\mathrm{Int}_r(D_0, D_1) \subseteq S'$. We will use this kind of base-change construction throughout the paper without explicit comment.

To make the above definitions more explicit, we will describe several different presentations of $\mathcal{O}_{D_0 \cap D_1}$ that can be used to determine its rank.

Construction 3.5 First, recall that we can form the divisor

$$D_0 + D_1 = \mathrm{spec}(\mathcal{O}_{\mathbb{G}}/f_0 f_1) = \mathrm{spec}(\mathcal{O}_S[\![x]\!]/f_0(x) f_1(x)).$$

This contains D_0 and D_1, so we have a pullback square of closed inclusions as shown on the left below. This gives a pushout square of $\mathcal{O}_S$-algebras as shown on the right.

$$\begin{array}{ccccccc} D_0 \cap D_1 & \rightarrowtail & D_0 & \qquad & \mathcal{O}_{D_0 \cap D_1} & \twoheadleftarrow & \mathcal{O}_{D_0} \\ \downarrow & & \downarrow & & \uparrow & & \uparrow \\ D_1 & \rightarrowtail & D_0 + D_1 & & \mathcal{O}_{D_1} & \twoheadleftarrow & \mathcal{O}_{D_0 + D_1}, \end{array}$$

which gives a presentation

$$\mathcal{O}_{D_0 + D_1} \to \mathcal{O}_{D_0} \oplus \mathcal{O}_{D_1} \to \mathcal{O}_{D_0 \cap D_1}.$$

Explicitly, this is just the presentation

$$\mathcal{O}_{\mathbb{G}}/(f_0 f_1) \xrightarrow{\phi} \mathcal{O}_{\mathbb{G}}/f_0 \oplus \mathcal{O}_{\mathbb{G}}/f_1 \xrightarrow{\psi} \mathcal{O}_{\mathbb{G}}/(f_0, f_1)$$

given by

$$\begin{aligned} \phi(g \bmod f_0 f_1) &= (g \bmod f_0, -g \bmod f_1) \\ \psi(g_0 \bmod f_0, g_1 \bmod f_1) &= g_0 + g_1 \bmod (f_0, f_1). \end{aligned}$$

Although this is probably the most natural presentation, it is not easy to write down the effect of ϕ on the obvious bases of $\mathcal{O}_{\mathbb{G}}/(f_0 f_1)$ and $\mathcal{O}_{\mathbb{G}}/f_i$. To remedy this, we give an alternate presentation.

Construction 3.6 Let J_i be the ideal generated by f_i and put $J = J_0 J_1$. Then J_i/J is free over $\mathcal{O}_S$ with basis $\{x^j f_i(x) \mid 0 \le j < d_{1-i}\}$ and the inclusion maps $J_i \to \mathcal{O}_{\mathbb{G}}$ give rise to a presentation

$$J_0/J \oplus J_1/J \xrightarrow{\zeta} \mathcal{O}_{\mathbb{G}}/J = \mathcal{O}_{D_0 + D_1} \xrightarrow{\xi} \mathcal{O}_{\mathbb{G}}/(J_0 + J_1) = \mathcal{O}_{D_0 \cap D_1}.$$

Let c_{ij} be the coefficient of x^{d_i-j} in $f_i(x)$, so that $c_{i0}=1$ and $f_i(x)=\sum_{d_i=j+k}c_{ij}x^k$. Then

$$\zeta(x^j f_0(x),0)=\sum_{k=j}^{d_0+j}c_{0,d_0+j-k}x^k \qquad \text{for } 0\leq j<d_1$$
$$\zeta(0,x^j f_1(x))=\sum_{k=j}^{d_1+j}c_{1,d_1+j-k}x^k \qquad \text{for } 0\leq j<d_0,$$

and this tells us the matrix for ζ in terms of the obvious bases of $J_0/J\oplus J_1/J$ and $\mathcal{O}_{\mathbb{G}}/J$. For example, if $d_0=2$ and $d_1=3$ the matrix is

$$\left(\begin{array}{ccc|cc} c_{02} & 0 & 0 & c_{13} & 0 \\ c_{01} & c_{02} & 0 & c_{12} & c_{13} \\ 1 & c_{01} & c_{02} & c_{11} & c_{12} \\ 0 & 1 & c_{01} & 1 & c_{11} \\ 0 & 0 & 1 & 0 & 1 \end{array}\right)$$

In general, we have a square matrix with d_0+d_1 rows and columns. The left hand block consists of d_1 columns, each of which contains d_1-1 zeros. The right hand block consists of d_0 columns, each of which contains d_0-1 zeros. Clearly $\mathrm{Int}_r(D_0,D_1)$ is the closed subscheme defined by the vanishing of the minors of this matrix of size d_0+d_1-r+1. In particular, $\mathrm{Int}_1(D_0,D_1)$ is defined by the vanishing of the determinant of the whole matrix, which is classically known as the *resultant* of f_0 and f_1. If $f_0(x)=\prod_i(x-a_i)$ and $f_1(x)=\prod_j(x-b_j)$ then the resultant is just $\prod_{i,j}(a_i-b_j)$. We do not know of any similar formula for the other minors.

Construction 3.7 For a smaller but less symmetrical presentation, we can just use the sequence $J_1/J\to\mathcal{O}_{\mathbb{G}}/J_0\to\mathcal{O}_{\mathbb{G}}/(J_0+J_1)$ induced by the inclusion of J_1 in $\mathcal{O}_{\mathbb{G}}$. This is isomorphic to the presentation $\mathcal{O}_{\mathbb{G}}/J_0\xrightarrow{\mu_1}\mathcal{O}_{\mathbb{G}}/J_0\to\mathcal{O}_{\mathbb{G}}/(J_0+J_1)$, where $\mu_1(g)=f_1g$. However, the isomorphism depends on a choice of coordinate on $\mathbb{G}$ (because the element f_1 does), so the previous presentation is sometimes preferable. There is of course a similar presentation $\mathcal{O}_{\mathbb{G}}/J_1\xrightarrow{\mu_0}\mathcal{O}_{\mathbb{G}}/J_1\to\mathcal{O}_{\mathbb{G}}/(J_0+J_1)$.

Finally, we give a presentation that depends only on the formal Laurent series f_0/f_1 and thus makes direct contact with the classical Thom-Porteous formula.

Construction 3.8 Write $\mathcal{M}_G=R((x))=\mathcal{O}_{\mathbb{G}}[x^{-1}]$. Note that $f_1(x)/x^{d_1}$ is a polynomial in x^{-1} whose constant term is 1 and whose other coefficients are nilpotent, so it is a unit in $R[x^{-1}]$. It follows that f_1 is a unit in $R(\!(x)\!)$. Put $Q = x^{-1}R[x^{-1}] \subseteq R(\!(x)\!)$, so that $R(\!(x)\!) = R[\![x]\!] \oplus Q$. Multiplication by the series $x^{d_1} f_0/f_1$ induces a map

$$P_1 = \frac{R[\![x]\!]}{f_1 R[\![x]\!]} \xrightarrow{\phi} P_0 = \frac{R(\!(x)\!)}{x^{d_1} R[\![x]\!] \oplus Q}.$$

We claim that the cokernel of ϕ is isomorphic to $R[\![x]\!]/(f_0, f_1) = \mathcal{O}_{D_0 \cap D_1}$, so we have yet another presentation of this ring. Indeed, the cokernel of ϕ is clearly given by $R(\!(x)\!)/(x^{d_1} f_0 f_1^{-1} R[\![x]\!] + x^{d_1} R[\![x]\!] + Q)$. The element f_1/x^{d_1} is invertible in $R[x^{-1}]$ so it is invertible in $R(\!(x)\!)$ and satisfies $(f_1/x^{d_1})Q = Q$. Thus, multiplication by this element gives an isomorphism

$$\frac{R(\!(x)\!)}{x^{d_1} f_0 f_1^{-1} R[\![x]\!] + x^{d_1} R[\![x]\!] + Q} \simeq \frac{R(\!(x)\!)}{f_0 R[\![x]\!] + f_1 R[\![x]\!] + Q}.$$

As $R(\!(x)\!) = R[\![x]\!] \oplus Q$, we see that the right hand side is just $R[\![x]\!]/(f_0, f_1)$ as claimed.

The elements $\{1, x, \dots, x^{d_1 - 1}\}$ give a basis for both P_0 and P_1, and the matrix elements of ϕ with respect to these bases are just the coefficients of f_0/f_1 (suitably indexed). More precisely, we have

$$f_0/f_1 = x^{d_0 - d_1} \sum_{i \geq 0} c_i x^{-i},$$

where $c_0 = 1$ and c_i is nilpotent for $i > 0$. We take $c_i = 0$ for $i < 0$ by convention. The matrix elements Φ_{ij} of ϕ are then given by $\Phi_{ij} = c_{d_0 + i - j}$ for $0 \leq i, j < d_1$. For example, if $d_0 = 3$ and $d_1 = 5$ then the matrix is

$$\Phi = \begin{pmatrix} c_3 & c_4 & c_5 & c_6 & c_7 \\ c_2 & c_3 & c_4 & c_5 & c_6 \\ c_1 & c_2 & c_3 & c_4 & c_5 \\ 1 & c_1 & c_2 & c_3 & c_4 \\ 0 & 1 & c_1 & c_2 & c_3 \end{pmatrix}.$$

Now suppose that our divisors D_i arise in the usual way from vector bundles V_i over a stably complex manifold X, and we have a generic linear map $g\colon V_0 \to V_1$. Let Z_r be the locus where the rank of g is at most r, and let $i\colon Z_r \to X$ be the inclusion. Generically, this will be a smooth stably complex submanifold of X, so we have a class $z_r = i_*[Z_r] \in E^0 X$. The Thom-Porteous formula says that $z_r = \det(\Psi_r)$, where Ψ_r is the square block of size $d_1 - r$ taken from the bottom left of Φ. More explicitly, the matrix elements are $(\Psi_r)_{ij} = c_{d_0 - k + i - j}$ for $0 \leq i, j < d_1 - r$. Clearly $\det(\Psi_r) \in I_{d_1 - r}(\phi) = I_r(\mathcal{O}_{D_0 \cap D_1})$. If Z_r is empty then $z_r = 0$. On the other hand, Proposition 5.3 will tell us that

$\mathrm{int}(D_0, D_1) > r$ and so $I_r(\mathcal{O}_{D_0 \cap D_1}) = 0$, so $\det(\Psi_r) = 0$, which is consistent with the Thom-Porteous formula. It is doubtless possible to prove the formula using the methods of this paper, but we have not yet done so.

Proposition 3.9 *We always have* $\mathrm{int}(D_0, D_1) \leq \min(d_0, d_1)$ *(unless the base scheme S is empty). If* $\mathrm{int}(D_0, D_1) = d_0$ *then $D_0 \leq D_1$, and if* $\mathrm{int}(D_0, D_1) = d_1$ *then $D_1 \leq D_0$.*

Proof The presentation $\mathcal{O}_{D_1} \xrightarrow{\mu_0} \mathcal{O}_{D_1} \to \mathcal{O}_{D_0 \cap D_1}$ shows that

$$\mathrm{int}(D_0, D_1) = \mathrm{rank}(\mathcal{O}_{D_0 \cap D_1}) \leq \mathrm{rank}(\mathcal{O}_{D_1}) = d_1.$$

If this is actually an equality we must have $\lambda^{d_1 - d_1 + 1}(\mu_0) = 0$ or in other words $\mu_0 = 0$, so $f_0 = 0 \pmod{f_1}$, so $D_1 \leq D_0$. The remaining claims follow by symmetry. □

Proposition 3.10 *If there is a divisor D of degree k such that $D \leq D_0$ and $D \leq D_1$, then* $\mathrm{int}(D_0, D_1) \geq k$.

Proof Clearly $\mathcal{O}_D$ is a quotient of the ring $\mathcal{O}_{D_0 \cap D_1}$, and it is free of rank k, so $\mathrm{int}(D_0, D_1) = \mathrm{rank}(\mathcal{O}_{D_0 \cap D_1}) \geq k$. □

Definition 3.11 Given two divisors D_0, D_1, we write $\mathrm{Sub}_r(D_0, D_1)$ for the scheme of divisors D of degree r such that $D \leq D_0$ and $D \leq D_1$. The proposition shows that the projection $\pi\colon \mathrm{Sub}_r(D_0, D_1) \to S$ factors through the closed subscheme $\mathrm{Int}_r(D_0, D_1)$.

Remark 3.12 Proposition 3.9 implies that $\mathrm{Int}_{d_0}(D_0, D_1)$ is just the largest closed subscheme of S over which we have $D_0 \leq D_1$. From this it is easy to see that $\mathrm{Sub}_{d_0}(D_0, D_1) = \mathrm{Int}_{d_0}(D_0, D_1)$.

It is natural to expect that the map $\pi\colon \mathrm{Sub}_r(D_0, D_1) \to \mathrm{Int}_r(D_0, D_1)$ should be surjective in some suitable sense. Unfortunately this does not work as well as one might hope: the map π is not faithfully flat or even dominant, so the corresponding ring map π^* need not be injective. However, it is injective in a certain universal case, as we shall show in Section 6.

We conclude this section with an example where π^* is not injective. Let $\mathbb{G}$ be the additive formal group over the scheme $S = \mathrm{spec}(\mathbb{Z}[a]/a^2)$. Let D_0 and D_1 be the divisors with equations $x^2 - a$ and x^2, respectively. Then $\mathcal{O}_{D_0 \cap D_1} = \mathcal{O}_S[x]/(x^2 - a, x^2) = \mathcal{O}_S[x]/(a, x^2)$, which is the cokernel of the map

$\mu\colon \mathcal{O}_S[x]/x^2 \to \mathcal{O}_S[x]/x^2$ given by $\mu(t) = at$. The matrix of μ is $\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}$ which is clearly nonzero, but $\lambda^2(\mu) = a^2 = 0$. It follows that $\operatorname{int}(D_0, D_1) = 1$, so $\operatorname{Int}_1(D_0, D_1) = S$. However, $\operatorname{Sub}_1(D_0, D_1)$ is just the scheme $D_0 \cap D_1 = \operatorname{spec}(\mathcal{O}_S[x]/(a, x^2))$, so $\pi^*(a) = 0$.

For a topological interpretation, let V_0 be the tautological bundle over $\mathbb{H}P^1 = S^4$, and let V_1 be the trivial rank two complex bundle. If we use the cohomology theory $E^*Y = (H^*Y)[u, u^{-1}]$ (with $|u| = 2$) and let a be the second Chern class of V_0 we find that $E^0X = \mathbb{Z}[a]/a^2$, and the equations of $D(V_0)$ and $D(V_1)$ are $x^2 - a$ and x^2. Using the theory to be developed in Section 5 and the calculations of the previous paragraph, we deduce that V_0 and V_1 cannot have a common subbundle of rank one, but there is no cohomological obstruction to finding a map $f\colon V_0 \to V_1$ with rank at least 1 everywhere. To see that such a map does in fact exist, choose a subspace $W < \mathbb{H}^2$ which is a complex vector space of dimension 2, but not an $\mathbb{H}$-submodule. We can then take the constant bundle with fibre $\mathbb{H}^2/W$ as a model for V_1. The bundle V_0 is by definition a subbundle of the constant bundle with fibre $\mathbb{H}^2$, so there is an evident projection map $f\colon V_0 \to V_1$. As W is not an $\mathbb{H}$-submodule, we see that f is nowhere zero and thus has rank at least one everywhere, as claimed.

4 Unitary bundles

In order to compare the constructions of the previous section with phenomena in topology, we need a topological interpretation of the exterior powers $\lambda^k\mathcal{O}_D$ when D is the divisor associated to a vector bundle.

Let V be a complex vector bundle of dimension d over a space X. We can thus form a bundle $U(V)$ of unitary groups in the evident way (so $U(V) = \{(x, g) \mid x \in X \text{ and } g \in U(V_x)\}$). The key point is that $E^*U(V)$ can be naturally identified with $\lambda^*_{E^*X} E^{*-1}PV$ (the exterior algebra over the ring E^*X generated by the module E^*PV). Moreover, we can use the group structure on $U(V)$ to make $E^*U(V)$ into a Hopf algebra over E^*X, and we can make $\lambda^*_{E^*X} E^{*-1}PV$ into a Hopf algebra by declaring E^*PV to be primitive. We will need to know that our isomorphism respects these structures. All this is of course well-known when X is a point and E represents ordinary cohomology. Kitchloo [5] has shown that if one chooses the right proof then the restriction on E can be removed. With just a few more words, we will be able to remove the restriction on X as well.

We start by comparing $U(V)$ with a suitable classifying space. First let V be a vector space rather than a bundle. We let $EU(V)$ denote the geometric realisation of the simplicial space $\{U(V)^{n+1}\}_{n\geq 0}$ and we put $BU(V)=EU(V)/U(V)$, which is the usual simplicial model for the classifying space of $U(V)$. There is a well-known map $\eta\colon U(V)\to\Omega BU(V)$, which is a weak equivalence of H-spaces. By adjunction we have a map $\zeta\colon \Sigma U(V)\to BU(V)$, which gives a map

$$\zeta^*\colon \widetilde{E}^*BU(V)\to\widetilde{E}^*\Sigma U(V)=\widetilde{E}^{*-1}U(V).$$

The fact that η is an H-map means that ζ is primitive, or in other words that

$$\zeta\circ\mu=\zeta\circ(\pi_0+\pi_1)\in[\Sigma U(V)^2,BU(V)].$$

We can also construct a tautological bundle $T=EU(V)\times_{U(V)}V$ over $BU(V)$.

We now revert to the case where V is a vector bundle over a space X, and perform all the above constructions fibrewise. Firstly, we construct the bundle $BU(V)=\{(x,e)\mid x\in X \text{ and } e\in BU(V_x)\}$. Note that each space $BU(V_x)$ has a canonical basepoint, and using these we get an inclusion $X\to BU(V)$.

A slightly surprising point is that there is a canonical homotopy equivalence $BU(V)\to X\times BU(d)$. Indeed, we can certainly perform the definition of T fibrewise to get a tautological bundle over $BU(V)$, which is classified by a map $q\colon BU(V)\to BU(d)$, which is unique up to homotopy. We can combine this with the projection $p\colon BU(V)\to X$ to get a map $f=(p,q)\colon BU(V)\to X\times BU(d)$. The map p is a fibre bundle projection, and the restriction of q to each fibre of p is easily seen to be an equivalence. It is now an easy exercise with the homotopy long exact sequence of p to see that f is a weak equivalence. (Nothing untoward happens with π_0 and π_1 because $BU(d)$ is simply connected.)

Remark 4.1 Let $q_0\colon X\to BU(d)$ be the restriction of q. Then q_0 classifies the bundle $T|_X\simeq V$, so in general it will be an essential map. Thus, if we just use the basepoint of $BU(d)$ to make $X\times BU(d)$ into a based space over X, then our equivalence $f\colon BU(V)\simeq X\times BU(d)$ does not preserve basepoints, and cannot be deformed to do so. If it did preserve basepoints we could apply the fibrewise loop functor Ω_X and deduce that $U(V)\simeq X\times U(d)$, but this is false in general.

It follows from the above that $E^*BU(V)$ is a formal power series algebra over E^*X, generated by the Chern classes of T. It will be convenient for us to modify this description slightly by considering the virtual bundle $T-V$ (where V is implicitly pulled back to $BU(V)$ by the map $p\colon BU(V)\to X$). We have $f_T(t) = t^d \sum_{k=0}^{d} a_k t^{-k}$ and $f_V(t) = t^d \sum_{k=0}^{d} b_k t^{-k}$ for some coefficients $a_k \in E^0BU(V)$ and $b_k \in E^0X$ so $f_{T-V}(t) = f_T(t)/f_V(t) = \sum_{k\geq 0} c_k t^{-k}$ for some $c_k \in E^0BU(V)$. For $k \leq d$ we have $c_k = a_k \pmod{b_1, \dots, b_d}$ and it follows easily that

$$E^*BU(V) = (E^*X)[\![c_1, \dots, c_d]\!].$$

Note that the restriction of $T - V$ to $X \subset BU(V)$ is trivial, so the classes c_k restrict to zero on X.

Next, consider the fibrewise suspension $\Sigma_X U(V)$. By dividing each fibre into two cones we obtain a decomposition $\Sigma_X U(V) = C_0 \cup C_1$ where the inclusion of X in each C_i is a homotopy equivalence, and $C_0 \cap C_1 = U(V)$. Using a Mayer-Vietoris sequence we deduce that $\widetilde{E}^*_X \Sigma_X U(V) \simeq \widetilde{E}^{*-1}U(V)$ and that this can be regarded as an ideal in $E^*\Sigma_X U(V)$ whose square is zero. Moreover, the construction of ζ can be carried out fibrewise to get a map $\Sigma_X U(V) \to BU(V)$ which is again primitive. It follows that ζ induces a map

$$\zeta^* \colon \operatorname{Ind}(E^*BU(V)) \to \operatorname{Prim}(E^{*-1}U(V)).$$

(Here Ind and Prim denote indecomposables and primitives over E^*X.) Note also that $\operatorname{Ind}(E^*BU(V))$ is a free module over E^*X generated by $\{c_1, \dots, c_d\}$.

To prove that ζ^* is injective, we need to consider the complex reflection map $\rho \colon \Sigma_X PV_{+X} \to U(V)$, which we define as follows. For $t \in S^1 = \mathbb{R} \cup \{\infty\}$ and $x \in X$ and $L \in PV_x$, the map $\rho(t, x, L)$ is the endomorphism of V_x that has eigenvalue $\gamma^{-1}(t)$ on the line L, and eigenvalue 1 on $L^\perp$. Here $\gamma^{-1}(t) = (it + 1)/(it - 1) \in U(1)$, as in Section 2.1. Using this we obtain a map $\xi = \zeta \circ \Sigma_X \rho \colon \Sigma_X^2 PV_{+X} \to BU(V)$.

Our next problem is to identify the virtual bundle $\xi^*(T - V)$ over $\Sigma_X^2 PV_{+X}$. For this it is convenient to identify S^2 with $\mathbb{C}P^1$ and thus $\Sigma^2 PV_{+X}$ with a quotient of $\mathbb{C}P^1 \times PV$. We have tautological bundles H and L over $\mathbb{C}P^1$ and PV, whose Euler classes we denote by y and x.

Lemma 4.2 *We have $\xi^*(T - V) \simeq (H - 1) \otimes L$. Moreover, there is a power series $g(s) \in E^0[\![s]\!]$ with $g(0) = 1$ such that $\xi^* c_k = -yx^{k-1}g(x)$ for $k = 1, \dots, d$. (If E^0 is torsion-free then $g(s) = 1/\log_F'(x)$.)*

Proof In the proof it will be convenient to write T_V and L_V instead of T and L, to display the dependence on V.

First consider the case where X is a point and $V = \mathbb{C}$. Then $\rho \colon S^1 \to U(1) = U(\mathbb{C})$ is a homeomorphism and $BU(\mathbb{C}) \simeq \mathbb{C}P^\infty$. It is a standard fact that

$\xi\colon S^2 \to BU(\mathbb{C})$ can be identified with the inclusion $\mathbb{C}P^1 \to \mathbb{C}P^\infty$, and thus that $\xi^* T_{\mathbb{C}} = H$.

In the general case, note that we have a map $\xi_L\colon \mathbb{C}P^1 \times PL \to BU(L)$ of spaces over PV. The projection $PL \to PV$ is a homeomorphism which we regard as the identity. If we let $\pi\colon PV \to X$ be the projection, we have a splitting $\pi^* V = L \oplus (\pi^* V \ominus L)$. The inclusion $L \to \pi^* V$ gives an inclusion $U(L) \to \pi^* U(V)$ and thus an inclusion $BU(L) \to \pi^* BU(V)$, or equivalently a map $\phi\colon BU(L) \to BU(V)$ covering π. As $T_V = V \times_{U(V)} EU(V)$ and $U(L)$ acts trivially on $\pi^* V \ominus L$ we see that $\phi^* T_V = T_L \oplus (\pi^* V \ominus L)$.

Next, we note that tensoring with L gives an isomorphism $\tau\colon U(\mathbb{C}) \times PV \to U(L)$ and thus an isomorphism $B\tau\colon BU(\mathbb{C}) \times PV \to BU(L)$ with $(B\tau)^* T_L = T_{\mathbb{C}} \otimes L$.

One can check that the following diagram commutes:

$$\begin{tikzcd}
\mathbb{C}P^1 \times PV \arrow[r, "1"] \arrow[d, "\xi_{\mathbb{C}} \times 1"'] & \mathbb{C}P^1 \times PV \arrow[r, "1"] \arrow[d, "\xi_L"'] & \mathbb{C}P^1 \times PV \arrow[d, "\xi_V"] \\
BU(\mathbb{C}) \times PV \arrow[r, "\simeq", "B\tau"'] & BU(L) \arrow[r, "\phi"'] & BU(V).
\end{tikzcd}$$

It follows that $\xi_V^* T_V \simeq (\xi_{\mathbb{C}} \times 1)^* (B\tau)^* \phi^* T_V$, and the previous discussion identifies this with $(H \otimes L) \oplus (\pi^* V \ominus L)$. It follows that $\xi_V^*(T_V - V) \simeq (H \otimes L) - L = (H - 1) \otimes L$, as claimed.

Now let $g(s)$ be the partial derivative of $t +_F s$ with respect to t evaluated at $t = 0$. This is characterised by the equation $t +_F s = tg(s) + s \pmod{t^2}$; it is clear that $g(0) = 1$, and by applying $\log_F$ we see that $g(s) = 1/\log_F'(s)$ in the torsion-free case. As $y^2 = 0$ we see that the Euler class of $H \otimes L$ is $x +_F y = x + yg(x)$. Thus, we have

$$\begin{aligned}
f_{H \otimes L - L}(t) &= (t - x - yg(x))/(t - x) \\
&= 1 - yg(x)t^{-1}/(1 - x/t) \\
&= 1 - \sum_{k>0} yg(x)x^{k-1}t^{-k}.
\end{aligned}$$

The k'th Chern class of $(H-1) \otimes L$ is the coefficient of t^{-k} in this series, which is $-yg(x)x^{k-1}$ as claimed. □

Corollary 4.3 *The induced map* $\xi^*\colon \mathrm{Ind}(E^* BU(V)) \to E^*(\Sigma^2_X PV_{+X}, X) = E^{*-2} PV$ *is an isomorphism.* □

Theorem 4.4 *There is a natural isomorphism $\lambda^* E^{*-1}PV \to E^*U(V)$ of Hopf algebras over E^*X.*

Proof Put $a_i = \xi^* c_i \in \mathrm{Prim}(E^*U(V))$ for $i = 1, \dots, d$. Given a sequence $I = (i_1, \dots, i_r)$ with $1 \leq i_1 < \dots < i_r \leq d$, put $a_I = \prod_j a_{i_j}$. We first claim that the elements a_I form a basis for $E^*U(V)$ over E^*X. This is very well-known in the case where X is a point (so $U(V) \simeq U(d)$) and E represents ordinary cohomology; it can proved using the Serre spectral sequence of the fibration $U(d-1) \to U(d) \to S^{2d-1}$. For a more general theory E we still have an Atiyah-Hirzebruch-Serre spectral sequence $H^p(S^{2d-1}; E^q U(d-1)) \Longrightarrow E^{p+q}U(d)$. It follows easily that the elements a_I form a basis whenever X is a point. A standard argument now shows that they form a basis for any X. Indeed, it follows easily from the above that they form a basis whenever V is trivialisable. We can give X a cell structure such that V is trivialisable over each cell, and then use Mayer-Vietoris sequences to check that the elements a_I form a basis whenever X is a finite complex. Finally, we can use the Milnor exact sequence to show that the elements a_I form a basis for all X.

The ring $E^*U(V)$ is graded-commutative so we certainly have $a_i a_j = -a_j a_i$ and in particular $2a_i^2 = 0$ for all i. Suppose we can show that $a_i^2 = 0$. Then ζ^* extends to give a map $\lambda^*_{E^*X} \mathrm{Ind}(E^*BU(V)) \to E^{*-1}U(V)$ of Hopf algebras, and from the previous paragraph we see that this is an isomorphism. Combining this with the isomorphism of Corollary 4.3 gives the required isomorphism $\lambda^* E^{*-1}PV \to E^*U(V)$.

All that is left is to check that $a_i^2 = 0$. For this we consider the case of the tautological bundle T over $BU(d)$, and take $E = MP = MU[u, u^{-1}]$. (We use this 2-periodic version of MU simply to comply with our standing assumptions on E; we could equally well use MU itself.) Here it is standard that $MP^*BU(d)$ is a formal power series algebra over MP^* and thus is torsion-free. The ring $MP^*U(T)$ is a free module over $MP^*BU(d)$ and thus is also torsion-free. As $2a_i^2 = 0$ we must have $a_i^2 = 0$ as required. More generally, for an arbitrary bundle V over a space X we have a classifying map $X \to BU(d)$ giving rise to a map $U(V) \to U(T)$. Moreover, for any E we can choose an orientation in degree zero and thus a ring map $MP \to E$. Together these give a ring map $MP^*U(T) \to E^*U(V)$, which carries a_i to a_i. As $a_i^2 = 0$ in $MP^*U(T)$, the same must hold in $E^*U(V)$. □

We will need to extend the above result slightly to give a topological interpretation of the quotient rings

$$\lambda^{\leq r} E^{*-1}PV = \lambda^* E^{*-1}PV / \lambda^{>r} E^{*-1}PV.$$

For this we recall Miller's filtration of $U(V)$:

$$\begin{aligned} F_kU(V) &= \{g \in U(V) \mid \operatorname{codim}(\ker(g-1)) \leq k\} \\ &= \{g \in U(V) \mid \operatorname{rank}(g-1) \leq k\}. \end{aligned}$$

More precisely, this is supposed to be interpreted fibrewise, so

$$F_kU(V) = \{(x,g) \mid x \in X \text{ and } g \in U(V_x) \text{ and } \operatorname{rank}(g-1) \leq k\}.$$

It is not hard to see that ρ gives a homeomorphism $\Sigma_X PV_{+X} \to F_1U(V)$. It is known from work of Miller [6] that when X is a point, the filtration is stably split. Crabb showed in [2] that the splitting works fibrewise; our outline of related material essentially follows his account.

We will need to recall the basic facts about the quotients in Miller's filtration. Consider the space

$$G_k(V) = \{(x,W) \mid x \in X \ , \ W \leq V_x \ , \ \dim(W) = k\}.$$

For each point $(x,W) \in G_k(V)$ we have a Lie group $U(W)$ and its associated Lie algebra $\mathfrak{u}(W) = \{\alpha \in \operatorname{End}(W) \mid \alpha + \alpha^* = 0\}$. These fit together to form a bundle over $G_k(V)$ which we denote by $\mathfrak{u}$. Given a point (x,W,α) in the total space of this bundle one checks that $\alpha - 1$ is invertible and that $g := (\alpha+1)(\alpha-1)^{-1}$ is a unitary automorphism of W without fixed points, so $g \oplus 1_{W^\perp} \in F_kU(V_x) \setminus F_{k-1}U(V_x)$. It is not hard to show that this construction gives a homeomorphism of the total space of $\mathfrak{u}$ with $F_kU(V) \setminus F_{k-1}U(V)$ and thus a homeomorphism of the Thom space $G_k(V)^{\mathfrak{u}}$ with $F_kU(V)/F_{k-1}U(V)$.

If $g \in F_jU(V_x)$ and $h \in F_kU(V_x)$ then $\ker(g-1) \cap \ker(h-1)$ has codimension at most $j+k$, so $gh \in F_{j+k}U(V)$, so the filtration is multiplicative. A less obvious argument shows that it is also comultiplicative, up to homotopy:

Lemma 4.5 *The diagonal map $\delta \colon U(V) \to U(V) \times_X U(V)$ is homotopic to a filtration-preserving map.*

Proof For notational convenience, we will give the proof for a vector space; it can clearly be done fibrewise for vector bundles.

We regard $U(1)$ as the set of unit complex numbers and define $p_0, p_1 \colon U(1) \to U(1)$ as follows:

$$p_0(z) = \begin{cases} z^2 & \text{if } \operatorname{Im}(z) \geq 0 \\ 1 & \text{otherwise} \end{cases}$$

$$p_1(z) = \begin{cases} z^2 & \text{if } \operatorname{Im}(z) \leq 0 \\ 1 & \text{otherwise.} \end{cases}$$

Thus $(p_0, p_1)\colon U(1) \to U(1) \times U(1)$ is just the usual pinch map $U(1) \to U(1) \vee U(1) \subset U(1) \times U(1)$.

Note that if $g \in U(V)$ and $r \in \{0,1\}$ then the eigenvalues of g lie in $U(1)$ so we can interpret $p_r(g)$ as an endomorphism of V as in Appendix A. As $p_r(U(1)) \subseteq U(1)$ we see that $\overline{p_r(z)} = p_r(z)^{-1}$ for all $z \in U(1)$ and thus that $p_r(g)^* = p_r(g)^{-1}$, so p_r gives a map from $U(V)$ to itself.

We now define $\delta'\colon U(V) \to U(V) \times U(V)$ by $\delta'(g) = (p_0(g), p_1(g))$. It is clear that the filtration of $p_0(g)$ is the number of eigenvalues of g (counted with multiplicity) lying in the open upper half-circle, and the filtration of $p_1(g)$ is the number in the open lower half-circle. Thus, the filtration of $\delta'(g)$ is the number of eigenvalues not equal to ± 1, which is less than or equal to the filtration of g.

On the other hand, each map $p_r\colon U(1) \to U(1)$ has degree 1 and thus is homotopic to the identity, so δ' is homotopic to δ. □

Theorem 4.6 *There is a natural isomorphism* $\lambda^{<k}_{E^*X} E^{*-1}PV \to E^*F_{k-1}U(V)$.

Proof For brevity we write $\lambda^k = \lambda^k_{E^*X} E^{*-1}PV$. We also write $\lambda^* = \bigoplus_k \lambda^k$ and $\lambda^{\geq k} = \bigoplus_{j \geq k} \lambda^j$ and $\lambda^{<k} = \lambda^*/\lambda^{\geq k} = \bigoplus_{j<k} \lambda^j$.

Because the filtration of $U(V)$ is stably split, the restriction map $\lambda^* E^{*-1}PV = E^*U(V) \to E^*F_{k-1}U(V)$ is a split surjection, with kernel J_k say. Note that λ^*/J_k and J_k are both projective over E^*X. We need to show that $J_k = \lambda^{\geq k}$.

First, we have $F_0U(V) = X$ and it follows easily that $J_1 = \lambda^{\geq 1}$.

We next claim that $J_j J_k \leq J_{j+k}$ for all j, k. Indeed, J_j is the image in cohomology of the map $U(V) \to U(V)/F_{j-1}$, and so $J_j J_k$ is contained in the image in cohomology of the map

$$\phi = (U(V) \xrightarrow{\delta} U(V) \times_X U(V) \to U(V)/F_{j-1} \wedge_X U(V)/F_{k-1}).$$

Note that δ is homotopic to the map δ', which sends F_{j+k-1} into $F_{j-1} \times_X U(V) \cup U(V) \times_X F_{k-1}$. It follows that the restriction of ϕ to F_{j+k-1} is null, and thus that $J_j J_k \leq J_{j+k}$ as claimed. It follows inductively that $\lambda^{\geq k} \leq J_k$ for all k. This gives us a natural surjective map $\lambda^{<k} \to E^*F_{k-1}U(V)$.

We previously gave a natural basis $\{a_I\}$ for λ^*, and it is clear that the subset $\{a_I \mid |I| < k\}$ is a basis for $\lambda^{<k}$. It will be enough to prove that the images of these form a basis for $E^*F_{k-1}U(V)$. The argument of Theorem 4.4 allows us to reduce to the case where X is a point, $V = \mathbb{C}^d$, and E represents

ordinary cohomology. A proof in this case has been given by Kitchloo [5] (and possibly by others) but we will sketch an alternate proof for completeness. As the map $\lambda^{<k} \to H^*F_{k-1}U(d)$ is surjective, it will suffice to show that the source and target have the same rank as free Abelian groups. For this, it will suffice to show that $\lambda^j H^*\mathbb{C}P^{d-1}$ has the same rank as $\widetilde{H}^*(F_jU(d)/F_{j-1}U(d))$ for $0 \le j \le d$. As $H^*\mathbb{C}P^{d-1}$ has rank d, it is clear that $\lambda^j H^*\mathbb{C}P^{d-1}$ has rank $\begin{pmatrix} d \\ j \end{pmatrix}$. On the other hand $F_jU(d)/F_{j-1}U(d)$ is the Thom space $G_j(\mathbb{C}^d)^{\mathfrak{u}}$. Note that although $\mathfrak{u}$ is not a complex bundle, it is necessarily orientable because $G_j(\mathbb{C}^d)$ is simply connected. Thus, the Thom isomorphism theorem tells us that the rank of $\widetilde{H}^*G_j(\mathbb{C}^d)^{\mathfrak{u}}$ is the same as that of $H^*G_j(\mathbb{C}^d)$. By counting Schubert cells we see that this is again $\begin{pmatrix} d \\ j \end{pmatrix}$, as required. (This will also follow from Proposition 7.3.) □

5 Intersections of bundles

Let X be a space, and let V_0 and V_1 be complex vector bundles over X. In Section 3 we defined divisors $D(V_i) = (PV_i)_E$ on $\mathbb{G}$ over X_E, and we also defined the intersection index $\operatorname{int}(V_0, V_1)$.

Theorem 5.1 *We have* $\operatorname{int}(V_0, V_1) \le \operatorname{int}(D(V_0), D(V_1))$.

Proof Suppose we have isometric linear embeddings $V_0 \xrightarrow{j_0} W \xleftarrow{j_1} V_1$ such that $\dim((j_0V_{0x}) \cap (j_1V_{1x})) \ge r$ for all x. We must show that $\operatorname{rank}(\mathcal{O}_{\mathbb{G}}/(f_{V_0}, f_{V_1})) \ge r$. Put $d_i = \dim(V_i)$ and $e = \dim(W)$. Recall that $E^0PV_i = \mathcal{O}_{\mathbb{G}}/f_{V_i}$ and that $E^0PW = \mathcal{O}_{\mathbb{G}}/f_W$. As each V_i embeds in W we see that f_{V_i} divides f_W and there is a natural surjection $E^0PW \to E^0PV_i$. By combining these maps we get a map $\phi\colon E^0PW \to E^0PV_0 \oplus E^0PV_1$, whose cokernel is $\mathcal{O}_{\mathbb{G}}/(f_{V_0}, f_{V_1})$. From the definition of the Fitting rank, we must prove that $\lambda^{d_0+d_1-r+1}\phi = 0$.

For this, we first note that an isometric embedding $j\colon V \to W$ of vector spaces gives rise to a homomorphism $j_*\colon U(V) \to U(W)$ by

$$j_*(g) = jgj^{-1} \oplus 1_{jV^\perp}\colon W = jV \oplus jV^\perp \to W.$$

The alternative description $j_*(g) = jgj^* + 1 - jj^*$ makes it clear that $j_*(g)$ depends continuously on j and g.

We now extend this definition fibrewise, and define $\gamma\colon U(V_0)\times_X U(V_1) \to U(W)$ by $\gamma(g_0,g_1) = (j_{0*}g_0)(j_{1*}g_1)$. We have $E^*U(W) = \lambda^* E^{*-1}PW$ and

$$\begin{aligned} E^*U(V_0)\times_X U(V_1) &= E^*U(V_0)\otimes_{E^*X} E^*U(V_1) \\ &= \lambda^* E^{*-1}PV_0 \otimes_{E^*X} \lambda^* E^{*-1}PV_1 \\ &= \lambda^*(E^{*-1}PV_0 \oplus E^{*-1}PV_1). \end{aligned}$$

Using the fact that $E^{*-1}PW$ is primitive in $E^*U(W)$, we find that $\gamma^* = \lambda^*\phi$. Next, observe that if $g_i \in U(V_{ix})$ for $i=0,1$ we have

$$\gamma(g_0,g_1) \in U(j_0V_{0x} + j_1V_{1x}) \subseteq U(W)$$

and $\dim(j_0V_{0x} + j_1V_{1x}) \le d_0 + d_1 - r$ so $\gamma(g_0,g_1) \in F_{d_0+d_1-r}U(W)$. Thus γ factors through $F_{d_0+d_1-r}U(W)$, and it follows that $\lambda^{d_0+d_1-r+1}E^{*-1}PW$ is mapped to zero by γ^*, as required. $\square$

As an addendum, we show that some natural variations of the definition of intersection index do not actually make a difference.

Lemma 5.2 *Let V and W be vector bundles over a space X, and let $j\colon V \to W$ be a linear embedding. Then j is an isometric embedding if and only if $j^*j = 1$ (where j^* is the adjoint of j). In any case, there is a canonical isometric embedding $\hat{\jmath}\colon V \to W$ with the same image as j.*

Proof If $j^*j = 1$ then $\|jv\|^2 = \langle jv, jv\rangle = \langle v, j^*jv\rangle = \langle v,v\rangle = \|v\|^2$, so j is an isometry. Conversely, if j is an isometry then it preserves inner products so $\langle v', j^*jv\rangle = \langle jv', jv\rangle = \langle v', v\rangle$ for all v, v' which means that $j^*jv = v$.

Even if j is not an isometry we have $\langle v, j^*jv\rangle = \|jv\|^2$ which implies that j^*j is injective. It is thus a strictly positive self-adjoint operator on V, so we can define $(j^*j)^{-1/2}$ by functional calculus (as in Appendix A). We then define $\hat{\jmath} = j\circ (j^*j)^{-1/2}$. This is the composite of j with an automorphism of V, so it has the same image as j. It also satisfies $\hat{\jmath}^*\hat{\jmath} = 1$, so it is an isometric embedding. $\square$

Proposition 5.3 *Let V_0 and V_1 be bundles over a space X. Consider the following statements:*

(a) *There exists a bundle V of dimension k and linear isometric embeddings $V_0 \xleftarrow{i_0} V \xrightarrow{i_1} V_1$.*

(a$'$) *There exists a bundle V of dimension k and linear embeddings $V_0 \xleftarrow{i_0} V \xrightarrow{i_1} V_1$.*

(b) *There exist a bundle W and isometric linear embeddings $V_0 \xrightarrow{j_0} W \xleftarrow{j_1} V_1$ such that $\dim((j_0V_{0x}) \cap (j_1V_{1x})) \geq k$ for all $x \in X$.*

(b$'$) *There exist a bundle W and linear embeddings $V_0 \xrightarrow{j_0} W \xleftarrow{j_1} V_1$ such that $\dim((j_0V_{0x}) \cap (j_1V_{1x})) \geq k$ for all $x \in X$.*

(c) *There is a linear map $f\colon V_0 \to V_1$ such that $\operatorname{rank}(f_x) \geq k$ for all $x \in X$.*

Then (a)$\Leftrightarrow$(a$'$)$\Rightarrow$(b)$\Leftrightarrow$(b$'$)$\Leftrightarrow$(c).

Proof It follows immediately from Lemma 5.2 that (a)$\Leftrightarrow$(a$'$) and (b)$\Leftrightarrow$(b$'$).

(a)$\Rightarrow$(b): Define W, j_0 and j_1 by the following pushout square:

$$\begin{tikzcd} V \arrow[r, tail, "i_0"] \arrow[d, tail, "i_1"'] & V_0 \arrow[d, tail, "j_0"] \\ V_1 \arrow[r, tail, "j_1"'] & W. \end{tikzcd}$$

Equivalently, we can write V_t' for the orthogonal complement of i_tV in V_t and then $W = V \oplus V_0' \oplus V_1'$.

(b)$\Rightarrow$(c): Put $f = j_1^*j_0\colon V_0 \to V_1$. By hypothesis, for each x we can choose an orthonormal sequence $u_1, \ldots, u_k$ in $(j_0V_{0x}) \cap (j_1V_{1x})$. We can then choose elements $v_p \in V_{0x}$ and $w_p \in V_{1x}$ such that $u_p = j_0v_p = j_1w_p$. We find that $\langle fv_p, w_q\rangle = \langle j_0v_p, j_1w_q\rangle = \langle u_p, u_q\rangle = \delta_{pq}$. This implies that the elements $fv_1, \ldots, fv_k$ are linearly independent, so $\operatorname{rank}(f) \geq k$ as required.

(c)$\Rightarrow$(b): Note that $f_x^*f_x\colon V_{0x} \to V_{0x}$ is a nonnegative self-adjoint operator with the same kernel as f_x, and thus the same rank as f_x. Similarly, $f_xf_x^*$ is a nonnegative self-adjoint operator on V_{1x} with the same rank as f_x. More basic facts about these operators are recorded in Proposition A.2.

As in Definition A.3 we let $\lambda_j = e_j(f_x^*f_x)$ be the j'th eigenvalue of $f_x^*f_x$ (listed in descending order and repeated according to multiplicity). We see from Proposition A.4 that λ_j is a continuous function of x. Moreover, as $f_x^*f_x$ has rank at least k we see that $\lambda_k > 0$. Now define $\tau_x\colon [0,\infty) \to [0,\infty)$ by $\tau_x(t) = \max(\lambda_k, t)$, and define $\mu_x = \tau_x(f_x^*f_x)$ and $\nu_x = \tau(f_xf_x^*)$. (Here we are using functional calculus as in Appendix A again.) One checks that $f_x\mu_x = \nu_xf_x$ and $\mu_xf_x^* = f_x^*\nu_x$. We now have maps

$$\begin{aligned} \mu^{1/2}\colon\ & V_0 \to V_0 \\ f\colon\ & V_0 \to V_1 \\ (\mu + f^*f)^{-1/2}\colon\ & V_0 \to V_0, \end{aligned}$$

which we combine to get a map

$$j_0 = (\mu^{1/2}, f) \circ (\mu + f^* f)^{-1/2} \colon V_0 \to V_0 \oplus V_1.$$

Similarly, we define

$$j_1 = (f^*, \nu^{1/2}) \circ (\nu + f f^*)^{-1/2} \colon V_1 \to V_0 \oplus V_1.$$

It is easy to check that $j_0^* j_0 = 1$ and $j_1^* j_1 = 1$, so j_0 and j_1 are isometric embeddings.

Now choose an orthonormal sequence $v_1, \dots, v_k$ of eigenvectors of $f_x^* f_x$, with eigenvalues $\lambda_1, \dots, \lambda_k$. Put $v_i' = f_x(v_i)/\sqrt{\lambda_i} \in V_1$; these vectors form an orthonormal sequence of eigenvectors of $f_x f_x^*$, with the same eigenvalues.

For $i \le k$ we have $\lambda_i \ge \lambda_k > 0$ so $\tau_x(\lambda_i) = \lambda_i$ so $(\mu + f^* f)^{-1/2}(v_i) = v_i/\sqrt{2\lambda_i}$ and $\mu^{1/2}(v_i) = \sqrt{\lambda_i} v_i$ so $j_0(v_i) = (v_i, v_i')/\sqrt{2}$. This is the same as $j_1(v_i')$, so it lies in $(j_0 V_{0x}) \cap (j_1 V_{1x})$. Thus, this intersection has dimension at least k, as required. $\square$

We conclude this section with a topological interpretation of the scheme $D(V_0) \cap D(V_1)$ itself.

Proposition 5.4 *Let V_0 and V_1 be vector bundles over a space X, and let L_0 and L_1 be the tautological bundles of the two factors in $PV_0 \times_X PV_1$. Then there is a natural map $S(\mathrm{Hom}(L_0, L_1))_E \to D(V_0) \cap D(V_1)$, which is an isomorphism if the map $E^* P(V_0 \oplus V_1) \to E^* PV_0 \oplus E^* PV_1$ is injective.*

Proof We divide the sphere bundle $S(V_0 \oplus V_1)$ into two pieces, which are preserved by the evident action of $U(1)$:

$$\begin{aligned} C_0 &= \{(v_0, v_1) \in S(V_0 \oplus V_1) \mid \|v_0\| \ge \|v_1\|\} \\ C_1 &= \{(v_0, v_1) \in S(V_0 \oplus V_1) \mid \|v_1\| \ge \|v_0\|\}. \end{aligned}$$

The inclusions $V_i \to V_0 \oplus V_1$ give inclusions $S(V_i) \to C_i$ which are easily seen to be homotopy equivalences. It follows that $C_i/U(1) \simeq PV_i$. We also have

$$C_0 \cap C_1 = \{(v_0, v_1) \mid \|v_0\| = \|v_1\| = 2^{-1/2}\} \simeq S(V_0) \times S(V_1).$$

Given a point in this space we have a map $\alpha \colon \mathbb{C} v_0 \to \mathbb{C} v_1$ sending v_0 to v_1. This has norm 1 and is unchanged if we multiply (v_0, v_1) by an element of $U(1)$. Using this we see that $(C_0 \cap C_1)/U(1) = S(\mathrm{Hom}(L_0, L_1))$. Of course, we also have $(C_0 \cup C_1)/U(1) = P(V_0 \oplus V_1)$. We therefore have a homotopy pushout square as shown on the left below, giving rise to a commutative square of formal schemes as shown on the right.

$$\begin{tikzcd} S(\mathrm{Hom}(L_0,L_1)) \arrow[r, "i_0"] \arrow[d, "i_1"'] & PV_0 \arrow[d, "j_0"] \\ PV_1 \arrow[r, "j_1"'] & P(V_0\oplus V_1) \end{tikzcd} \qquad \begin{tikzcd} S(\mathrm{Hom}(L_0,L_1))_E \arrow[r] \arrow[d] & D(V_0) \arrow[d, tail] \\ D(V_1) \arrow[r, tail] & D(V_0\oplus V_1). \end{tikzcd}$$

This evidently gives us a map $S(\mathrm{Hom}(L_0,L_1))_E \to D(V_0)\cap D(V_1)$.

To be more precise, we use the Mayer-Vietoris sequence associated to our pushout square. This gives a short exact sequence

$$\mathrm{cok}(f^0) \xrightarrow{p} E^0 S(\mathrm{Hom}(L_0,L_1)) \xrightarrow{q} \ker(f^{-1}),$$

where

$$f^k = (j_0^*, j_1^*)\colon\ E^k P(V_0\oplus V_1) \to E^k PV_0 \oplus E^k PV_1.$$

We have seen that $\mathrm{cok}(f^0) = \mathcal{O}_{D(V_0)\cap D(V_1)}$, and the map p just corresponds to our map

$$S(\mathrm{Hom}(L_0,L_1))_E \to D(V_0)\cap D(V_1).$$

This map will thus be an isomorphism if f^* is injective, as claimed. □

6 Algebraic universal examples

Let $\mathbb{G}$ be a formal group over a formal scheme S. Later we will work with bundles over a space X, and we will take $S = X_E$ and $\mathbb{G} = (\mathbb{C}P^\infty \times X)_E$. We write $\mathrm{Div}_d^+ = \mathrm{Div}_d^+(\mathbb{G}) \simeq \mathbb{G}^d/\Sigma_d$, so $\mathcal{O}_{\mathrm{Div}_d^+} = \mathcal{O}_S[\![c_1,\dots,c_d]\!]$.

Fix integers $d_0, d_1, r \geq 0$. We write $\mathrm{Int}_r(d_0,d_1)$ for the scheme of pairs (D_0,D_1) where D_0 and D_1 are divisors of degrees d_0 and d_1 on $\mathbb{G}$, and $\mathrm{int}(D_0,D_1)\geq r$. In other words, if D_i is the evident tautological divisor over $\mathrm{Div}_{d_0}^+\times\mathrm{Div}_{d_1}^+$ then $\mathrm{Int}_r(d_0,d_1) = \mathrm{Int}_r(D_0,D_1)$. We will assume that $r\leq \min(d_0,d_1)$ (otherwise we would have $\mathrm{Int}_r(d_0,d_1)=\emptyset$.)

For a more concrete description, put

$$R = \mathcal{O}_{\mathrm{Div}_{d_0}^+\times\mathrm{Div}_{d_1}^+} = \mathcal{O}_S[\![c_{0j} \mid j<d_0]\!][\![c_{1j}\mid j<d_1]\!].$$

Let A be the matrix of ζ over R as in Section 3, and let I be the ideal in R generated by the minors of A of size d_0+d_1-r+1. Then $\mathrm{Int}_r(d_0,d_1) = \mathrm{spf}(R/I)$.

We will also consider a "semi-universal" case. Suppose we have a divisor D_1 on $\mathbb{G}$ over S, with degree d_1. Let D_0 be the tautological divisor over $\mathrm{Div}^+_{d_0}$. We can regard D_0 and D_1 as divisors on $\mathbb{G}$ over $\mathrm{Div}^+_{d_0}$ and thus form the closed subscheme $\mathrm{Int}_r(D_0, D_1) \subseteq \mathrm{Div}^+_{d_0}$. We denote this scheme by $\mathrm{Int}_r(d_0, D_1)$.

We can also define schemes $\mathrm{Sub}_r(d_0, d_1)$ and $\mathrm{Sub}_r(d_0, D_1)$ in a parallel way.

Remark 6.1 $\mathrm{Sub}_r(d_0, d_1)$ is just the scheme of triples (D, D_0, D_1) for which $D \leq D_0$ and $D \leq D_1$. This is isomorphic to the scheme of triples $(D, D_0', D_1') \in \mathrm{Div}^+_r \times \mathrm{Div}^+_{d_0-r} \times \mathrm{Div}^+_{d_1-r}$, by the map $(D, D_0', D_1') \mapsto (D, D + D_0', D + D_1')$.

Definition 6.2 We write $\mathrm{Sub}_r(D)$ for the scheme of divisors D' of degree r such that $D' \leq D$. Using Remark 3.12 we see that $\mathrm{Sub}_r(D) = \mathrm{Sub}_r(r, D) = \mathrm{Int}_r(r, D)$.

Theorem 6.3 *The ring $\mathcal{O}_{\mathrm{Int}_r(d_0,d_1)}$ is freely generated over*

$$\mathcal{O}_S[\![c_{0i} \mid 0 < i \leq d_0 - r]\!][\![c_{1j} \mid 0 < j \leq d_1]\!]$$

by the monomials

$$c_0^\alpha := \prod_{i=d_0-r+1}^{d_0} c_{0i}^{\alpha_i}$$

for which $\sum_i \alpha_i \leq d_1 - r$. Moreover, if we let $\pi\colon \mathrm{Sub}_r(d_0, d_1) \to \mathrm{Int}_r(d_0, d_1)$ be the usual projection, then the corresponding ring map π^* is a split monomorphism of modules over $\mathcal{O}_{\mathrm{Div}^+_{d_1}}$ (so π itself is dominant).

The proof will be given after a number of intermediate results. It seems likely that the injectivity of π^* could be extracted from work of Pragacz [9, Section 3]. He works with Chow groups of varieties rather than generalised cohomology rings of spaces, and his methods and language are rather different; we have not attempted a detailed comparison.

We start by setting up some streamlined notation. We put $n = d_0 - r$ and $m = d_1 - r$. We use the following names for the coordinate rings of various schemes of divisors, and the standard generators of these rings:

$$\begin{aligned} C_0 &= \mathcal{O}_{\mathrm{Div}^+_{d_0}} = \mathcal{O}_S[\![u_1, \dots, u_{n+r}]\!] \\ C_1 &= \mathcal{O}_{\mathrm{Div}^+_{d_1}} = \mathcal{O}_S[\![v_1, \dots, v_{m+r}]\!] \\ A &= \mathcal{O}_{\mathrm{Div}^+_n} = \mathcal{O}_S[\![a_1, \dots, a_n]\!] \\ B &= \mathcal{O}_{\mathrm{Div}^+_m} = \mathcal{O}_S[\![b_1, \dots, b_m]\!] \\ C &= \mathcal{O}_{\mathrm{Div}^+_r} = \mathcal{O}_S[\![c_1, \dots, c_r]\!]. \end{aligned}$$

(In particular, we have renamed c_{0i} and c_{1i} as u_i and v_i.) We put $u_0 = v_0 = a_0 = b_0 = c_0 = 1$. We define $u_i = 0$ for $i < 0$ or $i > n+r$, and similarly for v_i, a_i, b_i and c_i. The equations of the various tautological divisors are as follows:

$$\begin{aligned} f_0(x) &= \sum_i u_i x^{n+r-i} \in C_0[x] \\ f_1(x) &= \sum_i v_i x^{m+r-i} \in C_1[x] \\ f(x) &= \sum_i a_i x^{n-i} \in A[x] \\ g(x) &= \sum_i b_i x^{m-i} \in B[x] \\ h(x) &= \sum_i c_i x^{r-i} \in C[x]. \end{aligned}$$

We write T_0 for the set of monomials of weight at most m in $u_{n+1}, \ldots, u_{n+r}$, and T for the set of monomials of weight at most m in $c_1, \ldots, c_r$. We also introduce the subrings

$$\begin{aligned} C_0' &= \mathcal{O}_S[\![u_1, \ldots, u_n]\!] \subseteq C_0 \\ C_0'' &= \mathcal{O}_S[\![u_1, \ldots, u_{n-1}]\!] \subseteq C_0'. \end{aligned}$$

We note that the ring $Q := \mathcal{O}_{\mathrm{Int}_r(d_0,d_1)}$ has the form $(C_0 \widehat{\otimes} C_1)/I$ for a certain ideal I. The theorem claims that Q is freely generated as a module over $C_0' \widehat{\otimes} C_1$ by T_0.

The map

$$\pi^* \colon C_0 \widehat{\otimes} C_1 \to A \widehat{\otimes} B \widehat{\otimes} C$$

sends $f_0(x)$ to $f(x)h(x)$ and $f_1(x)$ to $g(x)h(x)$. This induces a map $\pi^* \colon Q \to A \widehat{\otimes} B \widehat{\otimes} C$, and the theorem also claims that this is a split injection.

We will need to approximate certain determinants by calculating their lowest terms with respect to a certain ordering. More precisely, we consider monomials of the form $u^\alpha = \prod_{i=1}^{n+r} u_i^{\alpha_i}$, and we order these by $u^\alpha < u^\beta$ if there exists i such that $\alpha_i > \beta_i$ and $\alpha_j = \beta_j$ for $j > i$. The mnemonic is that $u_1 \ll \ldots \ll u_{n+r}$, so any difference in the exponent of u_i overwhelms any difference in the exponents of $u_1, \ldots, u_{i-1}$.

Lemma 6.4 *Suppose we have integers γ_i satisfying $0 \leq \gamma_0 < \ldots < \gamma_m < m+r$, and we put $M_{ij} = u_{n+r+i-\gamma_j}$ for $0 \leq i,j \leq m$, where u_k is interpreted* *as* 0 *if* $k<0$ *or* $k>n+r$. *Then the lowest term in* $\det(M)$ *is the product of the diagonal entries, so*

$$\det(M)=\prod_{i=0}^{m}u_{n+r+i-\gamma_i}+\textit{ higher terms }.$$

Remark 6.5 Determinants of this type are known as *Schur functions.*

Proof Put $\delta=\prod_{i=0}^{m}u_{n+r+i-\gamma_i}$. Let M'_i be obtained from M by removing the 0'th row and i'th column. The matrix M'_0 has the same general form as M so by induction we have $\det(M'_0)=\prod_{i=1}^{m}u_{n+r+i-\gamma_i}+$ higher terms . If we expand $\det(M)$ along the top row then the 0'th term is $u_{n+r-\gamma_0}\det(M'_0)=\delta+$ higher terms . As $0\leq\gamma_0<\ldots<\gamma_m$ we have $\gamma_i\geq i+\gamma_0$ and so δ only involves variables u_j with $j\leq n+r-\gamma_0$. The remaining terms in the row expansion of $\det(M)$ have the form $(-1)^i u_{n+r-\gamma_0+i}\det(M'_i)$ for $i>0$, and $u_{n+r-\gamma_0+i}$ is either zero (if $i>\gamma_0$) or a variable strictly higher than all those appearing in δ. The lemma follows easily. □

Lemma 6.6 *The ring* Q *is generated by* T_0 *as a module over* $C'_0\widehat{\otimes}C_1$.

Proof Let J be the ideal in $C'_0\widehat{\otimes}C_1$ generated by $u_1,\ldots,u_n$ and $v_1,\ldots,v_{m+r}$, so $(C'_0\widehat{\otimes}C_1)/J=\mathcal{O}_S$. We also put $C''_0=(C_0\widehat{\otimes}C_1)/J=\mathcal{O}_S[\![u_{n+1},\ldots,u_{n+r}]\!]$. As J is topologically nilpotent, it will suffice to prove the result modulo J. We will thus work modulo J throughout the proof, so that $f_1=x^{m+r}$, and we must show that Q/J is generated over $\mathcal{O}_S$ by T_0.

Let $\mu\colon C''_0[\![x]\!]/x^{m+r}\to C''_0[\![x]\!]/x^{m+r}$ be defined by $\mu(t)=f_0t$, and let M be the matrix of μ with respect to the obvious bases. It is then easy to see that $Q/J=C''_0/I$, where I is generated by the minors of M of size $m+1$. The entries in M are $M_{ij}=u_{n+r+i-j}$.

We next claim that all the generators u_k are nilpotent mod I, or equivalently that $u_k=0$ in the ring $R=C''_0/\sqrt{I}$ for all k. By downward induction we may assume that $u_l=0$ in R for $k<l\leq n+r$. We consider the submatrix M' of M given by $M'_{ij}=M_{i,n+r-k+j}=u_{i+k-j}$ for $0\leq i,j\leq m$. By the definition of I we have $\det(M')\in I$ and thus $\det(M')=0$ in R. On the other hand, we have $u_l=0$ for $l>k$ so M' is lower triangular so $\det(M')=\prod_i M'_{ii}=u_k^{m+1}$. Thus u_k is nilpotent in R but clearly $\mathrm{Nil}(R)=0$ so $u_k=0$ in R as required. It follows that Q/J is a quotient of the polynomial ring $\mathcal{O}_S[u_{n+1},\ldots,u_{n+r}]\subset\mathcal{O}_S[\![u_{n+1},\ldots,u_{n+r}]\!]$.

Now let W be the submodule of Q/J spanned over $\mathcal{O}_S$ by T_0; we must prove that this is all of Q/J. As $1 \in W$, it will suffice to show that W is an ideal. In the light of the previous paragraph, it will suffice to show that W is closed under multiplication by the elements $u_{n+1}, \dots, u_{n+r}$, or equivalently that W contains all monomials of weight $m+1$.

We thus let $\alpha = (\alpha_{n+1}, \dots, \alpha_{n+r})$ be a multiindex of weight $m+1$. There is then a unique sequence $(\beta_0, \dots, \beta_m)$ with $n+r \geq \beta_0 \geq \dots \geq \beta_m > n$ and $u^\alpha = \prod_i u_{\beta_i}$. Put $\gamma_i = n+r+i-\beta_i$, so that $0 \leq \gamma_0 < \dots < \gamma_m < m+r$. Let M_α be the submatrix of M consisting of the first $m+1$ columns of the rows of indices $\gamma_0, \dots, \gamma_m$, so the (i,j)'th entry of M_α is $u_{n+r+i-\gamma_j}$. Note that the elements $r_\alpha := \det(M_\alpha)$ lie in I.

Lemma 6.4 tells us that the lowest term in r_α is $\prod_i u_{n+r+i-\gamma_i} = \prod_i u_{\beta_i} = u^\alpha$. It is clear that the weight of the remaining terms is at most the size of M_α, which is $m+1$. By an evident induction, we may assume that their images in C_0'''/I lie in W. As $r_\alpha \in I$ we deduce that $u^\alpha \in W$ as well. □

Corollary 6.7 *Let D_1 be a divisor of degree d_1 on $\mathbb{G}$ over S', for some scheme S' over S. Then $\mathcal{O}_{\mathrm{Int}_r(d_0,D_1)}$ is generated over $\mathcal{O}_{S'}[\![c_{01}, \dots, c_{0,d_0-r}]\!]$ by the monomials $c_0^\alpha = \prod_{i=d_0-r+1}^{d_0} c_{0i}^{\alpha_i}$ for which $|\alpha| \leq d_1 - r$.*

Proof The previous lemma is the universal case. □

We next treat the special case of Theorem 6.3 where $n = 0$ and so $r = d_0$. As remarked in Definition 6.2, the map $\pi\colon\ \mathrm{Sub}_r(d_1) = \mathrm{Sub}_r(r, d_1) \to \mathrm{Int}_r(r, d_1)$ is an isomorphism in this case.

Lemma 6.8 *Let D be a divisor of degree d on $\mathbb{G}$ over S. For any $r \leq d$ we let $P_r(D)$ denote the scheme of tuples $(u_1, \dots, u_r) \in \mathbb{G}^r$ such that $\sum_{i=1}^r [u_r] \leq D$. Then $\mathcal{O}_{P_r(D)}$ is free of rank $d!/(d-r)!$ over $\mathcal{O}_S$.*

Proof There is an evident projection $P_r(D) \to P_{r-1}(D)$, which identifies $P_r(D)$ with the divisor $D - [u_1] - \dots - [u_{r-1}]$ on $\mathbb{G}$ over $P_{r-1}(D)$. This divisor has degree $d-r+1$, so $\mathcal{O}_{P_r(D)}$ is free of rank $d-r+1$ over $\mathcal{O}_{P_{r-1}(D)}$. It follows by an evident induction that $\mathcal{O}_{P_r(D)}$ is free over $\mathcal{O}_S$, with rank $d(d-1)\dots(d-r+1) = d!/(d-r)!$. □

Lemma 6.9 *Let D be a divisor of degree d on $\mathbb{G}$ over S, let D' be the tautological divisor of degree r over $\mathrm{Sub}_r(D)$, and let $f(x) = \sum_{i=0}^r c_i x^{r-i}$ be the equation of D. Then the set T of monomials of degree at most $d-r$ in $c_1, \dots, c_r$ is a basis for $\mathcal{O}_{\mathrm{Sub}_r(D)}$ over $\mathcal{O}_S$.*

Proof Put $K = |T|$; by elementary combinatorics we find that $K = \binom{d}{r}$. Put $R = \mathcal{O}_{\mathrm{Sub}_r(D)}$. Using T we obtain an $\mathcal{O}_S$-linear map $\beta\colon \mathcal{O}_S^K \to R$, which is surjective by Lemma 6.6; we must prove that it is actually an isomorphism.

Now consider the scheme $P_r(D)$; Lemma 6.8 tells us that the ring $R' := \mathcal{O}_{P_r(D)}$ is a free module over $\mathcal{O}_S$ of rank $d!/(d-r)! = r!K$. On the other hand, $P_r(D)$ can be identified with the scheme of tuples $(D', u_1, \ldots, u_r)$ where $D' \in \mathrm{Sub}_r(D)$ and $D' = [u_1] + \ldots + [u_r]$. In other words, if we change base to $\mathrm{Sub}_r(D)$ we can regard $P_r(D)$ as $P_r(D')$, and now Lemma 6.8 tells us that R' is free of rank $r!$ over R.

Now choose a basis $e_1, \ldots, e_{r!}$ for R' over R. We can combine this with β to get a map $\gamma\colon {\mathcal{O}_S}^{r!K} \to R'$. This is a direct sum of copies of β, so it is surjective. Both source and target of γ are free of rank $r!K$ over $\mathcal{O}_S$. Any epimorphism between free modules of the same finite rank is an isomorphism (choose a splitting and then take determinants). Thus γ is an isomorphism, and it follows that β is an isomorphism as required. □

Corollary 6.10 *The set T is a basis for $B\widehat{\otimes}C$ over C_1.*

Proof This is the universal case of the lemma. □

Corollary 6.11 *The set T is a basis for $A\widehat{\otimes}B\widehat{\otimes}C$ over $C_0'\widehat{\otimes}C_1$.*

Proof Note that $A\widehat{\otimes}B\widehat{\otimes}C = (B\widehat{\otimes}C)[\![a_1, \ldots, a_n]\!]$. For $0 < i \leq n$ we have

$$\pi^* u_i = \sum_{i=j+k} a_j c_k = a_i + c_i \text{ mod decomposables},$$

where c_i may be zero, but a_i is nonzero. It follows that our ring $A\widehat{\otimes}B\widehat{\otimes}C$ can also be described as $(B\widehat{\otimes}C)[\![u_1, \ldots, u_n]\!]$, or equivalently as $C_0'\widehat{\otimes}B\widehat{\otimes}C$. The claim now follows easily from the previous corollary. □

Now let T_1 be the set of monomials of the form $u_n^i c_1^{\alpha_1} \ldots c_n^{\alpha_n}$ for which $0 \leq i < |\alpha| \leq m$. These monomials can be regarded as elements of $A\widehat{\otimes}B\widehat{\otimes}C$, giving a map $(C_0''\widehat{\otimes}C_1)\{T_1\} \to A\widehat{\otimes}B\widehat{\otimes}C$. The map $\pi^*\colon C_0\widehat{\otimes}C_1 \to A\widehat{\otimes}B\widehat{\otimes}C$ also gives us a map $(C_0'\widehat{\otimes}C_1)\{T_0\} \to A\widehat{\otimes}B\widehat{\otimes}C$, and by combining these we get a map

$$\phi\colon (C_0'\widehat{\otimes}C_1)\{T_0\} \oplus (C_0''\widehat{\otimes}C_1)\{T_1\} \to A\widehat{\otimes}B\widehat{\otimes}C \simeq (C_0'\widehat{\otimes}C_1)\{T\}$$

of modules over $C_0''\widehat{\otimes}C_1$. Our main task will be to prove that this is an isomorphism. The proof will use the following lemma.

Lemma 6.12 *Let R be a ring, and let $\alpha\colon M \to N$ be a homomorphism of modules over $R[\![x]\!]$. Suppose that M can be written as a product of copies of $R[\![x]\!]$, and similarly for N. Suppose also that the induced map $M/xM \to N/xN$ is an isomorphism. Then α is also an isomorphism.*

Proof We have diagrams as shown below, in which the rows are easily seen to be exact:

$$\begin{tikzcd} M/x^kM \arrow[r, tail, "x"] \arrow[d, "\alpha"'] & M/x^{k+1}M \arrow[r, two heads] \arrow[d, "\alpha"'] & M/xM \arrow[d, "\alpha"] \\ N/x^kN \arrow[r, tail, "x"'] & N/x^{k+1}N \arrow[r, two heads] & N/xN \end{tikzcd}$$

We see by induction that the maps $M/x^kM \to N/x^kN$ are all isomorphisms, and the claim follows by taking inverse limits. □

Our map ϕ is a map of modules over the ring

$$C_0''\widehat{\otimes}C_1 = \mathcal{O}_S[\![u_1,\dots,u_{n-1},v_1,\dots,v_{m+r}]\!].$$

Moreover, we have $C_0'\widehat{\otimes}C_1 = (C_0''\widehat{\otimes}C_1)[\![u_n]\!] \simeq \prod_{k=0}^{\infty} C_0''\widehat{\otimes}C_1$. Now let J be the ideal in $C_0''\widehat{\otimes}C_1$ generated by $\{u_1,\dots,u_{n-1},v_1,\dots,v_{m+r}\}$, so $(C_0''\widehat{\otimes}C_1)/J = \mathcal{O}_S$ and ϕ induces a map

$$\overline{\phi}\colon\ \mathcal{O}_S[\![u_n]\!]\{T_0\} \oplus \mathcal{O}_S\{T_1\} \to \mathcal{O}_S[\![u_n]\!]\{T\}.$$

Note also that $\mathcal{O}_S\{T\}$ is the image of C in $(A\widehat{\otimes}B\widehat{\otimes}C)/J$ and is thus a subring of $\mathcal{O}_S[\![u_n]\!]\{T\}$. By an evident inductive extension of the lemma, it will suffice to show that $\overline{\phi}$ is an isomorphism.

Lemma 6.13 *We have $u_{n+j} = u_nc_j + w_j \pmod J$ for some polynomial w_j in $c_1,\dots,c_r$.*

Proof For any monic polynomial $p(x)$ of degree d we write $\hat{p}(y) = y^dp(1/y)$. If $p(x) = \sum_i r_ix^{d-i}$ then $\hat{p}(y) = \sum_i r_iy^i$. Note that $\widehat{pq} = \hat{p}\hat{q}$, and that $\hat{p}(0) = 1$. As we work mod $(u_i \mid i < n)$ we have $\hat{f}_0 = 1 \pmod{y^n}$. As we work mod $(v_j \mid j \leq m+r)$ we have $\hat{f}_1 = 1$. We also have $fh = f_0$ and $gh = f_1$, so $\hat{f}\hat{h} = \hat{f}_0 = 1 \pmod{y^n}$ and $\hat{g}\hat{h} = \hat{f}_1 = 1$. It follows easily that $\hat{f} = \hat{g} \pmod{y^n}$, so $a_i = b_i$ for $i < n$.

We now have to distinguish between the case $m < n$ and the case $m \geq n$. First suppose that $m < n$. Then for $i > n$ we have $a_i = b_i = 0$, and also $b_n = 0$, and

$a_i = b_i$ for $i < n$ by the previous paragraph. This implies that $\hat{f} - \hat{g} = a_n y^n$. We also have $(\hat{f} - \hat{g})\hat{h} = \hat{f}_0 - 1$, and by comparing coefficients we deduce that $a_n c_i = u_{n+i}$ for $i = 0, \dots, r$. The case $i = 0$ gives $u_n = a_n$, so $u_{n+i} = u_n c_i$ for $i = 1, \dots, r$, so the lemma is true with $w_i = 0$.

Now suppose instead that $m \geq n$. As $a_i = 0$ for $i > n$ we have

$$\hat{f} - \hat{g} - (a_n - b_n)y^n = -\sum_{i=n+1}^{m} b_i y^i \in C[y].$$

We now multiply this by $\hat{h}$ and use the fact that $(\hat{f} - \hat{g})\hat{h} = \hat{f}_0 - 1$. By comparing coefficients of y^n we find that $u_n = a_n - b_n$. In view of this, our equation reads

$$\hat{f}_0 - 1 - u_n y^n \hat{h} = -(\sum_{i=n+1}^{m} b_i y^i)\hat{h} \in C[y].$$

The right hand side has the form $\sum_{j>0} w_j y^{n+j}$ with $w_j \in C$, and by comparing coefficients we see that $u_{n+j} = u_n c_j + w_j$ as claimed. $\square$

Proof of Theorem 6.3 Lemma 6.13 tells us that $\overline{\phi}(u^\alpha)$ is $u_n^{|\alpha|} c^\alpha$ plus terms involving lower powers of u_n. It follows easily that if we filter the source and target of $\overline{\phi}$ by powers of u_n, then the resulting map of associated graded modules is a isomorphism. It follows that $\overline{\phi}$ is an isomorphism, and thus that ϕ is an isomorphism. It follows that the map $(C_0' \widehat{\otimes} C_1)\{T_0\} \to A \widehat{\otimes} B \widehat{\otimes} C$ is a split monomorphism of modules over $C_0'' \widehat{\otimes} C_1$ (and thus certainly of modules over C_1). We have seen that this map factors as

$$(C_0' \widehat{\otimes} C_1)\{T_0\} \xrightarrow{\psi} Q \xrightarrow{\pi^*} A \widehat{\otimes} B \widehat{\otimes} C,$$

where ψ is surjective by Lemma 6.6. It follows that ψ is an isomorphism and that π^* is a split monomorphism, as required. $\square$

7 Flag spaces

In the next section we will (in good cases) construct spaces whose associated formal schemes are the schemes $\mathrm{Sub}_r(D(V_0), D(V_1))$ and $\mathrm{Int}_r(D(V_0), D(V_1))$ considered previously. As a warm-up, and also as technical input, we will first consider the schemes associated to Grassmannian bundles and flag bundles. The results discussed are essentially due to Grothendieck [4]; we have merely adjusted the language and technical framework.

Let V be a bundle of dimension d over a space X. We write $P_r(V)$ for the space of tuples $(x, L_1, \ldots, L_r)$ where $x \in X$ and $L_1, \ldots, L_k \in PV_x$ and L_i is orthogonal to L_j for $i \neq j$. Recall also that in Lemma 6.8 we defined $P_r(D(V))$ to be the scheme over X_E of tuples $(u_1, \ldots, u_r) \in \mathbb{G}^r$ for which $[u_1] + \ldots + [u_r] \leq D(V)$.

Proposition 7.1 *There is a natural isomorphism $P_r(V)_E = P_r(D(V))$.*

Proof For each i we have a line bundle over $P_r(V)$ whose fibre over the point $(x, L_1, \ldots, L_r)$ is L_i. This is classified by a map $P_r(V) \to \mathbb{C}P^\infty$, which gives rise to a map $u_i\colon P_r(V)_E \to \mathbb{G}$. The direct sum of these line bundles corresponds to the divisor $[u_1] + \ldots + [u_r]$. This direct sum is a subbundle of V, so $[u_1] + \ldots + [u_r] \leq D(V)$. This construction therefore gives us a map $P_r(V)_E \to P_r(D(V))$.

In the case $r = 1$ we have $P_1(V) = PV$ and $P_1(D(V)) = D(V)$ so the claim is that $(PV)_E = D(V)$, which is true by definition. In general, suppose we know that $P_{r-1}(V)_E = P_{r-1}(D(V))$. We can regard $P_r(V)$ as the projective space of the bundle over $P_{r-1}(V)$ whose fibre over a point $(x, L_1, \ldots, L_{r-1})$ is the space $V_x \ominus (L_1 \oplus \ldots \oplus L_{r-1})$. It follows that $P_r(V)_E$ is just the divisor $D(V) - ([u_1] + \ldots + [u_{r-1}])$ over $P_{r-1}(D(V))$, which is easily identified with $P_r(D(V))$. The proposition follows by induction. □

Remark 7.2 One can easily recover the following more concrete statement. The ring $E^0 P_r(V) = \mathcal{O}_{P_r(D(V))}$ is the largest quotient ring of $(E^0X)[\![x_1, \ldots, x_r]\!]$ in which the polynomial $f_V(t)$ is divisible by $\prod_{i=1}^k (t - x_i)$. It is a free module over E^0X with rank $d!/(d-r)!$, and the monomials x^α with $0 \leq \alpha_i \leq d - i$ (for $i = 1, \ldots, r$) form a basis. More details about the multiplicative relations are given in Section 9.

We next consider the Grassmannian bundle

$$G_r(V) = \{(x, W) \mid x \in X\ ,\ W \leq V_x \text{ and } \dim(W) = r\}.$$

Proposition 7.3 *There is a natural isomorphism $G_r(V)_E = \mathrm{Sub}_r(D(V))$.*

Proof Let T denote the tautological bundle over $G_r(V)$. This is a rank r subbundle of the pullback of V so we have a degree r subdivisor $D(T)$ of the pullback of $D(V)$ over $G_r(V)_E$. This gives rise to a map $G_r(V) \to \mathrm{Sub}_r(D(V))$.

Next, consider the space $P_r(V)$. There is a map $P_r(V) \to G_r(V)$ given by $(x, L_1, \ldots, L_r) \mapsto (x, L_1 \oplus \ldots \oplus L_r)$. This lifts in an evident way to give a homeomorphism $P_r(V) \simeq P_r(T)$. Of course, this is exactly parallel to the proof of Lemma 6.9. Over $P_r(D(T))$ we have points $a_1, \dots, a_r$ of $\mathbb{G}$ with coordinate values $x_1, \dots, x_r \in \mathcal{O}_{P_r(D(T))}$ say. Let B be the set of monomials x^α with $0 \le \alpha_i \le r - i$ for $i = 1, \dots, r$. From our earlier analysis of $\mathrm{Sub}_r(D(V))$ and $P_r(D(T))$ we see that B is a basis for $\mathcal{O}_{P_r(D(T))}$ over $\mathcal{O}_{\mathrm{Sub}_r(D(V))}$. We also see from Remark 7.2 (applied to the bundle T) that B is a basis for $E^0 P_r(T)$ over $E^0 G_r(V)$. This means that our isomorphism $f\colon \mathcal{O}_{P_r(D(V))} \to E^0 P_r(V)$ is a direct sum (indexed by B) of copies of our map $g\colon \mathcal{O}_{\mathrm{Sub}_r(D(V))} \to E^0 G_r(V)$. It follows that g must also be an isomorphism. □

Remark 7.4 Lemma 6.9 now gives us an explicit basis for $E^0 G_r(V)$ over $E^0 X$, consisting of monomials in the Chern classes of the tautological bundle T.

8 Topological universal examples

In this section we construct spaces whose associated formal schemes are the algebraic universal examples considered in Section 6.

We first consider the easy case of the schemes $\mathrm{Sub}_r(D_0, D_1)$.

Definition 8.1 Given vector bundles V_0 and V_1 over X, we define $G_r(V_0, V_1)$ to be the space of quadruples (x, W_0, W_1, g) such that

(a) $x \in X$;

(b) W_i is an r-dimensional subspace of V_{ix} for $i = 0, 1$; and

(c) g is an isometric isomorphism $W_0 \to W_1$.

(We would obtain a homotopy equivalent space if we dropped the requirement that g be an isometry.)

If V_i is the evident tautological bundle over $BU(d_i)$ we write $G_r(d_0, d_1)$ for $G_r(V_0, V_1)$. More generally, if V is a bundle over X and $d_0 \ge 0$ we can let V_1 be the pullback of V to $BU(d_0) \times X$, and let V_0 be the pullback of of the tautological bundle over $BU(d_0)$; in this context we write $G_r(d_0, V)$ for $G_r(V_0, V_1)$.

Theorem 8.2 *There is a natural map $p\colon G_r(V_0, V_1)_E \to \mathrm{Sub}_r(D(V_0), D(V_1))$. In the universal case this is an isomorphism, so*

$$G_r(d_0, d_1)_E = \mathrm{Sub}_r(d_0, d_1).$$

More generally, there is a spectral sequence

$$\mathrm{Tor}^{**}_{E^*BU(d_0)\times BU(d_1)}(E^*X, E^*G_r(d_0,d_1)) \Longrightarrow E^*G_r(V_0,V_1),$$

whose edge map in degree zero is the map

$$p^*\colon\ \mathcal{O}_{\mathrm{Sub}_r(D(V_0),D(V_1))} \to E^0G_r(V_0,V_1).$$

The spectral sequence collapses in the universal case. (We do not address the question of convergence in the general case.)

Proof First, we can pull back the bundles V_i from X to $G_r(V_0,V_1)$ (without change of notation). We also have a bundle over $G_r(V_0,V_1)$ whose fibre over a point (x,W_0,W_1,g) is W_0; we denote this bundle by W, and note that there are natural inclusions $V_0 \xleftarrow{1} W \xrightarrow{g} V_1$. We then have divisors $D(W)$ and $D(V_i)$ on $\mathbb{G}$ over $G_r(V_0,V_1)_E$ with $D(W)\le D(V_0)$ and $D(W)\le D(V_1)$, so the triple $(D(V_0),D(V_1),D(W))$ is classified by a map $G_r(V_0,V_1)_E \to \mathrm{Sub}_r(D(V_0),D(V_1))$.

We next consider the universal case. As our model of $EU(d)$ we use the space of orthonormal d-frames in $\mathbb{C}^\infty$, so $BU(d)$ is just the Grassmannian of d-planes in $\mathbb{C}^\infty$. Given a point

$$(\underline{u},\underline{v}) = (u_1,\dots,u_{d_0},v_1,\dots,v_{d_1}) \in EU(d_0)\times EU(d_1)$$

we construct a point $((V_0,V_1),W_0,W_1,g)\in G_r(d_0,d_1)$ as follows:

(a) V_0 is the span of $u_1,\dots,u_{d_0}$

(b) V_1 is the span of $v_1,\dots,v_{d_1}$

(c) W_0 is the span of $u_1,\dots,u_r$

(d) W_1 is the span of $v_1,\dots,v_r$

(e) g is the map $W_0\to W_1$ that sends u_i to v_i.

This gives a map $f\colon\ EU(d_0)\times EU(d_1)\to G_r(d_0,d_1)$. Next, the group $U(d_0)\times U(d_1)$ has a subgroup $U(r)\times U(d_0-r)\times U(r)\times U(d_1-r)$, inside which we have the smaller subgroup Γ consisting of elements of the form (h,k_0,h,k_1). It is not hard to see that $\Gamma\simeq U(r)\times U(d_0-r)\times U(d_1-r)$, and that f gives a homeomorphism $(EU(d_0)\times EU(d_1))/\Gamma\to G_r(d_0,d_1)$. Moreover, $EU(d_0)\times EU(d_1)$ is contractible and Γ acts freely so $G_r(d_0,d_1)\simeq B\Gamma = BU(r)\times BU(d_0-r)\times BU(d_1-r)$, so $G_r(d_0,d_1)_E = \mathrm{Div}_r^+\times\mathrm{Div}_{d_0-r}^+\times\mathrm{Div}_{d_1-r}^+ = \mathrm{Sub}_r(d_0,d_1)$ as claimed.

In the general case we can choose maps $f_i\colon X \to BU(d_i)$ classifying V_i, and this gives rise to a pullback square as follows:

$$\begin{tikzcd} G_r(V_0,V_1) \arrow[r] \arrow[d] & G_r(d_0,d_1) \arrow[d] \\ X \arrow[r] & BU(d_0)\times BU(d_1). \end{tikzcd}$$

The vertical maps are fibre bundle projections so this is actually a homotopy pullback square. This give an Eilenberg-Moore spectral sequence as in the statement of the theorem. On the edge we have

$$E^*X \otimes_{E^*BU(d_0)\times BU(d_1)} E^*G_r(d_0,d_1),$$

which is the same as

$$E^*X \otimes_{E^0BU(d_0)\times BU(d_1)} E^*G_r(d_0,d_1).$$

We can now identify this as the tensor product of E^*X with $\mathcal{O}_{\mathrm{Sub}_r(d_0,d_1)}$ over $\mathcal{O}_{\mathrm{Div}^+_{d_0}\times\mathrm{Div}^+_{d_1}}$. The part in degree zero is easily seen to be $\mathcal{O}_{\mathrm{Sub}_r(D(V_0),D(V_1))}$ as claimed. □

We next show that our map $G_r(V_0,V_1)_E \to \mathrm{Sub}_r(D(V_0),D(V_1))$ is an isomorphism in the semiuniversal case as well as the universal case. We start by analysing the semiuniversal spaces $G_r(d_0,V)$ in more familiar terms.

Proposition 8.3 *There are natural homotopy equivalences*

$$G_r(d_0,V) \simeq G_r(V)\times BU(d_0-r)$$

(and in particular $G_r(r,V)\simeq G_r(V)$).

Proof A point of $G_r(d_0,V)$ is a tuple (V_0,x,W_0,W_1,g) where $V_0\in G_{d_0}(\mathbb{C}^\infty)$, $x\in X$, $W_0\in G_r(V_0)$, $W_1\in G_r(V_x)$ and $g\colon W_0\to W_1$. We can define a map $f\colon G_r(d_0,V)\to G_r(V)\times BU(d_0-r)$ by $f(V_0,x,W_0,W_1,g)=(x,W_1,V_0\ominus W_0)$. It is not hard to see that this is a fibre bundle projection, and that the fibre over a point (x,W,V') is the space of linear isometric embeddings from W to $\mathbb{C}^\infty\ominus V'$. This space is homeomorphic to the space of linear isometric embeddings of $\mathbb{C}^r$ in $\mathbb{C}^\infty$, which is well-known to be contractible. Thus f is a fibration with contractible fibres and thus is a weak equivalence. □

Corollary 8.4 *The map $G_r(d_0,V)_E \to \mathrm{Sub}_r(d_0,D(V))$ is an isomorphism.*

Proof Recall that $\mathrm{Sub}_r(d_0, D(V))$ is the scheme of pairs (D_1, D) where D_1 is a divisor of degree d_1, D is a divisor of degree r and $D \subseteq D_1 \cap D(V)$. There is an evident isomorphism $\mathrm{Sub}_r(D(V)) \times_S \mathrm{Div}^+_{d_0-r} \to \mathrm{Sub}_r(d_0, D(V))$ sending (D', D) to $(D + D', D)$. The proposition tells us that $G_r(d_0, V) = G_r(V) \times BU(d_0 - r)$. We already know that $BU(d_0 - r)_E = \mathrm{Div}^+_{d_0-r}$, and Proposition 7.3 tells us that $G_r(V)_E = \mathrm{Sub}_r(D(V))$. We therefore have an isomorphism $G_r(d_0, V)_E = \mathrm{Sub}_r(D(V)) \times_S \mathrm{Div}^+_{d_0-r} = \mathrm{Sub}_r(d_0, D(V))$. (This involves an implicit Künneth isomorphism, which is valid because $BU(d_0 - r)$ has only even-dimensional cells.) We leave it to the reader to check that this isomorphism is the same as the map considered previously. □

We now turn to parallel results for the schemes $\mathrm{Int}_r(D(V_0), D(V_1))$.

Definition 8.5 Given vector bundles V_0 and V_1 over a space X, we define $I_r(V_0, V_1)$ to be the space of pairs (x, f) where $f\colon V_{0x} \to V_{1x}$ is a linear map of rank at least r. We define the universal and semiuniversal spaces $I_r(d_0, d_1)$ and $I_r(d_0, V)$ by the evident analogue of Definition 8.1.

Remark 8.6 There is a natural map

$$G_r(V_0, V_1) \to I_r(V_0, V_1),$$

sending (x, W_0, W_1, g) to (x, f), where f is the composite

$$V_0 \xrightarrow{\mathrm{proj}} W_0 \xrightarrow{g} W_1 \xrightarrow{\mathrm{inc}} V_1.$$

This gives a homeomorphism of $G_r(V_0, V_1)$ with the subspace of $I_r(V_0, V_1)$ consisting of pairs (x, f) for which f^*f and ff^* are idempotent.

Definition 8.7 We define a natural map $q\colon I_r(V_0, V_1) \to \mathrm{Int}_r(D(V_0), D(V_1))$ as follows. If we let π denote the projection $I_r(V_0, V_1) \to X$ then we have a tautological map $f\colon \pi^*V_0 \to \pi^*V_1$ which has rank at least r everywhere. Proposition 5.3 now tells us that $\mathrm{int}(\pi^*V_0, \pi^*V_1) \geq r$. We can therefore apply Theorem 5.1 and deduce that the map $I_r(V_0, V_1)_E \to X_E$ factors through a map $q\colon I_r(V_0, V_1) \to \mathrm{Int}_r(D(V_0), D(V_1)) \subseteq X_E$ as required.

Later we will show that the map q is an isomorphism in the universal case. For this, it will be convenient to have an alternative model for the universal space $I_r(d_0, d_1)$.

Proposition 8.8 *Put*

$$I'_r(d_0, d_1) = \{(V_0, V_1) \in G_{d_0}(\mathbb{C}^\infty) \times G_{d_1}(\mathbb{C}^\infty) \mid \dim(V_0 \cap V_1) \geq k\}.$$

Then $I'_r(d_0, d_1)$ is homotopy equivalent to $I_r(d_0, d_1)$.

Proof The basic idea is to refine the proof of Proposition 5.3. We will take $G_d(\mathbb{C}^\infty)$ as our model for $BU(d)$. We write $I = I_r(d_0, d_1)$ and $I' = I'_r(d_0, d_1)$ for brevity.

We will need various isometries between infinite-dimensional vector spaces. We define $\delta\colon \mathbb{C}^\infty \to \mathbb{C}^\infty \oplus \mathbb{C}^\infty$ by $\delta(v) = (v, v)/\sqrt{2}$, and we define $\theta\colon \mathbb{C}^\infty \oplus \mathbb{C}^\infty \to \mathbb{C}^\infty$ by $\theta(v, w) = (v_0, w_0, v_1, w_1, \ldots)$. Next, it is well-known that the space of linear isometric embeddings of $\mathbb{C}^\infty$ in itself is contractible, so we can choose a continuous family of isometries ϕ_t with $\phi_0 = \theta\delta$ and $\phi_1 = 1$. Similarly, we can choose continuous families of isometric embeddings $\psi_0^t, \psi_1^t\colon \mathbb{C}^\infty \to \mathbb{C}^\infty \oplus \mathbb{C}^\infty$ with $\psi_0^0(v) = \theta(v, 0)$ and $\psi_1^0(v) = \theta(0, v)$ and $\psi_0^1(v) = \psi_1^1(v) = v$.

We now define a map $\alpha\colon I' \to I$ by $\alpha(V_0, V_1) = (V_0, V_1, f)$, where f is the orthogonal projection map from V_0 to V_1. This acts as the identity on $V_0 \cap V_1$ and thus has rank at least k. If we choose n large enough that $V_0 + V_1 \leq \mathbb{C}^n$ and let $V_0 \xrightarrow{i_0} \mathbb{C}^n \xleftarrow{i_1} V_1$ be the inclusions, then $f = i_1^* i_0$.

Next, we need to define a map $\beta\colon I \to I'$. Given $(V_0, V_1, f) \in I$ we can construct maps

$$\begin{aligned} \mu\colon & V_0 \to V_0 \\ \nu\colon & V_1 \to V_1 \\ j_0\colon & V_0 \to V_0 \oplus V_1 < \mathbb{C}^\infty \oplus \mathbb{C}^\infty \\ j_1\colon & V_1 \to V_0 \oplus V_1 < \mathbb{C}^\infty \oplus \mathbb{C}^\infty \end{aligned}$$

as in the proof of the implication (c)$\Rightarrow$(b) in Proposition 5.3, so $\dim(j_0 V_0 \cap j_1 V_1) \geq k$. We can thus define $\beta\colon I \to I'$ by $\beta(V_0, V_1, f) = (\theta j_0 V_0, \theta j_1 V_1)$.

Suppose we start with $(V_0, V_1) \in I'$, define $f\colon V_0 \to V_1$ to be the orthogonal projection, and then define j_0, j_1 as above so that $\beta\alpha(V_0, V_1) = (\theta j_0 V_0, \theta j_1 V_1)$. Observe that $f^* f\colon V_0 \to V_0$ decreases distances, and acts as the identity on $V := V_0 \cap V_1$. If we let $\lambda_1, \ldots, \lambda_{d_0}$ be the eigenvalues of $f^* f$ (listed in the usual way) we deduce that $\lambda_1 = \ldots = \lambda_k = 1$ and that $0 \leq \lambda_i \leq 1$ for all i. It follows from this that μ and ν are the respective identity maps, so

$$\begin{aligned} j_0 &= (1, f) \circ (1 + f^* f)^{-1/2} \\ j_1 &= (f^*, 1) \circ (1 + f f^*)^{-1/2}. \end{aligned}$$

In particular, we have $j_0(v) = j_1(v) = (v, v)/\sqrt{2}$ for $v \in V$, so $j_0|_V = j_1|_V = \delta|_V$.

Next, for $0 \leq t \leq 1$ we define $j_0^t\colon V_0 \to \mathbb{C}^\infty \oplus \mathbb{C}^\infty$ by

$$j_0^t = (i_0, t i_0 + (1 - t) f) \circ (1 + t^2 + (1 - t^2) f^* f)^{-1/2}.$$

One can check that this is an isometric embedding, with $j_0^0 = j_0$ and $j_0^1 = \delta|_{V_0}$ and $j_0^t|_V = \delta|_V$ for all t. Similarly, if we put

$$j_1^t = (i_1, ti_0 + (1-t)f^*) \circ (1 + t^2 + (1-t^2)ff^*)^{-1/2},$$

we find that this is an isometric embedding of V_1 in $\mathbb{C}^\infty \oplus \mathbb{C}^\infty$ with $j_1^0 = j_1$ and $j_1^1 = \delta|_{V_1}$ and $j_1^t|_V = \delta|_V$ for all t. It follows that $(\theta j_0^t V_0, \theta j_1^t V_1) \in I'$ for all t, and this gives a path from $\beta\alpha(V_0, V_1) = (\theta j_0 V_0, \theta j_1 V_1)$ to $(\theta\delta V_0, \theta\delta V_1)$. Recall that we chose a path $\{\phi_t\}$ from $\theta\delta$ to 1. The pairs $(\phi_t V_0, \phi_t V_1)$ now give a path from $(\theta\delta V_0, \theta\delta V_1)$ to (V_0, V_1) in I'. Both of the paths considered above are easily seen to depend continuously on the point $(V_0, V_1) \in I'$ that we started with, so we have constructed a homotopy $\beta\alpha \simeq 1$.

Now suppose instead that we start with a point $(V_0, V_1, f) \in I$; we need a path from $\alpha\beta(V_0, V_1, f)$ to (V_0, V_1, f). We have $\beta(V_0, V_1, f) = (\theta j_0 V_0, \theta j_1 V_1)$, so $\alpha\beta(V_0, V_1, f) = (\theta j_0 V_0, \theta j_1 V_1, f')$, where $f' \colon \theta j_0 V_0 \to \theta j_1 V_1$ is the orthogonal projection. One can check that this is characterised by $f'(\theta j_0(v)) = \theta j_1(j_1^* j_0(v))$. Next, for $0 \le t \le 1$ we define $k_0^t \colon V_0 \to V_0 \oplus V_1$ by

$$k_0^t = (\sqrt{1 - t^2 + t^2\mu}, tf) \circ (1 - t^2 + t^2\mu + t^2 f^* f)^{-1/2}.$$

This is an isometric embedding with $k_0^1 = j_0$ and $k_0^0(v) = (v, 0)$. Similarly, we define $k_1^t \colon V_1 \to V_0 \oplus V_1$ by

$$k_1^t = (tf^*, \sqrt{1 - t^2 + t^2\nu}) \circ (1 - t^2 + t^2\nu + t^2 ff^*)^{-1/2},$$

and we define $f_t' \colon \theta k_0^t V_0 \to \theta k_1^t V_1$ by

$$f_t'(\theta k_0^t(v)) = \theta k_1^t(j_1^* j_0(v)),$$

so $f_1' = f'$. The points $(k_0^t V_0, k_1^t V_1, f_t')$ give a path from $\alpha\beta(V_0, V_1, f)$ to $(\theta(V_0 \oplus 0), \theta(0 \oplus V_1), f_0')$ in I.

Next, we define $f_t'' \colon \psi_0^t V_0 \to \psi_1^t V_1$ by $f_t''(\psi_0^t(v)) = \psi_1^t(j_1^* j_0(v))$. The points $(\psi_0^t V_0, \psi_1^t V_1, f_t'')$ give a path from $(\theta(V_0 \oplus 0), \theta(0 \oplus V_1), f_0')$ to $(V_0, V_1, j_1^* j_0)$ in I. Using Proposition A.2 one can check that

$$\begin{aligned} j_1^* j_0 &= (\nu + ff^*) \circ (f\sqrt{\mu} + \sqrt{\nu} f) \circ (\mu + f^* f) \\ &= f \circ (2\mu^{1/2}(\mu + f^* f)^{-1}). \end{aligned}$$

The map $\zeta := 2\mu^{1/2}(\mu + f^* f)^{-1}$ is a strictly positive self-adjoint automorphism of V_0, so the same is true of $t + (1-t)\zeta$ for $0 \le t \le 1$. The points $(V_0, V_1, f \circ (t + (1-t)\zeta))$ form a path from $(V_0, V_1, j_1^* j_0)$ to (V_0, V_1, f). All the paths considered depend continuously on the point (V_0, V_1, f) that we started with, so we have defined a homotopy $\alpha\beta \simeq 1$. ∎

Theorem 8.9 *The map* $q\colon I_r(d_0,d_1)_E \to \mathrm{Int}_r(d_0,d_1)$ *is an isomorphism.*

Proof We first replace $I_r(d_0,d_1)$ by the homotopy-equivalent space $I'_r(d_0,d_1)$. We write $I_r = I'_r(d_0,d_1)$ and $G_r = G_r(d_0,d_1)$ for brevity, and similarly for Int_r and Sub_r. We first claim that there is a commutative diagram as follows.

$$\begin{tikzcd} \mathcal{O}_{\mathrm{Int}_r} \arrow[r, tail] \arrow[d, "q"'] & \mathcal{O}_{\mathrm{Sub}_r} \arrow[d, "p"', "\simeq"] \\ E^0 I_r \arrow[r] & E^0 G_r. \end{tikzcd}$$

Indeed, the isomorphism

$$p\colon \mathcal{O}_{\mathrm{Sub}_r(d_0,d_1)} \to E^0 G_r(d_0,d_1)$$

comes from Theorem 8.2, and the map q comes from Definition 8.7. It was proved in Theorem 6.3 that the top horizontal map is a split monomorphism of $\mathcal{O}_S$-modules, and it follows that the same is true of the map $q\colon \mathcal{O}_{\mathrm{Int}_r} \to E^0 I_r$.

We now specialise to the case where E is $H[u,u^{-1}]$, the two-periodic version of the integer Eilenberg-MacLane spectrum. We then have $E^0X = \prod_k H^{2k}X$ for all spaces X. This splits each of the rings on the bottom row of our diagram as a product of homogeneous pieces, and it is not hard to check that there is a unique compatible way to split the rings on the top row. We know that q is a split monomorphism; if we can show that the source and target have the same Poincaré series, it will follow that q is an isomorphism. If $r = \min(d_0,d_1)$ then $\mathrm{Int}_r = \mathrm{Sub}_r$ so the claim is certainly true. To work downwards from here by induction, it will suffice to show that

$$PS(H^*I_{r+1}) - PS(H^*I_r) = PS(\mathcal{O}_{\mathrm{Int}_{r+1}}) - PS(\mathcal{O}_{\mathrm{Int}_r})$$

for all r.

To evaluate the left hand side, we consider the space

$$I_r \setminus I_{r+1} = \{(V_0,V_1) \in G_{d_0}(\mathbb{C}^\infty) \times G_{d_1}(\mathbb{C}^\infty) \mid \dim(V_0 \cap V_1) = k\}.$$

Let G'_r be the space of triples (V,V'_0,V'_1) of mutually orthogonal subspaces of $\mathbb{C}^\infty$ such that $\dim(V) = r$ and $\dim(V_i) = d_i - r$. This is well-known to be a model of $BU(r) \times BU(d_0 - r) \times BU(d_1 - r)$ and thus homotopy-equivalent to G_r; the argument uses frames much as in the proof of Theorem 8.2. Let W be the bundle over G'_r whose fibre over (V,V'_0,V'_1) is $\mathrm{Hom}(V'_1,V'_0)$. If $\alpha \in \mathrm{Hom}(V'_0,V'_1)$ and we put $V_0 = V \oplus V'_1$ and $V_1 = V \oplus \mathrm{graph}(\alpha)$ then

$V_0 \cap V_1 = V$ and so $(V_0, V_1) \in I_r$. It is not hard to see that this construction gives a homeomorphism of the total space of W with $I_r \setminus I_{r+1}$. This in turn gives a homeomorphism of the Thom space $(G'_r)^W$ with the quotient space I_r/I_{r+1}. By induction we may assume that H^*I_{r+1} is concentrated in even degrees, and it is clear from the Thom isomorphism theorem that the same is true of $\widetilde{H}^*(G'_r)^W$. This implies that H^*I_r is in even degrees and that $PS(H^*I_r) - PS(H^*I_{r+1}) = PS(\widetilde{H}^*(G'_r)^W)$. As W has dimension $(d_0 - r)(d_1 - r)$, we see that $PS(\widetilde{H}^*(G'_r)^W) = t^{2(d_0-r)(d_1-r)} PS(H^*G'_r)$. We also know that $H^*G'_r \simeq \mathcal{O}_{\mathrm{Sub}_r}$. The conclusion is that

$$PS(H^*I_r) - PS(H^*I_{r+1}) = t^{2(d_0-r)(d_1-r)} PS(\mathcal{O}_{\mathrm{Sub}_r}).$$

We next evaluate $PS(\mathcal{O}_{\mathrm{Int}_{r+1}}) - PS(\mathcal{O}_{\mathrm{Int}_r})$. Put

$$R^*_r = \mathbb{Z}[\![c_{01}, \dots, c_{0,d_0-r}, c_{11}, \dots, c_{1,d_1}]\!].$$

We know from Theorem 6.3 that $\mathcal{O}_{\mathrm{Int}_r}$ is freely generated over R^*_r by the monomials $\prod_{i=1}^r c_{0,d_0-r+i}^{\alpha_i}$ for which $\sum_{i=1}^r \alpha_i \leq d_1 - r$. It follows that the monomials $\prod_{i=0}^r c_{0,d_0-r+i}^{\alpha_i}$ for which $\sum_1^r \alpha_i \leq d_1 - r$ form a basis for $\mathcal{O}_{\mathrm{Int}_r}$ over R^*_{r+1}. Similarly, those for which $\sum_{i=0}^r \alpha_i < d_1 - r$ form a basis for $\mathcal{O}_{\mathrm{Int}_{r+1}}$ over R^*_{r+1}. Thus, if we let M^* be the module generated over R^*_{r+1} by the monomials with $\sum_1^r \alpha_i \leq d_1 - r \leq \sum_0^r \alpha_i$, we find that $PS(\mathcal{O}_{\mathrm{Int}_{r+1}}) - PS(\mathcal{O}_{\mathrm{Int}_r}) = PS(M^*)$.

It is not hard to check that the monomials for which $\sum_0^r \alpha_i = d_1 - r$ form a basis for M^* over R^*_r. Next, let N^* be generated over $\mathbb{Z}$ by the monomials $\prod_{i=0}^r c_i^{\alpha_i}$ for which $\sum_0^r \alpha_i = d_1 - r$; note that this involves the variables $1 = c_0, \dots, c_r$ rather than the variables $c_{d_0-r}, \dots, c_{d_0}$ used in M^*. Because $\deg(c_{d_0-r+i}) = \deg(c_i) + 2(d_0 - r)$ we have

$$\deg(\prod_i c_{d_0-r+i}^{\alpha_i}) = \deg(\prod_i c_i^{\alpha_i}) + 2(d_0 - r) \sum_i \alpha_i.$$

Using this, we see that $PS(M^*) = t^{2(d_0-r)(d_1-r)} PS(N^*) PS(R^*_r)$. However, Corollary 6.11 essentially says that $\mathcal{O}_{\mathrm{Sub}_r} \simeq R^*_r \otimes N^*$ as graded Abelian groups, so $PS(N^*)PS(R^*_r) = PS(\mathcal{O}_{\mathrm{Sub}_r})$, so

$$\begin{aligned} PS(\mathcal{O}_{\mathrm{Int}_{r+1}}) - PS(\mathcal{O}_{\mathrm{Int}_r}) &= t^{2(d_0-r)(d_1-r)} PS(\mathcal{O}_{\mathrm{Sub}_r}) \\ &= PS(H^*I_r) - PS(H^*I_{r+1}). \end{aligned}$$

As explained previously, this implies that q is an isomorphism in the case $E = H[u, u^{-1}]$. We next consider the case $E = MU[u, u^{-1}]$. Let I be the kernel of the usual map $MU_* \to \mathbb{Z}$. Because H^*I_r is free of finite type and concentrated in even degrees, we see that the Atiyah-Hirzebruch spectral sequence collapses and that the associated graded ring $\mathrm{gr}_I MU^* I_r$ is isomorphic to $\mathrm{gr}_I(MU^*)\widehat{\otimes}H^*I_r$. Using this it is not hard to check that q is an isomorphism in the case $E = MU[u,u^{-1}]$ also. Finally, given an arbitrary even periodic ring spectrum E we can choose a complex orientation in $\widetilde{E}^0\mathbb{C}P^\infty$ and thus a ring map $MU[u,u^{-1}] \to E$. Using this, we deduce that q is an isomorphism for all E. □

Corollary 8.10 *Let V_0 and V_1 be bundles of dimensions d_0 and d_1 over a space X. Then there is a spectral sequence*

$$\mathrm{Tor}^{**}_{E^*BU(d_0)\times BU(d_1)}(E^*X, E^*I_r(d_0,d_1)) \Longrightarrow E^*I_r(V_0,V_1),$$

whose edge map in degree zero is the map

$$q^*\colon\ \mathcal{O}_{\mathrm{Int}_r(D(V_0),D(V_1))} \to E^0I_r(V_0,V_1).$$

The spectral sequence collapses in the semiuniversal and universal cases. (We do not address the question of convergence in the general case.)

Proof This is another Eilenberg-Moore spectral sequence. □

9 The schemes P_kD

Let D be a divisor of degree d on $\mathbb{G}$ over S, with equation

$$f(t) = f_D(t) = \sum_{i=0}^{d} c_i x^{d-i} \in \mathcal{O}_S[t],$$

say. In this section we assemble some useful facts about the scheme P_kD. This is a closed subscheme of $\mathbb{G}^k$, so $\mathcal{O}_{P_kD} = \mathcal{O}_S[\![x_0,\dots,x_{k-1}]\!]/J_k$ for some ideal J_k; our main task will be to find systems of generators for J_k. We put $p_i(t) = \prod_{j<i}(t-x_j)$, and we let $q_i(t)$ and $r_i(t)$ be the quotient and remainder when $f(t)$ is divided by $p_i(t)$. Thus $f(t) = q_i(t)p_i(t) + r_i(t)$ and $r_i(t)$ has the form $\sum_{j=0}^{i-1} a_{ij}t^j$ for some $a_{i0},\dots,a_{i,i-1} \in \mathcal{O}_S$. From the definitions it is clear that J_k is the smallest ideal modulo which $f(t)$ becomes divisible by $p_k(t)$, or in other words the smallest ideal modulo which $r_k(t) = 0$, so J_k is generated by $a_{k0},\dots,a_{k,k-1}$. Now put $b_i = a_{i+1,i}$ for $0 \le i < k$; we will show that these elements also generate J_k.

Lemma 9.1 *We have $b_i = q_i(x_i)$ and $r_{i+1} = b_ip_i + r_i$ for all i.*

Proof The polynomial $q_i(t)-q_i(x_i)$ is evidently divisible by $t-x_i$, say $q_i(t)-q_i(x_i)=(t-x_i)q'_{i+1}(t)$. If we put $r'_{i+1}(t)=q_i(x_i)p_i(t)+r_i(t)$ we find that r'_{i+1} is a polynomial of degree at most i and that $f(t)=q'_{i+1}(t)p_{i+1}(t)+r'_{i+1}(t)$, so we must have $q_{i+1}=q'_{i+1}$ and $r_{i+1}=r'_{i+1}$. Thus b_i is the coefficient of t^i in $r'_{i+1}(t)$. As r_i has degree less than i and p_i is monic of degree i we deduce that $b_i=q_i(x_i)$. □

Corollary 9.2 *The ideal J_k is generated by $b_0,\dots,b_{k-1}$.*

Proof Put $J'_k=(b_0,\dots,b_{k-1})$. If we work modulo J'_k then it is immediate from the lemma that $r_k=r_{k-1}=\dots=r_0=0$; this shows that $J_k\subseteq J'_k$. Conversely, if we work modulo J_k then f is divisible by p_k and hence by p_i for all $i\leq k$, so $r_0=\dots=r_k=0$. It follows from the lemma that $b_ip_i=0$ for all i, and p_i is monic so $b_i=0$. Thus $J'_k\subseteq J_k$. □

We now give a determinantal formula for the relators b_j. Consider the Vandermonde determinant

$$v_k:=\det(x_i^j)_{0\leq i,j<k}=\prod_{0\leq i<j<k}(x_j-x_i).$$

We also define a matrix B_k by

$$(B_k)_{ij}=\begin{cases}x_i^j & \text{if } 0\leq j<k-1\\ f(x_i) & \text{if } j=k-1.\end{cases}$$

Proposition 9.3 *We have $b_j=\det(B_j)/v_j$ for all j. (More precisely, we have $v_jb_j=\det(B_j)\in\mathcal{O}_S[\![x_0,\dots,x_{j-1}]\!]$, and v_j is not a zero-divisor in this ring.)*

Proof Define $\alpha\colon\mathcal{O}_S{}^j\to\mathcal{O}_S[\![t]\!]/p_j(t)$ by

$$\alpha(u_0,\dots,u_{j-2},w)=\sum_i u_it^i+wf(t)\pmod{p_j}=\sum_i u_it^i+wr_j(t)\pmod{p_j}.$$

Next, define $\beta\colon\mathcal{O}_S[\![t]\!]/p_j(t)\to\mathcal{O}_S{}^j$ by $\beta(g)=(g(x_0),\dots,g(x_{j-1}))$. We identify $\mathcal{O}_S[\![t]\!]/p_j(t)$ with $\mathcal{O}_S{}^j$ using the basis $\{t^i\mid 0\leq i<j\}$. It is easy to see that $\det(\beta)=v_j$ and $\det(\beta\alpha)=\det(B_j)$. Moreover, the matrix of α has the form $\left(\begin{array}{c|c} I & * \\ \hline 0 & b_j\end{array}\right)$ so $\det(\alpha)=b_j$. It follows immediately that $v_jb_j=\det(B_j)$. It is easy to see that none of the polynomials x_j-x_i (where $i<j$) are zero-divisors, so v_j is not a zero-divisor. □

We next need to relate the schemes P_kD to the exterior powers $\lambda^k\mathcal{O}_D$.

Lemma 9.4 *The ideal J_k maps to zero under the natural projection $\mathcal{O}_{D^k} = \mathcal{O}_D^{\otimes k} \to \lambda^k\mathcal{O}_D$.*

Proof It is enough to prove the corresponding result in the universal case, where D is the tautological divisor over Div_d^+. As the map $\mathbb{G}^d \to \mathbb{G}^d/\Sigma_d = \mathrm{Div}_d^+$ is faithfully flat, it is enough to prove the result after pulling back along this map. In other words, we need only consider the divisor over the ring $R := \mathcal{O}_{\mathbb{G}^d} = \mathcal{O}_S[\![y_i \mid i < d]\!]$ with equation $f(t) = \prod_i(t-y_i)$. Let w be the discriminant of this polynomial, so $w = \prod_{i\neq j}(y_i - y_j) \in R$. Put $N = \{0,\dots,d-1\}$, and let $F(N,R)$ denote the ring of functions from N to R, with pointwise operations. We can define $\phi\colon \mathcal{O}_D \to F(N,R)$ by $\phi(g)(i) = g(y_i)$, and the Chinese Remainder Theorem tells us that the resulting map $w^{-1}\mathcal{O}_D \to F(N, w^{-1}R)$ is an isomorphism, and it follows that $w^{-1}\mathcal{O}_{D^k} = F(N^k, w^{-1}R)$. We also have $\mathcal{O}_{D^k} = R[\![x_j \mid j < k]\!]/(f_D(x_j) \mid j < k)$; the element x_j corresponds to the function $\underline{n} \mapsto y_{n_j}$.

Now put

$$N_k = \{(n_0,\dots,n_{k-1}) \in N^k \mid n_i \neq n_j \text{ when } i \neq j\},$$

and $N_k^c = N^k \setminus N_k$. Let $r(t)$ be the remainder when the polynomial $f_D(t) := \prod_{i<d}(t-y_i)$ is divided by $f_{D'}(t) := \prod_{j<k}(t-x_j)$. This corresponds to the function $\underline{n} \mapsto r_{\underline{n}}(t)$, where $r_{\underline{n}}(t)$ is the remainder of $f_D(t)$ modulo $\prod_{j<k}(t - y_{n_j})$. As the discriminant is invertible in $w^{-1}R$ we see that $r_{\underline{n}}(t) = 0$ iff $\underline{n} \in N_k$, and otherwise some coefficient of $r_{\underline{n}}(t)$ is invertible. Using this, we deduce that $w^{-1}\mathcal{O}_{P_kD} = F(N_k, w^{-1}R)$ and $w^{-1}J_k = F(N_k^c, w^{-1}R)$. If we let $\{e_0,\dots,e_{k-1}\}$ be the evident basis of $F(N,R)$ over R, this means that $w^{-1}J_k$ is spanned over $w^{-1}R$ by the elements $e_{n_0}\otimes\dots\otimes e_{n_{k-1}}$ for which $n_i = n_j$ for some $i \neq j$, and these elements satisfy $e_{n_0}\wedge\dots\wedge e_{n_{k-1}} = 0$ so the map $w^{-1}J_k \to w^{-1}\lambda^k\mathcal{O}_D$ is zero. As w is not a zero-divisor we deduce that the map $J_k \to \lambda^k\mathcal{O}_D$ is zero, as claimed. □

Next note that the symmetric group Σ_k acts on D^k and P_kD and thus on the corresponding rings. In either case we define $\mathrm{alt}_k(a) = \sum_{\sigma\in\Sigma_k}\mathrm{sgn}(\sigma)\sigma.a$. We also let $\mu_k\colon \mathcal{O}_D^{\otimes k} \to \lambda^k\mathcal{O}_D$ be the usual projection, or equivalently the restriction of the product map $\mu_k\colon (\lambda^*\mathcal{O}_D)^{\otimes k} \to \lambda^*\mathcal{O}_D$. Dually, we let $\psi_k\colon \lambda^k\mathcal{O}_D \to \mathcal{O}_D^{\otimes k}$ be the component of the coproduct map $\psi_k\colon \lambda^*\mathcal{O}_D \to (\lambda^*\mathcal{O}_D)^{\otimes k}$. We also let $p^*\colon \mathcal{O}_D^{\otimes k} = \mathcal{O}_{D^k} \to \mathcal{O}_{P_kD}$ denote the usual projection, corresponding to the closed inclusion $P_kD \to D^k$.

Proposition 9.5 *There is a natural commutative diagram as follows.*

$$\begin{tikzcd}
\mathcal{O}_{P_kD} & & \mathcal{O}_{P_kD} \arrow[ll, "\mathrm{alt}_k"'] \arrow[dl, two heads, "\mu'_k"] \\
& \lambda^k\mathcal{O}_D \arrow[ul, "\psi'_k"'] \arrow[dl, hook, "\psi_k"'] & \\
\mathcal{O}_D^{\otimes k} \arrow[uu, two heads, "p^*"] & & \mathcal{O}_D^{\otimes k} \arrow[ll, "\mathrm{alt}_k"] \arrow[ul, two heads, "\mu_k"'] \arrow[uu, two heads, "p^*"']
\end{tikzcd}$$

Proof The main point to check is that $\psi_k\mu_k = \mathrm{alt}_k\colon \mathcal{O}_D^{\otimes k} \to \mathcal{O}_D^{\otimes k}$. Consider an element $a = a_0 \otimes \ldots \otimes a_{k-1} \in \mathcal{O}_D^{\otimes k}$. Let a_i^j denote the element $1^{\otimes j} \otimes a_i \otimes 1^{\otimes k-j-1} \in \mathcal{O}_D^{\otimes k}$, so that $\psi_k(a_i) = \sum_{j=0}^{k-1} a_i^j$ and $\psi_k\mu_k(a) = \prod_i \sum_j a_i^j$. We are interested in the component of this in $\mathcal{O}_D^{\otimes k} \subset (\lambda^*\mathcal{O}_D)^{\otimes k}$, which is easily seen to be $\sum_\sigma \prod_i a_i^{\sigma(i)}$. Moreover, one checks that

$$\prod_i a_i^{\sigma(i)} = \mathrm{sgn}(\sigma) a_{\sigma^{-1}(0)} \otimes \ldots \otimes a_{\sigma^{-1}(k-1)} = \mathrm{sgn}(\sigma)\sigma.a,$$

so the relevant component of $\psi_k\mu_k(a)$ is $\sum_\sigma \mathrm{sgn}(\sigma)\sigma.a = \mathrm{alt}_k(a)$, as claimed.

Let A be the set of multiindices $\alpha = (\alpha_0, \ldots, \alpha_{k-1})$ with $0 \leq \alpha_i < d$ for all i, and let A_0 be the subset of those for which $\alpha_0 > \ldots > \alpha_{k-1}$. Put $x^\alpha := x^{\alpha_0} \otimes \ldots \otimes x^{\alpha_{k-1}} \in \mathcal{O}_D^{\otimes k}$. Then $\{x^\alpha \mid \alpha \in A\}$ is a basis for $\mathcal{O}_D^{\otimes k}$, and $\{\mu_k(x^\alpha) \mid \alpha \in A_0\}$ is a basis for $\lambda^k\mathcal{O}_D$. Moreover, if $\alpha \in A_0$ and we write $\psi_k\mu_k(x^\alpha) = \mathrm{alt}_k(x^\alpha) = \sum_{\beta \in A} c_{\alpha\beta} x^\beta$ we see that $c_{\alpha\alpha} = 1$ and $c_{\alpha\beta} = 0$ if $\beta \in A_0$ and $\beta \neq \alpha$. It follows that μ_k is surjective and ψ_k is a split injection of $\mathcal{O}_S$-modules, as indicated in the diagram.

Lemma 9.4 tells us that μ_k factor as $\mu'_k p^*$ for some $\mu'_k\colon \mathcal{O}_{P_kD} \to \lambda^k\mathcal{O}_D$, and a diagram chase shows that μ'_k is surjective. This gives the right hand triangle of the diagram. We simply define $\psi'_k = p^*\psi_k$ to get the left hand triangle. As p^* is Σ_k-equivariant we have

$$\mathrm{alt}_k p^* = p^* \mathrm{alt}_k = p^*\psi_k\mu_k = \psi'_k\mu'_k p^*.$$

As p^* is surjective, this proves that $\psi'_k\mu'_k = \mathrm{alt}_k$, so the top triangle commutes. ∎

We next study certain orbit schemes for actions of Σ_k. Recall that $\mathcal{O}_{\mathbb{G}^k} = \mathcal{O}_S[\![x_i \mid i < k]\!]$ has a topological basis consisting of monomials in the variables

x_i. This basis is permuted by Σ_k, and the sums of the orbits form a topological basis for the invariant subring $\mathcal{O}_{\mathbb{G}^k}^{\Sigma_k} = \mathcal{O}_{\mathbb{G}^k/\Sigma_k} = \mathcal{O}_{\mathrm{Div}_k^+}$. It is clear from this analysis that our quotient construction commutes with base change, in other words $(S' \times_S \mathbb{G}^k)/\Sigma_k = S' \times_S (\mathbb{G}^k/\Sigma_k)$ for any scheme S' over S. Similarly, the set $\{x^\alpha \mid \alpha_i < d \text{ for all } i\}$ is a basis for $\mathcal{O}_{D^k}$ that is permuted by Σ_k, so the orbit sums give a basis for $\mathcal{O}_{D^k}^{\Sigma_k}$ and we have a quotient scheme $D^k/\Sigma_k = \mathrm{spf}(\mathcal{O}_{D^k}^{\Sigma_k})$ whose formation commutes with base change. By comparing our bases we see that the projection $\mathcal{O}_{\mathbb{G}} \to \mathcal{O}_D = \mathcal{O}_{\mathbb{G}}/f_D$ induces a surjective map $\mathcal{O}_{\mathbb{G}^k/\Sigma_k} \to \mathcal{O}_{D^k/\Sigma_k}$. In other words, we have a commutative square of schemes as shown, in which j and j' are closed inclusions, and q_2 is a faithfully flat map of degree $d!$.

$$\begin{tikzcd}
D^k \arrow[r, tail, "j"] \arrow[d, "q_1"'] & \mathbb{G}^k \arrow[d, two heads, "q_2"] \\
D^k/\Sigma_k \arrow[r, tail, "j'"'] & \mathbb{G}^k/\Sigma_k = \mathrm{Div}_k^+
\end{tikzcd}$$

One might hope to show that $P_k(D)/\Sigma_k = \mathrm{Sub}_k(D)$ in a similar sense, but this is not quite correct. For example if $D = 3[0]$ (so $f_D(t) = t^3$) and $k = 2$ then $\mathcal{O}_{P_2 D} = \mathcal{O}_S[\![x,y]\!]/(x^3, x^2 + xy + y^2)$. If we define a basis of this ring by

$$\{e_0, \dots, e_5\} = \{1, x, y, x^2, -x^2 - xy, x^2 y\},$$

we find that the generator of Σ_2 has the effect

$$e_0 \leftrightarrow e_0 \ , \ e_1 \leftrightarrow e_2 \ , \ e_3 \leftrightarrow e_4 \ , \ e_5 \leftrightarrow -e_5.$$

If $\mathcal{O}_S$ has no 2-torsion we find that $\mathcal{O}_{P_2 D}^{\Sigma_2}$ is spanned by $\{1, x+y, xy\}$ and thus is equal to $\mathcal{O}_{\mathrm{Sub}_2(D)}$. However, if $2 = 0$ in $\mathcal{O}_S$ we have an additional generator $x^2 y$, so $\mathcal{O}_{P_2 D}^{\Sigma_2}$ is strictly larger than $\mathcal{O}_{\mathrm{Sub}_2(D)}$. This example also shows that the formation of $\mathcal{O}_{P_2 D}^{\Sigma_2}$ is not compatible with base change.

The following proposition provides a substitute for the hope described above.

Proposition 9.6 *There is a commutative diagram as follows, in which i, i', j and j' are closed inclusions, and q_0 and q_2 are faithfully flat of degree $k!$. Moreover, the outer rectangle is a pullback, and if $J_k := \ker(i^*)$ then*

$\ker((i')^*) = J_k^{\Sigma_k}$.

$$\begin{tikzcd}
P_kD \arrow[r, tail, "i"] \arrow[d, two heads, "q_0"'] & D^k \arrow[r, tail, "j"] \arrow[d, "q_1"] & \mathbb{G}^k \arrow[d, two heads, "q_2"] \\
\mathrm{Sub}_k(D) \arrow[r, tail, "i'"'] & D^k/\Sigma_k \arrow[r, tail, "j'"'] & \mathbb{G}^k/\Sigma_k = \mathrm{Div}_k^+
\end{tikzcd}$$

Proof We have already produced the right hand square. The map i is just the obvious inclusion. The map q_0 sends $(a_0,\dots,a_{k-1}) \in P_kD$ to $[a_0]+\dots+[a_{k-1}] \in \mathrm{Sub}_k(D)$; it was observed in the proof of Lemma 6.9 that this makes $\mathcal{O}_{P_kD}$ into a free module of rank $k!$ over $\mathcal{O}_{\mathrm{Sub}_k(D)}$, so q_0 is faithfully flat of degree $k!$.

The points of $\mathrm{Sub}_k(D)$ are the divisors of degree k contained in D, so $\mathrm{Sub}_k(D)$ is a closed subscheme of Div_k^+; we write $m'\colon \mathrm{Sub}_k(D) \to \mathrm{Div}_k^+$ for the inclusion, and note that $m'q_0 = q_2ji$. As q_0 is faithfully flat and $m'q_0$ factors through D^k/Σ_k we see that m' factors through D^k/Σ_k, so there is a unique map $i'\colon \mathrm{Sub}_k(D) \to D^k/\Sigma_k$ such that $m' = j'i'$. As m' is a closed inclusion, the same is true of i'. A point of the pullback of m and q_2 is a list $a = (a_0,\dots,a_{k-1})$ of points of $\mathbb{G}$ such that the divisor $q_2(a) = \sum_r [a_r]$ lies in $\mathrm{Sub}_k(D)$, and thus satisfies $\sum_r [a_r] \le D$. It follows from the definitions that this pullback is just P_kD as claimed.

As q_0 is faithfully flat we have $\ker((i')^*) = \ker(q_0^*(i')^*) = \ker(i^*q_1^*)$. By construction, q_1^* is just the inclusion of the Σ_k-invariants in $\mathcal{O}_{D^k}$, so $\ker(i^*q_1^*) = \ker(i^*)^{\Sigma_k} = J_k^{\Sigma_k}$ as claimed. □

Corollary 9.7 *$\lambda^k\mathcal{O}_D$ is naturally a module over $\mathcal{O}_{\mathrm{Sub}_k(D)}$.*

Proof We can certainly regard $\mathcal{O}_{D^k}$ as a module over the subring $\mathcal{O}_{D^k/\Sigma_k} = \mathcal{O}_{D^k}^{\Sigma_k}$, and the map $\mathrm{alt}_k\colon \mathcal{O}_{D^k} \to \mathcal{O}_{D^k}$ respects this structure. This makes $\lambda^k\mathcal{O}_D = \mathrm{image}(\mathrm{alt}_k)$ into a module over $\mathcal{O}_{D^k}^{\Sigma_k}$. If $a \in J_k^{\Sigma_k}$ and $b \in \mathcal{O}_{D^k}$ then $a\,\mathrm{alt}_k(b) = \mathrm{alt}_k(ab)$ but $ab \in J_k$ so $\mathrm{alt}_k(ab) = 0$. This shows that $\lambda^k\mathcal{O}_D$ is annihilated by $J_k^{\Sigma_k}$, so it is a module over $\mathcal{O}_{D^k}^{\Sigma_k}/J_k^{\Sigma_k} = \mathcal{O}_{\mathrm{Sub}_k(D)}$, as claimed. □

We next identify $\lambda^k\mathcal{O}_D$ as a module over $\mathcal{O}_{\mathrm{Sub}_k(D)}$. Let D' be the tautological divisor of degree k over $\mathrm{Sub}_k(D)$. Then $\mathcal{O}_{D'}$ is naturally a quotient of the ring $\mathcal{O}_D \otimes_S \mathcal{O}_{\mathrm{Sub}_k(D)}$, which contains the subring $\mathcal{O}_D = \mathcal{O}_D \otimes 1$. This gives us a map $\mathcal{O}_D \to \mathcal{O}_{D'}$, which extends to give a map $\phi\colon \lambda_S^k\mathcal{O}_D \to \lambda^k_{\mathrm{Sub}_k(D)}\mathcal{O}_{D'}$.

Proposition 9.8 *The map $\phi\colon \lambda^k_S\mathcal{O}_D \to \lambda^k_{\mathrm{Sub}_k(D)}\mathcal{O}_{D'}$ is an isomorphism of free rank-one modules over $\mathcal{O}_{\mathrm{Sub}_k(D)}$.*

Proof Put $T = \mathrm{Sub}_k(D)$ for brevity. Note that

$$\begin{aligned} \mathcal{O}_D &= \mathcal{O}_S\{x^i \mid i < d\} \\ \lambda^k_S\mathcal{O}_D &= \mathcal{O}_S\{x^{i_0} \wedge \ldots \wedge x^{i_{k-1}} \mid d > i_0 > \ldots > i_{k-1}\} \\ \mathcal{O}_{D'} &= \mathcal{O}_T\{x^i \mid i < k\} \\ \lambda^k_T\mathcal{O}_{D'} &= \mathcal{O}_T\{x^{k-1} \wedge \ldots \wedge x^0\}. \end{aligned}$$

In particular, we see that $\lambda^k_S\mathcal{O}_D$ is free of rank $K := \binom{d}{k}$ over $\mathcal{O}_S$, and $\lambda^k_T\mathcal{O}_{D'}$ is free of rank one over $\mathcal{O}_T$. We also know from Lemma 6.9 that $\mathcal{O}_T$ is free of rank K over $\mathcal{O}_S$, so $\lambda^k_T\mathcal{O}_{D'}$ is also free of rank K over $\mathcal{O}_S$.

Suppose for the moment that ϕ is a homomorphism of $\mathcal{O}_T$-modules. It is clear that

$$\phi(x^{k-1} \wedge \ldots \wedge x^0) = x^{k-1} \wedge \ldots \wedge x^0,$$

and this element generates $\lambda^k_T\mathcal{O}_{D'}$, so ϕ is surjective. As the source and target are free of the same finite rank over $\mathcal{O}_S$, we deduce that ϕ is an isomorphism as claimed.

We still need to prove that ϕ is linear over $\mathcal{O}_T$. By the argument of Lemma 9.4 we reduce to the case where D is the divisor with equation $\prod_{i=0}^{d-1}(t-y_i)$ defined over the ring

$$R := \mathcal{O}_{\mathbb{G}^d} = \mathcal{O}_S[\![y_0, \ldots, y_{d-1}]\!],$$

and we can invert the discriminant $w = \prod_{i\neq j}(y_i - y_j)$. We reuse the notation in the proof of that lemma, so $w^{-1}\mathcal{O}_D = F(N, w^{-1}R)$ and $w^{-1}\mathcal{O}_{D^k} = F(N^k, w^{-1}R)$ and $w^{-1}\mathcal{O}_{P_kD} = F(N_k, w^{-1}R)$. We see from Proposition 9.6 that $w^{-1}\mathcal{O}_T$ is the image of $w^{-1}\mathcal{O}_{D^k}^{\Sigma_k}$ in $w^{-1}\mathcal{O}_{P_kD}$, which is the ring $F(N_k, w^{-1}R)^{\Sigma_k}$ of symmetric functions from N_k to $w^{-1}R$. If we write $N_k^+ = \{\underline{n} \in N^k \mid n_0 > \ldots > n_{k-1}\}$ then $N_k = \Sigma_k \times N_k^+$ as Σ_k-sets so $w^{-1}\mathcal{O}_{P_kD} = F(N_k^+, w^{-1}R)$. On the other hand, $\mathcal{O}_T$ is also a quotient of $R\widehat{\otimes}\mathcal{O}_{\mathrm{Div}_k^+}$, which is the ring of symmetric power series in k variables over R; a symmetric power series p corresponds to the function $\underline{n} \mapsto p(y_{\underline{n}}) := p(y_{n_0}, \ldots, y_{n_{k-1}})$.

If $\underline{n} \in N^k$ we put $e_{\underline{n}} = e_{n_0} \otimes \ldots \otimes e_{n_{k-1}}$, so these elements form a basis for $w^{-1}\mathcal{O}_D^{\otimes k}$ over $w^{-1}R$. Similarly, the set $\{\mu_k(e_{\underline{n}}) \mid \underline{n} \in N_k^+\}$ is a basis for $w^{-1}\lambda^k\mathcal{O}_D$. Using the previous paragraph we see that $p.\mu_k(e_{\underline{n}}) = p(y_{\underline{n}})\mu_k(e_{\underline{n}})$, which tells us the $\mathcal{O}_T$-module structure on $w^{-1}\lambda^k\mathcal{O}_D$.

We next analyse $w^{-1}\mathcal{O}_{D'}$. This is a quotient of the ring

$$w^{-1}\mathcal{O}_T \otimes \mathcal{O}_D = F(N_k^+, w^{-1}\mathcal{O}_D) = \prod_{\underline{n} \in N_k^+} w^{-1}R\{e_i \mid i < d\}.$$

It is not hard to check that the relevant ideal is a product of terms $I_{\underline{n}}$, where $I_{\underline{n}}$ is spanned by the elements e_i that do not lie in the list $e_{n_0}, \dots, e_{n_{k-1}}$. Thus

$$\begin{aligned} w^{-1}\mathcal{O}_{D'} &= \prod_{\underline{n}} w^{-1}R\{e_{n_j} \mid j < k\} \\ w^{-1}\lambda_T^k \mathcal{O}_{D'} &= \prod_{\underline{n}} w^{-1}R.e_{n_0} \wedge \dots \wedge e_{n_{k-1}}. \end{aligned}$$

Let $e'_{\underline{n}}$ be the element of this module whose $\underline{n}$'th component is $e_{n_0} \wedge \dots \wedge e_{n_{k-1}}$, and whose other components are zero. Clearly $\{e'_{\underline{n}} \mid \underline{n} \in N_k^+\}$ is a basis for $w^{-1}\lambda_T^k\mathcal{O}_{D'}$ over $w^{-1}R$. As a symmetric power series p corresponds to the function $\underline{n} \mapsto p(y_{\underline{n}})$ and $e'_{\underline{n}}$ is concentrated in the $\underline{n}$'th factor we have $p.e'_{\underline{n}} = p(y_{\underline{n}})e'_{\underline{n}}$. It is also easy to see that $\phi(\mu_k(e_{\underline{n}})) = e'_{\underline{n}}$, and it follows that ϕ is $\mathcal{O}_T$-linear as claimed. □

We next give a formula for ϕ in terms of suitable bases of $\lambda_S^k\mathcal{O}_D$ and $\lambda_{\mathrm{Sub}_r}^k\mathcal{O}_{D'}$. (This could be used to give an alternative proof that ϕ is an isomorphism.)

Proposition 9.9 *Suppose we have an element $x^{\beta_0} \wedge \dots \wedge x^{\beta_{k-1}} \in \lambda_S^k\mathcal{O}_D$, where $0 \le \beta_0 < \dots < \beta_{d-k-1}$. Let $\gamma_0, \dots, \gamma_{k-1}$ be the elements of $\{0, \dots, d-1\} \setminus \{\beta_0, \dots, \beta_{d-k-1}\}$, listed in increasing order. Then*

$$\phi(x^{\beta_0} \wedge \dots \wedge x^{\beta_{k-1}}) = \pm x^0 \wedge \dots \wedge x^{k-1} . \det(c_{k+i-\gamma_j})_{0 \le i,j < d-k},$$

where the elements c_i are the usual parameters of the divisor D'.

Proof For any increasing sequence $\alpha_0 < \dots < \alpha_{n-1}$ we write $x(\alpha) = x^{\alpha_0} \wedge \dots \wedge x^{\alpha_{n-1}}$. We also write $e' = x(0, 1, \dots, k-1)$ and $e = x(0, 1, \dots, d-1)$, and we put $T = \mathrm{Sub}_k(D)$.

We certainly have $\phi(x(\beta)) = b_\beta e'$ for some $b_\beta \in \mathcal{O}_T$. To analyse these elements, put $J' = \ker(\mathcal{O}_T \otimes \mathcal{O}_D \to \mathcal{O}_{D'}$, which is freely generated over $\mathcal{O}_T$ by $\{x^i f_{D'}(x) \mid i < d-k\}$. Consider the element

$$a = f_{D'} \wedge x f_{D'} \wedge \dots x^{d-k-1} f_{D'} \in \lambda^{d-k} J' \subset \mathcal{O}_T \otimes \lambda^{d-k}\mathcal{O}_D.$$

This clearly annihilates $J' \subset \mathcal{O}_T \otimes \lambda^1\mathcal{O}_D$, so multiplication by a induces a map $\lambda^k\mathcal{O}_{D'} \to \mathcal{O}_T \otimes \lambda^d\mathcal{O}_D$. As $f_{D'}$ is monic of degree k, we see that $e'a = e$. It follows that $x(\beta)a = b_\beta e' a = b_\beta e$.

On the other hand, we can expand a in the form $a = \sum_\gamma a_\gamma x(\gamma)$, where γ runs over sequences $0 \leq \gamma_0 < \ldots < \gamma_{d-k-1} < d$. We have $x(\beta)x(\gamma) = \pm e$ if β and γ are related as in the statement of the proposition, and $x(\beta)x(\gamma) = 0$ otherwise. It follows that $x(\beta)a = \pm a_\gamma e$, and thus that $b_\beta = \pm a_\gamma$.

Let A_γ be the matrix whose (i,j)'th entry is the coefficient of x^{γ_j} in $x^i f_{D'}(x)$; it is then clear that $a_\gamma = \det(A_\gamma)$. On the other hand, we have $x^i f_{D'}(x) = \sum_m c_m x^{k+i-m}$, so $(A_\gamma)_{ij} = c_{k+i-\gamma_j}$, and the proposition follows. □

10 Thom spectra of adjoint bundles

The following proposition is an immediate consequence of Theorem 4.6 and its proof (the first statement is just the case $k = d$ of the second statement).

Proposition 10.1 *Let V be a d-dimensional bundle over a space X. Then there are natural isomorphisms*

$$\begin{aligned} \widetilde{E}^0 \Sigma^{-d} X^{\mathrm{u}(V)} &= \lambda^d_{E^0X} E^0 PV \\ \widetilde{E}^0 \Sigma^{-k} G_k V^{\mathrm{u}} &= \lambda^k_{E^0X} E^0 PV \ \textit{for } 0 \leq k \leq d. \quad \square \end{aligned}$$

Remark 10.2 Note that the proposition gives two different descriptions of the module $\widetilde{E}^0 \Sigma^{-k} G_k V^{\mathrm{u}}$: the first statement with X replaced by $G_k(V)$ and V by T gives

$$\widetilde{E}^0 \Sigma^{-k} G_k V^{\mathrm{u}} = \lambda^k_{E^0 G_k V} E^0 PT,$$

whereas the second statement gives

$$\widetilde{E}^0 \Sigma^{-k} G_k(V)^{\mathrm{u}} = \lambda^k_{E^0X} E^0 PV.$$

We leave it to the reader to check that these two descriptions are related by the isomorphism $\phi\colon\ \lambda^k_S \mathcal{O}_D \to \lambda^k_{\mathrm{Sub}_k(D)} \mathcal{O}_{D'}$ of Proposition 9.8.

In the present section we examine the isomorphisms of Proposition 10.1 more carefully. We will construct a diagram as follows, whose effect in cohomology will be identified with the diagram in Proposition 9.5.

$$\begin{tikzcd} \Sigma^k P_k V_+ \arrow[rr, "s"] \arrow[dd, "p"'] \arrow[dr, "q'"] & & \Sigma^k P_k V_+ \arrow[dd, "p"] \\ & G_k(V)^{\mathrm{u}} \arrow[ur, "r'"] \arrow[dr, "r"] & \\ \Sigma^k P V^k_+ \arrow[ur, "q"] \arrow[rr, "s'"'] & & \Sigma^k P V^k_+ \end{tikzcd}$$

Here PV^k means the fibre product

$$PV^k = PV \times_X \ldots \times_X PV = \{(x, L_0, \ldots, L_{k-1}) \mid x \in X \ , \ L_0, \ldots, L_{k-1} \in PV_x\}.$$

Write $Q_k U(V) = F_k U(V)/F_{k-1}U(V)$, so that $Q_k U(V) \simeq G_k V^{\mathfrak{u}}$. As the filtration of $U(V)$ is multiplicative, the multiplication $U(V)^k \to U(V)$ induces a map $(Q_1 U(V))^{(k)} \to Q_k U(V)$, or equivalently $\Sigma^k PV_+ \to G_k(V)^{\mathfrak{u}}$. This is the map q in the diagram.

Recall that $P_k V$ is the set of points $(x, L_0, \ldots, L_{k-1}) \in PV^k$ such that the lines L_i are mutually orthogonal. The map $p\colon P_k V \to PV^k$ is just the inclusion. We also have a map $P_k V \to G_k V$ sending $(x, \underline{L})$ to $(x, \bigoplus_i L_i)$, and we note that $\mathfrak{u}(\bigoplus_i L_i)$ contains $\bigoplus_i \mathfrak{u}(L_i)$. Moreover, when L is one-dimensional there is a canonical isomorphism $\mathfrak{u}(L) \simeq i\mathbb{R} \simeq \mathbb{R}$, so $\bigoplus_i \mathfrak{u}(L_i) \simeq \mathbb{R}^k$, so we get an inclusion $\Sigma^k P_k V_+ \to G_k V^{\mathfrak{u}}$, which we call q'. It is not hard to see that this is the same as qp, so the left hand triangle commutes on the nose.

We next define the map $r'\colon G_k V^{\mathfrak{u}} \to \Sigma^k P_k V_+$ by a Pontrjagin-Thom construction. Let $N_0' \subset \mathbb{R}^k$ be the set of sequences $(t_0, \ldots, t_{k-1})$ such that $t_0 < \ldots < t_{k-1}$, and let N' be the space of triples (x, W, α) where $W \in G_k V_x$ and $\alpha \in \mathfrak{u}(W)$ and α has k distinct eigenvalues. This is easily seen to be an open subspace of the total space of the bundle $\mathfrak{u}$ over $G_k V$. Given such a triple, we note that the eigenvalues of α are purely imaginary, so we can write them as $it_0, \ldots, it_{k-1}$ with $t_0 < \ldots < t_{k-1}$. We also put $L_j = \ker(\alpha - it_j)$, so the spaces L_j are one-dimensional and mutually orthogonal, and their direct sum is W. Using this we see that the map $qp\colon \Sigma^k P_k V_+ \to G_k V^{\mathfrak{u}}$ induces a homeomorphism $N_0' \times P_k V \to N'$, and this gives a collapse map

$$G_k V^{\mathfrak{u}} \to N' \cup \{\infty\} \simeq (N_0' \times P_k V) \cup \{\infty\} \simeq (N_0' \cup \{\infty\}) \wedge P_k V_+.$$

On the other hand, the inclusion $N_0' \to \mathbb{R}^k$ gives a collapse map $S^k \to N_0' \cup \{\infty\}$ which is a homotopy equivalence; after composing with the inverse of this, we obtain a map $G_k V^{\mathfrak{u}} \to \Sigma^k P_k V_+$, which we denote by r'.

We now define a map $r\colon G_k V^{\mathfrak{u}} \to PV^k_+$. We first mimic Lemma 4.5 and define maps $m_j\colon U(1) \to U(1)$ (for $0 \leq j < k$) by

$$m_j(e^{i\theta}) = \begin{cases} e^{ik\theta} & \text{if } j/k \leq \theta/2\pi \leq (j+1)/k \\ 1 & \text{otherwise} \end{cases}$$

(where θ is assumed to be in the interval $[0, 2\pi]$). We then define $\delta_k'\colon U(V) \to U(V)^k$ by $\delta_k'(g) = (m_0(g), \ldots, m_{k-1}(g))$. This is homotopic to the diagonal and preserves filtrations so it induces a map $G_k V^{\mathfrak{u}} = Q_k U(V) \to Q_k(U(V)^k)$. The target of this map is the wedge of all the spaces $Q_{l_0} U(V) \wedge \ldots \wedge Q_{l_{k-1}} U(V)$

for which $\sum_i l_i = k$. We can thus project down to the factor $Q_1U(V) \wedge \ldots \wedge Q_1U(V) = \Sigma^k PV_+^k$ to get a map $G_kV^{\mathrm{u}} \to \Sigma^k PV_+^k$, which we call r.

It is not hard to recover the following more explicit description of r. Recall that we have a homeomorphism

$$\gamma\colon U(1) \to \mathbb{R} \cup \{\infty\}$$

given by $\gamma(z) = (z+1)(z-1)^{-1}/i$ and $\gamma^{-1}(t) = (it+1)/(it-1)$. One checks that $\gamma(e^{i\theta}) = -\cot(\theta/2)$, which is a strictly increasing function of θ for $0 < \theta < 2\pi$. Let A_j denote the arc $\{e^{i\theta} \mid j/k < \theta/2\pi < (j+1)/k\}$, so γA_j is the interval $(-\cot(\pi j/k), -\cot(\pi(j+1)/k))$. We also define $\overline{m}_j = \gamma m_j \gamma^{-1}$, which can be regarded as a homeomorphism $\gamma A_j \cup \{\infty\} \to \mathbb{R} \cup \{\infty\}$, homotopy inverse to the evident collapse map in the opposite direction. If we put $N_0 = \prod_j \gamma A_j \subset N_0' \subset \mathbb{R}^k$ then the maps $\overline{m}_j$ combine to give a homeomorphism $\overline{m}\colon N_0 \cup \{\infty\} \to \mathbb{R}^k \cup \{\infty\}$, which is again homotopy inverse to the evident collapse map in the opposite direction. Now let $N \subset N'$ be the space of triples (x, W, α) such that α/i has precisely one eigenvalue in γA_j for each j. If $(x, W, \alpha) \in N$ and t_j is the eigenvalue in γA_j and $L_j = \ker(\alpha - it_j)$ then we find that $\underline{t} \in N_0$ and $\underline{L} \in P_kV_x$ and $r(x, W, \alpha) = (\overline{m}(\underline{t}), x, \underline{L})$. On the other hand, if $(x, W, \alpha) \notin N$ we find that $r(x, W, \alpha) = \infty$. It follows that r is constructed in the same way as r', except that N' and N_0' are replaced by the smaller sets N and N_0. The projections $N' \cup \{\infty\} \to N \cup \{\infty\}$ and $N_0' \cup \{\infty\} \to N_0 \cup \{\infty\}$ are homotopy equivalences, and it follows that r is homotopic to pr'. This shows that the right hand triangle in our diagram commutes up to homotopy.

We now consider the composite $s = r'q'\colon \Sigma^k P_kV_+ \to \Sigma^k P_kV_+$, which is essentially obtained by collapsing out the complement of $(q')^{-1}(N')$. There is an evident action of the symmetric group Σ_k on the space $\Sigma^k P_kV_+$, given by

$$\sigma.(\underline{t}, \underline{L}) = (t_{\sigma^{-1}(0)}, \ldots, t_{\sigma^{-1}(k-1)}, L_{\sigma^{-1}(0)}, \ldots, L_{\sigma^{-1}(k-1)}).$$

One checks that $(q')^{-1}(N') = \coprod_\sigma \sigma.(N_0' \times P_kV)$, and using this one can see that s is just the trace map $\mathrm{tr}_{\Sigma_k} = \sum_{\sigma \in \Sigma_k} \sigma$.

Finally, we define

$$s' = rq\colon \Sigma^k PV_+^k \to \Sigma^k PV_+^k.$$

We can also define $\mathrm{tr}_{\Sigma_k}\colon \Sigma^k PV_+^k \to \Sigma^k PV_+^k$; we suspect that this is *not* the same as s', although we will see shortly that it induces the same map in cohomology.

We now apply the functor $\widetilde{E}^k(-) = \widetilde{E}^0\Sigma^{-k}(-)$ to our diagram of spaces and write $D = D(V)$ to get the following diagram:

$$\begin{tikzcd}
\mathcal{O}_{P_kD} & & \mathcal{O}_{P_kD} \arrow[ll, "s^*"'] \arrow[dl, "(r')^*"] \\
& \lambda^k\mathcal{O}_D \arrow[ul, "(q')^*"'] \arrow[dl, tail, "q^*"] & \\
\mathcal{O}_D^{\otimes k} \arrow[uu, two heads, "p^*"] & & \mathcal{O}_D^{\otimes k} \arrow[ll, "(s')^*"] \arrow[ul, "r^*"'] \arrow[uu, two heads, "p^*"']
\end{tikzcd}$$

The map $p^*\colon \mathcal{O}_D^{\otimes k} = \mathcal{O}_{D^k} \to \mathcal{O}_{P_kD}$ is the same as considered previously; this is the definition of our identification of $(P_kV)_E$ with P_kD. It follows from the Hopf algebra isomorphism of Theorem 4.4 that $r^* = \mu_k$ and $q^* = \psi_k$, and thus that $(s')^* = \psi_k\mu_k = \mathrm{alt}_k$. As μ_k factors uniquely through p^* we must have $(r')^* = \mu'_k$. As $q' = pq$ and $\psi'_k = p^*\psi_k$ we have $(q')^* = \psi'_k$. Finally, we know that $s = \mathrm{tr}_{\Sigma_k}$ and any permutation $\sigma \in \Sigma_k$ acts on the sphere S^k with degree equal to its signature so it follows that $s^* = \mathrm{alt}_k\colon E^0P_kV \to E^0P_kV$.

11 Fibrewise loop groups

We conclude the main part of this paper by studying the fibrewise loop space $\Omega_XU(V)$ and thereby providing a topological realisation of the diagram in Proposition 9.6.

First, the group structure on $U(V)$ gives a group structure on $\Omega_XU(V)$. We also have $\Omega_XU(V) \simeq \Omega_X^2BU(V)$, and there is a canonical homotopy showing that a double loop space is homotopy-commutative, so the proof goes through to show that $\Omega_XU(V)$ is fibrewise homotopy commutative.

We next recall certain subspaces of $\Omega_XU(V)$ which have been considered by a number of previous authors — we will mostly refer to Crabb's exposition [2], which cites on Mitchell's paper [7] and (apparently unpublished) work of Mahowald and Richter.

Let V be a vector space. Any finite Laurent series $f(z) = \sum_{i=-N}^{N} a_iz^i$ with coefficients $a_i \in \mathrm{End}(V)$ can be regarded as a map $U(1) \to \mathrm{End}(V)$ which we can compose with the standard homeomorphism $\gamma^{-1}\colon S^1 \to U(1)$ to get a map $\hat{f}\colon S^1 \to \mathrm{End}(V)$. We write $\Omega^{\mathrm{lau}}U(V)$ for the space of based loops

$u\colon S^1 \to U(V)$ that have the form $u = \hat{f}$ for some finite Laurent series f, and call this the space of Laurent loops. Similarly, we write $\Omega^{\mathrm{pol}}U(V)$ for the space of loops that have the form $\hat{f}$ for some polynomial f.

If $u = \hat{f}$ is a Laurent loop we have $f(z)^{-1} = f(z)^* = \sum_i a_i^* z^{-i}$ which is again a finite Laurent series. Using this we see that that $\Omega^{\mathrm{lau}}U(V)$ is a subgroup of $\Omega U(V)$ (but $\Omega^{\mathrm{pol}}U(V)$ is merely a submonoid). We also find that the function $d(z) = \det(f(z))$ is a finite Laurent series in $\mathbb{C}[z, z^{-1}]$ satisfying $d(z)\overline{d}(\overline{z}) = 1$ and $d(1) = 1$; it follows easily that $d(z) = z^n$ for some integer n, called the degree of u.

Definition 11.1 (a) We write S_kV for the space of polynomial loops of degree k on $U(V)$.

(b) The product structure on $\Omega U(V)$ induces maps $S_kV \times S_lV \to S_{k+l}V$, which we call μ_{kl}.

(c) Given $W \in G_kV$ and $z \in U(1)$ we have a polynomial $z\pi_W + (1-\pi_W) \in \mathrm{End}(V)[z]$ giving rise to a based loop in $U(V)$ which we call $\phi_k(W)$. This defines a map $\phi_k\colon G_kV \to S_kV$. It is not hard to show that $\phi_1\colon PV \to S_1V$ is a homeomorphism.

(d) By combining ϕ_1 with the product map we get a map $\nu_k\colon PV_X^k \to S_kV$.

If V is a bundle rather than a vector space, we make all these definitions fibrewise in the obvious way.

Note that ν_k induces a map $E^*S_kV \to (E^*PV)^{\otimes k}$, where the tensor product is taken over E^*X. We write $\mathrm{Sym}^k(E^*PV)$ for the submodule invariant under the action of Σ_k.

Proposition 11.2 *ν_k induces an isomorphism $E^*S_kV = \mathrm{Sym}^k(E^*PV)$, and thus an isomorphism $D(V)^k/\Sigma_k \to (S_kV)_E$.*

Proof Put $d = \dim(V)$ and let A be the set of lists $\alpha = (\alpha_i \mid i < k)$ with $0 \le \alpha_i < d$. We have

$$E^*PV_X^k = (E^*PV)^{\otimes k} = E^*[\![x_i \mid i < k]\!]/(f_V(x_i) \mid i < k),$$

and the set $\{x^\alpha \mid \alpha \in A\}$ is a basis for this ring over E^*X. Put $A_+ = \{\alpha \in A \mid \alpha_0 \le \ldots \le \alpha_{k-1}\}$ and $M_+ = \bigoplus_{\alpha \in A_+} E^*X$, and let $\pi\colon E^*PV_X^k \to M_+$ be the obvious projection. This clearly induces an isomorphism $\mathrm{Sym}^k E^*PV \to M_+$.

We take as our basic input (proved by Mitchell [7]) the fact that when X is a point, the map ν_k induces an isomorphism $(H_*PV)^{\otimes k}_{\Sigma_k} \to H_*S_kV$. In particular, this means that H_*S_kV is a finitely generated free Abelian group, concentrated in even degrees. By duality we see that $H^*S_kV = \mathrm{Sym}^k H^*PV$, and thus that the map $\pi\nu_k^*\colon H^*S_kV \to \bigoplus_{A_+} H^*X$ is an isomorphism. Using an Atiyah-Hirzebruch spectral sequence we see that $\pi\nu_k^*\colon E^*S_kV \to \bigoplus_{A_+} E^*X$ is an isomorphism for any E.

Now let X be arbitrary. If V is trivialisable with fibre V_0 then $S_kV = X \times S_kV_0$ and it follows from the above that $\pi\nu_k^*$ is an isomorphism. If V is not trivialisable, we can still give X a cell structure such that the restriction to any closed cell is trivialisable, and then use Mayer-Vietoris sequences, the five lemma, and the Milnor sequence to see that $\pi\nu_k^*$ is an isomorphism.

We next claim that the maps

$$\mu_{kl}^*\colon E^*S_{k+l}V \to E^*S_kV \otimes_{E^*X} E^*S_lV$$

give rise to a cocommutative coproduct. To see this, let $C(V)$ denote the following diagram:

$$\begin{tikzcd} S_kV \times S_lV \arrow[r, "\mu_{kl}"] \arrow[d, "\text{twist}"'] & S_{k+l}V \arrow[d, "1"] \\ S_lV \times S_kV \arrow[r, "\mu_{lk}"'] & S_{k+l}V. \end{tikzcd}$$

The claim is that the diagram $E^*C(V)$ commutes. Let $i_0, i_1\colon V \to V^2$ be the two inclusions. The map i_0 induces a map $E^*C(V^2) \to E^*C(V)$, and it follows easily from our previous discussion that this is surjective. It will thus be enough to show that the two ways round $E^*C(V^2)$ become the same when composed with the map

$$(S_k(i_0) \times S_l(i_0))^*\colon E^*(S_k(V^2) \times_X S_l(V^2)) \to E^*(S_kV \times_X S_lV).$$

It is standard that i_0 is homotopic to i_1 through linear isometries, so $S_l(i_0)$ is fibre-homotopic to $S_l(i_1)$. Similarly, the identity map of $S_{k+l}(V^2)$ is homotopic to $S_{k+l}(\text{twist})$. It is thus enough to check that the two composites $S_kV \times S_lV \to S_{k+l}(V^2)$ in the following diagram are the same:

$$\begin{tikzcd} S_k(V) \times S_l(V) \arrow[r, "S_k(i_0)\times S_k(i_1)"] & S_k(V^2) \times S_l(V^2) \arrow[r, "\mu_{kl}"] \arrow[d, "\text{twist}"'] & S_{k+l}(V^2) \arrow[d, "S_{k+l}(\text{twist})"] \\ & S_l(V^2) \times S_k(V^2) \arrow[r, "\mu_{lk}"'] & S_{k+l}(V^2). \end{tikzcd}$$

This is easy to see directly.

We now see that the map

$$\nu_k^*\colon E^*S_kV \to (E^*PV)^{\otimes k}$$

factors through $\mathrm{Sym}^k(E^*PV)$. As the map $\pi\colon \mathrm{Sym}^k(E^*PV) \to M_+$ and its composite with ν_k^* are both isomorphisms, we deduce that $\nu_k^*\colon E^*S_kV \to \mathrm{Sym}^k(E^*PV)$ is an isomorphism as claimed. □

Corollary 11.3 *The formal scheme $(\Omega_X U(V))_E$ is the free commutative formal group over X_E generated by the divisor DV.*

Proof We refer to [10, Section 6.2] for background on free commutative formal groups; the results there mostly state that the obvious methods for constructing such objects work as expected under some mild hypotheses. Given a formal scheme T over a formal scheme S, we use the following notation:

(a) M^+T is the free commutative monoid over S generated by T. This is characterised by the fact that monoid homomorphisms from M^+T to any monoid H over S biject with maps $T \to H$ of schemes over S. It is clear that if there exists an M^+T with this property, then it is unique up to canonical isomorphism. Similar remarks apply to our other definitions. In reasonable cases we can construct the colimit $\coprod_k T_S^k/\Sigma_k$ and this works as M^+T; see [10, Proposition 6.8] for technicalities.

(b) MT is the free commutative group over S generated by T.

(c) If T has a specified section $z\colon S \to T$, then N^+T is the free commutative monoid scheme generated by the based scheme T, so homomorphisms from N^+T to H biject with maps $T \to H$ such that the composite $S \xrightarrow{z} T \to H$ is zero. In reasonable cases N^+T can be constructed as $\varinjlim_k T_S^k/\Sigma_k$.

(d) If T has a specified section we also write NT for the free commutative group over S generated by the based scheme T.

The one surprise in the theory is that often $NT = N^+T$; this is analogous to the fact that a graded connected Hopf algebra automatically has an antipode. It is easy to check that $MT = \mathbb{Z} \times NT$, where $\mathbb{Z}$ is regarded as a discrete group scheme in an obvious way.

We first suppose that V has a one-dimensional summand, so $V = L \oplus W$ for some bundles L and W with $\dim(L) = 1$. Note that for each $x \in X$ there is a canonical isomorphism $\mathbb{C} \to \mathrm{End}(L)$ giving $U(1) \times X \simeq U(L)$. This gives an evident inclusion $\alpha\colon U(1)\times X\to U(V)$ with $\det\circ\alpha=1$. We define $\beta\colon U(V)\to SU(V)$ by $\beta(g)=\alpha(\det(g))^{-1}g$, and note that $\beta(\alpha(z)g)=\beta(g)$ for all z.

We also have an evident map $z\colon X\simeq PL\to PV\simeq S_1V$ splitting the projection $PV\to X$. Left multiplication by z gives a map $i_k\colon S_kV\to S_{k+1}V$. We also define $i_k\colon S_kV\to\Omega_XSU(V)$ to be the restriction of $\Omega_X\beta$ to $S_kV\subset\Omega_XU(V)$. Using the fact that $\beta(\alpha(z)g)=\beta(g)$ we see that $j_{k+1}i_k=j_k$. Thus, if we define $S_\infty V$ to be the homotopy colimit of the spaces S_kV, we get a map $j_\infty\colon S_\infty V\to\Omega_XSU(V)$ of spaces over X. Using the usual bases for $E^*S_kV=\mathrm{Sym}^k(E^*PV)$ we find that the maps $i_k^*\colon\ \mathrm{Sym}^{k+1}(E^*PV)\to\mathrm{Sym}^k(E^*PV)$ are surjective. It follows using the Milnor sequence that $E^*S_\infty V=\varprojlim_k\mathrm{Sym}^k(E^*PV)$ and thus that $(S_\infty V)_E=\varinjlim_k DV^k/\Sigma_k$. We claim that this is the same as N^+DV; this is clear modulo some categorical technicalities, which are covered in [10, Section 6.2]. In the case where X is a point, it is well-known and easy to check (by calculation in ordinary homology) that the map $S_\infty V\to\Omega_XSU(V)$ is a weak equivalence. In the general case we have a map between fibre bundles that is a weak equivalence on each fibre; it follows easily that the map is itself a weak equivalence, and thus that $\Omega_XSU(V)_E=N^+DV$. On the other hand, as $\Omega_XSU(V)$ is actually a group bundle, we see that $\Omega_XSU(V)_E$ is a formal group scheme, so $N^+DV=NDV$.

We now turn to the groups $\Omega_XU(V)$. We define $\mathbb{Z}_X=\mathbb{Z}\times X$, viewed as a bundle of groups over X in the obvious way. This can be identified with $\Omega_X(U(1)\times X)$ so the determinant map gives rise to a homomorphism $\delta\colon\Omega_XU(V)\to\mathbb{Z}_X$. Given $(n,x)\in\mathbb{Z}_X$ we have a homomorphism $U(1)\to U(V_x)$ given by $z\mapsto\alpha(z^n)$. This construction gives us a map $\sigma\colon\mathbb{Z}_X\to\Omega_XU(V)$ with $\delta\sigma=1$ and thus a splitting $\Omega_XU(V)\simeq\mathbb{Z}\times\Omega_XSU(V)$ and thus an isomorphism $\Omega_XU(V)_E=\mathbb{Z}\times NDV=MDV$. One can check that the various uses of the map α cancel out and that the standard inclusion $DV\to MDV$ is implicitly identified with the map coming from the inclusion $PV=S_1V\to\Omega_XU(V)$. This proves the corollary in the case where V has a one-dimensional summand.

Now suppose that V does not have such a summand. We have an evident coequaliser diagram $PV\times_XPV\rightrightarrows PV\to X$, giving rise to a coequaliser diagram $DV\times_{X_E}DV\to DV\to X_E$ of schemes over X_E, in which the map $DV\to X_E$ is faithfully flat. The pullback of V to PV has a tautological one-dimensional summand, which implies that $(PV\times_X\Omega_XU(V))_E$ has the required universal property in the category of formal group schemes over PV_E. Similar remarks apply to $PV\times_XPV\times_X\Omega_XU(V)$. It follows by a descent argument that $\Omega_XU(V)$ itself has the required universal property, as one sees easily from [10, Proposition 2.76 and Remark 4.52]. □

We next recall the standard line bundle over S_kV, which we will call T; see [2, 7] for more details. Write $A = \mathbb{C}[z]$ and $K = \mathbb{C}[z, z^{-1}]$. A point of S_kV has the form $(x, \hat{f})$ for some $f \in \mathrm{End}_{\mathbb{C}}(V_x)[z] \simeq \mathrm{End}_A(A \otimes V_x)$. Multiplication by f defines a surjective endomorphism $m(f)$ of $(K/A) \otimes V$, and we define $T_{(x,f)}$ to be the kernel of this endomorphism. One can check that this always has dimension k over $\mathbb{C}$ and that we get a vector bundle. This is classified by a map $\tau_k \colon S_kV \to BU(k) \times X$ of spaces over X. It is easy to see that the restriction of T to $G_kV \subset S_kV$ is just the tautological bundle.

There are evident short exact sequences

$$\ker(m(g)) \rightarrowtail \ker(m(fg)) \xrightarrow{m(g)} \ker(m(f)),$$

which can be split using the inner products to give isomorphisms $\mu_{kl}^*T \simeq \pi_0^*T \oplus \pi_1^*T$ over $S_kV \times_X S_lV$. This means that the map $\tau \colon \coprod_k S_kV \to (\coprod_k BU(k)) \times X$ is a homomorphism of H-spaces over X.

We now have a diagram of spaces as follows:

$$\begin{tikzcd}
P_kV \arrow[r, "p"] \arrow[d, "q'"'] & PV_X^k \arrow[r, "\tau_1^k"] \arrow[d, "\nu_k"'] & (\mathbb{C}P^\infty)^k \times X \arrow[d] \\
G_kV \arrow[r, "\phi_k"'] & S_kV \arrow[r, "\tau_k"'] & BU(k) \times X.
\end{tikzcd}$$

It is easy to identify the corresponding diagram of schemes with the diagram of Proposition 9.6.

A Appendix : Functional calculus

In this appendix we briefly recall some basic facts about functional calculus for normal operators. An endomorphism α of a vector space V is *normal* if it commutes with its adjoint. For us the relevant examples are Hermitian operators (with $\alpha^* = \alpha$), anti-Hermitian operators (with $\alpha^* = -\alpha$) and unitary operators (with $\alpha^* = \alpha^{-1}$).

For any operator α and any $\lambda \in \mathbb{C}$ we have

$$\ker(\alpha - \lambda)^\perp = \mathrm{image}((\alpha - \lambda)^*) = \mathrm{image}(\alpha^* - \overline{\lambda}).$$

If α is normal we deduce that $\ker(\alpha-\lambda)^{\perp}$ is preserved by α, and it follows easily that V is the orthogonal direct sum of the eigenspaces of α. It follows in turn that the operator norm of α (defined by $\|\alpha\|=\sup\{\|\alpha(v)\| : \|v\|=1\}$) is just the same as the spectral radius (defined as the maximum absolute value of the eigenvalues of α).

Now let X be a subset of $\mathbb{C}$ containing the eigenvalues of α, and let $f\colon X\to\mathbb{C}$ be a continuous function. We define $f(\alpha)$ to be the endomorphism of V that has eigenvalue $f(\lambda)$ on the space $\ker(\alpha-\lambda)$. From this definition it is clear that the following equations are valid whenever they make sense:

$$\begin{aligned}
c(\alpha) &= c.1_V \text{ if } c \text{ is constant}\\
\mathrm{id}(\alpha) &= \alpha\\
\mathrm{Re}(\alpha) &= (\alpha+\alpha^*)/2\\
\mathrm{Im}(\alpha) &= (\alpha-\alpha^*)/(2i)\\
(f+g)(\alpha) &= f(\alpha)+g(\alpha)\\
(fg)(\alpha) &= f(\alpha)g(\alpha)\\
\overline{f}(\alpha) &= f(\alpha)^*\\
(f\circ g)(\alpha) &= f(g(\alpha))\\
\|f(\alpha)\| &\leq \sup_{x\in X}|f(x)|.
\end{aligned}$$

The continuity properties of $f(\alpha)$ are less clear from our definition. However, they are provided by the following result.

Proposition A.1 *Let X be a closed subset of $\mathbb{C}$, and V a vector space. Let $N(X,V)$ be the set of normal operators on V whose eigenvalues lie in X, and let $C(X,\mathbb{C})$ be the set of continuous functions from X to $\mathbb{C}$ (with the topology of uniform convergence on compact sets). Define a function $E\colon C(X,\mathbb{C})\times N(X,V)\to\mathrm{End}(V)$ by $E(f,\alpha)=f(\alpha)$. Then E is continuous.*

Proof Let A be the set of functions $f\in C(X,\mathbb{C})$ for which the function $\alpha\mapsto f(\alpha)$ is continuous. Using the above algebraic properties, we see that A is a subalgebra of $C(X,\mathbb{C})$ containing the functions $z\mapsto\mathrm{Re}(z)$ and $z\mapsto\mathrm{Im}(z)$. By the Stone-Weierstrass theorem, it is dense in $C(X,\mathbb{C})$. Now suppose we have $f\in C(X,\mathbb{C})$, $\alpha\in N(X,V)$ and $\epsilon>0$. Put $Y=\{x\in X \mid |x|\leq\|\alpha\|+1\}$, which is compact. As A is dense we can choose $p\in A$ with $|f-p|<\epsilon/4$ on Y. As $p\in A$ can choose δ such that $\|p(\beta)-p(\alpha)\|<\epsilon/4$ whenever $\|\beta-\alpha\|<\delta$.

We may also assume that $\delta < 1$, which means that when $\|\beta - \alpha\| < \delta$ we have $\beta \in Y$. Now if $|f - g| < \epsilon/4$ on Y and $\|\alpha - \beta\| < \delta$ then

$$\begin{aligned}\|f(\alpha) - g(\beta)\| \leq & \|f(\alpha) - p(\alpha)\| + \|p(\alpha) - p(\beta)\| + \\ & \|p(\beta) - f(\beta)\| + \|f(\beta) - g(\beta)\| \\ < & \epsilon/4 + \epsilon/4 + \epsilon/4 + \epsilon/4 = \epsilon,\end{aligned}$$

as required. □

The following proposition is an elementary exercise in linear algebra.

Proposition A.2 *Let $\alpha\colon V \to W$ be a linear map. Then $\alpha^*\alpha$ and $\alpha\alpha^*$ are self-adjoint endomorphisms of V and W with nonnegative eigenvalues. For each $t > 0$ the map α gives an isomorphism of $\ker(\alpha^*\alpha - t)$ with $\ker(\alpha\alpha^* - t)$, so the nonzero eigenvalues of $\alpha^*\alpha$ and their multiplicities are the same as those of $\alpha\alpha^*$. If $f\colon [0, \infty) \to \mathbb{R}$ then $\alpha \circ f(\alpha^*\alpha) = f(\alpha\alpha^*) \circ \alpha$.* □

Definition A.3 We write $w(V) = \{\alpha \in \mathrm{End}(V) \mid \alpha^* = \alpha\}$ (the space of self-adjoint endomorphisms of V). If $\alpha \in w(V)$ then the eigenvalues of α are real, so we can list them in descending order, repeated according to multiplicity. We write $e_k(\alpha)$ for the k'th element in this list, so $e_1(\alpha) \geq \ldots \geq e_n(\alpha)$ and $\det(t - \alpha) = \prod_k (t - e_k(\alpha))$.

We will need the following standard result:

Proposition A.4 *The functions $e_k\colon w(V) \to \mathbb{R}$ are continuous.*

Proof Let γ be a simple closed curve in $\mathbb{C}$ and let m be an integer. Let U be the set of endomorphisms of V that have precisely m eigenvalues (counted according to multiplicity) inside γ, and no eigenvalues on γ. A standard argument with Rouché's theorem shows that U is open in $\mathrm{End}(V)$.

Given real numbers $r \leq R$, consider the rectangular contour $\gamma_{r,R}$ with corners at $r \pm i$ and $R \pm i$. Clearly $e_k(\alpha) > r$ iff α has at least k eigenvalues inside $\gamma_{r,R}$ for some R. It follows that $\{\alpha \mid e_k(\alpha) > r\}$ is open, as is $\{\alpha \mid e_k(\alpha) < r\}$ by a similar argument. This implies that e_k is continuous. □

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